Diversified homotopic behavior of closed orbits of some **R**-covered Anosov flows

Sérgio R. Fenley

March 2, 2014

Abstract – We produce infinitely many examples of Anosov flows in closed 3-manifolds where the set of periodic orbits is partitioned in two infinite subsets. In one subset every closed orbit is freely homotopic to infinitely other closed orbits of the flow. In the other subset every closed orbit is freely homotopic to only one other closed orbit. The examples are obtained by Dehn surgery on geodesic flows. The manifolds are toroidal and have Seifert pieces and atoroidal pieces in their torus decompositions.

1 Introduction

This article deals with the question of free homotopies of closed orbits of Anosov flows [An] in 3-manifolds. In particular we deal with the following question: how many closed orbits are freely homotopic to a given closed orbit of the flow? Suspension Anosov flows have the property that an arbitrary closed orbit is not freely homotopic to any other closed orbit. Geodesic flows have the property that every closed orbit (which corresponds to a geodesic in the surface) is only freely homotopic to one other closed orbit. The other orbit corresponds to the same geodesic in the surface, but being traversed in the opposite direction. About twenty years ago the author proved that there is an infinite class of Anosov flows in closed hyperbolic 3-manifolds satisfying the property that every closed orbit is freely homotopic to infinitely many other closed orbits [Fe1]. Obviously this was diametrically opposite to the behavior of the previous two examples and it was also quite unexpected. This property in these examples is strongly connected with large scale properties of Anosov flows when lifted to the universal cover. In particular in these examples the property of infinitely orbits which are freely homotopic to each other implies that the flows are not quasigeodesic [Fe1]. This provided the first examples of Anosov flows in hyperbolic 3-manifolds which are not quasigeodesic.

We define the *free homotopy class* of a closed orbit of a flow to be the collection of closed orbits which are freely homotopic to the original closed orbit. For an Anosov flow each free homotopy class is at most infinite countable as there are only countably many closed orbits of the flow [An]. In this article we are concerned with the cardinality of free homotopy classes. Suspensions have all free homotopy classes with cardinality one and geodesic flows have all free homotopy classes with cardinality two. In the hyperbolic examples mentioned above every free homotopy class has infinite cardinality. In addition if we do a finite cover of geodesic flows, where we "unroll the fiber direction" then we can get examples where every free homotopy class has cardinality $2n$ where n is a positive integer. The question we ask is whether we can have mixed behavior for an Anosov flow. In other words, can some free homotopy classes be infinite while others have finite cardinality? In this article we produce infinitely many examples where this indeed occurs.

Main theorem – There are infinitely many examples of Anosov flows Φ in closed 3-manifolds so that the set of closed orbits is partitioned in two infinite subsets A and B so that the following happens. Every closed orbit in A has infinite free homotopy class. Every closed orbit in B has free homotopy class of cardinality two. The examples are obtained by Dehn surgery on closed orbit of geodesic flows.

These flows are what is called **R**-*covered* Anosov flows [Fe1]. A foliation $\mathcal{F}$ is **R**-covered if the leaf space of the lifted foliation to the universal cover is homeomorphic to the real numbers **R**. An Anosov flow is **R**-covered if its stable foliation (or equivalently its unstable foliation [Ba, Fe1]) is **R**-covered.

2 Atoroidal submanifolds of unit tangent bundles of surfaces

Let S be a closed hyperbolic surface and let $M = T_1S$ be the unit tangent bundle of S. In the next section we will do Dehn surgery on a closed orbit of the geodesic flow to obtain our examples. Throughout the article we use the projection map

$$\tau : \ M = T_1S \ \rightarrow \ S$$

which is the projection of a unit tangent vector to its basepoint in S.

Let Φ be the geodesic flow in M. Let α be a closed geodesic in S. This geodesic of S generates two orbits of Φ, let α_1 be one such orbit. This is equivalent to picking an orientation along α. Let S_1 be a subsurface of S that α fills. If α is simple then S_1 is an annulus. If α fills S then $S_1 = S$. Let S_2 be the closure in S of $S - S_1$. Let $M_i = T_1S_i, \ i = 1, 2$.

The purpose of this section is to prove the following result.

Proposition 2.1. *(atoroidal) The submanifold* $M_1 - \alpha_1$ *is atoroidal.*

Proof. We will prove that $M_1 - \alpha$ is geometrically atoroidal: any π_1-injective embedded torus T in $M_1 - \alpha$ is homotopic, and hence isotopic, to the boundary. It could be isotopic to the boundary of a regular neighborhood of α_1. Notice that $M_1 - \alpha_1$ is irreducible [He]. Gabai [Ga] showed that since $M_1 - \alpha_1$ is not a small Seifert fibered space then $M_1 - \alpha_1$ is also homotopically atoroidal: any π_1-injective map from a torus into $M_1 - \alpha_1$ is homotopic to the boundary.

Let T be an incompressible torus in $M_1 - \alpha_1$. We think of T as contained in M. There are 2 possibilities:

Case 1 $-$ T is π_1-injective in M.

Here T is contained in $M - M_2$. We use that M is Seifert fibered. In addition T is incompressible, so it is an essential lamination in M. By Brittenham's theorem [Br] T is isotopic to either a vertical torus or a horizontal torus in M. Vertical torus means it is a union of S^1 fibers of the Seifert fibration. Horizontal torus means that it is transverse to these fibers. Since S is a hyperbolic surface, there is no horizontal torus in $M = T_1S$. It follows that T is isotopic to a vertical torus T'. In addition since T itself is disjoint from M_2 and M_2 is saturated by the Seifert fibration, we can push the isotopy away from M_2 and suppose it is contained in M_1. Finally the isotopy forces an isotopy of the orbit α into a curve α' disjoint from T'. This isotopy projects by τ to an homotopy from α to a curve α^* and this homotopy is contained in S_1. The curve α^* is disjoint from the projection $\tau(T')$ which is a simple closed curve β in S_1. Since α fills S_1 it now follows that β is a peripheral curve in S_1. By another isotopy we can assume that β does not intersect α or that T' does not intersect α_1.

In addition the isotopy from T to T' can be extended to an isotopy from M_1 to itself. The geometric intersection number of T, T' with α_1 is zero. So we can adjust the isotopy so that the images of T never intersect α_1, and consequently we can further adjust it so that it leaves α_1 fixed pointwise. In other words this induces an isotopy in $M_1 - \alpha_1$ from T to T'. This shows that T is peripheral in $M_1 - \alpha_1$. This finishes the proof in this case.

Case 2 $-$ T is not π_1-injective in M.

In particular since T is two sided (as M is orientable), then T is compressible. This means that there is a closed disk D which compresses T [He], chapter 6. Since T is incompressible in $M_1 - \alpha_1$, then D intersects α_1. Let D_1, D_2 be parallel isotopic copies of D very near D which also are compressing disks for T. Then D_1, D_2 intersect T in two curves which partition T into two annuli. One annulus is very near both D_1 and D_2, we call it A_1, let A be the other annulus which is almost all of T. Then $A \cup D_1 \cup D_2$ is an embedded two dimensional sphere W. Since M is irreducible then W bounds a 3-ball B. There are two possibilities for the sphere W and ball B. In addition $A_2 \cup D_1 \cup D_2$ also obviously bounds a ball B_1 which is very near the disk D.

Suppose first that the ball B contains T. This means that A_2 and consequently also B_1, are both contained in B. In addition B_1 is a regular tubular neighborhood of a properly embedded arc γ in B. The intersection of α_1 with B_1 is a collection of arcs which are isotopic to the core γ of B_1. Let δ_1 be one such arc. It was proved in [Ba-Fe], Proposition 3.1, that flow lines of Anosov flows lift to unknotted curves in $\widetilde{M}$. This means that γ is unknotted in B and it implies that $\pi_1(B - \gamma)$ is $\mathbf{Z}$. In particular the torus T is compressible in $B - B_1$ and hence compressible in $M_1 - \alpha_1$. This contradicts the assumption that T is incompressible in $M_1 - \alpha_1$.

The second possibility is that the ball B does not contain T. In particular B and B_1 have disjoint interiors and the the union $B \cup B_1$ is a solid torus V with boundary T. This solid torus lifts to an infinite solid tube $\widetilde{V}$ in $\widetilde{M}$ with boundary $\widetilde{T}$ which is an infinite cylinder. Notice that there is a lift $\widetilde{\alpha}_1$ of α_1 contained in $\widetilde{V}$ so $\widetilde{T}$ cannot be compact. Again by the result of Proposition 3.1 of [Ba-Fe] the infinite curve $\widetilde{\alpha}_1$ is unknotted in $\widetilde{M}$ and hence it is isotopic to the core of $\widetilde{V}$.

Let β be a simple closed curve in V which is isotopic to the core of V. If α_1 is not isotopic to β then it is homotopic to a power β^n where $n > 1$. Projecting β to $\tau(\beta)$ in S we obtain a closed curve in S so that $(\tau(\beta))^n$ is freely homotopic to α. But α is an <u>indivisible</u> closed geodesic and represents an indivisible element of $\pi_1(S)$. It follows that this cannot happen. We conclude that α_1 is isotopic to the core of V. It follows that T is isotopic to the boundary of a regular neighborhood of α in $M_1 - \alpha$ and hence again T is peripheral in $M_1 - \alpha_1$.

This finishes the proof of proposition 2.1. □

Since $M_1 - \alpha_1$ is atoroidal then the hyperbolic Dehn surgery theorem of Thurston [Th1, Th2] implies that for almost all Dehn fillings along α_1, the resulting manifold is hyperbolic.

3 Diversified homotopic behavior of closed orbits

First we prove the statements about free homotopy classes of suspension Anosov flows and geodesic flows mentioned in the introduction. Suppose first that Φ is the geodesic flow in $M = T_1 S$, where S is a closed, orientable hyperbolic surface. Suppose that α, β are closed orbits of Φ which are freely homotopic to each other. Then the projection $\tau(\alpha), \tau(\beta)$ of these orbits to the surface S are freely homotopic. But $\tau(\alpha), \tau(\beta)$ are closed geodesics in a hyperbolic surface, so they are freely homotopic if and only if they are the same geodesic. If α and β are distinct, this can only happen if they represent the same geodesic $\tau(\alpha)$ of S which is being traversed in opposite direction. Conversely if $\tau(\alpha) = \tau(\beta)$ and they are traversed in opposite directions, there is a free homotopy from α to β. This is achieved by considering all unit tangent vectors to $\tau(\alpha)$ in the direction of α and then at time t, $0 \leq t \leq 1$, rotating all these vectors by an angle of $t\pi$. At $t = \pi$ we obtain the tangent vectors to $\tau(\alpha)$ pointing in the opposite direction, that is, the direction of β. This shows that every free homotopic class of the geodesic flow has exactly two elements. The orientability of S is used because if S is not orientable and $\tau(\alpha)$ is an orientation reversing closed geodesic, one cannot continuously turn the angle along $\tau(\alpha)$.

Now consider a suspension Anosov flow Φ. It was proved in [Fe2, Fe3] that for an arbitrary Anosov flow, if there are 2 closed orbits which are freely homotopic to each other then the following happens. There are closed orbits α and β so that α is freely homotopic to β^{-1} as oriented periodic orbits. For suspension Anosov flows this is a problem as follows. This is because there is a cross section W which intersects all orbits of Φ. Suppose that the algebraic intersection number of α and W is positive. Then since α is freely homotopic to β^{-1} it follows that the algebraic intersection number of β and W is negative. But this is impossible as W is a cross section and transverse to Φ. This shows that every free homotopy class of a suspension is a singleton. Another proof of this fact is the following. There is a path metric in M which comes from a Riemannian metric in the universal cover $\widetilde{M} \cong \mathbf{R}^3$ with coordinates (x, y, t) given by the formula $ds^2 = \lambda_1^{2t} dx^2 + \lambda_2^{-2t} dy^2 + dt^2$ (1), where λ_1, λ_2 are real numbers > 1. The lifted flow $\widetilde{\Phi}$ has formula $\widetilde{\Phi}_t(x, y, t_0) = (x, y, t_0 + t)$ (2). If α, β are freely homotopic closed orbits of $\widetilde{\Phi}$, then they lift to two distinct orbits of $\widetilde{\Phi}$ which are a bounded distance from each other. But formulas (1) and (2) show that no two distinct entire orbits of $\widetilde{\Phi}$ are a bounded distance from each other. This also shows that free homotopy classes are singletons.

The property of free homotopy classes for the examples in hyperbolic 3-manifolds is proved in [Fe1].

We now proceed the construct the examples and we prove the Main theorem.

Let S be a hyperbolic surface and α a closed geodesic that does not fill S. As in the previous section let S_1 be a subsurface that α fills and let S_2 be the closure of $S - S_1$. Let $M = T_1 S$ and Φ the geodesic flow of S in M. Let α_1 be an orbit of Φ so that $\tau(\alpha_1) = \alpha$. Let $M_i = T_1 S_i$, $i = 1, 2$. In the previous section we proved that $M_1 - \alpha_1$ is atoroidal.

Now we will do flow Dehn surgery on α_1. For simplicity we will assume that the stable foliation of Φ (or equivalently the unstable foliation of Φ) is transversely orientable. This is equivalent to the surface S being orientable. In particular this implies that the stable leaf of α_1 is a annulus. Let Z be the boundary of a small tubular solid torus neighborhood Z_0 of α_1 contained in M_1. Then Z is a two dimensional torus and we will choose a base for $\pi_1(Z) = H_1(Z)$. We assume that Z is transverse to the local sheet of the stable leaf of α_1. Then this local sheet intersects Z in a pair of simple closed curves. Each of these defines a longitude $(0, 1)$ in $\pi_1(Z)$, choose the direction which is isotopic to the flow forward direction along α_1. The boundary of a meridian disk in Z_0 defines the meridian curve $(0, 1)$ in $\pi_1(Z)$. The meridian is well defined up to sign. If the stable foliation of Φ were not transversely orientable and α were an orientation reversing curve, then the stable leaf of α would be a Möbius band and the intersection of the local sheet with Z would be a single closed curve. This closed curve would intersect the meridian twice and could not form a basis of $H_1(Z)$ jointly with the meridian. We do not want that, hence one of the reasons to restrict to S orientable.

Now we will perform Fried's Dehn surgery on α_1 [Fr]. This was extensively analysed in [Fe1]. If one does $(1, n)$ Dehn surgery the resulting flow is Anosov in the Dehn surgery manifold M_α. In addition as proved in [Fe1], with one of the choices of the meridian then for any $n > 0$ the Dehn surgery flow Φ_α with new meridian the $(1, n)$ curve is an $\mathbf{R}$-covered Anosov flow. The $\mathbf{R}$-covered property means that that the stable foliation lifts to a foliation in the universal cover $\widetilde{M}_\alpha$ which has leaf space homeomorphic to the reals $\mathbf{R}$. The surgery of Fried [Fr] is obtained by blowing up the orbit α_1 into a two dimensional boundary torus with an induced flow and then blowing it back down to a closed orbit of Φ_α using the new meridian $(1, n)$. In particular there is a bijection between the orbits of the surgered flow Φ_α and the orbits of the original flow Φ. Given an orbit γ of Φ_α we let γ' be the corresponding orbit of Φ under this bijection.

We are now ready to prove the prove the main result of this article.

Theorem 3.1. *(diversified homotopic behavior) Let S be an orientable, closed hyperbolic surface with a closed geodesic α which does not fill S. Let S_1 be a subsurface of S which is filled by α and let S_2 be the closure of $S - S_1$. We assume also that S_2 is not a union of annuli. Let $M = T_1 S$ with geodesic flow Φ and let $M_i = T_1 S_i$, $i = 1, 2$. Let α_1 be a closed orbit of Φ which projects to α in S. Do $(1, n)$ Fried's Dehn surgery along α_1 to yield a manifold M_α and an Anosov flow Φ_α so that Φ_α is $\mathbf{R}$-covered. Since M_2 is disjoint from α_1 it is unaffected by the Dehn surgery and we consider it also as a submanifold of M_α. Let M_3 be the closure of $M_\alpha - M_2$. We still denote by α_1 the orbit of Φ_α corresponding to α_1 orbit of Φ. Proposition 2.1 implies that $M_3 - \alpha_1$ is atoroidal and for n big the hyperbolic Dehn surgery theorem [Th1, Th2] implies that M_3 is hyperbolic. Choose one such n. Consider the bijection $\beta \to \beta'$ between closed orbits of Φ_α and those of Φ. Then the following happens:*

- *i) Let γ be a closed orbit of Φ_α so that the corresponding orbit γ' of Φ is homotopic into the submanifold M_2. Equivalently γ' projects to a geodesic in S which is disjoint from α in S. Then γ*

is freely homotopic in M_α to just one other closed orbit of Φ_α.

- *ii) Let γ be a closed orbit of Φ_α which corresponds to a closed orbit γ' of Φ which is not homotopic into M_2. Equivalently γ' projects to a geodesic in S which transversely intersects α. Then γ is freely homotopic in M_α to infinitely many other closed orbits of Φ_α.*

- *In addition both classes i) and ii) have infinitely many elements.*

Proof. First we prove that both classes i) and ii) are infinite. Since orbits of Φ_α are in one to one correspondence with orbits of Φ this is just a statement about closed orbits of Φ. Any closed geodesic of S which intersects α is in class ii). Clearly there are infinitely many such geodesics so class ii) is infinite. On the other hand since S_2 is not a union of annuli, there is a component S' which is not an annulus. Any geodesic β of S which is homotopic into S' creates an orbit in class i). Since S' is not an annulus, there are infinitely many such geodesics β. This proves that i) and ii) are infinite subsets.

An orbit δ of Φ which projects in S to a geodesic intersecting α cannot be homotopic into M_2. Otherwise the homotopy projects in S to an homotopy from a geodesic intersecting α to a curve in S_2 and hence to a geodesic not intersecting α. This is impossible as closed geodesics in hyperbolic surfaces intersect minimally. Conversely if an orbit δ projects to a geodesic not intersecting α, then this geodesic is homotopic to a geodesic contained in S_2. This homotopy lifts to a homotopy in M from δ to a curve in M_2. This proves the equivalence of the first 2 statements in i) and in ii).

Now we prove that conditions i), ii) imply the respective conclusions about the size of the free homotopy classes. Let $\widetilde{\Phi}_\alpha$ be the lifted flow to the universal cover $\widetilde{M}_\alpha$. Since Φ_α is Anosov, then the orbit space $\mathcal{O}$ of $\widetilde{\Phi}_\alpha$ is homeomorphic to the plane $\mathbf{R}^2$ [Fe1]. Let Λ^s, Λ^u be the stable and unstable foliations of Φ_α. The lifted stable and unstable foliations in $\widetilde{M}_\alpha$ are denoted by $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$. They induce one dimensional foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O}$.

Since Φ_α is $\mathbf{R}$-covered there are two possibilities for the topological structure of the lifted stable and unstable foliations to $\widetilde{M}_\alpha$ [Ba, Fe1]:

- Suppose that every leaf of $\widetilde{\Lambda}^s$ intersects every leaf of $\widetilde{\Lambda}^u$. Then Barbot [Ba] showed that Φ_α is topologically equivalent to a suspension Anosov flow. This implies that M_α fibers over the circle with fiber a torus. But M_α has a torus decomposition with one hyperbolic piece M_3 and one Seifert piece M_2. Therefore it cannot fiber over the circle with fiber a torus. We conclude that this cannot happen.

- The other possibility is that Φ_α is a *skewed* $\mathbf{R}$-covered Anosov flow. This means that $\mathcal{O}$ has a model homeomorphic to an infinite strip $(0,1) \times \mathbf{R}$. In addition it satisfies the following properties. The stable foliation $\mathcal{O}^s$ in $\mathcal{O}$ is a foliation by horizontal segments in $\mathcal{O}$. The unstable foliation is a foliation by parallel segments in $\mathcal{O}$ which make and angle θ which is not $\pi/2$ with the horizontal. That is, they are not vertical and hence an unstable leaf does not intersect every stable leaf and vice versa. We refer to figure 1.

Let then β_0 be a closed orbit of Φ_α. since Φ_α is an skewed $\mathbf{R}$-covered Anosov flow we will produce orbits β_i, $i \in \mathbf{Z}$ which are all freely homotopic to β_0. However it is not true a priori that all β_i are distinct from each other, this will be analysed later.

Lift β_0 to an orbit $\widetilde{\beta}_0$ contained in a stable leaf l_0 of $\mathcal{O}^s$. Let g be the deck transformation of $\widetilde{M}_\alpha$ which corresponds to β_0 in the sense that it generates the stabilizer of $\widetilde{\beta}_0$. Then $\mathcal{O}^u(\widetilde{\beta}_0)$ intersects an open interval I of stable leaves. This is a strict subset of the leaf space of $\mathcal{O}^s$ (equal to leaf space of $\widetilde{\Lambda}^s$) by the skewed property. Let l_1 be one of the two stable leaves in the boundary of this interval. The fact that there are exactly two boundary leaves in this interval is a direct consequence of the fact that $\mathcal{O}^s$ (or $\widetilde{\Lambda}^s$) has leaf space $\mathbf{R}$ and this fact is not true in general. Since $g(\mathcal{O}^u(\widetilde{\beta}_0)) = \mathcal{O}^u(\widetilde{\beta}_0)$ and g preserves the orientation of $\mathcal{O}^s$ (because Λ^s is transversely orientable), then $g(l_1) = l_1$. But this implies that there

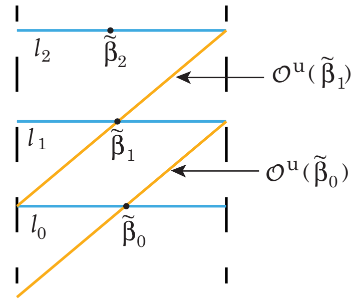

Figure 1: *The picture of the orbit space of a skewed* **R***-covered Anosov flow. It shows how to construct the orbits* $\widetilde{\beta}_i$ *starting with the orbit* $\widetilde{\beta}_0$.

is an orbit $\widetilde{\beta}_1$ of $\widetilde{\Phi}_\alpha$ in l_1 so that $g(\widetilde{\beta}_1) = \widetilde{\beta}_1$. We refer to fig. 1 which shows how to obtain leaf l_1 and hence the orbit $\widetilde{\beta}_1$. This orbit projects to a closed orbit β_1 of Φ_α in M_α. Since both are associated to g, it follows that β_0, β_1 are freely homotopic. More specifically if we care about orientations then the positively oriented orbit β_0 is freely homotopic to the inverse of the positively oriented orbit β_1.

Remark $-$ Transverse orientability of Λ^s is necessary for this. If for example Λ^s were not transversely orientable and the unstable leaf of β_0 were a Möbius band then the transformation g as constructed above does not preserve l_1 as constructed above. Therefore β_0 is not freely homotopic to β_1 as unoriented curves. But g^2 preserves l_1 and from this it follows that the square β_0^2 (as a non simple closed curve) is freely homotopic to β_1^2.

We proceed with the construction of freely homotopic orbits of Φ_α. From now on we iterate the procedure above: use $\mathcal{O}^u(\widetilde{\beta}_1)$ to produce a leaf l_2 of $\mathcal{O}^s$ invariant by g, and a closed orbit β_2 freely homotopic to β_1 $-$ if we consider them just as simple closed curves. We again refer to fig. 1. Now iterate and produce $\widetilde{\beta}_i, i \in \mathbf{Z}$ orbits of $\widetilde{\Phi}_\alpha$ so that they are all invariant under g and project to closed orbits β_i of Φ_α which are all freely homotopic to β_0 as unoriented curves.

In addition since β_0 is the only periodic orbit in $\Lambda^u(\beta_0)$, it follows that g only preserves the orbit $\widetilde{\beta}_0$ in $\mathcal{O}^u(\widetilde{\beta}_0)$. Therefore g does not leave invariant any stable leaf between $\mathcal{O}^s(\widetilde{\beta}_0)$ and $\mathcal{O}^s(\widetilde{\beta}_1)$ and similarly g does not leave invariant any stable leaf between $\mathcal{O}^s(\widetilde{\beta}_i)$ and $\mathcal{O}^s(\widetilde{\beta}_{i+1})$ for any $i \in \mathbf{Z}$. It follows that the collection $\{\mathcal{O}^s(\widetilde{\beta}_i), i \in \mathbf{Z}\}$ is exactly the collection of stable leaves left invariant by g.

Suppose now that δ is an orbit of Φ_α which is freely homotopic to β_0. We can lift the free homotopy so that β_0 lifts to $\widetilde{\beta}_0$ and δ lifts to $\widetilde{\delta}$. In particular g leaves invariant $\widetilde{\delta}$ and hence leaves invariant $\mathcal{O}^s(\widetilde{\delta})$. It follows that $\mathcal{O}^s(\widetilde{\delta}) = \mathcal{O}^s(\widetilde{\beta}_i)$ for some $i \in \mathbf{Z}$. As a consequence $\widetilde{\delta} = \widetilde{\beta}_i$ for $\widetilde{\beta}_i$ is the only orbit of $\widetilde{\Phi}_\alpha$ left invariant by g in $\mathcal{O}^s(\widetilde{\beta}_i)$. It follows that δ is one of $\{\beta_j, \ j \in \mathbf{Z}\}$.

Conclusion $-$ The free homotopy class of β_0 is finite if and only if the collection $\{\beta_i, i \in \mathbf{Z}\}$ is finite.

Suppose now that $\beta_i = \beta_j$ for some i, j distinct. Hence there is $f \in \pi_1(M_\alpha)$ with $f(\widetilde{\beta}_i) = \widetilde{\beta}_j$. Then f sends $\mathcal{O}^u(\widetilde{\beta}_i)$ to $\mathcal{O}^u(\widetilde{\beta}_j)$. Therefore f sends $\widetilde{\beta}_{i+1}$ to $\widetilde{\beta}_{j+1}$. Iterating this procedure shows that f preserves the collection $\{\widetilde{\beta}_k, \ k \in \mathbf{Z}\}$. In addition it follows easily that f sends $\widetilde{\beta}_0$ to $\widetilde{\beta}_k$ for $k = j - i$. The free homotopy from β_0 to $\beta_k = \beta_0$ produces a π_1-injective map of either the torus or the Klein bottle into M. We have to consider the Klein bottle because the free homotopy may be from β_0 to the inverse of β_0 when we account for orientations along orbits. Taking the square of this free homotopy if necessary we produce a π_1-injective map of the torus into M. The torus theorem [Ja, Ja-Sh] shows that the free homotopy is homotopic into a Seifert piece of the torus decomposition of M_α. Therefore the homotopy is freely homotopic into M_2. It follows that the orbit β'_0 of Φ associated to β_0 is freely homotopic into M_2. Therefore the geodesic $\tau(\beta'_0)$ of S does not intersect α.

This proves part ii) of the theorem: If the geodesic $\tau(\beta'_0)$ intersects α then the orbit β_0 of Φ_α is freely homotopic to infinitely many other closed orbits of Φ_α.

Consider now a closed orbit β_0 of Φ_α so that it corresponds to a geodesic in S which does not intersect α. This geodesic is $\tau(\beta'_0)$ which we denoted by γ. There is a non trivial free homotopy in $M = T_1S$ from β'_0 to itself with the same orientation, obtained by turning the angle along γ by a full turn, from 0 to 2π. Notice that this free homotopy at some point is exactly β'_0 and at another point it is exactly the orbit corresponding to the geodesic γ being traversed in the opposite direction. This free homotopy is entirely contained in M_2 and therefore this free homotopy survives in the Dehn surgered manifold M_α. By construction the image of the free homotopy in M_α contains two distinct closed orbits of Φ_α, one of which is β_0. In particular the free homotopy class of β_0 has at least two elements.

Let now δ be a closed orbit of Φ_α which is freely homotopic to β_0. In particular the free homotopy class of δ is the same as the free homotopy class of β_0 and this is finite. From the part we already proved in the theorem, it follows that δ is isotopic into M_2 and choosing M_2 appropriately we can assume that β_0, δ are contained in M_2. Let the free homotopy from β_0 to δ be realized by a π_1-injective annulus A which is in general position. The annulus A is a priori only immersed. Let $T = \partial M_3 = \partial M_2$ an embedded torus in M_α which is π_1-injective. Put A in general position with respect to T and analyse the self intersections. Any component which is null homotopic in T can be homotoped away because M_α is irreducible [He, Ja]. After this is eliminated each component of $A - T$ is an a priori only immersed annulus. But since M_3 is a hyperbolic manifold with a single boundary torus T it follows that M_3 is acylindrical [Th1, Th2]. This means that any π_1-injective properly immersed annulus is homotopic rel boundary into the boundary. This is because parabolic subgroups of the fundamental group of M_3 – as a Kleinian group, have an associated maximal $\mathbf{Z}^2$ parabolic subgroup [Th1, Th2]. In particular this implies that the annulus A can be homotoped away from M_3 to be entirely contained in M_2. Therefore the free homotopy represented by the annulus A survives if we undo the Dehn surgery on α. This produces a free homotopy between β'_0 and δ' in $M = T_1S$. Therefore there is only one possibility for δ if δ is distinct from β_0. This shows that the free homotopy class of β_0 has exactly two elements.

This finishes the proof of theorem 3.1 $\square$

4 Generalizations

There are a few ways to generalize the main result of this article. One construction is to first take a finite cover of order n of $M = T_1(S)$ unrolling the circle fibers. Then every closed orbit of the resulting flow is freely homotopic to $2n - 1$ other closed orbits of the flow. Do Dehn surgery on a closed orbit γ of the lifted flow so that this projects (eventually) to a closed geodesic in the surface which does not fill and some complementary component is not an annulus. Then the same proof as in the theorem 3.1 yields the result of theorem 3.1 except that some orbits are now freely homotopic to exactly $2n - 1$ other closed orbits.

Another constrution is to start with the geodesic flow in S and a finite collection of closed geodesics $\{\alpha_i, 1 \leq i \leq i_0\}$ which are pairwise disjoint and some component of the complement is not an annulus. Doing appropriate Fried Dehn surgeries on lifts of these geodesics yields $\mathbf{R}$-covered Anosov flows. The same arguments as in this article yield a result analogous to theorem 3.1.

One can also combine the two constructions above.

References

[An] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Inst. Math. **90** (1969).

[Ba] T. Barbot, *Caractérization des flots d'Anosov pour les feuilletages faibles*, Erg. Th. Dyn. Sys. **15** (1995) 247-270.

[Ba-Fe] T. Barthelmé and S. Fenley, *Knot theory of R-covered Anosov flows: homotopy versus isotopy of closed orbits*, to appear in Jour. Topol.

[Br] M. Brittenham, *Essential laminations in Seifert fibered spaces*, Topology **32** (1993) 61-85.

[Fe1] S. Fenley, *Anosov flows in 3-manifolds*, Ann. of Math. **139** (1994) 79-115.

[Fe2] S. Fenley, *Quasigeodesic Anosov flows and homotopic properties of flow lines*, Jour. Diff. Geom. **41** (1995) 479-514.

[Fe3] S. Fenley, *The structure of branching in Anosov flows of 3-manifolds*, Comm. Math. Helv. **73** (1998) 259-297.

[Fr] D. Fried, *Transitive Anosov flows and pseudo-Anosov maps*, Topology **22** (1983) 299-303.

[Ga] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. **136** (1992) 447-510.

[He] J. Hempel, *3-manifolds*, Ann. of Math. Studies **86**, Princeton Univ. Press, 1976.

[Ja] W. Jaco, *Lectures on three-manifold topology*, C.B.M.S. from A.M.S. **43**, 1980.

[Ja-Sh] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Memoirs A.M.S. **220**, 1979.

[Th1] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton University Lecture Notes, 1982.

[Th2] W. Thurston, *Three dimensional manifolds, Kleinian groups, and hyperbolic geometry*, Bull. A.M.S. **6** (1982) 357-381.

Florida State University
Tallahassee, FL 32306-4510