Projective Compactification of $\mathbb{R}^{1,1}$ and its Möbius Geometry

John A. Emanuello and Craig A. Nolder

February 1, 2014

Abstract

We examine the semi-Riemannian manifold $\mathbb{R}^{1,1}$, which is realized as the split complex plane, and its conformal compactifcation as an analogue of the complex plane and the Riemann sphere. We also consider conformal maps on the compactifcation and study some of their basic properties.

1 Introduction

There are many advantages of using the Riemann sphere instead of the complex plane as a domain for functions of a complex variable. As a compact space, it possesses some desirable topological properties. In terms of the analysis, one finds that many nice results are true on the sphere which are not true in the plane. For example the class of meromorphoric functions on the sphere are merely the rational functions, while in the plane the meormorphic functions are a larger class of functions. Also, the sphere has a rich conformal geometry.

In this work, we consider the split complex numbers $\mathbb{R}^{1,1}$, which is just the semi-Riemannian manifold $\mathbb{R}^2$ with the semi-Riemannian (or indefinite) metric

$$g\left(X,Y\right) = X_1Y_1 - X_2Y_2.$$

We put an algebraic structure on $\mathbb{R}^{1,1}$ by identifying it with the Clifford algebra $C\ell_{1,0}$. With this additional structure, we are able to define differential operators and develop a holomorphic function theory parallel to that of the complex plane. A natural question rises: *Can we find a better domain for functions of a split complex variable, like we do in complex analysis?*

This is a question which has been addressed over many years and in many places in the literature, although not always directly, but always in the affirmative [1–3]. M. Schottenloher linked the previous question with questions of conformality [4]. In particular, Schottenloher describes a model for compactifying $\mathbb{R}^{p,q}$ with an induced conformal structure, building on the work of Dirac, who used an analogous model for the compactification of four-dimensional spacetime several decades earlier.

Schottenloher also discusses the notion of conformal group on $\mathbb{R}^{1,1}$ and identifies it as a product of circle diffeomorphisms. It is not clear if the work extends these to the compactification. The authors have been unable to find a work which clearly identifies the collection of linear fractional transformations as a subgroup of the conformal group, even though this is the case. This work does make this clear and develops a Möbius transformation theory for $\mathbb{R}^{1,1}$.

Yaglom also defined a cross ratio on $\mathbb{R}^{1,1}$ in [2], which seems to have been largely overlooked or forgotten until [5]. This cross ratio has been previously shown to be well behaved with respect to the Möbius transformations, but this also seems to have been lost.

The purpose of this work is threefold. First, we wish to carefully construct the compactification of $\mathbb{R}^{1,1}$ in a conformal setting. Second, we use holomorphic and conformal conditions in $\mathbb{R}^{1,1}$ to extend these notions to the compactification in a simple way. Last, we use algebraic properties of the split complex numbers to show that the Möbius transformations are a direct product of real Möbius transformations and to discuss their conformality, transitivity, and fixed points carefully (something which has not been found in a rather thorough review of the literature).

Our paper is organized as follows. In Section 2, we give a quick overview of the split complex numbers and the analysis of functions of a split complex variable. We also introduce a notion of conformal mappings over $\mathbb{R}^{1,1}$ and briefly describe some involutions of the torus in $\mathbb{R}^4$. Throughout Section 3, we examine the compactification in some detail, which we denote $Q^{1,1}$, and its geometric properties. We also extend the definition of differentiablity to $Q^{1,1}$, and provide simple criteria for determining conformality. Last, Section 4 serves as an examination of Möbius transformations and their fixed points and transitivity. We also study the cross ratio and show that hyperbolae play a similar role here that circles do in complex analysis.

2 Preliminaries

2.1 Split Complex Numbers $\mathbb{R}^{1,1}$

The algebra of split complex numbers, or double numbers (as in [6]),

$$\mathbb{R}^{1,1} = \left\{ \zeta = x + yj \mid x, y \in \mathbb{R}, j^2 = 1 \right\},$$

is the Clifford algebra $\mathcal{C}\ell_{1,0}$ generated as an algebra over $\mathbb{R}$ by 1 and j. Even though $\mathbb{R}^{1,1}$ resembles $\mathbb{C}$, it has a different algebraic structure. In particular, $\mathbb{R}^{1,1}$ has zero divisors:

$$(1 + j)(1 - j) = 1 - j^2 = 0.$$

However, if we define

$$j_+ = \frac{(1 + j)}{2} \text{ and } j_- = \frac{(1 - j)}{2},$$

we get a useful basis for $\mathbb{R}^{1,1}$:

$$\zeta = uj_+ + vj_-,$$

where $u = x + y, v = x - y$. Notice that j_+ and j_- square to themselves and annihilate each other. Thus, multiplication simplifies in these new coordinates:

$$\begin{aligned}(u_1 j_+ + v_1 j_-)(u_2 j_+ + v_2 j_-) &= u_1 u_2 (j_+)^2 + u_1 v_2 j_+ j_- + v_1 u_2 j_- j_+ + v_1 v_2 (j_-)^2 \\ &= u_1 u_2 j_+ + v_1 v_2 j_-.\end{aligned}$$

This means $\mathbb{R}^{1,1}$ is isomorphic as an algebra to $\mathbb{R} \oplus \mathbb{R}$, which is also is the space of 2×2 diagonal matrices with real entries [7].

It is now clear that the zero divisors in $C\ell_{1,0}$ are precisely those elements which can be written

$$\zeta = \alpha j_+ \text{ or } \zeta = \alpha j_-,$$

for $\alpha \in \mathbb{R}$. These elements are called the light cone and we denote it by L. Clearly, L is the union of two one-dimensional subspaces,

$$L_+ = \{\alpha j_+ : \alpha \in \mathbb{R}\} \text{ and } L_- = \{\alpha j_- : \alpha \in \mathbb{R}\}.$$

When defined, the inverse is given by

$$\zeta^{-1} = u^{-1} j_+ + v^{-1} j_-.$$

It is also be useful to invert only one of the components:

$$\begin{aligned}\zeta_+^{-1} &= u^{-1} j_+ + v j_- \text{ and} \\ \zeta_-^{-1} &= u j_+ + v^{-1} j_-.\end{aligned}$$

We define a conjugation operation for the split complex numbers which resembles what we have in $\mathbb{C}$:

$$C(\zeta) = \bar{\zeta} = x - jy = v j_+ + u j_-.$$

Unlike in $\mathbb{C}$, however, the map

$$\begin{aligned}N(\zeta) &= \zeta\bar{\zeta} \\ &= x^2 - y^2 \\ &= uv.\end{aligned}$$

does not induce a norm in $C\ell_{1,0}$, since the positivity condition fails. However, $N(\zeta)$ is a pseudo-norm.

Remark 2.1. *The element $\zeta \in C\ell_{1,0}$ is also in L if and only if $N(\zeta) = 0$.*

2.2 Analysis of Functions of a Split-Complex Variable

Now, we consider functions of a split complex variable:

$$f : U \subseteq \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}.$$

Such functions have been examined in numerous places [6, 8, 9]. Here U is an open subset of $\mathbb{R}^{1,1}$.

These functions may be written $f(z) = f_1(x, y) + jf_2(x, y)$. Throughout this work, we shall assume that $f_1, f_2 \in C^1(U)$. Now just as in complex analysis, we must understand what it means for the limit of the difference quotient

$$\lim_{\substack{h\to 0\\ h\notin L}} \frac{f(z_0+h)-f(z_0)}{h}$$

to exist.

Now suppose this limit exists and assume that $h = h_x$ is real. Then

$$\begin{aligned}
&\lim_{h_x\to 0} \frac{f(z_0+h_x)-f(z_0)}{h_x}\\
=&\lim_{h_x\to 0} \frac{f_1(x_0+h_x,y_0)-f_1(x_0,y_0)}{h_x} + j\frac{f_2(x_0+h_x,y_0)-f_2(x_0,y_0)}{h_x}\\
=&\frac{\partial f_1}{\partial x}(x_0,y_0) + j\frac{\partial f_2}{\partial x}(x_0,y_0).
\end{aligned}$$

Similarly, if we assume $h = jh_y$ is purely imaginary then

$$\lim_{jh_y\to 0} \frac{f(z_0+jh_y)-f(z_0)}{jh_y} = j\frac{\partial f_1}{\partial y}(x_0,y_0) + \frac{\partial f_2}{\partial x}(x_0,y_0).$$

Remark 2.2. *This gives the Cauchy-Riemann equations in the split complex plane [8]:*

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \quad \textit{and} \quad \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}.$$

Conversely, if these equations are satisfied and the partial derivatives are continuous, then the limit of the difference quotient exists (just as in complex analysis). If either of these conditions are satisfied, then we say that f is $C\ell_{0,1}$-*differentiable* at z_0. If this is the case for all $z_0 \in U$ we say f is $C\ell_{0,1}$-differentiable on U.

Now consider the differential operator

$$\overline{\nabla} = \frac{1}{2}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right)$$

and its conjugate

$$\nabla = \frac{1}{2}\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)$$

which clearly satisfies

$$\nabla\overline{\nabla} = \overline{\nabla}\nabla = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right),$$

which is the wave operator in the plane. Functions which are annihilated by the wave operator are called *hyperharmonic.*

Remark 2.3. *This means that the components of a $C\ell_{0,1}$-differentiable function satisfy the wave equation in the plane [9]. Complex holomorphic functions have components which satisfy Laplace's equation [10].*

Theorem 2.4. *Let f be a split complex valued function of a split complex variable.*

$$\overline{\nabla} f(z_0) = 0 \iff f \text{ is } C\ell_{0,1}\text{-differentiable at } z_0.$$

Proof. Notice that

$$\begin{aligned}\overline{\nabla}(f_1 + jf_2) &= \frac{1}{2}\left[\frac{\partial f_1}{\partial x} + j\frac{\partial f_2}{\partial x} - j\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial y}\right] \\ &= \frac{1}{2}\left[\left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y}\right) + j\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right].\end{aligned}$$

Thus $\overline{\nabla}(f_1 + jf_2) = 0$ if and only if f is $C\ell_{0,1}$-differentiable. □

Definition 2.5. *We say a split complex valued function of a split complex variable f is $C\ell_{0,1}$-antidifferentiable at z_0 if*

$$\nabla f(z_0) = 0$$

Corollary 2.6. *Given a hyperharmonic function $g : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, the function $f = \nabla g$ is $C\ell_{0,1}$-differentiable and $h = \overline{\nabla} g$ is $C\ell_{0,1}$-antidifferentiable.*

If we use our alternative basis for split complex plane, we may rewrite

$$\begin{aligned}\overline{\nabla} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right) \\ &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right) - j\left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\right] \\ &= \frac{1}{2}\left[\frac{\partial}{\partial u} + \frac{\partial}{\partial v} - j\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}\right] \\ &= \frac{\partial}{\partial v}j_+ + \frac{\partial}{\partial u}j_-.\end{aligned}$$

It is easy to see that

$$\nabla = \frac{\partial}{\partial u}j_+ + \frac{\partial}{\partial v}j_-,$$

so that

$$\nabla\overline{\nabla} = \overline{\nabla}\nabla = \frac{\partial^2}{\partial u \partial v},$$

for reasonably behaved $f(u, v)$.

This gives us the following corollary [9].

Corollary 2.7. *The split complex valued function f is $C\ell_{0,1}$-differentiable if and only if*

$$f = f_1(u)j_+ + f_2(v)j_-.$$

Also, f is $C\ell_{0,1}$-antidifferentiable if and only if

$$f = f_1(v)j_+ + f_2(u)j_-.$$

With this new method for checking differentiability, we are able to check that analogues of some holomorphic functions in $\mathbb{C}$ are $C\ell_{0,1}$-differentiable in $\mathbb{R}^{1,1}$.

Example 2.8. *Let $f(\zeta) = \zeta^m$ for a positive integer m. Then f is $C\ell_{0,1}$-differentiable, and hence so are split-complex polynomials.*

Proof. With the simplified multiplication in the coordinates $\zeta = uj_+ + vj_-$, we have that

$$\zeta^m = u^m j_+ + v^m j_-.$$

$\square$

It will be useful to have a notion of meromorphic functions.

Definition 2.9. *Let f be a split complex valued function of a split complex variable. We say f is $C\ell_{0,1}$-meromorphic if*

$$f = f_1(u)j_+ + f_2(v)j_-,$$

and f_1, f_2 are real meromorphic functions. We say f is $C\ell_{0,1}$-antimeromorphic if

$$f = f_1(v)j_+ + f_2(u)j_-,$$

and f_1, f_2 are real meromorphic functions.

2.3 Analogues of Complex Analysis

The notion of $C\ell_{0,1}$-differentiablity yields some analogues of theorems from complex analysis. In particular, we have an analogue of Cauchy's Theorem.

Proposition 2.10. *Let $U \subseteq C\ell_{1,0}$ be open (in the Euclidean topology of $\mathbb{R}^2$). Suppose $S \subseteq U$ is bounded, orientable subdomain with a piecewise differentiable boundary. If f is $C\ell_{1,0}$-differentiable on U , then*

$$\int_{\partial S} f dz = 0,$$

where $dz = dx + jdy$

The proof, which can be found in [9], uses Stoke's Theorem.

There is also an analogue of the Cauchy Integral formula, which is presented in [9]. However, we find that because $C\ell_{1,0}$ is not a field, we do not use a $C\ell_{1,0}$-valued kernel. Rather, we will use a kernel which takes values in $C\ell_{1,0} \otimes \mathbb{C}$.

Lemma 2.11. *Define $K : (C\ell_{1,0})^\times \to (C\ell_{1,0})^\times$ by*

$$K(z) = z^{-1} = \frac{\bar{z}}{N(z)}.$$

Then $\overline{\nabla} K(z) = 0$ for every $z \in (C\ell_{1,0})^\times$.

To obtain the kernel we seek, we simply "complexify" K. Define

$$K_\epsilon(x + jy) = \frac{1+j}{2} \cdot \frac{1}{x + y + i\epsilon \cdot \operatorname{sign}(x - y)} + \frac{1-j}{2} \cdot \frac{1}{x - y + i\epsilon \cdot \operatorname{sign}(x + y)}.$$

Proposition 2.12 (Libine's Integral Formula)**.** *Let $R > 0$ and define $S = \{z \in C\ell_{1,0} : |N(z)| < R\}$. Let U be an open neighborhood of $\overline{S}$. Suppose $f : U \to C\ell_{1,0}$ is smooth and $\overline{\nabla} f = 0$. Then for any $\zeta \in S$,*

$$f(\zeta) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\{|N(z)|=R\}} K_\epsilon(z - \zeta) f(z) dz.$$

The proof is found in [9].

2.4 Conformal Mappings of $\mathbb{R}^{1,1}$

Recall that $\mathbb{R}^{1,1}$ is a semi-Riemannian manifold with indefinite metric $g^{1,1}$ of signature $(1, 1)$:

$$g^{1,1}(X, Y) = X_1 Y_1 - X_2 Y_2.$$

In the literature, a smooth map $f : U \subset X \to X$ on a semi-Riemannian manifold (X, g) of maximal rank (that is, f is a local diffeomorphism) is called (locally) conformal on U if there is a smooth map $\Omega : U \to \mathbb{R}_{>0}$ such that

$$f^* g = \Omega g,$$

where $f^* g(X, Y) = g(df(X), df(Y)$, and df is the tangent map of f [4].

However, we will opt for a global definition of conformal mappings on $\mathbb{R}^{1,1}$. Indeed, we shall require such maps to be globally one-to-one. Also, we will not require C^∞ smoothness; rather, C^1 smooth will be enough. Given that we are using an indefinite metric, we find the condition that Ω be positive to be too restrictive. Moreover, we will only require Ω to be non-zero.

For $\mathbb{R}^{1,1}$, Schottenloher shows that the conformal factor has a simple form [4]:

$$\Omega = \left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_2}{\partial x}\right)^2 = \left(\frac{\partial f_2}{\partial y}\right)^2 - \left(\frac{\partial f_1}{\partial y}\right)^2.$$

Moreover, we get a simple condition to check for conformality.

Theorem 2.13. *A one-to-one C^1 map $f : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$, where $f = f_1(x, y) + j f_2(x, y)$, is (locally) conformal when either*

1. $\overline{\nabla} f = 0$ *and* $\left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_2}{\partial x}\right)^2 \neq 0$*, or*

2. $\nabla f = 0$ *and* $\left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_2}{\partial x}\right)^2 \neq 0$*,*

everywhere on $\mathbb{R}^{1,1}$*.*

If we use the alternative basis, these conditions are

1. $\overline{\nabla} f = 0$ *and* $\left(\frac{\partial g_1}{\partial u}\right)\left(\frac{\partial g_2}{\partial v}\right) \neq 0$*, or*

2. $\nabla f = 0$ *and* $\left(\frac{\partial g_1}{\partial v}\right)\left(\frac{\partial g_2}{\partial u}\right) \neq 0$*,*

where $f = g_1(u,v)j_+ + g_2(u,v)j_-$*.*

Proof. The proof in the original basis for positive conformal factor can be found in [4]. Requiring a non-vanishing conformal factor does not change the proof. Thus, it only remains to show that the first set of conditions (that is those in the original basis) are equivalent to the second set of conditions.

Suppose $\overline{\nabla} f = 0$. Thus,

$$f(\zeta) = f_1(x,y) + f_2(x,y)j = g_1(u)j_+ + g_2(v)j_-,$$

so that

$$f_1 = \frac{g_1(u) + g_2(v)}{2} \text{ and } f_2 = \frac{g_1(u) - g_2(v)}{2}.$$

Recall,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}.$$

Then,

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= \frac{1}{2}\left[\frac{\partial}{\partial u}\left(g_1(u) + g_2(v)\right) + \frac{\partial}{\partial v}\left(g_1(u) - g_2(v)\right)\right] \\ &= \frac{1}{2}\left[\frac{\partial g_1}{\partial u} + \frac{\partial g_2}{\partial v}\right]. \end{aligned}$$

Similarly,

$$\frac{\partial f_2}{\partial x} = \frac{1}{2}\left[\frac{\partial g_1}{\partial u} - \frac{\partial g_2}{\partial v}\right].$$

Thus,

$$\left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_2}{\partial x}\right)^2 = \left(\frac{\partial g_1}{\partial u}\right)\left(\frac{\partial g_2}{\partial v}\right).$$

That is, the first conditions in each set are equivalent. A similar argument shows the equivalence of the second ones. $\square$

Soon, we will see that Möbius transformations of $\mathbb{R}^{1,1}$ are a large, but not exhaustive class of conformal mappings.

2.5 The Torus in $\mathbb{R}^4$

Our study of the torus is motivated by our desire to find a compactification of $\mathbb{R}^{1,1}$. That is, we are looking for a space where we can embed $\mathbb{R}^{1,1}$ and the analysis, topology, and geometry are analogous to the Riemann sphere. The torus does not provide the model we need for the compactifcation, but it is close to what we need.

We denote the torus $S^1 \times S^1$ embedded in $\mathbb{R}^4$ as follows:

$$T^{1,1} = \{(x_0, x_1, x_2, x_3)|\, x_0^2 + x_1^2 = 1,\, x_2^2 + x_3^2 = 1\}.$$

The torus $T^{1,1}$ is the union of two disjoint open sets along with their common boundary:

$$T^{1,1} = T_+^{1,1} \cup T_-^{1,1} \cup T_0^{1,1}$$

where $T_+^{1,1} = \{x \in T^{1,1}|x_0 + x_3 > 0\}, T_-^{1,1} = \{x \in T^{1,1}|x_0 + x_3 < 0\}$ and $T_0^{1,1} = \{x \in T^{1,1}|x_0 + x_3 = 0\}$.

We define some involutions of $T^{1,1}$ which are also bijective diffeomorphisms. They will also play an important role in developing the compactification of $\mathbb{R}^{1,1}$.

Definition 2.14. *a. We define some involutions of* $T^{1,1}$ *:*

i. Left Inversion: $J_+(x) = (x_2, x_3, x_0, x_1)$*;*

ii. Right Inversion: $J_-(x) = (-x_2, x_3, -x_0, x_1)$*;*

iii. Inversion: $J(x) = (-x_0, x_1, -x_2, x_3) = J_+(x) \circ J_-(x)$*.*

b. The following involutions preserve $T_0^{1,1}$*:*

iv. Left Negation: $N_+(x) = (x_3, -x_2, -x_1, x_0)$*;*

v. Right Negation: $N_-(x) = (x_3, x_2, x_1, x_0)$*;*

vi. Negation: $N(x) = (x_0, -x_1, -x_2, x_3) = N_+(x) \circ N_-(x)$*;*

vii. Conjugation: $C(x) = (x_0, x_1, -x_2, x_3)$*;*

viii. Reflection: $R(x) = (-x_0, -x_1, -x_2, -x_3) = J_+ \circ N \circ J_+ \circ N(x)$*.*

3 The Conformal Compactification of $\mathbb{R}^{1,1}$

The notion of compactifying $\mathbb{R}^{1,1}$ is well known (see [3,11]) and is of some interest to physicists [4, 12,13]. One finds that this problem has been explained in numerous places in the literature, though there are some differences in the models used. For example, Kisil and others use a hyperboloid model in extended 3-space [6,14], and in others as a quadric in projective space [4]. In this work, we utilize the quadric model, which we denote $Q^{1,1}$. However, we follow the largely forgotten method presented in Segal's book (see [3]): we construct the compactification by way of torus on which we embed $\mathbb{R}^{1,1}$ and then quotient by a projection.

We shall define the conformal compactification of $\mathbb{R}^{1,1}$ as follows.

Definition 3.1. *A conformal compactification, up to isomorphism, of $\mathbb{R}^{1,1}$ is a compact semi-Riemannian manifold M with a conformal embedding ι such that $\iota(\mathbb{R}^{1,1})$ is dense in M.*

Remark 3.2. *In this context, the sphere S^2 is the conformal compactification of $\mathbb{C}$ with the stereographic projection as a conformal embedding.*

3.1 Embedding of $\mathbb{R}^{1,1}$ onto $Q^{1,1}$

A preliminary step to constructing the conformal compactification is to find a way to embed $\mathbb{R}^{1,1}$ in the torus. This embedding $\tau : \mathbb{R}^{1,1} \to T_+^{1,1}$ is given by

$$\tau(\zeta) = \tau(u,v) = \frac{(1-uv, u+v, u-v, 1+uv)}{\sqrt{(1+u^2)(1+v^2)}},$$

and is a bijective diffeomorphism of $\mathbb{R}^{1,1}$ onto $T_+^{1,1}$. Later, we shall see that τ is a conformal mapping with respect to the semi-Riemannian metric on $\mathbb{R}^{1,1}$ and $Q^{1,1}$. We remark the $R(\tau(\zeta))$ is a bijective diffeomorphism of $\mathbb{R}^{1,1}$ onto $T_-^{1,1}$.

Notice that for $\zeta \in \mathbb{R}^{1,1}$, when the inverses exist,

$$\begin{aligned} \tau(\bar{\zeta}) &= C(\tau(\zeta)), \\ \tau(\zeta^{-1}) &= \pm J(\tau(\zeta)), \\ \tau(\zeta_+^{-1}) &= \pm J_+(\tau(\zeta)), \\ \tau(\zeta_-^{-1}) &= \pm J_-(\tau(\zeta)). \end{aligned}$$

Similar formulas hold for N, N_+, N_-:

$$\begin{aligned} \tau(-\zeta) &= N(\tau(\zeta)), \\ \tau(-uj_+ + vj_-) &= N_+(\tau(\zeta)), \\ \tau(uj_+ - vj_-) &= N_-(\tau(\zeta)). \end{aligned}$$

Also notice that J_+ and N do not commute on $T^{1,1}$:

$$N \circ J_+(x) = (x_2, -x_3, -x_0, x_1) \neq J_+ \circ N(x) = (-x_2, x_3, x_0, -x_1),$$

even though the corresponding involutions on $\mathbb{R}^{1,1}$ commute. As such we descend to the projective space of the torus, which turns out to be the conformal compactification we seek (and we shall see why this is true in the subsequent subsections).

Definition 3.3. *We define a quadric surface $Q^{1,1}$ in the projective space $\mathbb{P}^3$ by*

$$Q^{1,1} = T^{1,1}/\sim$$

under the equivalence

$$x \sim y \quad \text{if and only if} \quad x = \pm y.$$

Remark 3.4. *We denote this identification by π.*

Notice that this equivalence identifies x and $R(x)$ and hence points in $T_+^{1,1}$ with points in $T_-^{1,1}$ and points in $T_0^{1,1}$ with points in $T_0^{1,1}$. These identified points are pairs of antipodal points on the 3-sphere containing $T^{1,1}$.

We denote points in $Q^{1,1}$ by $(x_0 : x_1 : x_2 : x_3)$. As a matter of notation, if $(x_0 : x_1 : x_2 : x_3) \in Q^{1,1}$, then we assume that $(\lambda x_0 : \lambda x_1 : \lambda x_2 : \lambda x_3)$ represents this point for all non-zero $\lambda \in \mathbb{R}$.

The inverse mapping $\tau^{-1} : T_+^{1,1} \to \mathbb{R}^{1,1}$ is given by

$$u = \frac{x_1 + x_2}{x_0 + x_3},$$
$$v = \frac{x_1 - x_2}{x_0 + x_3}.$$

Notice that τ^{-1} extends to $T_+^{1,1} \cup T_-^{1,1}$ as a two to one cover of $\mathbb{R}^{1,1}$.

For notational convenience, we shall denote the composition $\pi \circ \tau$ by

$$\eta : \mathbb{R}^{1,1} \to Q^{1,1}.$$

We shall also extend our definition of inversion to $Q^{1,1}$:

$$(x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : x_1 : -x_2 : x_3).$$

For simplicity we shall also denote this J (there should be no ambiguity since $\pi \circ J \circ \tau = J \circ \eta$.)

3.2 Added Points

The conformal compactification of $\mathbb{C}$ has one more point than the plane, namely the point at infinity. For $\mathbb{R}^{1,1}$, we must add an additional point for every point in the light cone plus two additional points which compactify the light cone. We calculate the coordinates of these additional points.

Suppose $v = 0$. Then

$$\eta(\zeta) = (1 : u : u : 1),$$

which goes to $(0 : 1 : 1 : 0)$ as $u \to \infty$, and

$$J\eta(\zeta) = (-1 : u : -u : 1),$$

goes to $(0 : 1 : -1 : 0)$ as $u \to \infty$. When $u = 0$,

$$\eta(\zeta) = (1 : v : -v : 1),$$

goes to $(0 : 1 : -1 : 0)$ as $v \to \infty$, and

$$J\eta(\zeta) = (-1 : v : v : 1),$$

tends to $(0 : 1 : 1 : 0)$ as $v \to \infty$.

Throughout $-\infty < \alpha < \infty$, define

$$\mathcal{L}_+ = \eta(L_+) = \{(1 : \alpha : \alpha : 1)\} \quad \text{and} \quad \mathcal{L}_- = \eta(L_-) = \{(1 : \alpha : -\alpha : 1)\}.$$

The above observations give us the following.

Remark 3.5. *If we define*

$$\begin{aligned} \mathcal{L}_+^{-1} &:= \overline{J\eta(L_+)} = \{(-1 : \alpha : -\alpha : 1) : \ \alpha \in \mathbb{R}\} \cup (0 : 1 : -1 : 0) \quad \textit{and} \\ \mathcal{L}_-^{-1} &:= \overline{J\eta(L_-)} = \{(-1 : \alpha : \alpha : 1) : \ \alpha \in \mathbb{R}\} \cup (0 : 1 : 1 : 0). \end{aligned}$$

Then these intersect at $\pi J(\tau(0)) = (-1 : 0 : 0 : 1)$ *and we shall define* $Q_0^{1,1} := \mathcal{L}_+^{-1} \cup \mathcal{L}_-^{-1}$.

Remark 3.6. *To avoid an unnecessarily cumbersome notation, we shall adopt the following:*

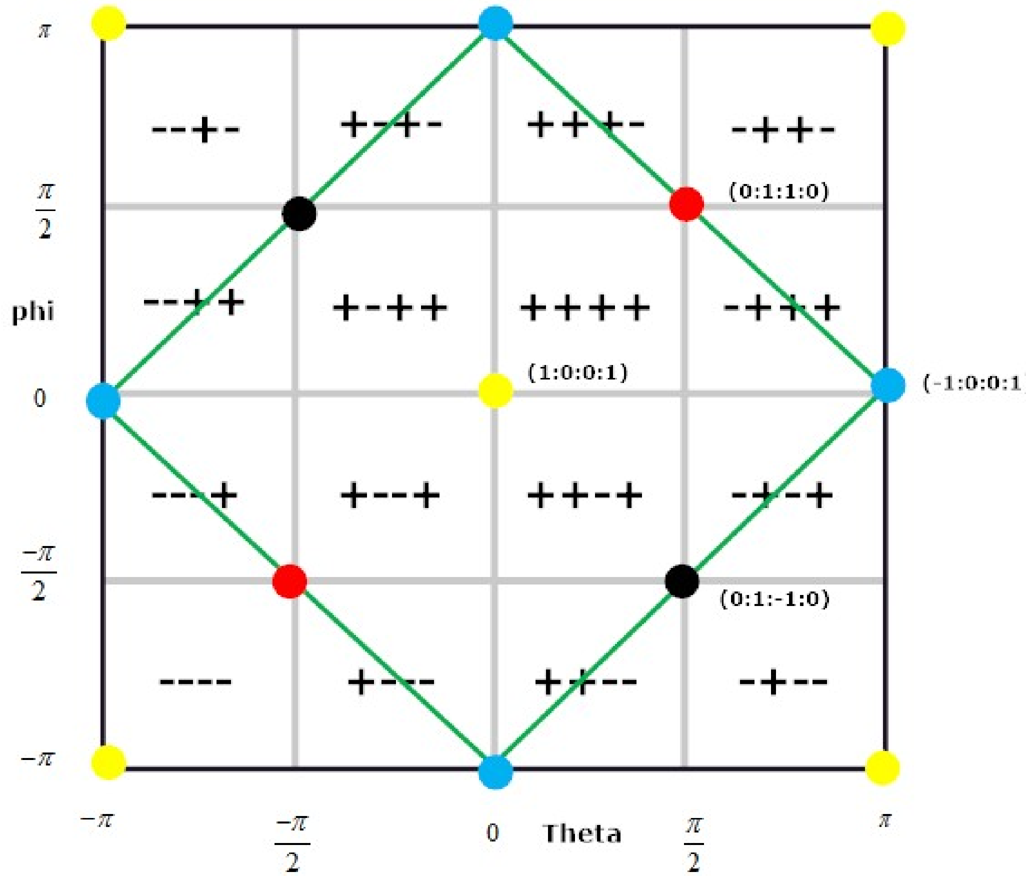
1. $\frac{1}{\alpha} j_+ + \infty j_-$ *shall denote* $(-1 : \alpha : -\alpha : 1)$.
2. $\infty j_+ + \frac{1}{\alpha} j_-$ *shall denote* $(-1 : \alpha : \alpha : 1)$.
3. ∞j_+ *shall denote* $(0 : 1 : 1 : 0)$.
4. ∞j_- *shall denote* $(0 : 1 : -1 : 0)$.
5. ∞ *shall denote* $\infty j_+ + \infty j_- = \frac{1}{0}$, *the inversion of zero.*

In this manner, it is clear that elements of $Q^{1,1}$ *can be regarded as elements* $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$, *where* $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. *In fact,* $Q^{1,1}$ *is locally isomorphic to* $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ *(in the sense of conformal geometry) [4]. These are sometimes referred to as the "extended double numbers" [6].*

Other rays through the origin embed as follows. If $u = \beta v, \beta \neq 0$, then

$$\eta(\zeta) = (1 - \beta u^2 : (1 + \beta)u : (1 - \beta)u : 1 + \beta u^2),$$

Figure 1: We parametrize $T^{1,1} = \{(\theta, \phi)|\, x_0 = \cos\theta,\, x_1 = \sin\theta,\, x_2 = \sin\phi,\, x_3 = \cos\phi, -\pi \leq \theta, \phi \leq \pi\}$. This give a parametrization of $Q^{1,1}$. The plus and minus signs indicate the signs of the cosines and sines in the parametrization.



which goes to $(-1:0:0:1)$ as $u \to \infty$.

Hyperbolas, $uv = R$, $R \in \mathbb{R}, \neq 0$ have the following embeddings :

$$(1 - R : u + R/u : u - R/u : 1 + R).$$

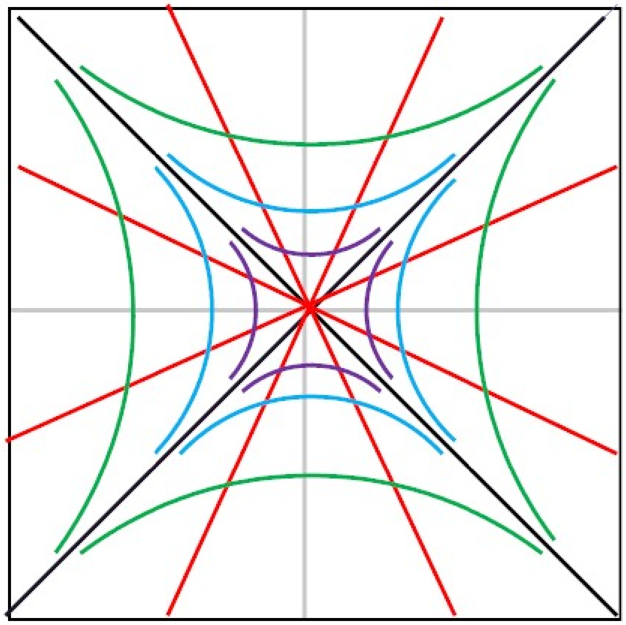
Notice that as $u \to \pm\infty$, this curve goes to $(0:1:1:0)$. Alternatively, we can write

$$(1 - R : R/v + v : R/v - v : 1 + R),$$

which goes to $(0:1:-1:0)$ as $v \to \pm\infty$.

The figures below give a nice picture of $Q^{1,1}$ and show what the above curves look like in $\mathbb{R}^{1,1}$ and what there embeddings look like in $Q^{1,1}$.

Figure 2: The $\mathbb{R}^{1,1}$ plane and some curves.



3.3 $Q^{1,1}$ as a Conformal Compactification

We claim that $Q^{1,1}$ is the conformal compactifcation we seek [4]. This means we have two things to check:

1. $\eta\left(\mathbb{R}^{1,1}\right)$ is dense;
2. η is a conformal map.

The first is true by construction and the second is rather simple:

Lemma 3.7. *The quotient map $\pi : T^{1,1} \to Q^{1,1}$ is conformal.*

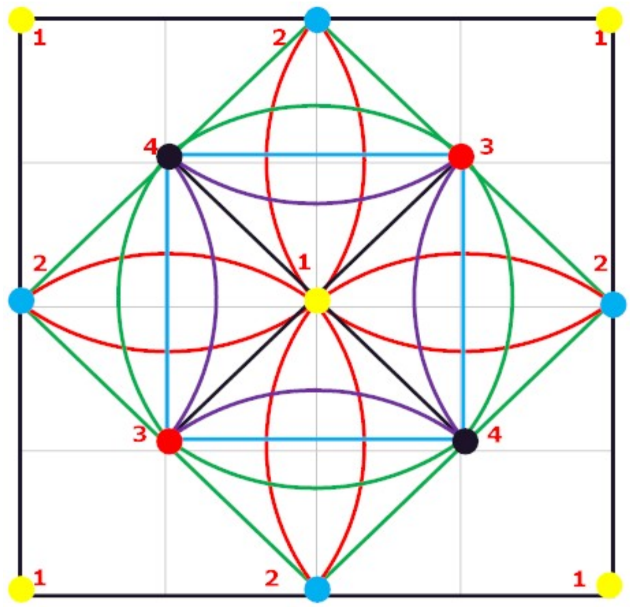
Proof. Since $T^{1,1} \subseteq \mathbb{R}^{2,2}$, it inherits the semi-Riemannian metric $g^{2,2}$ of signature $(2,2)$. We also pass this indefinite metric to $Q^{1,1}$.

It is clear that since π is surjective and a local diffeomorphism, it is also a local isometry, and hence conformal. $\square$

We also have the following [4].

Proposition 3.8. *The map $\eta = \pi \circ \tau : \mathbb{R}^{1,1} \to Q^{1,1}$ is conformal.*

Figure 3: The embedded curves. Points with the same numeric label are identified.



Remark 3.9. *Under certain conditions we can take a conformal map*

$$f : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$$

and extend it to a conformal map

$$\hat{f} : Q^{1,1} \to Q^{1,1}.$$

In particular, when $\zeta, \hat{f}(\zeta) \in \eta\left(\mathbb{R}^{1,1}\right)$, $\hat{f}(\zeta) = \eta \circ f \circ \eta^{-1}$. *In such a case we continue to write* $f(\zeta)$ *for* $\hat{f}(1 - uv : u + v : u - v : 1 + uv)$.

At $Q_0^{1,1}$, $\hat{f}$ *is continuous, and this means that the values of such points can be defined as limits. For example,*

$$\hat{f}(0 : 1 : 1 : 0) = \lim_{u \to \infty} \hat{f}(1 : u : u : 1).$$

3.4 Differentiable Functions and Conformal Mappings on $Q^{1,1}$

We want to define a notion of differentiability on $Q^{1,1}$ which is consistent the notion of differentiability on $\mathbb{R}^{1,1}$ defined in Section 2.2. This has been explored on the equivalent space of extended double numbers in [6]. We shall proceed in a similar fashion and borrow some ideas from complex analysis (as in [10]). In complex analysis one adds a point at infinity to invert the origin and discusses the behavior of functions at this added point using the inversion. In a similar way, we invert the added points to investigate functions at these added points. We define the following sets and maps for this purpose. *They do not serve as an atlas.*

$$
\begin{aligned}
&A_1 = \eta\left(\mathbb{R}^{1,1}\right),\ \phi_1 = \eta^{-1} \\
&A_2 = \{\infty\},\ \phi_2 = \eta^{-1} \circ J \\
&A_3 = \mathcal{L}_+^{-1} \setminus \{\infty\},\ \phi_3 = \eta^{-1} \circ J_+ \\
&A_4 = \mathcal{L}_-^{-1} \setminus \{\infty\},\ \phi_4 = \eta^{-1} \circ J_-
\end{aligned}
$$

Definition 3.10. *Consider the function*

$$f : Q^{1,1} \to Q^{1,1},$$

and $\zeta_0 \in Q^{1,1}$. Suppose $\zeta_0 \in A_i$ and $f(\zeta_0) \in A_j$. We say f is $Q^{1,1}$-differentiable at ζ_0 if

$$\phi_j \circ f \circ \phi_i^{-1}$$

is $C\ell_{1,0}$-differentiable at $\eta^{-1}(\zeta_0)$. If f is $Q^{1,1}$-differentiable everywhere, we will say f is $Q^{1,1}$-differentiable.

We also define $Q^{1,1}$-meromorphic (and $Q^{1,1}$-antimeromorphic) as functions of the form $f_1(u)j_+ + f_2(v)j_-$ (resp. $f_1(v)j_+ + f_2(u)j_-$), where f_1 and f_2 are meromorphic functions on $\widehat{\mathbb{R}}$.

We recall that $Q^{1,1}$ inherits the semi-Riemannian metric $g^{2,2}$. Hence, a notion of conformal map is defined. By using the conformality of η, J, and $J_\pm$ on their respective domains to get a shortcut definition for conformality of maps $f : Q^{1,1} \to Q^{1,1}$, :

Theorem 3.11. *A function*

$$f : Q^{1,1} \to Q^{1,1}$$

is (globally) conformal iff the maps

$$\phi_j \circ f \circ \phi_i^{-1}$$

are conformal (in the sense of Theorem 2.13) everywhere they are defined.

First we need some lemmas:

Lemma 3.12. *The maps $J,\ J_+,\ J_- : Q^{1,1} \to Q^{1,1}$ are conformal.*

Proof. Write $p = (x_0 : x_1 : x_2 : x_3)$. Then $J(p) = (-x_0 : x_1 : -x_2 : x_3)$, and

$$dJ = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $X_i = \frac{\partial}{\partial x_i}$. Then

$$dJ(X_i) = \begin{cases} -X_i & \text{if } i = 0, 2 \\ X_i & \text{if } i = 1, 3. \end{cases}$$

Thus,
$$J^*g(X_i, X_j) = g(X_i, X_j).$$
This means J is conformal.

Similar calculations show
$$(J_+)^*g(X_i, X_j) = -g(X_i, X_j)$$
and
$$(J_-)^*g(X_i, X_j) = -g(X_i, X_j),$$
$i, j = 0, ..., 3$. So, J_+ and J_- are also conformal. □

The next lemma is trivial.

Lemma 3.13. *Let (M, g) be a semi-Riemannian manifold. Let $f : M \to M$ and $h : M \to M$ be smooth maps of maximal rank.*

(i) Suppose f and $h \circ f$ are conformal maps with non-vanishing factors Ω_1 and Ω_2, respectfully. Then h is conformal.

(ii) Suppose h and $h \circ f$ are conformal maps with non-vanishing factors Ω_1 and Ω_2, respectfully. Then f is conformal.

Proof of Theorem 3.11. Let $f : Q^{1,1} \to Q^{1,1}$ be a smooth map of maximal rank.

Suppose f is conformal. Then by lemma 3.12, we conclude that
$$\phi_j \circ f \circ \phi_i^{-1},$$
is conformal for every i, j.

Conversely, suppose the $\phi_j \circ f \circ \phi_i^{-1}$ are conformal. Then $\phi_j \circ f$ and $f \circ \phi_i^{-1}$ are conformal. By Lemma 3.13, this means f is conformal.

□

In the next section, we shall see that the Möbius transformations form a large class of (globally) conformal mappings. However, these are not the only such mappings.

Example 3.14. *Consider*
$$f(u, v) = (\arctan(u) + u)\, j_+ + v j_-.$$

It is easy to check that f is conformal everywhere. Yet, it is not a Möbius transformation.

4 Möbius Transformations

With $A, B, C, D \in \mathbb{R}^{1,1}$, we consider the actions of Möbius transformations expressed as

$$\mathcal{M}(\zeta) = \frac{(A\zeta + B)}{(C\zeta + D)}.$$

Also referred to as linear fractional transformations, these have been the subject of recent works on analogues of theorems in complex analysis [6, 15].

As is done in the complex plane, we can represent such functions via a matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in PGL\left(2, C\ell_{1,0}\right).$$

Notice that if we write $A = a_1 j_+ + a_2 j_-, B = b_1 j_+ + b_2 j_-, C = c_1 j_+ + c_2 j_-, D = d_1 j_+ + d_2 j_-$, then

$$\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} j_+ + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} j_- = \mathcal{A}_1 j_+ + \mathcal{A}_2 j_-,$$

It is easy to check that

$$\det \mathcal{A} = \det \mathcal{A}_1 j_+ + \det \mathcal{A}_2 j_- \quad \text{and} \quad N(\det \mathcal{A}) = \det \mathcal{A}_1 \det \mathcal{A}_2.$$

Interestingly, the literature does not contain such a decomposition for Möbius transformations. We feel that this makes proving interesting facts about these functions much easier.

Notice that $N(\det \mathcal{A}) \neq 0$ if and only if $\det \mathcal{A}_1 \neq 0$ and $\det \mathcal{A}_2 \neq 0$. Thus, $\mathcal{A}$ is invertible if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are invertible so that

$$\mathcal{A}^{-1} = \mathcal{A}_1^{-1} j_+ + \mathcal{A}_2^{-1} j_-.$$

That is,

$$PGL(2, C\ell_{1,0}) \cong PGL(2, \mathbb{R}) \times PGL(2, \mathbb{R}).$$

We also have the decomposition

$$\mathcal{M}(\zeta) = \frac{A\zeta + B}{C\zeta + D} = \mathcal{M}_1(u) j_+ + \mathcal{M}_2(v) j_-.$$

We call a Möbius transformation real when $A, B, C, D \in \mathbb{R}$. In this case we have $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.

We also consider conjugate Möbius transformations of the form

$$(A\overline{\zeta} + B)(C\overline{\zeta} + D)^{-1} = \frac{a_1 v + b_1}{c_1 v + d_1} j_+ + \frac{a_2 u + b_2}{c_2 u + d_2} j_-.$$

Proposition 4.1. *Möbius Transformations of the form*

$$\mathcal{M}(\zeta) = \frac{A\zeta + B}{C\zeta + D} \text{ or } \mathcal{M}(\zeta) = \frac{A\overline{\zeta} + B}{C\overline{\zeta} + D}$$

such that $N(AD - BC) \neq 0$ *are conformal on* $Q^{1,1}$.

Proof. Let $\mathcal{M}(\zeta) = \dfrac{A\zeta + B}{C\zeta + D} = \dfrac{A_1 u + B_1}{C_1 u + D_1} j_+ + \dfrac{A_2 v + B_2}{C_2 v + D_2} j_-$. It is clear that $\mathcal{M}$ is $Q^{1,1}$-differentiable since combinations of compositions of J and $J_\pm$ on either side of $\mathcal{M}$ are differentiable at appropriate places.

Notice that $N(AD - BC) = (A_1 D_1 - B_1 C_1)(A_2 D_2 - B_2 C_2) \neq 0$. Also notice that the compositions of J and $J_\pm$ on either side of $\mathcal{M}$ correspond to a possible swapping of rows and or columns of the corresponding matrices of $\mathcal{M}_1$ and $\mathcal{M}_2$, and hence change at most the sign of $N(AD-BC)$. Thus, the corresponding partial derivatives at appropriate points are still non-zero. Hence, we have conformality everywhere.

A similar argument works for $\mathcal{M}(\zeta) = \dfrac{A\overline{\zeta} + B}{C\overline{\zeta} + D}$, using ∇. □

4.1 Fixed Points and Transitivity

The notion of a fixed point under $\mathcal{M}$ now makes sense everywhere, since we understand what it means for a real Möbius transformation to have a fixed point in $\mathbb{R} \cup \{\infty\}$. As in complex analysis (see [10]), we find an association between conjugacy classes of Möbius transformations and their fixed points. However, the situation is a little more complicated.

Theorem 4.2. *Let $\mathcal{M}$ be a Möbius transformation on $Q^{1,1}$ where neither component is the identity. Then $\mathcal{M}$ has either zero fixed points, one fixed point, two fixed points or four fixed points. When the Möbius transformation is real, there cannot be two fixed points.*

Proof. The fixed points of the Möbius transformation $\mathcal{M}$ are determined by those of the component transformations $\mathcal{M}_1$ and $\mathcal{M}_2$. If one component has zero fixed points, then $\mathcal{M}$ has zero fixed points. If both have one fixed point, the $\mathcal{M}$ has one fixed point. If one component has one fixed point and the other two, then $\mathcal{M}$ has two fixed points. Finally, if both components have two fixed points, then $\mathcal{M}$ has four.

If $\mathcal{M}$ is a real Möbius transformation, then both components are the same and so have the same number of fixed points. Hence in this case two fixed points cannot occur. □

Remark 4.3. *Fixed points at an infinity occur when one or both components have infinity as a fixed point.*

Remark 4.4. *Consider the first component of a Möbius transformation:*

$$\frac{au + b}{cu + d}$$

where $a, b, c, d \in \mathbb{R}$.

Case 1. *Infinity is a fixed point if and only if $c = 0$. In this case, when $a = d$, infinity is the only fixed point.*

Case 2. *If $c = 0$ and $a \neq d$, there is a second fixed point namely $b/(d - a)$.*

Case 3. *When $c \neq 0$ solutions to*

$$\frac{au + b}{cu + d} = u$$

have the form

$$u = \frac{(a - d) \pm \sqrt{\Delta}}{2c}.$$

where $\Delta = (a - d)^2 + 4bc$. If $\Delta < 0$, then there are no fixed points. When $\Delta = 0$, there is one fixed point. If $\Delta > 0$, then there are two fixed points.

Similar calculations hold for $\mathcal{M}_2$.

Remark 4.5. *We can actually rewrite Δ in terms of* Tr *and* det*:*

$$\begin{aligned}
\Delta &= \Delta + 4ad - 4ad \\
&= a^2 - 2ad + d^2 + 4ad - 4ad + 4bc \\
&= (a + d)^2 - 4(ad - bc) \\
&= \mathrm{Tr}^2(\mathcal{M}_1) - 4\det(\mathcal{M}_1).
\end{aligned}$$

Given the above discussion and the fact that Tr and det completely determine eigenvalues (see [16]), we get a link between the number of eigenvalues of the component matrices and the number of fixed points a Möbius transformation has.

Proposition 4.6. *Let $\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, \mathbb{R})$ be a non-identity element.*

a. If $\mathcal{M}$ has one (real) eigenvalue, then $\mathcal{M}(u)$ has one fixed point.

b. If $\mathcal{M}$ has two real eigenvalues, then $\mathcal{M}(u)$ has two fixed points.

c. If $\mathcal{M}$ has two complex eigenvalues, then $\mathcal{M}(u)$ has no fixed points.

Proof. The Δ defined above is precisely the discriminant of the characteristic polynomial of $\mathcal{M}$. $\square$

Example 4.7. *a. The mapping $1/\zeta$ has four fixed points, $\pm 1, \pm j$,.*

b. The mapping $-1/\zeta$ has zero fixed points,

c. The mapping $\zeta + 1$ has one fixed point, ∞.

d. The mapping $2\zeta + 1$ has four fixed points, ∞, -1, $-j_+ + \infty j_-$ and $\infty j_+ - j_-$.

e. The mapping $\frac{1}{u} j_+ + (v + 1) j_-$ has two fixed points, $j_+ + \infty j_-$ and $-j_+ + \infty j_-$.

We can also have attracting and repelling fixed points.

Example 4.8. *Consider $\mathcal{M}(\zeta) = \alpha u j_+ + \beta v j_-$ with $\alpha, \beta \neq 0$. Of course, then $\mathcal{M}$ is conformal.*

Notice that $\mathcal{M}$ fixes $0, \infty, \infty j_+, \infty j_-$. Now, $\mathcal{M}^n(\zeta) = \alpha^n u j_+ + \beta^n v j_-$ and we have four cases:

Case 1 *Let $|\alpha|, |\beta| > 1$. Then for all $u \neq 0, \infty$,*

$$\alpha^n u \to \infty \text{ as } n \to \infty,$$

and $v \neq 0, \infty$,

$$\beta^n v \to \infty \text{ as } n \to \infty.$$

Thus ∞ is an attracting fixed point and $0, \infty j_+, \infty j_-$ are repelling.

Case 2 *Let $|\alpha| > 1, |\beta| < 1$. Then for all $u \neq 0, \infty$*

$$\alpha^n u \to \infty \text{ as } n \to \infty,$$

and $v \neq 0, \infty$,

$$\beta^n v \to 0 \text{ as } n \to 0.$$

Thus ∞j_+ is an attracting fixed point and $0, \infty, \infty j_-$ are repelling.

Case 3 *Let $|\alpha| < 1, |\beta| > 1$. Then for all $u \neq 0, \infty$,*

$$\alpha^n u \to \infty \text{ as } n \to 0,$$

and $v \neq 0, \infty$,

$$\beta^n v \to \infty \text{ as } n \to \infty.$$

Thus ∞j_- is an attracting fixed point and $0, \infty, \infty j_+$ are repelling.

Case 4 *Let $|\alpha| < 1, |\beta| < 1$. Then for all $u \neq 0, \infty$,*

$$\alpha^n u \to \infty \text{ as } n \to 0,$$

and $v \neq 0, \infty$,

$$\beta^n v \to \infty \text{ as } n \to 0.$$

Thus 0 is an attracting fixed point and $\infty, \infty j_+, \infty j_-$ are repelling.

Now we see that the transitivity of the Möbius transformations of $Q^{1,1}$ are only one transitive, a property which is quite different from what we find on the Riemann sphere, where the Möbius transformations are three transitive [10].

Theorem 4.9. *The Möbius group acts transitively on $Q^{1,1}$. It is not two transitive in general.*

Proof. Given $\zeta_1 = u_1 j_+ + v_1 j_-$ and $\zeta_2 = u_2 j_+ + v_2 j_-$, we know that by transitivity of Möbius transformations of $\widehat{\mathbb{R}}$ that there is $\mathcal{N} \in PGL(2, \mathbb{R})$ such that $\mathcal{N}(u_1) = u_2$ and $\mathcal{P} \in PSL(2, \mathbb{R})$ such that $\mathcal{P}(v_1) = v_2$. Therefore, $\mathcal{M}(\zeta) = \mathcal{N}(u) j_+ + \mathcal{P}(v) j_-$ maps ζ_1 to ζ_2.

To see that Möbius group is not two-transitive, we give a proof by counterexample. Consider $\{j_+, 2j_+\}$ and $\{j_-, 2j_-\}$. Notice that in order for $\mathcal{M}(j_+) = j_-$ and $\mathcal{M}(2j_+) = 2j_-$, we would need a real Möbius transformation which maps 0 to 1 and 0 to 2, an obvious contradiction.

$\square$

Definition 4.10. *For our purposes, we shall define a* ***hyperbola*** *to be a subset H of $Q^{1,1}$ which is Möbius equivalent to the closure of the set*

$$\{uj_+ + vj_- \ : uv = 1\}.$$

That is there exists real Möbius transformations $\mathcal{M}_1$, $\mathcal{M}_2$ such that

$$\mathcal{M}_1(u)\mathcal{M}_2(v) = 1,$$

for every $u, v \in H \cap \mathbb{R}^{1,1}$.

In a similar fashion, we define a ***degenerate hyperbola*** *to be a subset D of $Q^{1,1}$ that is Möbius equivalent to the closure of the light cone*

$$L = \{uj_+ + vj_- \ : uv = 0\}.$$

Remark 4.11. *By definition, the closure of the light cone $\overline{L} := \overline{\{uv = 0\}}$ is a degenerate hyperbola. It contains two branches, namely $\left\{vj_- \ : v \in \widehat{\mathbb{R}}\right\}$ and $\left\{uj_+ \ : u \in \widehat{\mathbb{R}}\right\}$. It is clear that the same is true for any degenerate hyperbola.*

Notice that conjugation $\zeta \mapsto vj_+ + uj_-$ maps a degenerate hyperbola to itself, interchanging branches.

In a similar way, $\left\{\frac{u}{v} = 0\right\}$ consists of two branches: $\left\{vj_- \ : v \in \widehat{\mathbb{R}}\right\}$ and $\left\{uj_+ + \infty j_- \ : u \in \widehat{\mathbb{R}}\right\}$. Again, conjugation interchanges these branches.

Remark 4.12. *Let $\alpha, \beta, \mu, R \in \mathbb{R}$ with $\mu, R \neq 0$. The closure of curves of the form $(u-\alpha)(v-\beta) = R$, $u = \mu v + \beta$, $v = \mu u + \beta$ are hyperbolae. They are also the only hyperbolae.*

Let $\gamma, \delta \in \widehat{\mathbb{R}}$. Then it is clear that the degenerate hyperbolae are of the form $\{u = \gamma\} \cup \{v = \delta\}$.

The following is clearly true.

Proposition 4.13. *The Möbius groups act transitively on hyperbolae. That is, a Möbius transformation will map any hyperbola to another. The Möbius groups also acts transitively on degenerate hyperbolae.*

4.2 Cross Ratio

Yaglom defines a cross ratio on $\mathbb{R}^{1,1}$, and shows that it can be used to define hyperbola and lines [2]. Recent works have brought further analogues of the cross ratio in the complex plane to new spaces [5, 17].

We shall extend Yaglom's ratio to $Q^{1,1}$ and use the j_+, j_- basis to understand it as a direct product of real cross ratios. This gives more natural proofs of geometric ideas brought forth in [2].

Four points in $\mathbb{R}^{1,1}$, $\zeta_i = u_i j_+ + v_i j_-$, $i = 1, 2, 3, 4$, with distinct u_is and distinct v_is, are called *completely distinct*.

The following lemmas, though rather trivial, are useful.

Lemma 4.14. *The image of a set of completely distinct points under a Möbius transformation is a set of completely distinct points.*

Lemma 4.15. *Hyperbolae contain infinite sets of completely distinct points. In particular, they contain sets of four completely distinct points.*

Remark 4.16. *Because of their form, degenerate hyperbolae cannot contain three completely distinct points; they may have at most two and they must lie on separate branches.*

This implies that hyperbolae and degenerate hyperbolae are not Möbius equivalent, since Möbius transformations are one-to-one maps.

Given a 4-tuple of completely distinct points, we define the cross ratio as follows :

$$\lambda = [\zeta_1, \zeta_2; \zeta_3, \zeta_4] = \frac{(\zeta_1 - \zeta_3)\,(\zeta_2 - \zeta_4)}{(\zeta_2 - \zeta_3)\,(\zeta_1 - \zeta_4)}.$$

We then have

$$\begin{aligned}
[\zeta_1, \zeta_2; \zeta_3, \zeta_4] &= \frac{(u_1 - u_3)\,(u_2 - u_4)}{(u_2 - u_3)\,(u_1 - u_4)} j_+ + \frac{(v_1 - v_3)\,(v_2 - v_4)}{(v_2 - v_3)\,(v_1 - v_4)} j_- \\
&= [u_1, u_2; u_3, u_4]\, j_+ + [v_1, v_2; v_3, v_4]\, j_- \\
&= \lambda_1 j_+ + \lambda_2 j_-.
\end{aligned}$$

By taking limits, this is defined when one of the u_is or v_is is infinite.

The following two theorems and proposition are mentioned in Yaglom's book without proof [2].

Theorem 4.17. *Let $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and $\xi_1, \xi_2, \xi_3, \xi_4$ be 4-tuples of completely distinct points. There is a Möbius transformation sending $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ to $\xi_1, \xi_2, \xi_3, \xi_4$ if and only if their cross ratios are equal. As such, the cross ratio is a bijection on orbits of 4-tuples of completely distinct points.*

Proof. The above lemma implies that for any Möbius transformation $\mathcal{M}$, $[\mathcal{M}(\zeta_1), \mathcal{M}(\zeta_2); \mathcal{M}(\zeta_3), \mathcal{M}(\zeta_4)]$ is defined if and only if $[\zeta_1, \zeta_2; \zeta_3, \zeta_4]$ is defined.

Suppose there exists such a Möbius transformation $\mathcal{M} = \mathcal{M}_1 j_+ + \mathcal{M}_2 j_-$. By a simple calculation, we have

$$\mathcal{M}_1(u_i) - \mathcal{M}_1(u_j) = \frac{(u_i - u_j)(a_1 d_1 - b_1 c_1)}{(c_1 u_i + d_1)(c_1 u_j + d_1)}.$$

Then,

$$
\begin{aligned}
[\mathcal{M}_1(u_1), \mathcal{M}_1(u_2); \mathcal{M}_1(u_3), \mathcal{M}_1(u_4)] &= \frac{\frac{(u_1-u_3)(a_1d_1-b_1c_1)}{(c_1u_1+d_1)(c_1u_3+d_1)}\frac{(u_2-u_4)(a_1d_1-b_1c_1)}{(c_1u_2+d_1)(c_1u_4+d_1)}}{\frac{(u_2-u_3)(a_1d_1-b_1c_1)}{(c_1u_2+d_1)(c_1u_3+d_1)}\frac{(u_1-u_4)(a_1d_1-b_1c_1)}{(c_1u_1+d_1)(c_1u_4+d_1)}} \\
&= \frac{(u_1-u_3)(u_2-u_4)}{(u_2-u_3)(u_1-u_4)}.
\end{aligned}
$$

Similar calculations work for $\mathcal{M}_2$.

Thus, Möbius transformations preserve cross ratios.

Conversely, assume that the points are finite and suppose the cross ratios are equal. Then,

$$S(\zeta) = \frac{(\zeta-\zeta_3)(\zeta_2-\zeta_4)}{(\zeta_2-\zeta_3)(\zeta-\zeta_4)}$$

maps $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ to $[\zeta_1, \zeta_2; \zeta_3, \zeta_4], 0, 1, \infty$ and

$$T(\xi) = \frac{(\xi-\xi_3)(\xi_2-\xi_4)}{(\xi_2-\xi_3)(\xi-\xi_4)}$$

maps $\xi_1, \xi_2, \xi_3, \xi_4$ to $[\xi_1, \xi_2; \xi_3, \xi_4], 0, 1, \infty$.

Hence $T^{-1} \circ S$ is a Möbius transformation sending $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ to $\xi_1, \xi_2, \xi_3, \xi_4$. A similar calculation holds when points are an infinity.

□

Proposition 4.18. *If $\zeta_1, \zeta_2, \zeta_3$, are completely distinct points on some hyperbola H, then the cross ratio $[\zeta_1, \zeta_2; \zeta_3, \zeta_4]$ is in $\widehat{\mathbb{R}}$ if and only if ζ_4 is another point on H.*

Proof. Suppose that ζ_4 also lies on H, then by definition of hyperbola and Möbius invariance of cross ratios, we may assume that H is the hyperbola $uv = 1$.

Hence,

$$
\begin{aligned}
[\zeta_1, \zeta_2; \zeta_3, \zeta_4] &= \frac{(u_1-u_3)(u_2-u_4)}{(u_2-u_3)(u_1-u_4)} j_+ + \frac{(v_1-v_3)(v_2-v_4)}{(v_2-v_3)(v_1-v_4)} j_- \\
&= \frac{\left(\frac{1}{v_1}-\frac{1}{v_3}\right)\left(\frac{1}{v_2}-\frac{1}{v_4}\right)}{\left(\frac{1}{v_2}-\frac{1}{v_3}\right)\left(\frac{1}{v_1}-\frac{1}{v_4}\right)} j_+ + \frac{(v_1-v_3)(v_2-v_4)}{(v_2-v_3)(v_1-v_4)} j_- \\
&= \left(\frac{\frac{1}{v_1v_2v_3v_4}}{\frac{1}{v_1v_2v_3v_4}}\right) \frac{(v_3-v_1)(v_4-v_2)}{(v_3-v_2)(v_4-v_1)} j_+ + \frac{(v_1-v_3)(v_2-v_4)}{(v_2-v_3)(v_1-v_4)} j_- \\
&= \frac{(v_1-v_3)(v_2-v_4)}{(v_2-v_3)(v_1-v_4)} \in \widehat{\mathbb{R}}.
\end{aligned}
$$

For the converse, it suffices to show that if $u_iv_i = 1$ for $i = 1, 2, 3$ and $[\zeta_1, \zeta_2; \zeta_3, \zeta_4] \in \widehat{\mathbb{R}}$, then $u_4v_4 = 1$.

Then by hypothesis,

$$\frac{\left(\frac{1}{v_1}-\frac{1}{v_3}\right)\left(\frac{1}{v_2}-u_4\right)}{\left(\frac{1}{v_2}-\frac{1}{v_3}\right)\left(\frac{1}{v_1}-u_4\right)}=\frac{(v_1-v_3)(v_2-v_4)}{(v_2-v_3)(v_1-v_4)}.$$

After some algebra and using the fact that none of the factors vanish, we see that

$$\frac{(1-u_4v_2)}{(1-u_4v_1)}=\frac{(v_2-v_4)}{(v_1-v_4)},$$

which immediately implies that

$$v_1(1-u_4v_4)=v_2(1-u_4v_4).$$

But since $v_1\neq v_2$, this must mean that

$$1-u_4v_4=0.$$

□

Theorem 4.19. *If the cross ratio $[\zeta_1,\zeta_2;\zeta_3,\zeta_4]\in\widehat{\mathbb{R}}$, then there exists a Möbius transformation sending $\zeta_1,\zeta_2,\zeta_3,\zeta_4$ to any hyperbola.*

Proof. By assumption we have

$$[u_1,u_2;u_3,u_4]=[v_1,v_2;v_3,v_4]=\lambda\in\widehat{\mathbb{R}}.$$

Define

$$U(u):=\frac{(u-u_3)(u_2-u_4)}{(u_2-u_3)(u-u_4)}\text{ and }V(v)=\frac{(v-v_3)(v_2-v_4)}{(v_2-v_3)(v-v_4)},$$

which are Möbius transformations. Then

$$\begin{aligned}U(u_1)&=V(v_1)=\lambda\\U(u_2)&=V(v_2)=1\\U(u_3)&=V(v_3)=0\\U(u_4)&=V(v_4)=\infty.\end{aligned}$$

Thus, $\mathcal{M}(\zeta)=U(u)j_++V(v)j_-$ maps $\zeta_1,\zeta_2,\zeta_3,\zeta_4$ to a line $\{u=v\}$, which can be mapped by way of a Möbius transformation (namely J_+ or J_-) to a hyperbola.

□

References

[1] P. A. M. Dirac. Wave equations in conformal space. *Ann. of Math*, 37, 1936.

[2] I.M. Yaglom. *A Simple Non-Euclidean Geometry and Its Physical Basis.* Springer-Verlag.

[3] Irving Ezra Segal. *Mathematical Cosmology and Extragalactic Astronomy*, volume 68 of *Pure and Applied Mathematics*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976.

[4] Martin Schottenloher. *A Mathematical Introduction to Conformal Field Theory.* Springer, Berlin Heidelberg, 2008.

[5] Sky Brewer. Projective cross-ratio on hypercomplex numbers. *Adv. Appl. Clifford Algebras*, 23(1):1–14, March 2013. arXiv:1203.2554.

[6] Kyle DenHartigh and Rachel Flim. Liouville theorems in the dual and double planes. *Rose-Hulman Undergraduate Mathematics Journal*, 12, 2011.

[7] Ian Porteous. Mathematical structure of clifford algebras. In Rafal Ablamowicz and Garret Sobczyk, editors, *Lectures on Clifford(Geometric) Algebras and Applications.*

[8] M.A.B Deakin. Functions of a dual or duo variable. *Mathematics Magaizne*, 1966.

[9] Matvei Libine. Hyperbolic cauchy integral formula for the split complex numbers. arXiv:0712.0375, 12.

[10] Lars V. Ahlfors. *Complex Analysis : An Introduction to the Theory of Analytic Functions of One Complex Variable.* McGraw-Hill, New York, 1979.

[11] Vladimir V. Kisil. Analysis in $\mathbf{R}^{1,1}$ or the principal function theory. *Complex Variables Theory Appl.*, 40(2):93–118, 1999. arXivfunct-an/9712003.

[12] Francisco J. Herranz and Mariano Santander. Conformal compactification of spacetimes. *J. Phys. A*, 35(31):6619–6629, 2002.

[13] Vladimir V. Kisil. Two-dimensional conformal models of space-time and their compactification. *J. Math. Phys.*, 48(7), 2007. arXivmath-ph/0611053.

[14] Vladimir V. Kisil. *Geometry of Möbius Transformations: Elliptic, Parabolic and Hyperbolic Actions of* $\mathrm{SL}_2(\mathbf{R})$. Imperial College Press, London, 2012. Includes a live DVD.

[15] Joshua Keilman and Andrew Jullian Mis. A beckman-quarles type theorem for linear fractional transformations of the extended double plane. *Rose-Hulman Undergraduate Mathematics Journal*, 12, 2011.

[16] Roger A. Horn and Charles R. Johnson. *Matrix Analysis.* Cambridge University Press., 1985.

[17] Ewain Gwynne and Matvei Libine. On a quaternionic analogue of the cross-ratio. *Adv. Appl. Clifford Algebras*, 22, 2012.