CLASSIFICATION AND RIGIDITY OF TOTALLY PERIODIC PSEUDO-ANOSOV FLOWS IN GRAPH MANIFOLDS

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Abstract − In this article we analyze totally periodic pseudo-Anosov flows in graph three manifolds. This means that in each Seifert fibered piece of the torus decomposition, the free homotopy class of regular fibers has a finite power which is also a finite power of the free homotopy class of a closed orbit of the flow. We show that each such flow is topologically equivalent to one of the model pseudo-Anosov flows which we previously constructed in [Ba-Fe]. A model pseudo-Anosov flow is obtained by glueing standard neighborhoods of Birkhoff annuli and perhaps doing Dehn surgery on certain orbits. We also show that two model flows on the same graph manifold are isotopically equivalent (ie. there is a isotopy of M mapping the oriented orbits of the first flow to the oriented orbits of the second flow) if and only if they have the same topological and dynamical data in the collection of standard neighborhoods of the Birkhoff annuli.

1. Introduction

Pseudo-Anosov flows are extremely common amongst three manifolds, for example[1]: 1) Suspension pseudo-Anosov flows [Th1, Th2, Th3], 2) Geodesic flows in the unit tangent bundle of negatively curved surfaces [An], 3) Certain flows transverse to foliations in closed atoroidal manifolds [Mo3, Cal1, Cal2, Cal3, Fe4]; flows obtained from these by either 4) Dehn surgery on a closed orbit of the pseudo-Anosov flow [Go, Fr], or 5) Shearing along tori [Ha-Th]; 6) Non transitive Anosov flows [Fr-Wi] and flows with transverse tori [Bo-La].

The purpose of this article is to analyse the question: how many essentially different pseudo-Anosov flows are there in a manifold? Two flows are essentially the same if they are *topologically equivalent*. This means that there is a homeomorphism between the manifolds which sends orbits of the first flow to orbits of the second flow preserving orientation along the orbits. In this article, we will also consider the notion of *isotopic equivalence*, i.e. a topological equivalence induced by an isotopy, that is, a homeomorphism isotopic to the identity.

We will restrict to closed, orientable, toroidal manifolds. In particular they are sufficiently large in the sense of [Wald3], that is, they have *incompressible surfaces* [He, Ja]. Manifolds with pseudo-Anosov flows are also irreducible [Fe-Mo]. It follows that these manifolds are Haken [Ja]. We have recently extended a result of the first author ([Ba2]) to the case of general pseudo-Anosov flows: if the ambient manifold is Seifert fibered, then the flow is up to finite cover topologically equivalent to a geodesic flow in the unit tangent bundle of a closed hyperbolic surface [Ba-Fe, Theorem A]. In addition we also proved that if the ambient manifold is a solvable three manifold, then the flow is topologically equivalent to a suspension Anosov flow [Ba-Fe, Theorem B]. Notice that in both cases the flow does not have singularities, that is, the type of the manifold strongly restricts the type of pseudo-Anosov that it can admit. This is in contrast with the strong flexibility in the construction of pseudo-Anosov flows − that is because many flows are constructed in atoroidal manifolds or are obtained by flow Dehn surgery on the pseudo-Anosov flow, which changes the topological type of the manifold. Therefore in many constructions one cannot expect that the underlying manifold is toroidal.

In this article we will mainly study pseudo-Anosov flows in graph manifolds. A *graph manifold* is an irreducible three manifold which is a union of Seifert fibered pieces. In a previous article [Ba-Fe] we produced a large new class of examples in graph manifolds. These flows are totally periodic. This means that each Seifert piece of the torus decomposition of the graph manifold is *periodic*, that is, up to finite powers, a regular fiber is freely homotopic to a closed orbit of the flow. More recently, Russ Waller [Wa]

[1] We also mention a recent work in progress by F. Béguin, C. Bonatti and Bin Yu, constructing a wide family of new Anosov flows; which can be seen as an extension of the construction in [Ba-Fe] ([BBB]).

has been studying how common these examples are, that is, the existence question for these type of flows. He showed that these flows are as common as they could be (modulo the necessary conditions).

In this article we will analyse the question of the classification and rigidity of such flows. In order to state and understand the results of this article we need to introduce the fundamental concept of a *Birkhoff annulus.* A Birkhoff annulus is an a priori only immersed annulus, so that the boundary is a union of closed orbits of the flow and the interior of the annulus is transverse to the flow. For example consider the geodesic flow of a closed, orientable hyperbolic surface. The ambient manifold is the unit tangent bundle of the surface. Let α be an oriented closed geodesic - a closed orbit of the flow - and consider a homotopy that turns the angle along α by π. The image of the homotopy from α to the same geodesic with opposite orientation is a Birkhoff annulus for the flow in the unit tangent bundle. If α is not embedded then the Birkhoff annulus is not embedded. In general Birkhoff annuli are not embedded, particularly in the boundary. A Birkhoff annulus is transverse to the flow in its interior, so it has induced stable and unstable foliations. The Birkhoff annulus is *elementary* if these foliations in the interior have no closed leaves.

In [Ba-Fe, Theorem F] we proved the following basic result about the relationship of a pseudo-Anosov flow and a periodic Seifert piece P: there is a finite collection of elementary Birkhoff annuli $A_1, \ldots, A_k$, which is unique up to homotopy along the flow (and isotopy restricted to the union of the interiors of the annuli A_i), such that:

– the interior of the A_i's are embedded and two-by-two disjoint;

– the periodic orbits which are boundary components of the Birkhoff annuli A_i are disjoint from the union of the interiors of the A_i's. These orbits are called the *vertical periodic orbits*,

– the *spine* is the union Z of all (closed) Birkhoff annuli A_i. It is a deformation retract of P,

– the spine is minimal with respect to these conditions.

In particular, Z is connected and the vertical orbits are uniquely determined because of uniqueness up to flow homotopy. Without the minimality condition the spine may not be unique. In general it may happen that there is a Birkhoff annulus A' so that $Z \cup A'$ is connected, and $Z \cup A'$ also satisfies all the properties of the definition of the spine, except for the minimality condition. See the details in [Ba-Fe] – such an annulus A' is associated with a lozenge which is not in the axis of any element of $\pi_1(P)$ acting freely in the orbit space. We will also abuse terminology in this article and say that Z is unique up to flow isotopy to mean that it is unique up to flow homotopy and unique up to flow isotopy in the interior of the Birkhoff annuli.

In general the Birkhoff annuli are not embedded in Z: it can be that the two boundary components of the same Birkhoff annulus are the same vertical periodic orbit of the flow. It can also occur that the annulus wraps a few times around one of its boundary orbits. These are not exotic occurrences, but rather fairly common.

In the sequel, we denote by $N(Z)$ a *representative* tubular neighborhood of Z, i.e. a tubular neighborhhod such that the local flow induced by Φ admits the following properties:

– Z is a deformation retract of $N(Z)$ (hence is isotopic to P),

- every orbit of the local flow crossing one the A_i enters in $N(Z)$ at a point where it is transverse, then crosses Z at only one point, and exits from $N(Z)$ at a point where it is transverse,

– for every vertical periodic orbit α in $N(Z)$, the local stable leaf of α is a finite union of annuli, called *stable vertical annuli,* tangent to the flow, each transverse to $\partial N(Z)$ and realizing a homotopy between (a power of) α and a closed loop in $\partial N(Z)$. One defines similarly *unstable* vertical annuli in $N(Z)$.

These conditions still allow that some orbits of the local flow in $N(Z)$ do not cross the spine, or that the flow is not transverse to the boundary of $N(Z)$, but it will follow from Theorems B and C below that these situations do not arise in the case of totally periodic pseudo-Anosov flows and that in this situation every orbit of the local flow either meet Z or lies in a vertical annulus.

We first analyse periodic Seifert pieces. The first theorems (Theorem A and B) are valid for any closed orientable manifold M, not necessarily a graph manifold. The first result is (see Proposition 3.2):

Theorem A – Let Φ be a pseudo-Anosov flow in M^3. If $\{P_i\}$ is the (possibly empty) collection of periodic Seifert pieces of the torus decomposition of M, then the spines Z_i and neighborhoods $N(Z_i)$ can be chosen to be pairwise disjoint.

We prove that the $\{Z_i\}$ can be chosen pairwise disjoint. Roughly this goes as follows: we show that the vertical periodic orbits in Z_i cannot intersect Z_j for $j \neq i$, because fibers in different Seifert pieces cannot have common powers which are freely homotopic. We also show that the possible interior intersections are null homotopic and can be isotoped away.

The next result (Proposition 3.4) shows that the boundary of the pieces can be put in good position with respect to the flow:

Theorem B – Let Φ be a pseudo-Anosov flow and P_i, P_j be periodic Seifert pieces with a common boundary torus T. Then T can be isotoped to a torus transverse to the flow.

The main property used to prove this result is that regular fibers restricted to both sides of T (from P_i and P_j) cannot represent the same isotopy class in T.

Finally we prove the following (Proposition 3.5):

Theorem C – Let Φ be a totally periodic pseudo-Anosov flow with periodic Seifert pieces $\{P_i\}$. Then neighborhoods $\{N(Z_i)\}$ of the spines $\{Z_i\}$ can be chosen so that their union is M and they have pairwise disjoint interiors. In addition each boundary component of every $N(Z_i)$ is transverse to the flow. Each $N(Z_i)$ is flow isotopic to an arbitrarily small neighborhood of Z_i.

We stress that for general periodic pieces it is not true that the boundary of $N(Z_i)$ can be isotoped to be transverse to the flow. There are some simple examples as constructed in [Ba-Fe]. The point here is that we assume that *all* pieces of the JSJ decomposition are periodic Seifert pieces.

Hence, according to Theorem C, totally periodic pseudo-Anosov flow are obtained by gluing along the boundary a collection of small neighborhoods $N(Z_i)$ of the spines. There are several ways to perform this gluing which lead to pseudo-Anosov flows. The main result of this paper is that the resulting pseudo-Anosovs flows are all topologically equivalent one to the other. More precisely (see section 5.1):

Theorem D – Let Φ, Ψ be two totally periodic pseudo-Anosov flows on the same orientable graph manifold M. Let P_i be the Seifert pieces of M, and let $Z_i(\Phi)$, $Z_i(\Psi)$ be spines of Φ, Ψ in P_i. Then, Φ and Ψ are topologically equivalent if and only if there is a homeomorphism of M mapping the collection of spines $\{Z_i(\Phi)\}$ onto the collection $\{Z_i(\Psi)\}$ and preserving the orientations of the vertical periodic orbits induced by the flows.

Theorem D is a consequence of the following Theorem, more technical but slightly more precise (see section 5.2):

Theorem D' – Let Φ, Ψ be two totally periodic pseudo-Anosov flows on the same orientable graph manifold M. Let P_i be the Seifert pieces of M, and let $Z_i(\Phi)$ be spines of Φ, with tubular neighborhoods $N(Z_i(\Phi))$ as in the statement of Theorem C. Then, Φ and Ψ are isotopically equivalent if and only if, there is an isotopy in M mapping every spine $Z_i(\Phi)$ onto spines $Z_i(\Psi)$ of Ψ, mapping every stable/unstable vertical annulus of Φ in $N(Z_i(\Phi))$ to a stable/vertical annulus in $N(Z_i(\Psi))$ and preserving the orientations of the vertical periodic orbits induced by the flows.

The main ideas of the proof are as follows. One implication is obvious: if the two flows are isotopically equivalent, the isotopy from the definition of isotopically equivalent, maps every $Z_i(\Phi)$ onto a spine $Z_i(\Psi)$ of Ψ, with all the required properties.

Conversely, assume that up to isotopy Φ and Ψ admit the same decomposition in neighborhoods $N(Z_i)$ of spines Z_i, so that they share exactly the same oriented vertical periodic orbits and the same stable/unstable vertical annuli. Consider all the lifts to the universal cover of the tori in $\partial N(Z_i)$ for all i. This is a collection $\mathcal{T}$ of properly embedded topological planes in $\widetilde{M}$, which is transverse to the lifted flows $\widetilde{\Phi}$ and $\widetilde{\Psi}$. We show that an orbit of $\widetilde{\Phi}$ or $\widetilde{\Psi}$ (if not the lift of a vertical periodic orbits) is completely determined by its itinerary up to shifts: the itinerary is the collection of planes it intersects. One thus gets a map between orbits of $\widetilde{\Phi}$ and orbits of $\widetilde{\Psi}$. This extends to the lifts of the vertical periodic orbits. This is obviously group equivariant. The much harder step is to prove that this is continuous, which we do using the exact structure of the flows and the combinatorics. Using this result we can then show that the flow Φ is topologically equivalent to Ψ. Since the action on the fundamental group level is trivial, this topological equivalence is homotopic to the identity, hence, by a Theorem by Waldhausen ([Wald3]), isotopic to the identity: it is an isotopic equivalence.

We then show (section 6.2) that for any totally periodic pseudo-Anosov flow Φ there is a model pseudo-Anosov flow as constructed in [Ba-Fe] which has precisely the same data Z_i, $N(Z_i)$ that Φ has. This proves the following:

Main theorem – Let Φ be a totally periodic pseudo-Anosov flow in a graph manifold M. Then Φ is topologically equivalent to a model pseudo-Anosov flow.

Model pseudo-Anosov flows are defined by some combinatorial data (essentially, the data of some fat graphs and Dehn surgery coefficients; see section 6.1 for more details) and some parameters λ_i (one for each Seifert piece P_i). A nice corollary of Theorem D' is that, up to isotopic equivalence, the model flows actually do not depend on the choice of the λ_i's, nor on the choice of the selection of the particular gluing map between the model periodic pieces.

In the last section, we make a few remarks on the action of the mapping class group of M on the space of isotopic equivalence classes of totally periodic pseudo-Anosov flows on M.

2. Background

Pseudo-Anosov flows – definitions

Definition 2.1. *(pseudo-Anosov flow) Let Φ be a flow on a closed 3-manifold M. We say that Φ is a pseudo-Anosov flow if the following conditions are satisfied:*

- For each $x \in M$, the flow line $t \to \Phi(x,t)$ is C^1, it is not a single point, and the tangent vector bundle $D_t\Phi$ is C^0 in M.

- There are two (possibly) singular transverse foliations Λ^s, Λ^u which are two dimensional, with leaves saturated by the flow and so that Λ^s, Λ^u intersect exactly along the flow lines of Φ.

- There is a finite number (possibly zero) of periodic orbits $\{\gamma_i\}$, called singular orbits. A stable/unstable leaf containing a singularity is homeomorphic to $P \times I/f$ where P is a p-prong in the plane and f is a homeomorphism from $P \times \{1\}$ to $P \times \{0\}$. In addition p is at least 3.

- In a stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backwards asymptotic.

Basic references for pseudo-Anosov flows are [Mo1, Mo2] and [An] for Anosov flows. A fundamental remark is that the ambient manifold supporting a pseudo-Anosov flow is necessarily irreducible - the universal covering is homeomorphic to $\mathbf{R}^3$ ([Fe-Mo]). We stress that in our definition one prongs are not allowed. There are however "tranversely hyperbolic" flows with one prongs:

Definition 2.2. *(one prong pseudo-Anosov flows) A flow Φ is a one prong pseudo-Anosov flow in M^3 if it satisfies all the conditions of the definition of pseudo-Anosov flows except that the p-prong singularities can also be 1-prong ($p = 1$).*

Torus decomposition

Let M be an irreducible closed 3–manifold. If M is orientable, it has a unique (up to isotopy) minimal collection of disjointly embedded incompressible tori such that each component of M obtained by cutting along the tori is either atoroidal or Seifert-fibered [Ja, Ja-Sh] and the pieces are isotopically maximal with this property. If M is not orientable, a similar conclusion holds; the decomposition has to be performed along tori, but also along some incompressible embedded Klein bottles.

Hence the notion of maximal Seifert pieces in M is well-defined up to isotopy. If M admits a pseudo-Anosov flow, we say that a Seifert piece P is *periodic* if there is a Seifert fibration on P for which, up to finite powers, a regular fiber is freely homotopic to a periodic orbit of Φ. If not, the piece is called *free.*

Remark. In a few circumstances, the Seifert fibration is not unique: it happens for example when P is homeomorphic to a twisted line bundle over the Klein bottle or P is $T^2 \times I$. We stress out that our convention is to say that the Seifert piece is free if no Seifert fibration in P has fibers homotopic to a periodic orbit.

Orbit space and leaf spaces of pseudo-Anosov flows

Notation/definition: We denote by $\pi : \widetilde{M} \to M$ the universal covering of M, and by $\pi_1(M)$ the fundamental group of M, considered as the group of deck transformations on $\widetilde{M}$. The singular foliations lifted to $\widetilde{M}$ are denoted by $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$. If $x \in M$ let $W^s(x)$ denote the leaf of Λ^s containing x. Similarly one defines $W^u(x)$ and in the universal cover $\widetilde{W}^s(x), \widetilde{W}^u(x)$. Similarly if α is an orbit of Φ define $W^s(\alpha)$, etc... Let also $\widetilde{\Phi}$ be the lifted flow to $\widetilde{M}$.

We review the results about the topology of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ that we will need. We refer to [Fe2, Fe3] for detailed definitions, explanations and proofs. The orbit space of $\widetilde{\Phi}$ in $\widetilde{M}$ is homeomorphic to the plane $\mathbf{R}^2$ [Fe-Mo] and is denoted by $\mathcal{O} \cong \widetilde{M}/\widetilde{\Phi}$. There is an induced action of $\pi_1(M)$ on $\mathcal{O}$. Let

$$\Theta : \ \widetilde{M} \ \to \ \mathcal{O} \ \cong \ \mathbf{R}^2$$

be the projection map: it is naturally $\pi_1(M)$-equivariant. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then $\Theta(L) \subset \mathcal{O}$ is a tree which is either homeomorphic to $\mathbf{R}$ if L is regular, or is a union of p-rays all with the same starting point if L has a singular p-prong orbit. The foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ induce $\pi_1(M)$-invariant singular 1-dimensional foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O}$. Its leaves are $\Theta(L)$ as above. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then a *sector* is a component of $\widetilde{M} - L$. Similarly for $\mathcal{O}^s, \mathcal{O}^u$. If B is any subset of $\mathcal{O}$, we denote by $B \times \mathbf{R}$ the set $\Theta^{-1}(B)$. The same notation $B \times \mathbf{R}$ will be used for any subset B of $\widetilde{M}$: it will just be the union of all flow lines through points of B. We stress that for pseudo-Anosov flows there are at least 3-prongs in any singular orbit ($p \geq 3$). For example, the fact that the orbit space in $\widetilde{M}$ is a 2-manifold is not true in general if one allows 1-prongs.

Definition 2.3. *Let L be a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. A slice of L is $l \times \mathbf{R}$ where l is a properly embedded copy of the reals in $\Theta(L)$. For instance if L is regular then L is its only slice. If a slice is the boundary of a sector of L then it is called a line leaf of L. If a is a ray in $\Theta(L)$ then $A = a \times \mathbf{R}$ is called a half leaf of L. If ζ is an open segment in $\Theta(L)$ it defines a* flow band *L_1 of L by $L_1 = \zeta \times \mathbf{R}$. We use the same terminology of slices and line leaves for the foliations $\mathcal{O}^s, \mathcal{O}^u$ of $\mathcal{O}$.*

If $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$ then F and G intersect in at most one orbit.

We abuse convention and call a leaf L of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ *periodic* if there is a non trivial covering translation g of $\widetilde{M}$ with $g(L) = L$. This is equivalent to $\pi(L)$ containing a periodic orbit of Φ. In the same way an orbit γ of $\widetilde{\Phi}$ is *periodic* if $\pi(\gamma)$ is a periodic orbit of Φ. Observe that in general, the stabilizer of an element α of $\mathcal{O}$ is either trivial, or a cyclic subgroup of $\pi_1(M)$.

Perfect fits, lozenges and scalloped chains

Recall that a foliation $\mathcal{F}$ in M is $\mathbf{R}$-covered if the leaf space of $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ is homeomorphic to the real line $\mathbf{R}$ [Fe1].

Definition 2.4. *(*[Fe2, Fe3]*) Perfect fits - Two leaves $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$, form a perfect fit if $F \cap G = \emptyset$ and there are half leaves F_1 of F and G_1 of G and also flow bands $L_1 \subset L \in \widetilde{\Lambda}^s$ and $H_1 \subset H \in \widetilde{\Lambda}^u$, so that the set*

$$\overline{F}_1 \cup \overline{H}_1 \cup \overline{L}_1 \cup \overline{G}_1$$

separates M and forms an a rectangle R with a corner removed: The joint structure of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in R is that of a rectangle with a corner orbit removed. The removed corner corresponds to the perfect of F and G which do not intersect.

We refer to fig. 1, a for perfect fits. There is a product structure in the interior of R: there are two stable boundary sides and two unstable boundary sides in R. An unstable leaf intersects one stable boundary side (not in the corner) if and only if it intersects the other stable boundary side (not in the corner). We also say that the leaves F, G are *asymptotic*.

Definition 2.5. *(*[Fe2, Fe3]*) Lozenges - A lozenge R is an open region of $\widetilde{M}$ whose closure is homeomorphic to the product of the real line with a closed rectangle with two corners removed. More specifically two points p, q (possibly singular) define the corners of a lozenge if there are half leaves A, B of $\widetilde{W}^s(p), \widetilde{W}^u(p)$*

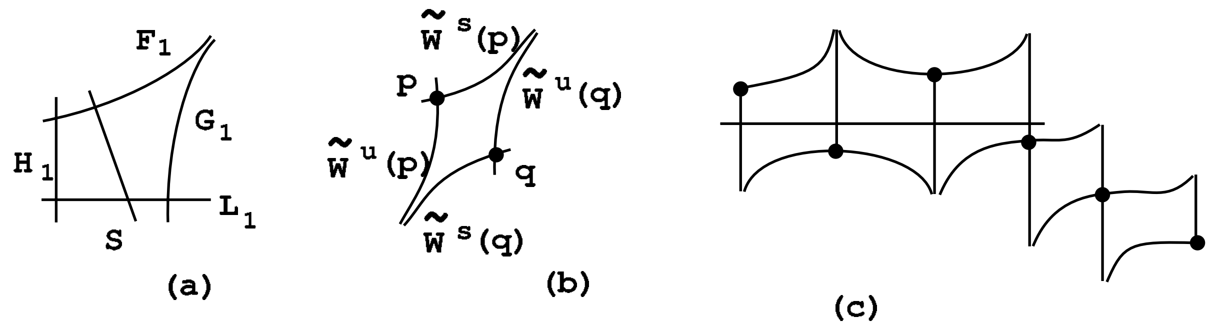

Figure 1: *a. Perfect fits in $\widetilde{M}$, b. A lozenge, c. A chain of lozenges.*

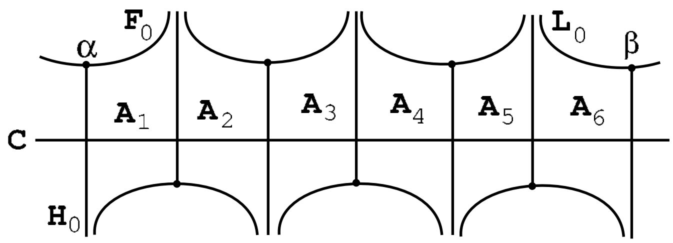

Figure 2: *A partial view of a scalloped region. Here C, F_0, L_0 are stable leaves, so this is a s-scalloped region.*

defined by p and C, D half leaves of $\widetilde{W}^s(q), \widetilde{W}^u(q)$ defined by p, q, so that A and D form a perfect fit and so do B and C. The lozenge R does not have any singularities. The sides of R are A, B, C, D. The sides are not contained in the lozenge, but are in the boundary of the lozenge. See fig. 1, b.

There may be singularities in the boundary of the lozenge: on the sides A, B, C and D, or in the corner orbits.

Two lozenges are *adjacent* if they share a corner and there is a stable or unstable leaf intersecting both of them, see fig. 1, c. Therefore they share a side. A *chain of lozenges* is a collection $\{\mathcal{C}_i\}, i \in I$, where I is an interval (finite or not) in $\mathbf{Z}$; so that if $i, i+1 \in I$, then $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ share a corner, see fig. 1, c. Consecutive lozenges may be adjacent or not. The chain is finite if I is finite.

Definition 2.6. *(scalloped chain) Let $\mathcal{C}$ be a chain of lozenges. If any two successive lozenges in the chain are adjacent along one of their unstable sides (respectively stable sides), then the chain is called* s-scalloped *(respectively* u-scalloped*) (see fig. 2 for an example of a s-scalloped chain). Observe that a chain is s-scalloped if and only if there is a stable leaf intersecting all the lozenges in the chain. Similarly, a chain is u-scalloped if and only if there is an unstable leaf intersecting all the lozenges in the chain. The chains may be infinite. A scalloped chain is a chain that is either s-scalloped or u-scalloped.*

For simplicity when considering scalloped chains we also include any half leaf which is a boundary side of two of the lozenges in the chain. The union of these is called a *scalloped region* which is then a connected set.

We say that two orbits γ, α of $\widetilde{\Phi}$ (or the leaves $\widetilde{W}^s(\gamma), \widetilde{W}^s(\alpha)$) are connected by a chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$, if γ is a corner of $\mathcal{C}_1$ and α is a corner of $\mathcal{C}_n$.

Remark 2.7. A key fact, first observed in [Ba3], and extensively used in [Ba-Fe], is that the lifts in $\widetilde{M}$ of elementary Birkhoff annuli are related to lozenges invariant by some cyclic subgroup of $\pi_1(M)$ (see [Ba3, Proposition 5.1] for the case of embedded Birkhoff annuli). It will also play a crucial role in the sequel. More precisely: let A be an elementary Birkhoff annulus. We say that A lifts to the lozenge C in $\widetilde{M}$ if the saturation under $\widetilde{\Phi}$ of the interior of A is contained in C. It follows that this lift intersects every orbit in C exactly once and also that the two boundary closed orbits of A lift

to the full corner orbits of C.

In particular the following important property also follows: if α and β are the periodic orbits in ∂A (traversed in the flow forward direction), then there are positive integers n, m so that α^n is freely homotopic to $(\beta^m)^{-1}$. We emphasize the free homotopy between inverses.

Remark 2.8. According to remark 2.7, chains of lozenges correspond to sequences of Birkhoff annuli, every Birkhoff annulus sharing a common periodic orbit with the previous element of the sequence, and also a periodic orbit with the next element in the sequence. When the sequence closes up, it provides an immersion $f : T^2$ (or K) $\to M$, which is called a *Birkhoff torus* (if the cyclic sequence contains an even number of Birkhoff annuli), or a *Birkhoff Klein bottle* (in the other case).

3. **Disjoint pieces and transverse tori**

As mentioned in the introduction, in [Ba-Fe] section 7, we proved that if P is a periodic Seifert fibered piece of M with a pseudo-Anosov flow Φ then the following happens: there is a connected, finite union Z of elementary Birkhoff annuli, which is weakly embedded $-$ this means that restricted to the union of the interiors of the Birkhoff annuli it is embedded and the periodic orbits are disjoint from the interiors. We call such a Z a *spine* for the Seifert piece P.

In addition a compact neighborhood $N(Z)$ is a *representative* for the Seifert fibered piece P if:

– Z is a deformation retract of $N(Z)$ (hence is isotopic to P),

– every orbit of the local flow crossing one the A_i enters in $N(Z)$ at a point where it is transverse, then crosses Z at only one point, and exits from $N(Z)$ at a point where it is transverse,

– for every vertical periodic orbit α in $N(Z)$, the local stable leaf of α is a finite union of annuli, called *stable vertical annuli,* tangent to the flow, each transverse to $\partial N(Z)$ and realizing a homotopy between (a power of) α and a closed loop in $\partial N(Z)$. One defines similarly *unstable* vertical annuli in $N(Z)$.

We select one initial representative $N(Z)$, which will be modified all along the section.

In this section we prove several important results concerning the relative position of spines of distinct periodic Seifert pieces (if there are such), and we prove that the representatives $N(Z)$ can be chosen so that every boundary component of $N(Z)$ are transverse to the flow and that the retraction of $N(Z)$ to Z can be performed along the flow Φ.

Let us recall briefly how the spine Z is constructed in [Ba-Fe]: let h be the element of $\pi_1(P)$ corresponding to regular fibers of P: h lies in the pseudo-center of $\pi_1(P)$ $-$ actually, it generates the pseudo-center except in a few elementary cases. We consider the graph $\mathcal{T}$ whose vertices are fixed points of h in $\mathcal{O}$, and whose edges correspond to projections in $\mathcal{O}$ of h-invariant lozenges (cf. definition 2.11 in [Ba-Fe]). Then, $\mathcal{T}$ is connected, $\pi_1(P)$-invariant. We define the subtree $\mathcal{T}'$ of $\mathcal{T}$ which is the union of axes of elements of $\pi_1(P)$ acting freely on $\mathcal{T}$: it is a connected $\pi_1(P)$-invariant subtree (see the paragraph "Pruning the tree" in [Ba-Fe][section 7]). Every edge of $\mathcal{T}'$ corresponds to a h-invariant lozenge which is the lift of a weakly embedded elementary Birkhoff annulus. According to the following lemma, this elementary Birkhoff annulus is well-defined up to homotopy along the flow:

Lemma 3.1. *Let A_1, A_2 be two elementary Birkhoff annuli which lift to the same lozenge C in $\widetilde{M}$ and so that the cores of A_1, A_2 are freely homotopic. Then A_1 is flow homotopic to A_2 in the interior. That is, there is a homotopy f_t from A_1 to A_2 so that for any x in the interior of A_1, $f_{[0,1]}(x)$ is contained in a flow line of Φ. In addition if x is a point where A_1 does not self intersect, then $f_t([0,1])(x)$ is a set of no self intersections, of the homotopy.*

Proof. Choose fixed lifts $\widetilde{A}_1, \widetilde{A}_2$ so that the interiors intersect exactly the orbits in C. Let g in $\pi_1(M)$ so that it generates $Stab(\widetilde{A}_i)$. The fact that a single g generates both stabilizers uses the condition on the cores of A_1, A_2.

Let E be the interior of $\widetilde{A}_1$. For any p in E, there is a unique real number $t(p)$ so that $\widetilde{\Phi}_{t(p)}(p) = \eta_i(p)$ is a point in $\widetilde{A}_2$. This map $t(p)$ is continuous and clearly equivariant under g: $\widetilde{\Phi}_{t(g(p))}\ (g(p)) = g(\eta_i(p))$. Since this is equivariant, it projects to a map from the interior of A_1 to the interior of A_2. The linear homotopy along the orbits in the required homotopy. The homotopy is an isotopy where A_1 is embedded. $\square$

The fact that all these Birkhoff annuli can be selected so that their union is weakly embedded is proved in the paragraph "Weaky embedded union of Birkhoff annuli" of [Ba-Fe][section 7]. It follows easily from this description and from lemma 3.1 that *the spine Z is unique up to homotopy along the flow and unique up to isotopy restricted to the union of the interiors of the Birkhoff annuli.* In particular, the *vertical* *orbits*, i.e. the periodic orbits of each Z_i are uniquely determined by P_i and Φ. In this article we abuse terminology and say that Z is unique up to flow isotopy.

Proposition 3.2. *Let Φ be a pseudo-Anosov flow and let $\{P_i,\ 1 \leq i \leq n\}$ (where n may be 0) be the periodic Seifert pieces of Φ. Then we may choose the spines Z_i of P_i so that they are pairwise disjoint.*

Proof. We prove the proposition in several steps.

I) For any $i \neq j$, a periodic orbit in Z_i does not intersect Z_j.

In this part there will be no need to make adjustments to the Birkhoff annuli. Suppose α is a closed orbit in Z_i which intersects Z_j with j, i distinct. The first situation is that α intersects a closed orbit β in Z_j, in which case $\alpha = \beta$. Recall that in a periodic Seifert piece some power α^{n_i} represents a regular fiber in Z_i and similarly some power β^{n_j} represents a regular fiber in Z_j. But then the regular fibers in Z_i, Z_j have common powers $-$ this is impossible for distinct Seifert fibered pieces of M [He].

The second situation is that α intersects the interior of a Birkhoff annulus A in Z_j. Since Z_j is a spine for the Seifert piece P_j, there is an immersed Birkhoff torus T in Z_j containing A. In addition choose T to be π_1-injective. This can be achieved by looking at a lift $\widetilde{T}$ to $\widetilde{M}$ and the sequence of lozenges intersected by $\widetilde{T}$. If there is no backtracking in the sequence of lozenges then T is π_1-injective. It is easy to choose one such T with no backtracking.

Fix a lift $\widetilde{A}$ of A contained in a lift $\widetilde{T}$ of T and let $\widetilde{\alpha}$ be a lift of α intersecting $\widetilde{A}$. Since T is incompressible, $\widetilde{T}$ is a properly embedded plane in $\widetilde{M}$. The topological plane $\widetilde{T}$ is contained (except for the lifts of the periodic orbits) in a bi-infinite chain of lozenges $\mathcal{C}$. Any orbit in the interior of one of the lozenges in $\mathcal{C}$ intersects $\widetilde{T}$ exactly once.

Since α corresponds to a closed curve in P_i and Z_j is isotopic into P_j, then α can be homotoped to be disjoint from Z_j. Lift this homotopy to $\widetilde{M}$ from $\widetilde{\alpha}$ to a bi-infinite curve in $\widetilde{M}$ disjoint from $\widetilde{T}$. Recall that $\widetilde{\alpha}$ intersects $\widetilde{T}$ in a single point. This implies that a whole ray r^+ of $\widetilde{\alpha}$ has to move across $\widetilde{T}$ by the homotopy. Hence this ray is at bounded distance a_0 from $\widetilde{T}$. As α is compact this implies that a power of α is freely homotopic into T. More precisely: let g be the covering translation which is a generator of $Stab(\widetilde{\alpha})$ and such that $g^{-1}(r^+) \subset r^+$. Let p be the initial point of r^+. For every $n > 0$ let $q_n = g^{-n}(p)$ which is a point in r^+ and hence $d(q_n, \widetilde{T}) < a_0$ for all n. Then $g^n(q_n) = p$ is at a distance $< a_0$ from $g^n(\widetilde{T})$ for any $n > 0$. But there are only finitely many translates of $\widetilde{T}$ which intersect any compact set in $\widetilde{M}$. It follows that there is $n > 0$ so that $g^n(\widetilde{T}) = \widetilde{T}$, meaning that α^n is freely homotopic into T. In addition g^n is in $Stab(\widetilde{T})$.

This is now a contradiction because g^n leaves invariant the bi-infinite chain of lozenges $\mathcal{C}$. In addition g leaves $\widetilde{\alpha}$ invariant, so $g^n(\widetilde{\alpha}) = \widetilde{\alpha}$. Since g^n also leaves $\mathcal{C}$ invariant, then g^n leaves invariant the lozenge B of $\mathcal{C}$ containing $\widetilde{\alpha}$. But then g^n would leave invariant the pair of corners of B, contradiction to leaving invariang $\widetilde{\alpha}$. We conclude that this cannot happen. This finishes part I).

II) Suppose that for some $i \neq j$ there are Birkhoff annuli $A \subset Z_i, B \subset Z_j$ so that $A \cap B \neq \emptyset$.

Notice that the intersections are in the interior by part I). Recall also that the interiors of the Birkhoff annuli are embedded. By a small perturbation put the collection $\{Z_k\}$ in general position with respect to itself. Let δ be a component of $A \cap B$.

Suppose first that δ is not null homotopic in A. Since A is π_1-injective, then the same is true for δ in M and δ in B. Then δ is homotopic in A to a power of a boundary of A, which itself has a common power with the regular fiber of P_i. This implies that the fibers in P_i, P_j have common powers, contradiction as in part I).

It follows that δ is null homotopic in A and hence bounds a disc D in A. Notice that δ is embedded as both A and B have embedded interiors. We proceed to eliminate such intersections by induction. We assume that δ is innermost in D: the interior of D does not intersect any $Z_k, k \neq i$ (switch j if necessary). In addition δ also bounds a disc D' in B whose interior is disjoint from D $-$ by choice of D. Hence $D \cup D'$ is an embedded sphere which bounds a ball B in M $-$ because M is irreducible. We can use this ball to isotope Z_j to replace a neighborhood of D' in B by a disc close to D and disjoint from D, eliminating the intersection δ and possibly others. Induction eliminates all intersections so we can assume that all $\{Z_k\}$ are disjoint (for a more detailed explanation of this kind of argument, see [Ba3, section 7]).

Notice that the modifications in $\{Z_k\}$ in part II) were achieved by isotopies. This finishes the proof of the proposition. $\square$

Recall that we are assuming the manifold M to be orientable, so that we can use [Ba-Fe, Theorem F]; however the following Lemma holds in the general case, hence we temporally drop the orientability hypothesis.

Lemma 3.3. *(local transversality) Let V be an immersed Birkhoff torus or Birkhoff Klein bottle with no backtracking – this means that for any lift of V to $\widetilde{M}$, the sequence of lozenges associated to it has no backtracking. Let $\widetilde{V}$ be a fixed lift of V to $\widetilde{M}$ and let $\widetilde{\alpha}$ a lift to $\widetilde{V}$ of a closed orbit α in V. There are well defined lozenges B_1, B_2 in $\widetilde{M}$ which contain a neighborhood of $\widetilde{\alpha}$ in $\widetilde{V}$ (with $\widetilde{\alpha}$ removed): $\widetilde{\alpha}$ is a corner of both B_1 and B_2. If B_1, B_2 are adjacent lozenges then V can be homotoped to a torus or Klein bottle V' transverse to Φ near α, which furthermore satisfies the following additional property: for every element x of V', either x lies in the local stable leaf leaf of α (if B_1, B_2 are adjacent along a stable leaf), the local unstable leaf of α (if B_1, B_2 are adjacent along an unstable leaf), or x lies in the Φ-orbit of an element of V. In other words, any neighborhood of V contains a torus or Klein bottle flow homotopic to V'.*

Conversely, if V can be homotoped to be transverse to Φ near α then B_1 and B_2 are adjacent. This is independent of the lift $\widetilde{V}$ of V and of $\widetilde{\alpha}$.

Proof. Formally we are considering a map $f : T^2$ (or K) $\rightarrow M$ so that the image is the union V of (immersed) Birkhoff annuli. The homotopy is a homotopy of the map f and it may peel off pieces of V which are glued together. This occurs for instance if the orbit α is traversed more than once in V, the image of f. An example of this is a Birkhoff annulus that wraps around its boundary a number of times. Another possibility is that many closed curves in T^2 or K may map to α and we are only modifying the map near one of these curves.

Let S be the domain of f which is either the torus T^2 or the Klein bottle K. There is a simple closed curve β in S and a small neighborhood E of β in S so that $f(E)$ is also the projection of a small neighborhood of $\widetilde{\alpha}$ in $\widetilde{V}$ to M. Notice that E may be an annulus or Mobius band. The statement "V can be homotoped to be transverse to Φ in a neighborhood of α" really means that $f|E$ can be homotoped so that its final image $f'|E$ is transverse to Φ. We will abuse terminology and keep referring to this as "V can be homotoped ...".

Let g be a covering translation associated to $f(\beta)$. It follows that $g(\widetilde{\alpha}) = \widetilde{\alpha}$. In addition since g is associated to a loop coming from S (and not just a loop in V), then g preserves $\widetilde{V}$ and more to the point here g preserves the pair B_1, B_2. It may be that g switches B_1, B_2, for example if β is one sided in a Mobius band. This is crucial here: if we took g associated to α for instance, then $g(\widetilde{\alpha}) = \widetilde{\alpha}$, but g could scramble the lozenges with corner $\widetilde{\alpha}$ in an unexpected manner and one could not guarantee that B_1, B_2 would be preserved by g. We also choose β so that $f(\beta) = \alpha$.

Suppose first B_1, B_2 are adjacent and wlog assume they are adjacent along a half leaf Z of $\widetilde{W^u}(\alpha)$. The crucial fact here is that since g preserves the pair B_1, B_2 then g leaves Z invariant. Let U be a neighborhood of α in M. Choose it so the intersection with $f(E)$ is either an annulus or Mobius band (in general only immersed). Using the image $\pi(Z)$ of the half leaf Z we can homotope the power of α corresponding to g (that is, corresponding to $f(\beta)$ as a parametrized loop) away from α so that its image in $\pi(Z)$ is transverse to the flow and closes up. In the universal cover g preserves the set B_1, B_2 then the pushed curve from $\widetilde{\alpha}$ returns to the same sector of $\widetilde{M} - \widetilde{W^s}(\widetilde{\alpha})$ and this curve can be closed up when mapped to M. Once that is done we can also homotope a neighborhood of α in V as well to be transverse to Φ. Observe that this homotopy can be done so that α is homotoped inside $W^s(\alpha)$ and that points of $V \setminus \alpha$ are homotoped along the flow, see figure 3.

In the most general situation that neighborhood of α in V could be an annulus which is one sided in M, then the push away from α could not close up. In our situation it may be that this annulus goes around say twice over α and going once around α sends the lozenges B_1, B_2 to other lozenges. But going around twice over α (corresponding to g) returns $B_1 \cup B_2$ to itself. If the neighborhood is a Mobius band,

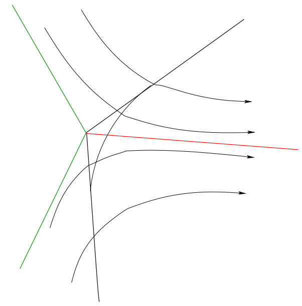

Figure 3: *A transverse view of the homotopy. Lines with arrows correspond to flowlines, green lines represent local stable semi leaves and the red line represents the local unstable semi leaf along which B_1 and B_2 are adjacent.*

we want to consider the core curve as it generates the fundamental group of this neighborhood. This finishes the proof of the first statement of the lemma.

Suppose now that V can be homotoped to be transverse to Φ is a neighborhood of α. We use the same setup as in the first part. Let U be a neighborhood of α in M so that the pulled back neighborhood of β in S is either an annulus or Mobius band. We assume that V can be perturbed near α to V' in U, keeping it fixed in ∂U, and to be transverse to Φ in a neighborhood of α. Let A be the the part of V' which is the part of V perturbed near α.

Consider all prongs of $(\widetilde{W}^s(\widetilde{\alpha}) \cup \widetilde{W}^u(\widetilde{\alpha})) - \widetilde{\alpha}$. By way of contradiction we are assuming that the lozenges B_1, B_2 are not adjacent. Then there are at least 2 such prongs as above separating B_1 from B_2 in $\widetilde{M} - \widetilde{\alpha}$ on either component of $\widetilde{M} - (B_1 \cup B_2 \cup \widetilde{\alpha})$. Let $\widetilde{A}$ be the lift of A near $\widetilde{\alpha}$.

We first show that $\widetilde{A} \cap \widetilde{\alpha}$ is empty. Suppose not and let p in the intersection. Since A is transverse to Φ then $\widetilde{A}$ is transverse say to $\widetilde{\Lambda}^u$ so we follow the intersection $\widetilde{A} \cap \widetilde{W}^u(p)$ from p. This projects to a compact set in A, contained in the interior of A as $\widetilde{W}^u(p)$ does not intersect $\partial \widetilde{A}$. This is because $\partial \widetilde{A}$ is contained in the union of the lozenges B_1, B_2 and they are disjoint from any prong of p. So the original curve in $\widetilde{W}^u(p)$ has to return to $\widetilde{\alpha}$ and looking at this curve in $\widetilde{W}^u(\alpha)$ this transverse curve has to intersect $\widetilde{\alpha}$ twice, which makes it impossible to be transverse to the flow.

Since $\widetilde{A}$ cannot intersect $\widetilde{\alpha}$ and it has boundaries in B_1 and B_2 then it has to intersect at least two prongs from $\widetilde{\alpha}$, at least one stable and one unstable prong in $\widetilde{U}$. Project to M. Then A cannot be transverse to the flow Φ. This is because in a stable prong of α the flow is transverse to A in one direction and in an unstable prong of α the flow is transverse to A in the opposite direction. This finishes the proof of lemma 3.3. □

Proposition 3.4. *(transverse torus) Let Φ be a pseudo-Anosov flow in M^3. Suppose that P_i, P_j are periodic Seifert fibered pieces which are adjacent and let T be a torus in the common boundary of P_i, P_j. Then we can choose $N(Z_i), N(Z_j)$ representative neighborhoods of P_i, P_j so that the components T_i, T_j of $\partial N(Z_i), \partial N(Z_j)$ isotopic to T are the same set and this set is transverse to Φ.*

Proof. By proposition 3.2 we may assume that $N(Z_i), N(Z_j)$ are disjoint. Since Z_i is a spine for P_i, the torus T is homotopic to a Birkhoff torus T_1 contained in Z_i. We assume that T_1 has no backtracking. Quite possibly T_1 is only an immersed torus, for example there may be Birkhoff annuli in Z_i which are covered twice by T_1. The torus T_1 lifts to a properly embedded plane $\widetilde{T}_1$ which intersects a unique bi-infinite chain of lozenges $\mathcal{B}_1$. With appropriate choices we may assume that $\pi_1(T) \cong \mathbf{Z}^2$ corresponds to a subgroup G of covering translations leaving $\mathcal{B}_1$ invariant. The corners of the lozenges in $\mathcal{B}_1$ project to closed orbits in T_1. These have powers which are freely homotopic to the regular fiber in P_i because P_i is a periodic Seifert piece. Similarly P_j produces a Birkhoff torus T_2 homotopic to T with T_2 contained in Z_j and a lift $\widetilde{T}_2$ contained in a bi-infinite chain of lozenges $\mathcal{B}_2$, which is also invariant under the same G. The corners of the lozenges in $\mathcal{B}_2$ project to closed orbits of the flow with powers freely homotopic to a regular fiber in P_j. If these two collections of corners are the same, they have the same isotropy group, which would imply the fibers in P_i, P_j have common powers, impossible as seen before.

We conclude that $\mathcal{B}_1, \mathcal{B}_2$ are distinct and both invariant under $G \cong \mathbf{Z}^2$. This is an exceptional situation and proposition 5.5 of [Ba-Fe] implies that both chains of lozenges are contained in a scalloped region and one of them (say $\mathcal{B}_1$) is s-scalloped and the other ($\mathcal{B}_2$) is u-scalloped. The lozenges in the s-scalloped region all intersect a common stable leaf, call it E and the corners of these lozenges are in stable leaves in the boundary of the scalloped region.

Let then α be a periodic orbit in T_1 with lift $\widetilde{\alpha}$ to $\widetilde{T_1}$ and lozenges B_1, B_2 of $\mathcal{B}_1$ which have corner $\widetilde{\alpha}$. Then B_1, B_2 are adjacent along an unstable leaf. Further if $f(\beta)$ is the curve as in the proof of the previous lemma, which is homotopic to a power of α, then $f(\beta)$ is in T and so the covering translation associated to $f(\beta)$ preserves $\widetilde{T_1}$ and hence also B_1, B_2. By the previous lemma we can homotope T_1 slightly near α to make it transverse to the flow near α. When lifting to the universal cover, the corresponding lift of the perturbed torus will not intersect $\widetilde{\alpha}$, but will intersect all orbits in the half leaf of $\widetilde{W^u}(\widetilde{\alpha})$ which is in the common boundary of B_1 and B_2. Do this for all closed orbits of Φ in T_1. Notice we are pushing T_1 along unstable leaves. Consider now a lozenge B_1 in $\mathcal{B}_1$ and A the Birkhoff annulus contained in the closure of $\pi(B_1)$ which is contained in T_1. Both boundaries have been pushed away along unstable leaves. The unstable leaves are on the same side of the Birkhoff annulus A. Therefore one can also push in the same direction the remainder of T_1 – to make it disjoint from Z_i. This produces a new torus T'_1 satisfying

- T'_1 is a contained in a small neighborhood of Z_i and is transverse to the flow Φ,
- T'_1 is disjoint from every Z_k (including Z_i),
- There is a fixed lift $\widetilde{T'_1}$ which is invariant under G and that it intersects exactly the orbits in the scalloped region,

The much more subtle property to prove is the following:

Claim – T'_1 can be chosen embedded.

To prove this claim we fix the representative $N(Z_i)$ of P_i and a Seifert fibration $\eta_i : P_i \to \Sigma_i$ so that Z_i is a union of fibers: each Birkhoff annulus A of Z_i is a union of fibers and it is embedded in the interior. Since M is orientable, the orbifold Σ_i is a surface with a finite number of singular points, which are the projections by η_i of the vertical periodic orbits. Moreover, $\eta_i(Z_i)$ is a *fat graph,* i.e. is a graph embedded in Σ_i which is a deformation retract of Σ_i. One can furthermore select η_i so that the stable and unstable vertical annuli in $N(Z_i)$ are Seifert saturated, ie. project to arcs in Σ_i with one boundary in Σ_i, the other being a vertex of $\eta_i(Z_i)$. Finally, one can assume that the retraction $r : \Sigma_i \to \eta_i(Z_i)$ is constant along stable and unstable arcs, mapping each of them on the vertex of $\eta_i(Z_i)$ lying in their boundary.

Since the Birkhoff annuli in Z_i are transverse to the flow, one can distinguish the two sides of every edge of $\eta_i(Z_i)$, one where the flow is "incoming", and the other "outgoing". The stable arcs are contained in the incoming side, whereas the unstable arcs are contained in the outgoing side. It follows that the set of boundary components of Σ can be partitioned in two subsets so that for every edge e of $\eta_i(Z_i)$, the two sides of e in Σ lie in different sets of this partition.

The immersed Birkhoff torus T_1 is a sequence of Birkhoff annuli A_1, A_2, ... , A_k, $A_{k+1} = A_1$. It corresponds to a sequence e_1, ... , e_k, $e_{k+1} = e_1$ of edges in $\eta_i(Z_i)$. As described above, since $\mathcal{B}_1$ is s-scalloped, T'_1 is obtained by pushing every A_i along the unstable annuli so that T'_i intersects no stable annulus. It follows that we always push on the "outgoing" side. Let c_i be the unique segment in the outgoing boundary of Σ_i whose image by the retraction r is e_i: it follows that the sequence of segments c_1, c_2, ... , c_k describe an outgoing component C of $\partial\Sigma_i$. In other words, $\eta_i(T_1)$ is the retraction of a boundary component of $N(Z_i)$.

Hence, if we have $e_i = e_j$ for some $i < j$, we have $e_{i+1} = e_{j+1}$, and so on, so that the sequence e_1, ... , e_k is the repetition of a single loop in $\eta_i(Z_i)$. Then, T_1 is homotopic to the boundary component $\eta_i^{-1}(C)$ of $N(Z_i)$ repeated at least twice. But it would mean that the JSJ torus T is homotopic to the JSJ torus $\eta_i^{-1}(C)$ repeated several times, which is a clear contradiction.

Therefore, e_1, ... , e_k is a simple loop: T_1 can pass through a Birkhoff annulus in Z_i at most once. Then the homotopy from T_1 to T'_1 does the following: the interiors of the Birkhoff annuli are homotoped to an embedded collection. The neighborhoods of the periodic orbits also satisfy that. We conclude that T'_1 can be chosen embedded. This finishes the proof of the claim.

As T_1' is embedded and homotopic to T and M is irreducible, then T_1' is in fact isotopic to T [He]. The same is true for T_2 to produce T_2' with similar properties. Notice that $\widetilde{T}_1'$ and $\widetilde{T}_2'$ intersect exactly the same set of orbits in $\widetilde{M}$. Hence their projections T_1', T_2' to M bound a closed region F in M with boundary $T_1' \cup T_2'$, homeomorphic to $T_1' \times [0,1]$ and so that the flow is a product in F. We can then isotope T_1' and T_2' along flow lines to collapse them together.

In this way we produce representatives $N(Z_i), N(Z_j)$ of P_i, P_j respectively; with boundary components T_1', T_2' isotopic to T (they are the same set) and transverse to the flow Φ. This finishes the proof of proposition 3.4. □

Proposition 3.5. *(good position) Suppose that Φ is a totally periodic pseudo-Anosov flow in a graph manifold M. Let P_i be the Seifert fibered spaces in the torus decomposition of M. Then there are spines Z_i made up of Birkhoff annuli for P_i and representative compact neighborhoods $N(Z_i)$ so that:*

- *$N(Z_i)$ is isotopic to P_i,*
- *The union of $N(Z_i)$ is M and the interiors of $N(Z_i)$ are pairwise disjoint,*
- *Each $\partial N(Z_i)$ is a union of tori in M all of which are transverse to the flow Φ and flow isotopic to tori in arbitrarly small neighborhoods of Z_i.*

Proof. If P_i and P_j are adjoining, the previous proposition explains how to adjust the corresponding components of $\partial N(Z_i)$ and $\partial N(Z_j)$ to satisfy the 4 properties for that component without changing any of the $\{Z_k\}$ or the other components of $\partial N(Z_i), \partial N(Z_j)$ (the fact that the tori in $\partial N(Z_i)$ can be pushed under the flow in any arbitrary small neighborhood of Z_i follows from the way they are constructed, see lemma 3.3). We can adjust these tori in boundary of the collection $\{N(Z_i)\}$ one by one. This finishes the proof. This actually shows that any component of $M - \cup\, Z_i$ is homeomorphic to $T^2 \times [0,1]$. □

Remarks – 1) This proposition shows that given any boundary torus of (the original) $N(Z_i)$ it can isotoped to be transverse to Φ. Fix a component $\widetilde{Z}_i$ of the inverse image of Z_i in $\widetilde{M}$ and let B_1 be a lozenge with a corner $\widetilde{\alpha}$ in $\widetilde{Z}_i$ and so that B_1 contains a lift $\widetilde{A}$ of an (open) Birkhoff annulus A in Z_i. Let α be the projection of $\widetilde{\alpha}$ to M. The proof of proposition 3.4 shows that for each side of A in M there is a torus which is a boundary component of a small neighborhood $N(Z_i)$ and which contains an annulus very close to A. Going to the next Birkhoff annulus on each torus beyond α, proposition 3.4 shows that the corresponding lozenge B is adjacent to B_1. Hence we account for the two lozenges adjacent to B_1. This can be iterated. This shows that for any corner $\widetilde{\alpha}$, *every* lozenge with corner $\widetilde{\alpha}$ contains the lift of the interior of a Birkhoff annulus in $\partial N(Z_i)$. Hence there are no more lozenges with a corner in $\widetilde{Z}_i$. Hence the pruning step done in section 7 of [Ba-Fe] is inexistent: the subtree $\mathcal{T}'$ is the entire $\mathcal{T}$. In other words, the collection of lozenges which are connected by a chain of lozenges to any corner in $\widetilde{Z}_i$ is already associated to $N(Z_i)$. In particular, the minimality condition in the definition of the spines Z_i is not necessary in the case of totally periodic pseudo-Anosov flows. The other conditions determine the spine up to flow isotopy.

2) These properties also imply that every periodic orbit of the flow admitting a power which is freely homotopic to a finite power of a regular fiber of P_i is a vertical periodic orbit (hence contained in Z_i). To see this: let α be such an orbit and $n > 0$ so that α^n is freely homotopic to a regular fiber of P_i. Lift coherently to $\widetilde{M}$ to get $\widetilde{\alpha}$ left invariant by g, so that g also leaves invariant the tree $\mathcal{T}'$. In particular $\widetilde{\alpha}$ is in $\mathcal{T}$, which in this case is equal to $\mathcal{T}'$. Projecting to M this shows that α is a vertical periodic orbit and hence contained in Z_i.

4. **Itineraries**

In the previous section, we proved that M admits a JSJ decomposition so that every Seifert piece P_i is a neighborhood $N(Z_i)$ of the spine Z_i and whose boundary is a union of tori transverse to Φ. Denote by $T_1, \ldots, T_{k_0}$ the collection of all these tori: for every k, there is a Seifert piece P_i such that Φ points outward P_i along T_k, and another piece P_j such that Φ points inward P_j along T_k (observe that we may have $i = j$, and also P_i and P_j may have several tori T_k in common). It follows from the description of

$N(Z_i)$ that the only full orbits of Φ contained in $N(Z_i)$ are closed orbits and they are the vertical orbits of $N(Z_i)$. Moreover, for every x in T_k, the future orbit of x either accumulates on a vertical periodic orbit of P_j, or intersects a Birkhoff annulus in Z_j and then exits from P_j through one other torus T_l.

From now on in this section, we fix one such JSJ decomposition of M associated to the flow Φ. In this section when we consider a Birkhoff annulus without any further specification we are referring to a Birkhoff annulus in one of the fixed spines Z_i. In the same way a *Birkhoff band* is a lift to $\widetilde{M}$ of the interior of one of the fixed Birkhoff annuli.

Let $\mathcal{G}^s(T_k)$, $\mathcal{G}^u(T_k)$ be the foliations induced on T_k by Λ^s, Λ^u. It follows from the previous section (and also by the Poincaré-Hopf index theorem) that $\mathcal{G}^s(T_k)$ and $\mathcal{G}^u(T_k)$ are regular foliations, i.e. that the orbits of Φ intersecting T_k are regular. Moreover, $\mathcal{G}^s(T_k)$ admits closed leaves, which are the intersections between T_k and the stables leaves of the vertical periodic orbits contained in $N(P_j)$. Observe that all the closed leaves of $\mathcal{G}^s(T_k)$ are obtained in this way: it follows from the fact that T_k can be retracted along the flow to an union of Birkhoff annuli in the spine Z_i.

Hence there is a cyclic order on the set of closed leaves of $\mathcal{G}^s(T_k)$, two successive closed leaves for this order are the boundary components of an annulus in T_k which can be pushed forward along the flow to a Birkhoff annulus contained in the spine Z_j. We call such a region of T_k an *elementary $\mathcal{G}^s$-annulus of T_k.*

Notice that an elementary annulus is an open subset of the respective torus T_k – the boundary closed orbits are not part of the elementary annulus.

Similarly, there is a cyclic order on the set of closed leaves of $\mathcal{G}^u(T_k)$ so that the region between two successive closed leaves (an *elementary $\mathcal{G}^u(T_k)$-annulus*) is obtained by pushing forward along Φ a Birkhoff annulus appearing in Z_i. The regular foliations $\mathcal{G}^s(T_k)$ and $\mathcal{G}^u(T_k)$ are transverse one to the other and their closed leaves are not isotopic. Otherwise P_i, P_j have Seifert fibers with common powers, contradiction. Hence none of these foliations admits a Reeb component. It follows that leaves in an elementary $\mathcal{G}^s(T_k)$ or $\mathcal{G}^u(T_k)$-annulus spiral from one boundary to the other boundary so that the direction of "spiralling" is the opposite at both sides. It also follows that the length of curves in one leaf of these foliations not intersecting a closed leaf of the other foliation is uniformly bounded from above. In other words:

Lemma 4.1. *There is a positive real number L_0 such that any path contained in a leaf of $\mathcal{G}^u(T_k)$ (respectively $\mathcal{G}^s(T_k)$) and contained in an elementary $\mathcal{G}^s(T_k)$-annulus (respectively $\mathcal{G}^u(T_k)$-annulus) has length $\leq L_0$.* □

The sets $\mathcal{T}$, Δ and $\mathcal{T}_\sharp$ – Let $\mathcal{T}$ be the collection of all the lifts in $\widetilde{M}$ of the tori T_k. Every element of $\mathcal{T}$ is a properly embedded plane in $\widetilde{M}$. We will also abuse notation and denote by $\mathcal{T}$ the union of the elements of $\mathcal{T}$. Let Δ be the union of the lifts of the vertical orbits of Φ. Finally let $\mathcal{T}_\sharp = \mathcal{T} \cup \Delta$.

Observe that there exists a positive real number η such that the $\eta/2$-neighborhoods of the T_k are pairwise disjoint. Therefore:

$$\forall \widetilde{T}, \widetilde{T}' \in \mathcal{T}, \;\; \widetilde{T} \neq \widetilde{T}' \Rightarrow d(\widetilde{T}, \widetilde{T}') \geq \eta \tag{1}$$

Here $d(\widetilde{T}, \widetilde{T}')$ is the minimum distance between a point in $\widetilde{T}$ and a point in $\widetilde{T}'$.

What we have proved concerning the foliations $\mathcal{G}^s(T_k)$, $\mathcal{G}^u(T_k)$ implies the following: for every $\widetilde{T} \in \mathcal{T}$, the restrictions to $\widetilde{T}$ of $\widetilde{\Lambda}^s$ and $\widetilde{\Lambda}^u$ are foliations by lines, that we denote by $\widetilde{\mathcal{G}}^s(\widetilde{T})$, $\widetilde{\mathcal{G}}^u(\widetilde{T})$. These foliations are both product, i.e. the leaf space of each of them is homeomorphic to the real line. Moreover, every leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T})$ intersects every leaf of $\widetilde{\mathcal{G}}^u(\widetilde{T})$ in one and only one point. Therefore, we have a natural homeomorphism $\widetilde{T} \approx \mathcal{H}^s(\widetilde{T}) \times \mathcal{H}^u(\widetilde{T})$, identifying every point with the pair of stable/unstable leaf containing it (here, $\mathcal{H}^{s,u}(\widetilde{T})$ denotes the leaf space of $\widetilde{\mathcal{G}}^{s,u}(\widetilde{T})$).

Bands and elementary bands – Some leaves of $\widetilde{\mathcal{G}}^{s,u}(\widetilde{T})$ are lifts of closed leaves: we call them *periodic leaves*. They cut $\widetilde{T}$ in bands, called (stable or unstable) *elementary bands*, which are lifts of elementary annuli (cf. fig. 4). Observe that the intersection between a stable elementary band and an unstable elementary band is always non-trivial: such an intersection is called a *square*. Finally, any pair of leaves (ℓ_1, ℓ_2) of the same foliation $\widetilde{\mathcal{G}}^s(\widetilde{T})$ or $\widetilde{\mathcal{G}}^u(\widetilde{T})$ bounds a region in $\widetilde{T}$ that we will call a *band* (elementary

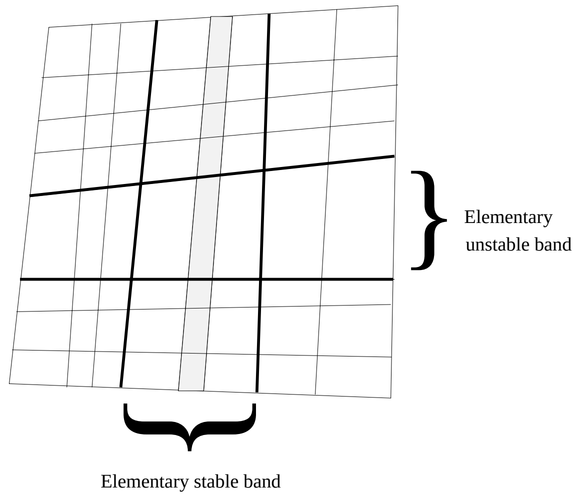

Figure 4: *Bands and elementary bands. Nearly vertical lines are leaves of $\widetilde{\mathcal{G}}^s(\widetilde{T})$, and nearly horizontal lines are leaves of $\widetilde{\mathcal{G}}^u(\widetilde{T})$. Thicker lines are periodic leaves. The shaded region is a band which is not an elementary band.*

bands defined above is in particular a special type of band). A priori bands and elementary bands can be open, closed or "half open" subsets of $\widetilde{T}$.

Remark 4.2. We arbitrarily fix a transverse orientation of each foliation $\mathcal{G}^s(T_k)$, $\mathcal{G}^u(T_k)$. It induces (in a $\pi_1(M)$-equivariant way) an orientation on each leaf space $\mathcal{H}^{s,u}(\widetilde{T})$. Since every leaf of $\widetilde{\mathcal{G}}^u(\widetilde{T})$ is naturally identified with $\mathcal{H}^s(\widetilde{T})$, the orientation of $\mathcal{H}^s(\widetilde{T})$ induces an orientation on every leaf of $\widetilde{\mathcal{G}}^u(\widetilde{T})$. When one describes successively the periodic leaves of $\widetilde{\mathcal{G}}^u(\widetilde{T})$, this orientation alternatively coincide and not with the orientation induced by the direction of the flow. This is because such leaves are lifts of closed curves isotopic to periodic orbits.

For every $\tilde{x}$ in $\widetilde{M}$, let $(\widetilde{T}_1(\tilde{x}), ..., \widetilde{T}_n(\tilde{x}), ...)$ be the list of the elements of $\mathcal{T}$ successively met by the positive $\widetilde{\Phi}$-orbit of $\tilde{x}$ (including an initial $\widetilde{T}$ if $\tilde{x}$ is contained in an element $\widetilde{T}$ of $\mathcal{T}$). Observe that this sequence can be finite, even empty: it happens precisely when the positive orbit remains trapped in a connected component of $\widetilde{M} \setminus \mathcal{T}$, i.e. the lift of a Seifert piece P_i. In this case, the projection of the orbit lies in the stable leaf of a vertical periodic orbit θ of P_i. In other words, $\tilde{x}$ lies in $\widetilde{W}^s(\alpha)$ where α is a lift of θ. In that case, we denote by $I^+(\tilde{x})$ the sequence $(\widetilde{T}_1(\tilde{x}), ..., \widetilde{T}_n(\tilde{x}), \alpha, \alpha, ...)$, where $\widetilde{T}_n(\tilde{x})$ is the last element of $\mathcal{T}$ intersecting the positive $\widetilde{\Phi}$-orbit of $\tilde{x}$, and all the following terms are all equal to α. We say then that $I^+(\tilde{x})$ is *finite.* In the other case, i.e. when the sequence $(\widetilde{T}_1(\tilde{x}), ..., \widetilde{T}_n(\tilde{x}), ...)$ is infinite, $I^+(\tilde{x})$ will denote this infinite sequence. In both situations, $I^+(\tilde{x})$ is called the *positive itinerary* of $\tilde{x}$.

Similarly, one can define the *negative itinerary* $I^-(\tilde{x})$ has the sequence of elements of $\mathcal{T}$ successively crossed by the negative orbit of $\tilde{x}$. Once more, such a sequence can be finite if $\tilde{x}$ lies in the unstable leaf of the lift of a periodic vertical orbit β, in which case we repeatedly add this information at the end of the sequence. Actually, we consider $I^-(\tilde{x})$ as a sequence indexed by $0, -1, -2, ...$

Total itinerary and itinerary map – The sequence $I^-(\tilde{x})$ together with $I^+(\tilde{x})$ defines a sequence indexed by $\mathbb{Z}$ called the *total itinerary,* denoted by $I(\tilde{x})$. This defines a map $I : \widetilde{M} \to \mathcal{T}_\sharp^{\mathbf{Z}}$ called the *itinerary map.*

4.1. **Characterization of orbits by their itineraries.** A very simple but crucial fact for the discussion here is the following: if $\widetilde{T}$ is an element of $\mathcal{T}$, then $\widetilde{T}$ is a properly embedded plane transverse to $\widetilde{\Phi}$. Hence it separates $\widetilde{M}$ and intersects an arbitrary orbit of $\widetilde{\Phi}$ at most once.

Lemma 4.3. *Let $\widetilde{T}$ be an element of $\mathcal{T}$. Let $\tilde{x}$, $\tilde{y}$ be two elements of $\widetilde{T}$ such that $\widetilde{W}^s(\tilde{x}) = \widetilde{W}^s(\tilde{y})$. Then $I^+(\tilde{x}) = I^+(\tilde{y})$.*

Proof. Clearly:

$$\widetilde{T}_1(\tilde{x}) = \widetilde{T} = \widetilde{T}_1(\tilde{y})$$

For every integer i such that $\widetilde{T}_i(\tilde{x})$ is well-defined, let $\widetilde{T}_i^+$ be the connected component of $\widetilde{M} \setminus \widetilde{T}_i(\tilde{x})$ not containing $\tilde{x}$. One easily observes that if $i < j$, then $\widetilde{T}_j^+ \subset \widetilde{T}_i^+$.

We first consider the case where $I^+(\tilde{x})$ is finite:

$$I^+(\tilde{x}) = (\widetilde{T}_1(\tilde{x}), ..., \widetilde{T}_n(\tilde{x}), \alpha, \alpha, ...)$$

Then, $\alpha \subset \widetilde{T}_n^+$. Moreover, $\tilde{y}$ lies in $\widetilde{W}^s(\tilde{x}) = \widetilde{W}^s(\alpha)$, hence $I^+(\tilde{y})$ is finite, and the $\widetilde{\Phi}$-orbit of $\tilde{y}$ accumulates on α. It must therefore enter in $\widetilde{T}_n^+$, hence intersects $\widetilde{T}_n(\tilde{x})$. But for that purpose, it must enter in $\widetilde{T}_{n-1}^+$, hence intersect $\widetilde{T}_{n-1}(\tilde{x})$. Inductively, we obtain that $(\widetilde{T}_1(\tilde{x}), ..., \widetilde{T}_n(\tilde{x}))$ is a subsequence (in that order), of $I^+(\tilde{y})$.

Since we can reverse the role of $\tilde{x}$ and $\tilde{y}$, we also prove in a similar way that $I^+(\tilde{y})$ is a subsequence of $I^+(\tilde{x})$. By the remark above the equality $I^+(\tilde{x}) = I^+(\tilde{y})$ follows.

We consider now the other case; the case where $I^+(\tilde{x})$ is infinite. Then, by what we have just proved above, $I^+(\tilde{y})$ is an infinite sequence too. Recall that there is a positive real number η bounding from below the distance between elements of $\mathcal{T}$. In particular:

$$\forall i \in \mathbb{N}, \quad d(\widetilde{T}_i(\tilde{x}), \widetilde{T}_{i+1}(\tilde{x})) \geq \eta$$

Now, any length minimizing path between $\widetilde{T}_i(\tilde{x})$ and $\widetilde{T}_{i+2}(\tilde{x})$ must intersect $\widetilde{T}_{i+1}(\tilde{x})$. It follows easily that:

$$\forall i \in \mathbb{N}, \quad d(\widetilde{T}_i(\tilde{x}), \widetilde{T}_{i+2}(\tilde{x})) \geq 2\eta$$

Inductively, one gets:

$$\forall i, p \in \mathbb{N}, \quad d(\widetilde{T}_i(\tilde{x}), \widetilde{T}_{i+p}(\tilde{x})) \geq p\eta$$

On the other hand, since $\tilde{y}$ lies in $\widetilde{W}^s(\tilde{x})$, there is a positive real number R such that:

$$\forall t > 0, \quad d(\widetilde{\Phi}^t(\tilde{x}), \widetilde{\Phi}^t(\tilde{y})) \leq R$$

For every positive integer n, select $t \in \mathbb{R}^+$ such that $\widetilde{\Phi}^t(\tilde{x})$ lies in $\widetilde{T}_{n+p}^+(\tilde{x})$, where $p \geq 2R/\eta$. Then:

$$d(\widetilde{\Phi}^t(\tilde{x}), \widetilde{T}_n(\tilde{x})) \geq p\eta \geq 2R$$

Since $d(\widetilde{\Phi}^t(\tilde{x}), \widetilde{\Phi}^t(\tilde{y})) \leq R$, $\widetilde{\Phi}^t(\tilde{x})$ and $\widetilde{\Phi}^t(\tilde{y})$ lie on the same side of $\widetilde{T}_n(\tilde{x})$, i.e. $\widetilde{T}_n^+$. Hence $\widetilde{T}_n(\tilde{x})$ appears in the positive itinerary of $\tilde{y}$. Since n is arbitrary, it follows that $I^+(\tilde{x})$ is a subsequence of $I^+(\tilde{y})$, the order in the sequence being preserved.

Switching the roles of $\tilde{x}$ and $\tilde{y}$, we also get that $I^+(\tilde{y})$ is a subsequence of $I^+(\tilde{x})$. Hence, the two itineraries must coincide. □

In order to prove the reverse statement, we need the following result:

Proposition 4.4. *Let $\widetilde{T}$, $\widetilde{T}'$ be two elements of $\mathcal{T}$, intersected successively by a $\widetilde{\Phi}$-orbit, i.e. such that for some $\tilde{z} \in \widetilde{T}$, the first intersection of the forward orbit of $\tilde{z}$ with $\mathcal{T}$ (after $\widetilde{T}$) is some $\widetilde{\Phi}^t(\tilde{z}) \in \widetilde{T}'$, $t > 0$. Then, the subset $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ comprised of elements of $\widetilde{T}$ such that the positive itinerary starts by $(\widetilde{T}, \widetilde{T}', ...)$ is a stable elementary band; more precisely, the stable elementary band of $\widetilde{T}$ containing $\tilde{z}$.*

Proof. The orbit of $\tilde{z}$ between the times 0 and t lies in a connected component of $\widetilde{M} \setminus \mathcal{T}$, hence projects into a Seifert piece P_i. Due to the previous section, this orbit in M starts in a connected component of ∂P_i (projection T of $\widetilde{T}$), intersects one of the Birkhoff annuli A_0 contained in the spine Z_i, and then crosses the projection T' of $\widetilde{T}'$. In other words, there is a lift $\widetilde{A}_0$ of A_0 intersected by the orbit of $\tilde{z}$ and contained between $\widetilde{T}$ and $\widetilde{T}'$. The boundary of A_0 is the union of two periodic orbits (maybe equal one to the other), and any element of A_0 has a negative orbit intersecting T, and a positive orbit intersecting T'. At the universal covering level, the boundary of $\widetilde{A}_0$ is the union of two distinct orbits α and β; the $\widetilde{\Phi}$-saturation of the Birkhoff band $\widetilde{A}_0$ intersects $\widetilde{T}$ (respectively $\widetilde{T}'$) along an elementary band $\widetilde{A}$

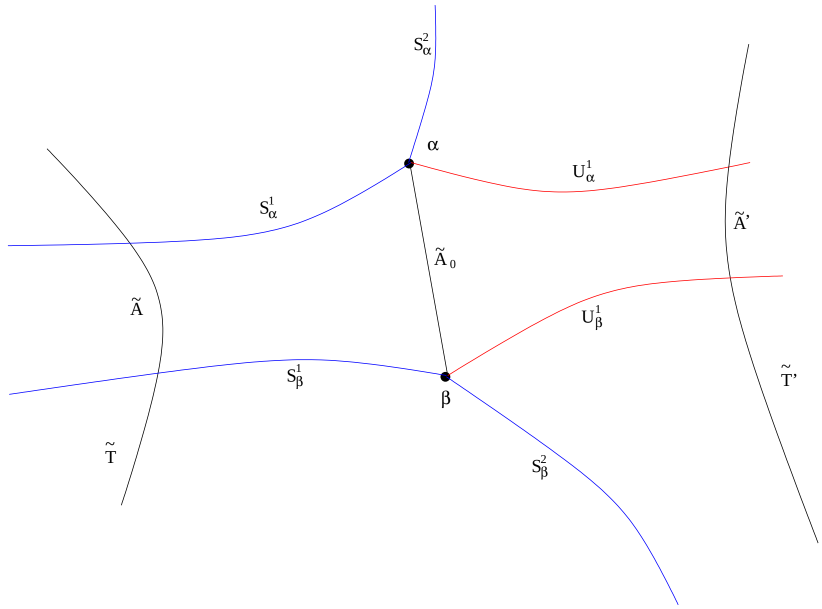

Figure 5: *Isolating $\widetilde{T}'$ and $\widetilde{A}_0$ by stable prongs.*

(respectively $\widetilde{A}'$). Moreover, the boundary of $\widetilde{A}$ is the union of two leaves of $\widetilde{\mathcal{G}}^s(\widetilde{T})$. More precisely, one of these leaves is contained in the intersection $\widetilde{T} \cap S^1_\alpha$, and the other in the intersection $\widetilde{T} \cap S^1_\beta$, where S^1_α is a component of $\widetilde{W}^s(\alpha) \setminus \alpha$ and S^1_β a component of $\widetilde{W}^s(\beta) \setminus \beta$.

Similarly, $\widetilde{A}'$ is an elementary band in $\widetilde{T}'$ bounded by two leaves of $\widetilde{\mathcal{G}}^u(\widetilde{T}')$, which are contained in some components U^1_α, U^1_β of $\widetilde{W}^u(\alpha) \setminus \alpha$, $\widetilde{W}^u(\beta) \setminus \beta$.

Clearly $\widetilde{A} \subset \widetilde{A}(\widetilde{T}, \widetilde{T}')$.

Claim 1: the intersection $S^1_\alpha \cap \widetilde{T}$ is connected. If not, there would be a segment of orbit of $\widetilde{\Phi}$ with extremities in $\widetilde{T}$ but not intersecting $\widetilde{T}$. It would be in contradiction with the fact that $\widetilde{T}$ disconnects $\widetilde{M}$ and is transverse to $\widetilde{\Phi}$.

Similarly, $S^1_\beta \cap \widetilde{T}$ is connected (i.e. is reduced to a boundary component of the elementary band $\widetilde{A}$, and the intersections $U^1_\alpha \cap \widetilde{T}'$, $U^1_\beta \cap \widetilde{T}'$ are connected, i.e. precisely the boundary components of $\widetilde{A}'$.

Key fact: by our definition of pseudo-Anosov flows, α and β are not 1-prong orbits. It follows that there is a component S^2_α of $\widetilde{W}^s(\alpha) \setminus \alpha$ different from S^1_α. We select this component to be the one just after U^1_α, i.e. such that the following is true: the union $S(\alpha) = S^1_\alpha \cup \alpha \cup S^2_\alpha$ is a 2-plane such that the connected component $C(\alpha)$ of $\widetilde{M} \setminus S(\alpha)$ containing $\widetilde{A}_0$ does not intersect $\widetilde{W}^s(\alpha)$. Similarly, we define a 2-plane $S(\beta) = S^1_\beta \cup \beta \cup S^2_\beta$ contained in $\widetilde{W}^s(\beta)$ such that the connected component $C(\beta)$ of $\widetilde{M} \setminus S(\beta)$ containing $\widetilde{A}_0$ does not intersect $\widetilde{W}^s(\beta)$ (see fig. 5).

Claim 2: S^2_α and S^2_β are disjoint from $\widetilde{T}'$: indeed, the positive orbit of any point in $\widetilde{T}'$ is trapped into the component of $\widetilde{M} \setminus \widetilde{T}'$ disjoint from $\widetilde{A}_0$, hence cannot accumulate on α or β: $\widetilde{T}'$ is disjoint from $\widetilde{W}^s(\alpha) \cup \widetilde{W}^s(\beta)$. The claim follows.

Claim 3: S^2_α and S^2_β are disjoint from $\widetilde{T}$: Assume by contradiction that S^2_α intersects $\widetilde{T}$. In the same way we have proved that $S^1_\alpha \cap \widetilde{T}$ is connected, one can prove that $S^2_\alpha \cap \widetilde{T}$ is a single leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T})$. It bounds together with $S^1_\alpha \cap \widetilde{T}$ a $\widetilde{\mathcal{G}}^s(\widetilde{T})$-band $\widetilde{B}$. Let g be the indivisible element of $\pi_1(M)$ corresponding to a generator of the fundamental group of the projection of α: it preserves $\widetilde{\alpha}$, $\widetilde{T}$, hence also $S^1_\alpha \cap \widetilde{T}$, $S^2_\alpha \cap \widetilde{T}$ and the band $\widetilde{B}$. It follows that the union of $\widetilde{B}$ with the regions in S^1_α, S^2_α between $S^1_\alpha \cap \widetilde{T}$, $S^2_\alpha \cap \widetilde{T}$ and α projects in M as a torus contained in the Seifert piece P_i. This torus bounds a solid torus in P_i. Every orbit of ϕ entering in this solid torus from the projection of $\widetilde{B}$ cannot exit from this solid torus, since it cannot further intersect the projection of $\widetilde{B}$, nor the projections of S^1_α, S^2_α. It has to remain in the Seifert piece containing the projection of α. It is a contradiction since the set of positive orbits trapped in P_i has empty interior. Claim 3 is proved.

We now focus our attention to the region $C(\alpha)\cap C(\beta)$, whose boundary is the disjoint union $S(\alpha)\bigsqcup S(\beta)$. According to Claim 2, $\widetilde{T}'$ is contained in $C(\alpha)\cap C(\beta)$. Now it follows from Claim 3 that the intersection $C(\alpha)\cap C(\beta)\cap \widetilde{T}$ is the elementary band $\widetilde{A}$.

Consider now the positive orbit of an element p of $\widetilde{T}\setminus\widetilde{A}$: if p lies in $\partial\widetilde{A}$, then this orbit accumulates on α or β and therefore does not intersect $\widetilde{T}'$. If not, then this orbit is disjoint from $\widetilde{W}^s(\alpha)\cup\widetilde{W}^s(\beta)$, hence never enters in $C(\alpha)\cap C(\beta)$. In particular, it never crosses $\widetilde{T}'$.

It follows that $\widetilde{A}(\widetilde{T},\widetilde{T}')\subset\widetilde{A}$ and so proposition 4.4 is proved. ☐

We can now prove the converse of Lemma 4.3:

Lemma 4.5. *Let $\widetilde{T}$ be an element of $\mathcal{T}$. Let $\tilde{x}$, $\tilde{y}$ be two elements of $\widetilde{T}$ such that $I^+(\tilde{x})=I^+(\tilde{y})$. Then $\tilde{y}$ and $\tilde{x}$ lie in the same leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T})$.*

Proof. Let $\tilde{x}_1$, $\tilde{x}_2$, ... and $\tilde{y}_1$, $\tilde{y}_2$, ... be the elements of the positive orbits of $\tilde{x}$, $\tilde{y}$ belonging in $\widetilde{T}_1 := \widetilde{T}_1(\tilde{x})=\widetilde{T}_1(\tilde{y})$, $\widetilde{T}_2 := \widetilde{T}_2(\tilde{x})=\widetilde{T}_2(\tilde{y})$, ... According to Proposition 4.4, for every positive integer i, the iterates $\tilde{x}_i$, $\tilde{y}_i$ lie in the same stable elementary band $\widetilde{A}_i := \widetilde{A}(\widetilde{T}_i,\widetilde{T}_{i+1})\subset\widetilde{T}_i$.

Consider first the case where the common itinerary $I^+(\tilde{x})=I^+(\tilde{y})$ is finite, of length $n+1$: $\tilde{x}_n$ and $\tilde{y}_n$ lie in the elementary band of $\widetilde{T}_n$, and $I_k(\tilde{x})=I_k(\tilde{y})=\alpha$ for some periodic orbit α, for all $k>n$. Hence, th positive orbits of $\tilde{x}_n$, $\tilde{y}_n$ accumulate on α. It follows from the arguments used in the proof of Claim 3 of Proposition 4.4 that the intersection $\widetilde{W}^s(\alpha)\cap\widetilde{T}_n$ is a single leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T}_n)$. Proposition 4.3 follows easily in this case.

We are left with the case where $I^+(\tilde{x})=I^+(\tilde{y})$ is infinite. Let $\widetilde{\mathcal{G}}^s_i$, $\widetilde{\mathcal{G}}^u_i$ denote the restriction to $\widetilde{A}_i$ of $\widetilde{\Lambda}^s$, $\widetilde{\Lambda}^u$. Observe that every leaf of $\widetilde{\mathcal{G}}^u_i$ intersects every leaf of $\widetilde{\mathcal{G}}^s_i$. In particular, the $\widetilde{\mathcal{G}}^s_1$-leaf of $\tilde{x}$ intersects the $\widetilde{\mathcal{G}}^u_1$-leaf of $\tilde{y}$. Therefore, according to Lemma 4.3, one can assume without loss of generality that $\tilde{x}$ and $\tilde{y}$ lies in the same leaf of $\widetilde{\mathcal{G}}^u_1$. More precisely, there is a path $c:[a,b]\to\widetilde{A}_1$ contained in a leaf of $\widetilde{\mathcal{G}}^u_1$ and joining $\tilde{x}$ to $\tilde{y}$.

Assume by way of contradiction that c is not a trivial path reduced to a point. If we push c forward by the flow $\widetilde{\Phi}$, one get a path $c_2:[a,b]\to\widetilde{A}_2$, contained in a leaf of $\widetilde{\mathcal{G}}^u_2$, connecting $\tilde{x}_2$ to $\tilde{y}_2$. By induction, pushing along $\widetilde{\Phi}$, we get a sequence of unstable paths $c_i:[a,b]\to\widetilde{A}_i$. Now, since all these paths are obtained from c by pushing along $\widetilde{\Phi}$, the length of c_i is arbitrarily long if i is sufficiently big. This contradicts Lemma 4.1.

This contradiction shows that c is reduced to a point, i.e. that $\tilde{x}$ and $\tilde{y}$ lie in the same leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T})$. ☐

Applying Lemmas 4.3 and 4.5 to the reversed flow one obtains:

Proposition 4.6. *Let $\widetilde{T}$ be an element of $\mathcal{T}$. Let $\tilde{x}$, $\tilde{y}$ be two elements of $\widetilde{T}$. Then $I^-(\tilde{x})=I^-(\tilde{y})$ if and only if $\tilde{y}$ and $\tilde{x}$ lie in the same leaf of $\widetilde{\mathcal{G}^u}(\widetilde{T})$.* ☐

Itineraries are elements of $\mathcal{I}:=\mathcal{T}_\sharp^{\mathbb{Z}}$ where $\mathcal{T}_\sharp$ is the disjoint union of $\mathcal{T}$ with the set Δ of lifts of vertical periodic orbits of Φ. We define the *shift map* $\sigma:\mathcal{I}\to\mathcal{I}$ which send any sequence $(\zeta_i)_{i\in\mathbb{Z}}$ to the sequence $(\zeta_{i+1})_{i\in\mathbb{Z}}$. Clearly, if $\tilde{x}$ and $\tilde{y}$ are two elements of $\widetilde{M}$ lying on the same orbit of $\widetilde{\Phi}$, then $I(\tilde{y})$ is the image of $I(\tilde{x})$ under some iterate σ^k. Conversely:

Corollary 4.7. *Let $\tilde{x}$, $\tilde{y}$ be two elements of $\widetilde{M}$. Then $\tilde{x}$ and $\tilde{y}$ lie in the same orbit of $\widetilde{\Phi}$ if and only if $I(\tilde{y})=\sigma^k(I(\tilde{x}))$ for some $k\in\mathbb{Z}$.*

Proof. Assume that $I(\tilde{y})=\sigma^k(I(\tilde{x}))$ for some $k\in\mathbb{Z}$. Suppose first that the $\widetilde{\Phi}$ orbit of $\tilde{x}$ intersects no element of $\mathcal{T}$. Then $\tilde{x}$ is in Δ and similarly $\tilde{y}$ is also in Δ. The hypothesis immediately imply that $\tilde{x},\tilde{y}$ are in the same orbit of $\widetilde{\Phi}$. If $\tilde{x}$ intersects an element $\mathcal{T}$, then after a shift if necessary, we may assume that $T_1(\tilde{x})$ is an element of $\mathcal{T}$. Then, by replacing $\tilde{y}$ by the element of its $\widetilde{\Phi}$-orbit in the element $T_1(\tilde{x})=T_{1+k}(\tilde{y})$ of $\mathcal{T}$, and $\tilde{x}$ by its iterate in $T_1(\tilde{x})$, one can assume that $\tilde{x}$ and $\tilde{y}$ both lie in $T_1(\tilde{x})$, and that $I(\tilde{x}) = I(\tilde{y})$. In particular, $I^+(\tilde{x}) = I^+(\tilde{y})$ and $I^-(\tilde{x}) = I^-(\tilde{y})$. Then, according to Lemma 4.3 and proposition 4.6, $\tilde{x}$ and $\tilde{y}$ have the same stable leaf and the same unstable leaf. The corollary follows. □

In the same way one can prove that for any $\tilde{x}, \tilde{y}$ in $\widetilde{M}$, then $\tilde{x}, \tilde{y}$ are in the same stable leaf of $\widetilde{\Phi}$ if and only if the positive iteneraries of $\tilde{x}, \tilde{y}$ are eventually equal up to a fixed shift.

We can now extend Proposition 4.4:

Proposition 4.8. *Let $\widetilde{T}$, $\widetilde{T}'$ be two elements of $\mathcal{T}$. Then, the subset $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ of $\widetilde{T}$ comprised of elements of $\widetilde{T}$ whose positive orbits intersects $\widetilde{T}'$, if non-empty, is a stable band. Furthermore, let $\tilde{x}$ be an element of $\widetilde{A}(\widetilde{T}, \widetilde{T}')$; its positive itinerary has the form $(\widetilde{T}_1 := \widetilde{T}, \widetilde{T}_2, ..., \widetilde{T}_n, ...)$ where $\widetilde{T}_n = \widetilde{T}'$. Then, for any other element $\tilde{y}$ of $\widetilde{A}(\widetilde{T}, \widetilde{T}')$, the first n-terms of $I^+(\tilde{y})$ are also $(\widetilde{T}_1 := \widetilde{T}, \widetilde{T}_2, ..., \widetilde{T}_n)$. The elements $\widetilde{T}_2$, ... , $\widetilde{T}_{n-1}$ are precisely the elements of $\mathcal{T}$ that separate $\widetilde{T}$ from $\widetilde{T}'$ in $\widetilde{M}$.*

Proof. Assume that $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ is non-empty, and let $\tilde{x}$ be an element of $\widetilde{A}(\widetilde{T}, \widetilde{T}')$. Its positive itinerary contains $\widetilde{T}'$, hence has the form $(\widetilde{T}_1 := \widetilde{T}, \widetilde{T}_2, ..., \widetilde{T}_n, ...)$ described in the statement. As we have observed in the proof of Lemma 4.3, we have $\widetilde{T}_j^+ \subset \widetilde{T}_i^+$ for every $1 \leq i < j \leq n$, hence every $\widetilde{T}_i$ for $1 < i < n$ disconnects $\widetilde{T}$ from $\widetilde{T}'$. On the other hand, every element of $\mathcal{T}$ disconnecting $\widetilde{T}$ and $\widetilde{T}'$ must appear in the positive itinerary of elements of $\widetilde{A}(\widetilde{T}, \widetilde{T}')$. It follows easily that the first n-terms of the positive itinerary of elements of $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ coincide as stated in Proposition 4.8. Moreover, if an element of $\widetilde{T}$ has a positive itinerary of the form $(\widetilde{T}_1 := \widetilde{T}, \widetilde{T}_2, ..., \widetilde{T}_n, ...)$, it obviously belongs to $\widetilde{A}(\widetilde{T}, \widetilde{T}')$.

The only remaining point to check is that $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ is a stable band. But this follows immediatly from Lemma 4.3.

It is very useful to give more information here: if $n = 2$ then $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ is a stable elementary band which projects to an open annulus in T. If $n > 2$ then $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ is a stable band which is not elementary. For simplicity we describe the case $n = 3$. With the notation above, then $\widetilde{A}(\widetilde{T}_2, \widetilde{T}_3)$ is a stable elementary band as proved in Proposition 4.4. This band in $\widetilde{T}_2$ intersects the unstable elementary bands of $\widetilde{T}_2$ in open squares. The one which has points flowing back to $\widetilde{T}_1$ has boundary made up of two stable sides a_1, a_2 which are contained in leaves of $\widetilde{\mathcal{G}}^s(\widetilde{T}_2)$, and unstable sides b_1, b_2 contained in leaves of $\widetilde{\mathcal{G}}^u(\widetilde{T}_2)$. Flowing back to $\widetilde{T} = \widetilde{T}_1$ (in this case) produces $\widetilde{A}(\widetilde{T}, \widetilde{T}')$. The arcs b_1, b_2 flow back towards two vertical periodic orbits, without ever reaching them. The arcs a_1, a_2 flow back to two full stable leaves a'_1, a'_2 of $\widetilde{\mathcal{G}}^s(\widetilde{T})$. Then $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ is the stable band with boundary a'_1, a'_2. This is not an elementary stable band. In fact more is true: this band is strictly contained in a unique elementary band and does not share a boundary component with this elementary band. Finally this stable band projects injectively to T – unlike what happens for elementary bands.

If $n > 3$ this process can be iterated. Using the notation above the stable band bounded by a'_1, a'_2 intersects the unstable elementary bands in their lifted torus in squares. When flowing backwards, the same behavior described above occurs. □

Definition 4.9. *Let $\widetilde{T}$, $\widetilde{T}'$ be two elements of $\mathcal{T}$. We define the (signed) distance $n(\widetilde{T}, \widetilde{T}')$ as follows:*

– *if $\widetilde{T} = \widetilde{T}'$, then $n(\widetilde{T}, \widetilde{T}') = 0$,*

– *if $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ is non empty, then $n(\widetilde{T}, \widetilde{T}')$ is the integer n such that for every element $\tilde{x}$ of $\widetilde{A}(\widetilde{T}, \widetilde{T}')$, we have $\widetilde{T}_{n+1}(\tilde{x}) = \widetilde{T}'$; and $n(\widetilde{T}', \widetilde{T}) = -n(\widetilde{T}, \widetilde{T}')$,*

– *if $\widetilde{A}(\widetilde{T}, \widetilde{T}')$ and $\widetilde{A}(\widetilde{T}', \widetilde{T})$ are both empty, then $n(\widetilde{T}, \widetilde{T}') = n(\widetilde{T}', \widetilde{T}) = \infty$.*

Notice that every orbit γ of $\widetilde{\Phi}$ is either contained in $\widetilde{T}$ (a vertical orbit) or intersects it at most once. This was explained in the proof of Proposition 3.2. It follows that the number $n(\widetilde{T}, \widetilde{T}')$ is uniquely defined.

4.2. **Behavior of the first return map.** Let us consider once more two successive elements $\widetilde{T}$, $\widetilde{T}'$ of $\mathcal{T}$, i.e. such that $n(\widetilde{T}, \widetilde{T}') = 1$. Recall that there is a stable elementary band $\widetilde{A} := \widetilde{A}(\widetilde{T}, \widetilde{T}') \subset \widetilde{T}$ bounded by two stable leaves l_1, l_2, and an unstable elementary band $\widetilde{A}' \subset \widetilde{T}'$ bounded by two unstable leaves l'_1,

l'_2, such that orbit of $\widetilde{\Phi}$ intersecting $\widetilde{T}$ and $\widetilde{T}'$ intersects them in precisely $\widetilde{A}$, $\widetilde{A}'$, respectively. The union of all these orbits is a region of $\widetilde{M}$ bounded by (see fig. 5):

- $\widetilde{A}$ and $\widetilde{A}'$;
- two stable bands S^1_α, S^1_β where α, β are lifts of periodic orbits,
- two unstable bands U^1_α, U^1_β.

More specifically here S^1_α denotes the unique component of $\widetilde{W}^s(\alpha) - (\alpha \cup \widetilde{T})$ whose closure intersects both α and $\widetilde{T}$. Its boundary is the union of α and a stable leaf in $\widetilde{T}$. We call such a 3-dimensional region the *block defined by $\widetilde{T}$, $\widetilde{T}'$;* the orbits α, β are the *corners* of the block.

The map $f_{\widetilde{T},\widetilde{T}'}$ – We have a well-defined map $f_{\widetilde{T},\widetilde{T}'} : \widetilde{A} \to \widetilde{A}'$, mapping every point to the intersection between its positive $\widetilde{\Phi}$-orbit and $\widetilde{T}'$. As long as there is no ambiguity on $\widetilde{T}$ and $\widetilde{T}'$, we will denote $f_{\widetilde{T},\widetilde{T}'}$ by f.

Clearly, if two elements of $\widetilde{A}$ lie on the same leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T})$, then $f(\tilde{x})$ and $f(\tilde{x}')$ lie on the same leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T}')$. In other words, f induces a map

$$f^s := f^s_{\widetilde{T},\widetilde{T}'} : \ (l_1, l_2) \ \to \ \mathcal{H}^s(\widetilde{T}'),$$

where (l_1, l_2) is the open segment of the leaf space $\widetilde{\mathcal{H}}^s(\widetilde{T})$ delimited by the boundary leaves l_1 and l_2 of $\widetilde{A}$.

Since every leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T}')$ intersects every leaf of $\widetilde{\mathcal{G}}^u(\widetilde{T}')$, it follows that f^s is <u>surjective</u>: if v is a leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T}')$ then the intersection property implies that v intersects l'_1. Then v intersects $\widetilde{A}'$ so v is in the image of f. Moreover, the intersection between every leaf of $\widetilde{\mathcal{G}}^s(\widetilde{T}')$ and $\widetilde{A}'$ is connected, hence f^s is one-to-one.

Now assume that as in fig. 6 the flow along the "periodic orbit" α is going up. Then, β is going down. It follows that the map f has the following behavior: points in $\widetilde{A}$ close to l_1 are sent by f in the top direction of $\widetilde{A}'$, meaning that the closer to l_1 is the point $\tilde{x}$, the upper is the image $f(\tilde{x})$. Indeed, the closer to l_1 is $\tilde{x}$, the longest is the period of time the positive orbit of $\tilde{x}$ will follow the vertical direction of α. On the other hand, when $\tilde{x}$ is going near to l_2, the image $f(\tilde{x})$ will be going the closer to l'_2, and in the bottom direction.

We already know that stable leaves in $\widetilde{T}'$ cross the two sides l'_1 and l'_2 of $\widetilde{A}'$, hence can be drawn as in the picture in a nearly horizontal way (since we have drawn $l'_{1,2}$ as vertical lines). Therefore, stable leaves in $\widetilde{A}$, which are the pull-back by f of stable leaves in $\widetilde{T}$, are as depicted in fig. 6: if we describe such a leaf s from the bottom to the top, the image $f(s)$ will go from the left (l'_1) to the right (l'_2). It follows that what is at the right (respectively, at the left) of s is mapped under f below (respectively, above) $f(s)$.

Observe that if we reverse the direction of the flow on α (and hence also on β), when we would have the opposite behavior: f would map what is on the right of s above $f(s)$.

Now recall that we have arbitrarily fixed an orientation of $\mathcal{H}^s(\widetilde{T})$ and $\mathcal{H}^s(\widetilde{T}')$ (Remark 4.2). It is equivalent to prescribe a total order $\prec$ on $\mathcal{H}^s(\widetilde{T})$ and $\mathcal{H}^s(\widetilde{T}')$. Assume e.g. that the positive orientation on $\mathcal{H}^s(\widetilde{T})$ - which is a space of roughly vertical lines in $\widetilde{A}$ - is from the left to the right; and assume that the orientation on $\mathcal{H}^s(\widetilde{T}')$ is from the top to the bottom. Then, if α is oriented from the bottom to the top (as in the figure), the map f^s preserves the orientation, whereas if α as the inverse orientation, f^s reverses the orientations.

The map $f_{\widetilde{T},\widetilde{T}'}$ when $n(\widetilde{T},\widetilde{T}') > 1$ – Now if $\widetilde{T}$, $\widetilde{T}'$ are elements of $\mathcal{T}$ with $n(\widetilde{T},\widetilde{T}') = n > 1$, we still have a map $f_{\widetilde{T},\widetilde{T}'} : \widetilde{A}(\widetilde{T},\widetilde{T}') \to \widetilde{A}'(\widetilde{T},\widetilde{T}')$ where $\widetilde{A}(\widetilde{T},\widetilde{T}')$ is a stable band in $\widetilde{T}$ and $\widetilde{A}'(\widetilde{T},\widetilde{T}')$ an unstable

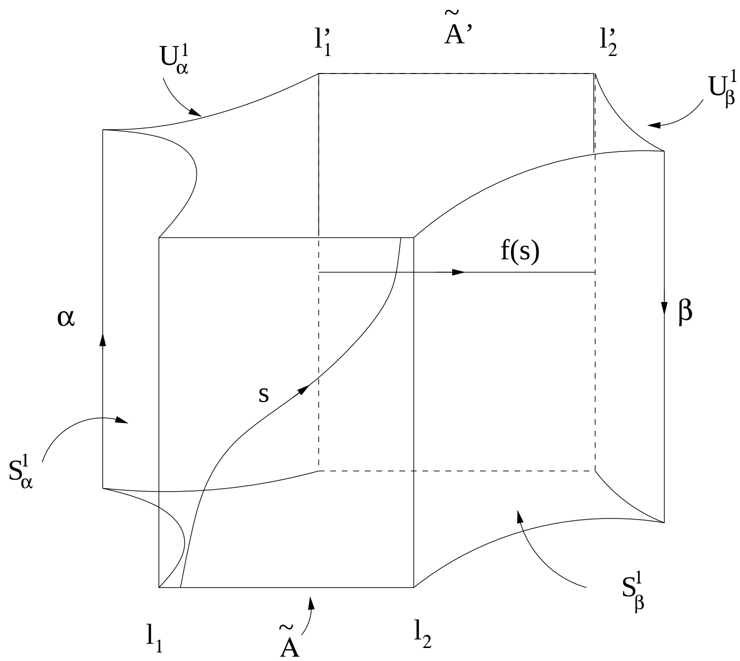

Figure 6: *Pushing along the flow a stable leaf in an elementary band.*

band in $\widetilde{T}'$. More precisely, the positive itinerary of orbits starting from $\widetilde{T}$ and crossing $\widetilde{T}'$ starts by $\widetilde{T}_1 := \widetilde{T}$, $\widetilde{T}_2$, ... , $\widetilde{T}_n := \widetilde{T}'$. Then $f_{\widetilde{T},\widetilde{T}'}$ is the composition of all the $f_{\widetilde{T}_i,\widetilde{T}'_{i+1}}$. The fact that the domain of $f_{\widetilde{T},\widetilde{T}'}$ is a stable band was proved in the end of the proof of proposition 4.8. To get that the image is an unstable band notice that flowing backwards instead of forwards shows this fact (as the image is the domain of the inverse map).

Exactly as in the case that $n(\widetilde{T},\widetilde{T}') = 1$, it follows that the map $f_{\widetilde{T},\widetilde{T}'}$ also induces a **surjective** map $f^s_{\widetilde{T},\widetilde{T}'}$ from a segment of $\mathcal{H}^s(\widetilde{T})$ onto the entire $\mathcal{H}^s(\widetilde{T}')$. This map can preserve the orientation or not; this property depends on the orientation of the corners of the Birkhoff annuli successively crossed.

Corollary 4.10. *Let $\widetilde{T}$, $\widetilde{T}'$, $\widetilde{T}''$ three elements of $\mathcal{T}$ such that $n(\widetilde{T},\widetilde{T}')$ and $n(\widetilde{T}',\widetilde{T}'')$ are positive. Then:*

$$n(\widetilde{T},\widetilde{T}'') = n(\widetilde{T},\widetilde{T}') + n(\widetilde{T}',\widetilde{T}'')$$

Proof. Let $\widetilde{T}_1 := \widetilde{T}$, $\widetilde{T}_2$, ... , $\widetilde{T}_n := \widetilde{T}'$ the initial terms of future itineraries of elements of $A(\widetilde{T},\widetilde{T}')$, and $\widetilde{T}'_1 := \widetilde{T}'$, $\widetilde{T}'_2$, ... , $\widetilde{T}'_m := \widetilde{T}''$ the initial terms of future itineraries of elements of $A(\widetilde{T}',\widetilde{T}'')$. Since the map $f^s_{\widetilde{T},\widetilde{T}'}$ is surjective, the unstable band $A'(\widetilde{T},\widetilde{T}')$ intersects the stable band $A(\widetilde{T}',\widetilde{T}'')$, $A(\widetilde{T},\widetilde{T}'')$ is not empty. Furthermore, the initial terms of itineraries of elements in $A(\widetilde{T},\widetilde{T}'')$ are:

$$\widetilde{T}_1 := \widetilde{T}, \widetilde{T}_2, ..., \widetilde{T}_n := \widetilde{T}' = \widetilde{T}'_1, \widetilde{T}'_2, ..., \widetilde{T}'_m$$

Hence $n(\widetilde{T},\widetilde{T}'') = n + m$. The corollary follows. □

4.3. **Realization of itineraries.** We define an oriented graph $\widetilde{\mathfrak{G}}$ as follows:

- vertices are elements of $\mathcal{T}$,
- edges are Birkhoff bands,
- the initial vertex and the final vertex of an oriented edge E are the elements $\widetilde{T}$, $\widetilde{T}'$ of $\mathcal{T}$ such that there are orbits of $\widetilde{\Phi}$ intersecting $\widetilde{T}$ at a point $\tilde{x}$, then crossing E, and crossing afterwards $\widetilde{T}'$ at

a point $\tilde{x}'$. We require furthermore that E is the unique Birkhoff band intersected by the orbit between $\tilde{x}$ and $\tilde{x}'$; in other words, that $n(\widetilde{T}, \widetilde{T}') = 1$.

We add to $\widetilde{\mathfrak{G}}$ some vertices: the set Δ of lifts of vertical periodic orbits. These new vertices will not be connected one to the other, but only to vertices of $\widetilde{\mathfrak{G}}$: we add an edge oriented from $\widetilde{T}$ to α (respectively from α to $\widetilde{T}$) if some element $\tilde{x}$ of $\widetilde{T}$ has a positive (respectively negative) orbit accumulating on α without intersecting any element of $\mathcal{T}$. The result is the *augmented graph* $\widetilde{\mathfrak{G}}^\sharp$.

Lemma 4.11. *The graphs $\widetilde{\mathfrak{G}}$ and $\widetilde{\mathfrak{G}}^\sharp$ are (weakly) connected.*

Proof. Recall that an oriented graph is weakly connected if the underlying non-oriented graph is connected, i.e. if any pair of vertices can be connected by a sequence of edges. It is quite obvious that $\widetilde{\mathfrak{G}}^\sharp$ is weakly connected as soon as $\widetilde{\mathfrak{G}}$ is weakly connected.

For every element $\widetilde{T}$ of $\mathcal{T}$ let $U_1(\widetilde{T})$ be the union of the points in $\widetilde{M}$ whose orbits of $\widetilde{\Phi}$ intersect $\widetilde{T}$: $U_1(\widetilde{T})$ is an open domain in $\widetilde{M}$ which is $\widetilde{\Phi}$ invariant. Let $W_1(\widetilde{T})$ be the set of elements of $\mathcal{T}$ which can be joined to $\widetilde{T}$ by an orbit of $\widetilde{\Phi}$, i.e. the elements of $\mathcal{T}$ which intersects $U_1(\widetilde{T})$. Notice that we can connect $\widetilde{T}$ to an element of $U_1(\widetilde{T})$ by a flow segment <u>either going forwards or backwards</u>. Define then inductively:

$$
\begin{aligned}
U_{i+1}(\widetilde{T}) &= \bigcup_{\widetilde{T}' \in W_i(\widetilde{T})} U_1(\widetilde{T}') \\
W_{i+1}(\widetilde{T}) &= \bigcup_{\widetilde{T}' \in W_i(\widetilde{T})} W_1(\widetilde{T}')
\end{aligned}
$$

We obtain an increasing sequence of domains $U_i(\widetilde{T})$ whose union $U_\infty(\widetilde{T})$ is an open subset of $\widetilde{M}$; more precisely, of $\widetilde{M} \setminus \Delta$ (recall that Δ is the union of lifts of vertical periodic orbits). In a less formal way, one can define $U_\infty(\widetilde{T})$ as the set of elements of $\widetilde{M} \setminus \Delta$ which are attainable from $\widetilde{T}$ through concatenations of segments of orbits of $\widetilde{\Phi}$ and paths in elements of $\mathcal{T}$. For example suppose $\widetilde{T}$ and $\widetilde{T}'$ are "connected via a singular vertical orbit γ". This means that say $\widetilde{W}^s(\gamma)$ has an annulus connecting γ to $\widetilde{T}$ and say $\widetilde{W}^u(\gamma)$ has an annulus B connecting it to $\widetilde{T}'$. Suppose that A and B are separated by similar annuli $A_1, A_2, ...$ which are alternatively unstable and stable (so A_1 is not B). Start from $\widetilde{T}$ near A and flow <u>forwards</u> to $\widetilde{T}_1$ close to A and then along A_1. Then $\widetilde{T}_1$ is in $W_1(\widetilde{T})$. Then in $\widetilde{T}_1$ move accross A_1. Then flow <u>backwards</u> tracking A_1 and then A_2 (notice A_2 is a stable annulus) until it hits $\widetilde{T}_2$ which then is in $W_2(\widetilde{T})$. Proceeding this way we get that $\widetilde{T}'$ is in $W_j(\widetilde{T})$ for some $j > 0$.

An easy property is that the domains $U_\infty(\widetilde{T})$ where $\widetilde{T}$ are elements of $\mathcal{T}$ are either pairwise disjoint or equal. Since any orbit of $\widetilde{\Phi}$ which is not the lift of a vertical periodic orbit intersects one element of $\mathcal{T}$, it follows that the union of these domains is the entire set $\widetilde{M} \setminus \Delta$. This domain is connected, hence $\widetilde{M} \setminus \Delta = U_\infty(\widetilde{T})$ for every $\widetilde{T}$ in $\mathcal{T}$. In particular, for every $\widetilde{T}$, $\widetilde{T}'$ in $\mathcal{T}$, there is an integer i such that $\widetilde{T}'$ lies in $W_i(\widetilde{T})$.

Now observe that the sequence of elements of $\mathcal{T}$ successively crossed by an orbit of $\widetilde{\Phi}$ defines a path in $\widetilde{\mathfrak{G}}$. It follows that $\widetilde{\mathfrak{G}}$ is weakly connected, as required. □

As we have observed in the previous proof, every oriented path in $\widetilde{\mathfrak{G}}^\sharp$ defines naturally an element of $\mathcal{I} = \mathcal{T}_\sharp^{\mathbb{Z}}$. Recall that $\mathcal{T}_\sharp = \mathcal{T} \cup \Delta$.

Let $\mathcal{I}_0 \subset \mathcal{I}$ be the subset of $\mathcal{I}$ comprising sequences $(\zeta_i)_{i\in\mathbb{Z}}$ corresponding to oriented paths in $\widetilde{\mathfrak{G}}^\sharp$ satisfying the following additional property:

if ζ_i is an element of Δ, then $\zeta_j = \zeta_i$ for either all $j \leq i$, or all $j \geq i$.

Proposition 4.12. *The image of the itinerary map $I : \widetilde{M} \to \mathcal{T}_\sharp^{\mathbb{Z}}$ is precisely $\mathcal{I}_0$.*

Proof. The fact that the image of I is contained in $\mathcal{I}_0$ is quite obvious since if an itinerary $I(\tilde{x}) = (\zeta_i)_{i\in\mathbb{Z}}$ has a term ζ_i equal to $\alpha \in \Delta$, then either $\tilde{x} \in \widetilde{W}^u(\alpha)$, or $\tilde{x} \in \widetilde{W}^s(\alpha)$. In the first case, $\zeta_j = \alpha$ for all $j \leq i$, whereas in the second case $\zeta_j = \alpha$ for all $j \geq i$.

Let now $(\zeta_i)_{i\in\mathbb{Z}}$ be an element of $\mathcal{I}_0$. If every ζ_i lies in Δ, then they are all equal (since there is no edge in $\widetilde{\mathfrak{G}}^\sharp$ connecting two different elements of Δ): every ζ_i is equal to $\alpha \in \Delta$. Then the sequence is the itinerary of any element of α.

Assume now that some ζ_i is an element $\widetilde{T}$ of $\mathcal{T}$, suppose this is ζ_1. Consider positive integers n: as long as ζ_n is an element of $\mathcal{T}$ (and not of Δ), then the signed distance $n(\zeta_1, \zeta_n)$ (in the sense of Definition 4.9) is $+n$. It follows that $\widetilde{A}(\zeta_1, \zeta_n)$ is a (non-empty!) stable band (usually not elementary). At the leaf space level $(\widetilde{\mathcal{H}}^s(\widetilde{T}))$, the projection of $\widetilde{A}(\zeta_1, \zeta_n)$ is an open segment $J(\zeta_1, \zeta_n)$ in $\widetilde{\mathcal{H}}^s(\widetilde{T}) \approx \mathbb{R}$.

Assume first the case where the positive itinerary is finite: there is an integer $n > 0$ such that $\zeta_i \in \mathcal{T}$ for all $i \leq n$, and such that ζ_{n+1} is an element α of Δ. Then, there is one (and only one) stable leaf s_0 of ζ_n whose elements has positive itinerary $(\alpha, \alpha, ...)$. Since $f^s_{\zeta_1,\zeta_n}$ is surjective, there is a stable leaf s in $J(\zeta_1, \zeta_n)$ whose image by $f^s_{\zeta_1,\zeta_n}$ is s_0. Then, the positive itinerary of elements of s is, as required, $(\zeta_i)_{i\geq 1}$.

Consider now the other case, that is, the case where every ζ_i $(i > 0)$ is an element of $\mathcal{T}$. Then, the segments $(J(\zeta_1, \zeta_i)_{i\geq 1})$ form a decreasing (for the inclusion) sequence of intervals in $\widetilde{\mathcal{H}}^s(\widetilde{T}) \approx \mathbb{R}$. In fact more is true. The explanation at the end of the proof of proposition 4.8 shows that when n increases by one, then <u>both</u> endpoints of $J(\zeta_1, \zeta_n)$ change. This follows from the fact in that explanation that the band inside the elementary band did not share a boundary component with the elementary band. Given this fact, it follows that the intersection of the $J(\zeta_1, \zeta_n)$ is non-empty. Every $\tilde{x}$ in ζ_1 whose projection lies in this intersection will admit as positive itinerary $(\zeta_i)_{i\geq 1}$.

In both situations, we have a non-empty stable band $\widetilde{A}((\zeta_i)_{i\geq 1})$ comprising elements of $\mathcal{T}$ with positive itinerary $(\zeta_i)_{i\geq 1}$. Observe that according to Lemma 4.5, $\widetilde{A}((\zeta_i)_{i\geq 1})$ is a single stable leaf – that is, it is a degenerate stable band.

By applying this argument to the reversed flow, one gets that the set of elements of ζ_0 whose negative itinerary coincide with $(\zeta_i)_{i\leq 0}$ is an *unstable* leaf. Since in the plane ζ_1 every unstable leaf intersects every stable leaf, we obtain that ζ_1 contains exactly one element whose itinerary is precisely $(\zeta_i)_{i\in\mathbb{Z}}$. □

5. **Topological and isotopic equivalence**

Let Φ, Ψ be two totally periodic pseudo-Anosov flows, and let $\{Z_i(\Phi)\}$, $\{Z_i(\Psi)\}$ be respective (chosen) spine collections.

5.1. **Topological equivalence.** In this section we show how to deduce Theorem D from Theorem D'.

Proof of Theorem D – First suppose that Φ and Ψ are topologically equivalent. Let f be a self homeomorphism of M realizing this equivalence. Then $f(N(Z_i(\Phi)))$ is a representative for a Seifert fibered piece P_j of M and $\partial f(N(Z_i(\Phi)))$ is transverse to Ψ. In addition the image of every spine $Z_i(\Phi)$ satisfies all the the defining properties for a spine of Ψ in P_j. Hence we can assume that this image is a spine $Z'_j(\Psi)$ for Ψ in the Seifert piece P_j, and every $N(Z'_j(\Psi)) = f(N(Z_i(\Phi)))$ is a tubular neighborhood for this spine. Obviously, f maps vertical orbits for Φ to vertical orbits of Ψ, preserving the orientation. Since the spine decomposition of Ψ is unique up to isotopy along the flow, this finishes the proof of this direction.

Conversely, assume that up to a homeomorphism, we have the equality $\{Z_i(\Phi)\} = \{Z_j(\Psi)\}$, and that the two flows define the same orientation on the vertical orbits. Up to reindexing the collection $\{Z_j(\Psi)\}$ we can assume that for all i, $Z_i(\Phi) = Z_i(\Psi)$. Since the JSJ decomposition of M is unique up to isotopy [Ja-Sh, Jo], it follows that $N(Z_i(\Phi))$ is isotopic to $N(Z_i(\Psi))$ for all i. A torus T boundary of $N(Z_i(\Phi))$ and $N(Z_j(\Phi))$ is isotopic to a corresponding boundary torus T' between $N(Z_i(\Psi))$ and $N(Z_j(\Psi))$. We can then change the flow Ψ by an isotopy so that $T' = T$. Hence we can assume that $N(Z_i(\Phi)) = N(Z_i(\Psi))$. We simplify the notations by setting $Z_i = Z_i(\Phi) = Z_i(\Psi)$ and $N(Z_i) = N(Z_i(\Phi)) = N(Z_i(\Psi))$.

For each component T_k of $\partial N(Z_i)$, let $\mathcal{G}^{s,u}_\Phi(T_k)$ and $\mathcal{G}^{s,u}_\Psi(T_k)$ be the foliations on T_k induced by the stable/unstable foliations of respectively Φ, Ψ. Of course, a priori there is no reason for $\mathcal{G}^{s,u}_\Phi(T_k)$ and $\mathcal{G}^{s,u}_\Psi(T_k)$ to be equal. However we prove the following crucial properties:

Claim – The two foliations $\mathcal{G}^{s,u}_\Phi(T_k)$ and $\mathcal{G}^{s,u}_\Psi(T_k)$ have the same number of closed leaves, and these leaves are all vertical. It follows that these closed curves are isotopic. In addition one can assume that the elementary $\mathcal{G}^{s,u}_\Phi$-annuli are exactly the elementary $\mathcal{G}^{s,u}_\Psi$-annuli.

Consider the component W_k of $N(Z_i) - Z_i$ containing T_k in its boundary. Then W_k is homeomorphic to $T^2 \times [0,1)$, where $T_k = T^2 \times \{0\}$ is a boundary component of $N(Z_i)$ and is therefore entering or exiting $N(Z_i)$. Suppose without loss of generality that T_k is an outgoing component. By the description of the flow in $N(Z_i)$ every point in W_k flows backward to intersect Z_i or be asymptotic to a vertical orbit in Z_i. Let α be a vertical orbit of Z_i. Since T_k is outgoing it follows that the unstable leaf of α intersects W_k and consequently this unstable leaf intersects T_k, and in a closed leaf. It follows that for any such α there is a closed leaf of $\mathcal{G}^s_\Phi(T_k)$ and a closed leaf of $\mathcal{G}^s_\Psi(T_k)$. These closed curves in T_k have powers which are freely homotopic to powers of the regular fiber in P_i and hence they have powers which are freely homotopic to each other. Since both are simple closed curves in T_k and $N(Z_i)$ is Seifert fibered, it now follows that these closed leaves are isotopic in T_k. This proves that that $\mathcal{G}^{s,u}_\Phi(T_k)$ and $\mathcal{G}^{s,u}_\Psi(T_k)$ have the same number of closed leaves and they are all isotopic.

The manifold M is obtained from the collection of Seifert pieces $\{P_i\}$ by glueings along the collection of tori $\{T_k\}$. Isotopic glueing maps of the $\{T_k\}$ generate the same manifold. Hence we can change the glueing maps and have a homeomorphism of M which sends the closed leaves of $\mathcal{G}^{s,u}_\Phi(T_k)$ to the closed leaves of $\mathcal{G}^{s,u}_\Psi(T_k)$.

The unstable vertical annuli for Φ (respectively for Ψ) connect the vertical periodic orbits to the closed leaves in T_k. Now the point is that the vertical annuli for Ψ may not be isotopic rel boundary to the vertical annuli for Φ: they may wrap around T_k, intersecting several times the vertical annuli for Φ. However, there is an orientation preserving homeomorphism which is the identity outside $\overline{W}_k$, inducing a Dehn twist around T_k in the mapping class group of M, which maps every Φ-unstable annulus to the corresponding Ψ-unstable annulus. The completion of W_k is a manifold which is a quotient of $T^2 \times [0,1]$. The only identifications are in "vertical" orbits in $T^2 \times \{1\}$. The maps above induce a homeomorphism of this quotient of $T^2 \times [0,1]$ which is the identity in the boundary. It follows that this is isotopic to a "Dehn twist" in the boundary $T^2 \times \{0\}$. We leave the details to the reader.

After these steps we have a homeomorphism h of M so that $h(Z_i(\Phi)) = Z_i(\Psi)$, $h(N(Z_i(\Phi))) = N(Z_i(\Psi))$, vertical annuli of Φ in $N(Z_i)$ are mapped to vertical annuli of Ψ in $N(Z_j)$, and h preserves the orientation of the vertical orbits.

Now the conjugate of the flow Φ by the homeomorphism h has precisely the same spine decomposition $\{N(Z_i(\Psi))\}$, the same orientation on vertical periodic orbits as Ψ, and the same stable/vertical annuli as Ψ in each $N(Z_i(\Psi))$. According to Theorem D' (to be proved in the next section), using the identity as the homeomorphism used in the statement of Theorem D'; this conjugate flow $h\Phi h^{-1}$ is isotopically equivalent to Ψ. This finishes the proof of Theorem D.

5.2. **Isotopic equivalence.** In this section we prove Theorem D'. As we have observed in the introduction, one implication is obvious: if Φ and Ψ are isotopically equivalent, the isotopy realizing this equivalence maps a spine decomposition of Φ to a spine decomposition of Ψ with all the required properties.

Conversely we assume as in the previous section that for every i we have $Z_i(\Phi) = Z_i(\Psi) = Z_i$ (after reindexing), that Φ, Ψ define the same spine decomposition of M in periodic Seifert pieces $N(Z_i)$, and that they induce the same orientation on the vertical periodic orbits in each Z_i. We furthermore assume that in each $N(Z_i)$ they have exactly the same stable/unstable vertical annuli.

Recall the following several objects we introduced in section 4:

- the set $\mathcal{T}$ of lifts of boundary components of $N(Z_i)$,
- the set Δ of lifts of vertical periodic orbits,
- elementary bands for $\widetilde{\Phi}$,

The hypothesis imply that these objects coincide precisely with the similar objects associated to Ψ.

Observe that in every Seifert piece $N(Z_i)$, the "blocks" connecting one incoming boundary torus to an outgoing torus are delimited by vertical stable/unstable annuli, hence are exactly the same for the two flows. It follows that the graph $\widetilde{\mathfrak{G}}$ and the augmented graph $\widetilde{\mathfrak{G}}^\sharp$ are precisely the same for the two flows. Indeed: let $\widetilde{T}$, $\widetilde{T}'$ are two vertices of the graph $\widetilde{\mathfrak{G}}$ for Φ, connected by one edge, then there is a lift $\widetilde{N(Z_i)}$ such that $\widetilde{T}$ and $\widetilde{T}'$ are boundary component of $\widetilde{N(Z_i)}$, and an orbit of $\widetilde{\Phi}$ inside $\widetilde{N(Z_i)}$ joining a point $\tilde{x}$ in $\widetilde{T}$ to a point $\tilde{y}$ in $\widetilde{T}'$. This orbit is trapped in the lift $\widetilde{U}$ of a block as described in section 4, delimited by two stable bands, two unstable bands, and two elementary bands, one in $\widetilde{T}$, the other in $\widetilde{T}'$ (we refer once again to figure 5). It follows that the orbit of $\widetilde{\Psi}$ starting at $\tilde{x}$ is trapped in the same lifted block $\widetilde{U}$ until it reachs the same elementary band as the one attained by the $\widetilde{\Phi}$-orbit, so that we have also $n(\widetilde{T}, \widetilde{T}') = +1$ from the view point of Ψ.

Hence, Proposition 4.12 implies that any itinerary of $\widetilde{\Psi}$ is realized by $\widetilde{\Phi}$, and vice-versa.

Since the flows Φ and Ψ have precisely the same blocks, with the same orientation on the periodic corner orbits, we can apply section 4.2 and its conclusion. More precisely: for every lifted torus $\widetilde{T}$, let $\mathcal{H}^s_\Psi(\widetilde{T})$ denote the leaf space of the restriction to $\widetilde{T}$ of the stable foliation of $\widetilde{\Psi}$. If $\widetilde{T}'$ is another lifted torus such that $n(\widetilde{T}, \widetilde{T}') > 0$, we can define a map $g^s_{\widetilde{T},\widetilde{T}'}$, analogous to $f^s_{\widetilde{T},\widetilde{T}'}$, from an interval of $\mathcal{H}^s_\Psi(\widetilde{T})$ onto $\mathcal{H}^s_\Psi(\widetilde{T}')$. This map is obtained using the flow $\widetilde{\Psi}$.

We also need to define the transverse orientation to the foliations $\mathcal{G}^s_\Psi(T_k)$ and $\mathcal{G}^u_\Psi(T_k)$. These are the stable and unstable foliations induced by the flow Ψ in the JSJ tori $\{T_k\}$. As observed in the previous section, the closed leaves of these foliations are the same as the closed leaves of the foliations $\mathcal{G}^s_\Phi(T_k)$ and $\mathcal{G}^u_\Phi(T_k)$ induced by Φ on T_k. Choose the transverse orientation of (say) $\mathcal{G}^s_\Psi(T_k)$ to have it agree with the transverse orientation of $\mathcal{G}^s_\Phi(T_k)$ across the closed leaves.

By hypothesis, the flow directions of corresponding vertical periodic orbits of Φ and Ψ in any given Seifert piece P_i agree. This implies that the holonomy of the foliations $\mathcal{G}^s_\Phi(T_k)$ and $\mathcal{G}^s_\Psi(T_k)$ along the closed leaves agrees with each other, that is, they are either both contracting or both repelling.

The important conclusion is that with this choice of orientations and the above remark, $g^s_{\widetilde{T},\widetilde{T}'}$ is orientation preserving if and only if $f^s_{\widetilde{T},\widetilde{T}'}$ is orientation preserving.

Let $\mathcal{O}_\Phi$, $\mathcal{O}_\Psi$ denote the orbit spaces of $\widetilde{\Phi}$, $\widetilde{\Psi}$, respectively. According to Corollary 4.7 applied to Φ and to Ψ as well, there is a natural bijection $\varphi : \mathcal{O}_\Phi \to \mathcal{O}_\Psi$: the one mapping an orbit of $\widetilde{\Phi}$ to the unique orbit of $\widetilde{\Psi}$ admitting the same itinerary up to the shift map. Observe that the map φ is naturally $\pi_1(M)$-equivariant. More precisely, φ commutes with the action of $\pi_1(M)$: for every θ in $\mathcal{O}_\Phi$ and every γ in $\pi_1(M)$, we have:

$$\varphi(\gamma\theta) = \gamma\varphi(\theta) \tag{2}$$

Lemma 5.1. *The map $\varphi : \mathcal{O}_\Phi \to \mathcal{O}_\Psi$ is a homeomorphism.*

Proof. The only remaining point to prove is the continuity of φ (the continuity of the inverse map φ^{-1} is obtained by reversing the arguments below). We already know that two orbits lie in the same stable leaf if and only if their itineraries, up to a shift, coincide after some time. Hence, φ maps the foliation $\mathcal{O}^s_\Phi$ onto the foliation $\mathcal{O}^s_\Psi$, and similarly, φ maps $\mathcal{O}^u_\Phi$ onto $\mathcal{O}^u_\Psi$.

Observe that the projection of Δ by the map $\Theta_\Phi : \widetilde{M} \to \mathcal{O}_\Phi$ is a closed discrete subset of $\mathcal{O}_\Phi$ that we denote by Δ_Φ, and the image of Δ by φ is a closed discrete subset Δ_Ψ of $\mathcal{O}_\Psi$ corresponding to the lifts of vertical periodic orbits of Ψ.

We first show the continuity of φ on $\mathcal{O}_\Phi \setminus \Delta_\Phi$. Let θ be an element of $\mathcal{O}_\Phi \setminus \Delta_\Phi$. This is an orbit of $\widetilde{\Phi}$ which crosses some element $\widetilde{T}$ of $\mathcal{T}$ at a point $\tilde{x}$. The restriction to $\widetilde{T}$ of the projection map $\Theta_\Phi : \widetilde{M} \to \mathcal{O}_\Phi$ is injective as remarked before. Let $\mathcal{P}_\Phi$ be the projection of $\widetilde{T}$ to $\mathcal{O}_\Phi$; it is an open 2-plane contained in $\mathcal{O}_\Phi \setminus \Delta$. Elements of $\mathcal{P}_\Phi$ are characterized by the property that their itinerary (which is well-defined up to the shift map) contains $\widetilde{T}$. It follows that the image of $\mathcal{P}_\Phi$ by φ is the projection $\mathcal{P}_\Psi$ of $\widetilde{T}$ in $\mathcal{O}_\Psi \setminus \Delta_\Psi$. We will show that the restriction of φ to $\mathcal{P}_\Phi$ is continuous, which will prove as required that

φ is continuous on $\mathcal{O}_\Phi \setminus \Delta_\Phi$. The restrictions $\mathcal{P}^s_\Phi$, $\mathcal{P}^u_\Phi$ of $\mathcal{O}^s_\Phi$ and $\mathcal{O}^u_\Phi$ to $\mathcal{P}_\Phi$ are the projections of $\widetilde{\mathcal{G}}^s_\Psi(\widetilde{T})$, $\widetilde{\mathcal{G}}^u_\Psi(\widetilde{T})$. They are regular foliations; more precisely, they are product foliations, transverse to each other. Every leaf of $\mathcal{P}^s_\Phi$ intersects every leaf of $\mathcal{P}^u_\Phi$ in one and only one point. In summary, $\mathcal{P}_\Phi$ is homeomorphic to $\mathcal{H}^s_{\mathcal{P}_\Phi} \times \mathcal{H}^u_{\mathcal{P}_\Phi} \approx \mathbb{R}\times\mathbb{R}$, where $\mathcal{H}^{s,u}_{\mathcal{P}_\Phi}$ denotes the leaf space of $\mathcal{O}^{s,u}_\Phi$ restricted to $\mathcal{P}_\Phi$ respectively.

Similarly, $\mathcal{P}_\Psi$ is homeomorphic to $\mathcal{H}^s_{\mathcal{P}_\Psi} \times \mathcal{H}^u_{\mathcal{P}_\Psi}$. Furthermore, since φ maps $\mathcal{O}^s_\Phi$, $\mathcal{O}^u_\Phi$ onto $\mathcal{O}^s_\Psi$, $\mathcal{O}^u_\Psi$, the restriction of φ to $\mathcal{P} \approx \mathcal{H}^s_{\mathcal{P}_\Phi} \times \mathcal{H}^u_{\mathcal{P}_\Phi}$ has the form:

$$(S,U) \to (\varphi^s(S), \varphi^u(U))$$

where S, U denote leaves of $\mathcal{P}^s_\Phi$, $\mathcal{P}^u_\Phi$ and $\varphi^s(S), \varphi^u(U)$ are the induced images in the leaf space level.

Each leaf space $\mathcal{H}^s_{\mathcal{P}_\Phi}$, $\mathcal{H}^s_{\mathcal{P}_\Psi}$ admits a subdivision in *elementary segments* $J_\Phi(\widetilde{T}, \widetilde{T}')$ (respectively $J_\Psi(\widetilde{T}, \widetilde{T}')$), which are the projections of elementary bands $A(\widetilde{T}, \widetilde{T}')$ where $n(\widetilde{T}, \widetilde{T}') = 1$. Since the elementary bands in $\widetilde{T}$ for $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are exactly the same, the map φ^s preserves the order between elementary segments. This uses the choice of transverse orientations for $\mathcal{G}^s_\Phi(T)$ and $\mathcal{G}^s_\Psi(T)$.

Let S, S' be arbitrary elements of $\mathcal{H}^s_{\mathcal{P}_\Phi}$, such that $S \prec S'$ for the order defined on $\mathcal{H}^s_{\mathcal{P}_\Phi} \approx \mathcal{H}^s_\Phi(\widetilde{T})$ in Remark 4.2. Then, if S, S' lie in different elementary bands, it follows from what we have just seen that $\varphi^s(S) \prec \varphi^s(S')$, since φ^s preserves the order between the elementary intervals $J_\Phi(\widetilde{T}, \widetilde{T}')$ and $J_\Psi(\widetilde{T}, \widetilde{T}')$.

Assume now that S, S' lie in the same elementary band. Since $S \neq S'$, there is an element $\widetilde{T}'$ of $\mathcal{T}$ such that $f^s_{\widetilde{T},\widetilde{T}'}(S)$ and $f^s_{\widetilde{T},\widetilde{T}'}(S')$ either lie in different elementary segments of $\mathcal{H}^s_\Phi(\widetilde{T}')$ or lie in the closure of the same elementary segment. The second case is equivalent to S or S' being in the stable manifold of a lift of a vertical periodic orbit. We will deal with the first case, the second case being simpler.

As above we have a map $\varphi^s_* : \mathcal{H}^s_{\mathcal{P}'_\Phi} \to \mathcal{H}^s_{\mathcal{P}'_\Psi}$ between the corresponding leaf spaces of foliations in $\widetilde{T}'$. Here $\mathcal{P}'_\Phi$ and $\mathcal{P}'_\Psi$ are the projections of $\widetilde{T}'$ to the orbit spaces of $\widetilde{\Phi}$ and $\widetilde{\Psi}$ respectively. We clearly have

$$g^s_{\widetilde{T},\widetilde{T}'}(\varphi^s(S)) \;=\; \varphi^s_*(f^s_{\widetilde{T},\widetilde{T}'}(S)) \quad \text{and} \quad g^s_{\widetilde{T},\widetilde{T}'}(\varphi^s(S')) \;=\; \varphi^s_*(f^s_{\widetilde{T},\widetilde{T}'}(S')).$$

Hence, $g^s_{\widetilde{T},\widetilde{T}'}(\varphi^s(S))$ and $g^s_{\widetilde{T},\widetilde{T}'}(\varphi^s(S)')$ lie in different elementary segments of $\mathcal{H}^s_\Phi(\widetilde{T}')$. We have the following alternatives:

If $f^s_{\widetilde{T},\widetilde{T}'}$ preserves orientation: then $f^s_{\widetilde{T},\widetilde{T}'}(S) \prec f^s_{\widetilde{T},\widetilde{T}'}(S')$. In other words, the elementary segment containing $f^s_{\widetilde{T},\widetilde{T}'}(S)$ is above the elementary segment containing $f^s_{\widetilde{T},\widetilde{T}'}(S')$. Since φ^s_* preserves the order between elementary segments, we obtain:

$$g^s_{\widetilde{T},\widetilde{T}'}(\varphi^s(S)) \;=\; \varphi^s_*(f^s_{\widetilde{T},\widetilde{T}'}(S)) \;\prec\; \varphi^s_*(f^s_{\widetilde{T},\widetilde{T}'}(S')) \;=\; g^s_{\widetilde{T},\widetilde{T}'}(\varphi^s(S'))$$

But in this case, $g^s_{\widetilde{T},\widetilde{T}'}$ is also orientation preserving; hence $\varphi^s(S) \prec \varphi^s(S')$.

If $f^s_{\widetilde{T},\widetilde{T}'}$ reverses orientation: then $f^s_{\widetilde{T},\widetilde{T}'}(S') \prec f^s_{\widetilde{T},\widetilde{T}'}(S)$, therefore:

$$g^s_{\widetilde{T},\widetilde{T}'}(\varphi^s(S')) = \varphi^s_*(f^s_{\widetilde{T},\widetilde{T}'}(S')) \prec \varphi^s_*(f^s_{\widetilde{T},\widetilde{T}'}(S)) = g^s_{\widetilde{T},\widetilde{T}'}(\varphi^s(S))$$

Since $g^s_{\widetilde{T},\widetilde{T}'}$ reverses orientation, we deduce $\varphi^s(S) \prec \varphi^s(S')$.

In both cases, we have proved $\varphi^s(S) \prec \varphi^s(S')$. Therefore, φ^s preserves the total orders on $\mathcal{H}^s_{\mathcal{P}_\Phi}$ and $\mathcal{H}^s_{\mathcal{P}_\Psi}$; since we already know that it is a bijection, it follows that φ^s is a homeomorphism.

We can reproduce the same argument, but this time for the negative itineraries, and this time involving the unstable leaves: we then obtain that φ^u is a homeomorphism. Therefore, the restriction of φ to $\mathcal{P}_\Phi$ is continuous. As previously observed, it proves that φ is continuous on $\mathcal{O}_\Phi \setminus \Delta_\Phi$.

Permute the role of Φ and Ψ: the restriction to $\mathcal{O}_\Psi \setminus \Delta_\Psi$ of the inverse map φ^{-1} is continuous. Hence the restriction of φ to $\mathcal{O}_\Phi \setminus \Delta_\Phi$ is a homeomorphism - in particular, proper.

Now the continuity of φ on the entire $\mathcal{O}_\Phi$ follows easily: indeed, on the one hand, the space of ends of $\mathcal{O}_\Phi \setminus \Delta_\Phi$ (respectively $\mathcal{O}_\Psi \setminus \Delta_\Psi$) is naturally the union of Δ_Φ and the end ∞ of the plane $\mathcal{O}_\Phi$. Since $\mathcal{O}_\Phi$ is a two dimensional plane, and Δ_Φ is discrete, it now follows that the restriction of φ to $\mathcal{O}_\Phi \setminus \Delta_\Phi$ admits a unique continuous extension $\bar{\varphi} : \mathcal{O}_\Phi \to \mathcal{O}_\Psi$, mapping Δ_Φ onto Δ_Ψ. On the other hand, φ maps bijectively Δ_Φ onto Δ_Ψ. The problem is to prove that $\bar{\varphi}$ and φ coincide on Δ_Φ. This follows easily from the fact that for every element θ of Δ_Φ, φ maps an open half leaf of the stable leaf of θ onto an open half leaf of the stable leaf of $\varphi(\theta)$, and that these stable leaves do not accumulate at other elements of Δ_Ψ.

We have proved that $\varphi : \mathcal{O}_\Phi \to \mathcal{O}_\Psi$ is continuous. The proof of the Lemma is completed. □

Proposition 5.2. *(Isotopic equivalence) The flows Φ and Ψ are isotopically equivalent.*

Proof. Given Lemma 5.1 this follows from established results: First, Haefliger [Hae] showed that Lemma 5.1 implies that the homeomorphism φ lifts as a map $\tilde{f} : \widetilde{M} \to \widetilde{M}$, satisfying:

$$\varphi \circ \Theta_\Phi = \Theta_\Psi \circ \tilde{f}$$

where $\Theta_\Phi : \widetilde{M} \to \mathcal{O}_\Phi$ and $\Theta_\Psi : \widetilde{M} \to \mathcal{O}_\Psi$ are the projection maps on the leaf spaces. In other words, $\tilde{f}$ maps the orbits of $\tilde{\Phi}$ onto orbits of $\tilde{\Psi}$. In addition the map $\tilde{f}$ also projects to a map in the quotient M, $f : M \to M$, so that f is a homotopy equivalence of M which takes orbits of Φ to orbits of Ψ.

Actually, according to equation (2) (and according to [Hae]), we have for every γ in $\pi_1(M)$:

$$\gamma \circ \tilde{f} = \tilde{f} \circ \gamma$$

It means that the map $f : M \to M$ is homotopic to the identity.

Since f is a homotopy equivalence, f is surjective, but it still may fail to be injective along the orbits. Ghys [Gh] explained how to produce a homeomorphism with the same properties, by an average process along the orbits. This was explicitly done by the first author, Theorem of [Ba1], for Anosov flows. In [Ma-Tsu, Proposition 5.1], S. Matsumoto and T. Tsuboi proved a more general version suitable for our purpose here. Their results imply that we eventually get the existence of a topological equivalence between Φ and Ψ, which is homotopic to the identity. According to [Wald3, Theorem 7.1], using that M is irreducible and has an incompressible surface, this homeomorphism f is isotopic to the identity. This proves Theorem D'. □

6. Model flows

6.1. **Construction of model pseudo-Anosov flows.** In this section, we recall the construction in [Ba-Fe], section 8, of model pseudo-Anosov flows. Actually, we will define a family of model flows $\Psi_{\lambda,\ldots,\lambda_k}$ depending on real parameters $\lambda_1, \ldots, \lambda_k$; the flow $\Psi_{\lambda,\ldots,\lambda_k}$ is pseudo-Anosov if the absolute value of every λ_i is sufficiently large. In [Ba-Fe], we only considered the case where all the λ_i have the same value λ, but it is quite obvious that it need not to be so. The point here is that the choice of these parameters is irrelevant, since it leads to the same flow up to isotopic equivalence.

First, fix a real number λ. The flow will be obtained from building blocks, which are standard neighborhoods of intrinsic elementary Birkhoff annuli. Such a neighborhood is homeomorphic to $[0,1] \times \mathbf{S}^1 \times [0,1]$ (with corresponding (x,y,z) coordinates).

The Birkhoff annulus is $[0,1] \times \mathbf{S}^1 \times \{1/2\}$, where $\{0\} \times \mathbf{S}^1 \times \{1/2\}$ and $\{1\} \times \mathbf{S}^1 \times \{1/2\}$ are the only closed orbits of the semiflow in this block and they are the boundaries of the Birkhoff annulus (see figure 7).

The flow is tangent to the side boundaries $\{0\} \times \mathbf{S}^1 \times [0,1]$ and $\{1\} \times \mathbf{S}^1 \times [0,1]$. The flow is incoming along $[0,1] \times \mathbf{S}^1 \times \{0\}$ and outgoing along $[0,1] \times \mathbf{S}^1 \times \{1\}$. The vertical orbits are the periodic orbits in the block. For example, the stable manifold of $\{0\} \times \mathbf{S}^1 \times \{1/2\}$ is $\{0\} \times \mathbf{S}^1 \times [0,1/2]$ and the unstable manifold is $\{0\} \times \mathbf{S}^1 \times [1/2,1]$. Every orbit entering in the interior of the incoming side reach the opposite outgoing side. Such an orbit has a vertical deviation in the y coordinate, the bigger λ is, the bigger is this vertical deviation.

In [Ba-Fe] we prescribe an explicit formula for the flow, denoted by Ψ_λ, in the block B, depending on the parameter λ.

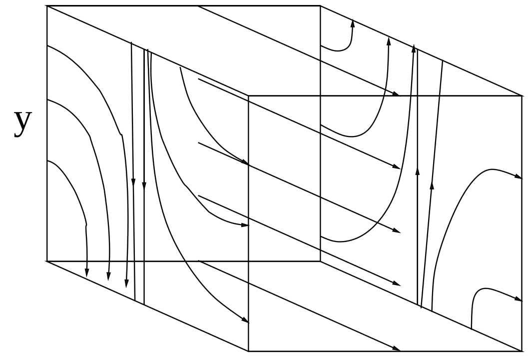

Figure 7: *Standard neighborhood of a Birkhoff annulus.*

Several copies of these blocks can be glued one to the other along annuli which are half of a tangential boundary annulus. These are either the local stable or unstable manifolds of one of the periodic orbits. For example one may glue $\{0\} \times \mathbf{S}^1 \times [0, 1/2]$ to a similar half annulus in another copy of the block. In particular we are glueing stable or unstable manifolds of certain vertical orbits to similar sets of other vertical orbits. The glueings preserve the flow. One can do this in a very flexible way, so that in the end all tangential boundary components are eliminated. At this point one obtains a semiflow in a manifold P' which is a circle bundle over a surface with boundary Σ. In order to define model flows, we actually specify particular glueings between the stable/unstable annuli, so that every boundary component admits a natural coordinate system (x, y) ([Ba-Fe, Section 8]). In particular, the coordinate y defines a function on P', whose level sets are sections of the fibration over Σ_i. All this process is encoded by the data of a fat graph X embedded in Σ, whose vertices correspond to the (vertical) periodic orbits, whose edges correspond to embedded elementary Birkhoff annuli, and satisfying the following properties:

(1) X is a deformation retract of Σ.
(2) The valence of every vertex of X is an even number.
(3) The set of boundary components of Σ is partitioned in two subsets so that for every edge e of X, the two sides of e in Σ lie in different sets of this partition.
(4) Each loop in X corresponding to a boundary component of Σ contains an even number of edges.

We also allow Dehn surgery along the vertical orbits so that the resulting manifold is a Seifert bundle over the surface Σ. This operation is encoded by the data of the fat graph (Σ, X) and also the Dehn surgery coefficients D on vertices of X, i.e. on vertical orbits (these coefficients are well-defined with the convention that the meridians of the Dehn fillings to be the loops contained in the section $\{y = Cte\}$ mentioned above). We denote the resulting manifold with semi-flow by $(P(\Sigma, X, D), \Psi_\lambda)$. Observe that $P(\Sigma, X, D)$ is a Seifert manifold. Moreover, the (x, y) coordinates in the initial building block provides natural coordinates on every boundary component of $P(\Sigma, X, D)$. The last item above ensures that each boundary component of $P(\Sigma, X, D)$ is a torus, as opposed to being a Klein bottle.

Finally, we consider several copies $P_1, \ldots, P_k$ of such Seifert fibered manifolds along their transversal boundaries, each equipped with a local model flow Ψ_{λ_i} for some real parameters λ_i. Let us be more precise: for each P_i and each component T of P_i, select a *vertical/horizontal basis* of $H_1(T, \mathbb{Z})$, i.e. a basis whose first element is represented by vertical loops (i.e. regular fibers of P_i), and whose second element is represented by the intersection between T and the preferred section $\{y = Cte\}$ we have defined above. One could think at first glance that we have defined by this way a canonical basis of $H_1(T, \mathbb{Z})$, but the point is that these homology classes are defined only up to sign: $H_1(T, \mathbb{Z})$ admits four vertical/horizontal basis.

Once such a basis is selected in each boundary torus, select a pairing between these boundary tori, and for each such pair (T, T') choose a two-by-two matrix $M(T, T')$ with integer coefficients. It defines an isomorphism between $H_1(T, \mathbb{Z})$ and $H_1(T', \mathbb{Z})$; hence an isotopy class of homeomorphisms between T and T'. In order to obtain a pseudo-Anosov flow in the resulting manifold, it is necessary that the glueing maps do not map fibers to curves homotopic to fibers, i.e. that no $M(T,T')$ is upper triangular. In [Ba-Fe] we show[2] that the the resulting flow, denoted by $\Psi_{\lambda_1,...,\lambda_i}$, is pseudo-Anosov as soon as the real parameters λ_i are all sufficiently large. We proved it for very particular glueing maps between the boundary tori, which are linear in the natural coordinates defined by the (x,y) coordinates (even if the proof in [Ba-Fe] can be easily extended to less restrictive choices). It is clear that for the resulting flow $\Psi_{\lambda_1,...,\lambda_i}$ - called a *model flow* - the spine in every piece P_i, as stated in Theorem C, is the preimage by the Seifert fibration of the fat graph X_i.

An immediate consequence of Theorem D' is that the resulting pseudo-Anosov flow is insensitive up to isotopic equivalence to several choices: it does not depend on the real parameters λ_1, ... , λ_k, nor to the choice of the glueing map in the isotopy class defined by the matrices $M(T,T')$, as long as these data are chosen so that the resulting flow is pseudo-Anosov.

In summary, the model flow is uniquely defined up to isotopic equivalence by:

(1) The data of a family of fat graphs (Σ_i, X_i) and Dehn filling coefficients D_i $(i = 1, ..., k)$.
(2) A choice of vertical/horizontal basis in each $H_1(T,\mathbb{Z})$,
(3) A pairing between the boundary tori of the $P_i = P(\Sigma_i, X_i, D_i)$ (i.e. a pairing between the boundary components of the Σ_i's).
(4) For each such a pair (T,T'), a two-by-two matrix with integer coefficients which is not upper triangular.

Observe that item (2) is not innocuous: Suppose that for some P_i we replace the first vector in the vertical/horizontal basis of each component of P_i (but we do not modify the basis of the boundary components of the other pieces P_j, nor any of the other combinatorial data). Then we obtain a model flow on the same manifold, with the same spine decomposition and vertical stable/unstable annuli, *but where the orientation of the vertical periodic orbits has been reversed.* Therefore, this second model pseudo-Anosov flow is *not* isotopically equivalent to the initial model flow.

We could formulate a statement establishing precisely when two initial combinatorial data provide topologically equivalent model pseudo-Anosov flows, but it would require a detailed presentation of F. Waldhausen's classification Theorem of graph manifolds ([Wald1, Wald2]). We decided that it would be an unnecessary complication, and that Theorems D and D' already provide a convenient formulation for the solution of the classification problem of totally periodic pseudo-Anosov flows. For example one issue we do not address is the choice of section in each Seifert fibered piece minus small neighborhoods of singular fibers. The boundary curves of these sections are essential to the classification of totally periodic pseudo-Anosov flows. These boundary curves are determined up to Dehn twists along vertical curves [Wald1, Wald2, BNR].

6.2. **Topological equivalence with model pseudo-Anosov flows.** In this section we present the proof of the Main Theorem, i.e. we show why every totally periodic pseudo-Anosov flow (M, Φ) is topologically equivalent to one of the model pseudo-Anosov flows constructed in the previous section. More precisely, we will show that is topologically equivalent with one model flow $\Psi_{\lambda_1,...,\lambda_i}$ where all the λ_i's are equal to the same real parameter λ - we denote such a model flow by Ψ_λ.

According to section 3, the manifold M is obtained by glueing the Seifert pieces $N(Z_i)$ along (transverse) tori. Moreover, every $N(Z_i)$ can be obtained by glueing appropriate sides of a collection of "blocks" $U(A_p)$, where each A_p is a Birkhoff annulus. The union of these Birkhoff annuli is the spine Z_i of $N(Z_i)$. Every block $U(A_p)$ can be described as follows: its boundary contains two annuli, one inward and the other outward, whose lifts correspond to what have been called elementary bands in the previous section. The block $U(A_p)$ also contains two periodic orbits (the boundary components of A_p; which may be identified through the glueing), two stable annuli (one for each vertical periodic orbit) and two unstable annuli (also one for each periodic orbit) connecting the periodic orbits to the entrance/exit transverse annuli in the boundary. All these annuli constitute the boundary of $U(A_p)$. The remaining part of $U(A_p)$ is a union of orbits crossing A_p, and joining the entrance annulus to the exit annulus. Of course, the lifts in $\widetilde{M}$ of these blocks are nothing but what have been called *blocks* in section 4.2.

[2]Actually, in this reference we only considered the case where all the λ_i's are equal, but exactly the same proof therein applies in the slightly more general case we consider here.

The entire Seifert piece $N(Z_i)$ is obtained by glueing all these $U(A_p)$ along their stable and unstable sides. The way this glueing has to be performed is encoded by a fat graph whose vertices correspond to the vertical periodic orbits, and the edges are the Birkhoff annuli A_p. Let us be slightly more precise: remove around every vertical periodic orbit a small tubular neighborhood. The result is a compact 3-manifold $N(Z_i)^*$ which is a circle bundle over a surface Σ_i^* with boundary. Notice that, as any other circle bundle (orientable or not) over a surface (orientable or not) with non-empty boundary, this circle bundle admits a section. In other words, we can consider Σ_i^* as a surface embedded in $N(Z_i)^*$; the edges of X_i then are the intersection between Σ_i^* and the the Birkhoff annuli A_p.

There are three types of boundary components:

- boundary components corresponding to exit transverse tori,
- boundary components corresponding to entrance transverse tori,
- boundary components corresponding to (the boundary of tubular neighborhoods of) vertical periodic orbits.

Let us define the surface Σ_i obtained by shrinking the last type of boundary components to points, that we call *special points*. One can select the circle fibration $\eta_i^* : N(Z_i)^* \to \Sigma_i^*$ so that the restrictions of the Birkhoff annuli A_p are vertical. Then their projections define in Σ_i a collection of segments which are the edges of a graph X_i embedded in Σ_i. Moreover, since every orbit in $N(Z_i)$ crosses a Birkhoff annulus or accumulates on a vertical orbit, X_i is a retract of Σ_i, in other words, (Σ_i, X_i) is a fat graph. Observe that it satisfies the four properties required in the definition of model flows in the previous section: we have just established the first and third items; the second item corresponds to the fact that at each vertical periodic orbit there is an equal number of stable and unstable vertical annuli, so that each of them is adjacent to an even number of Birkhoff annuli. The last item corresponds to the fact that the component of $\partial P'$ must be tori, not Klein bottles.

Therefore, there is a model semi flow Ψ_i on a circle bundle $P_i' = P(\Sigma_i, X_i) \to \Sigma_i$ as described in the previous section. This partial flow has essentially the same properties that the restriction Φ_i of Φ to $N(Z_i)$ has: P_i' is a circle bundle over Σ_i; its boundary components are transverse to Ψ_i; moreover the inward (respectively outward) boundary components for Ψ_i correspond to the boundary components of Σ_i that are the entrance (respectively exit) components as defined previously, i.e. with respect to Φ_i. Moreover, the preimage in P_i' of edges of X_i are annuli A_p^0 transverse to Ψ_i. Every orbit of Ψ_i either crosses one (and only one) A_p^0, or accumulates on one vertical periodic orbit.

The main difference is that P_i' is a circle bundle over Σ_i, whereas $N(Z_i)$ is merely a Seifert manifold with a Seifert fibration. In particular near the vertical periodic orbits, the fibration of P_i is a product fibration. Moreover, some of the vertical periodic orbit of Ψ_i might be 1-prong orbits. There is an embedding $N(Z_i)^* \hookrightarrow P_i'$ preserving the fibers, and mapping Σ^*, considered as a section in $N(Z_i)^*$, into the canonical section $\{y = Cte\}$ of $N(X_i) \to \Sigma_i$. We can furthermore choose this embedding so that it is coherent relatively to the orientation of vertical orbits: we require that the orientation of the regular fibers in each component C of $\partial N(Z_i)^*$ defined by the oriented periodic orbit of Φ surrounded by C coincide with the orientation defined by the periodic orbit of Ψ_i surrounded by the image of C in P_i'.

Although $N(Z_i)$ is not always homeomorphic to P_i', it is obtained from it by a Dehn surgery along the vertical orbits. This Dehn surgery is encoded by the data of Dehn coefficients D_i, i.e. the data at each vertex of X_i of a pair (p_i, q_i) of relatively prime integers. This data is well-defined once one selects the section Σ_i^*, since meridians around every vertical orbit can be defined as the loops contained in Σ^*. Hence, there is a homeomorphism between $N(Z_i)$ and the model piece $P_i = P(\Sigma_i, X_i, D_i)$, which maps oriented vertical orbits of Φ contained in Z_i to oriented vertical orbits of the model semi-flow Ψ_i. Observe that this homeomorphism maps stable/unstable vertical annuli into stable/unstable vertical annuli of Ψ_i.

Now M is obtained by glueing exit transverse tori to entrance transverse tori of the various Seifert pieces $N(Z_i)$ through identification homeomorphisms φ_k (where k describes the set of transverse tori T_k as denoted previously). The isotopy classes of these homeomorphisms can be characterized by two-by-two matrices, once selected in each $H_1(T, \mathbb{Z})$ vertical/horizontal basis, where the second (horizontal) element of the basis is now determined according to the section Σ_i^*. These two-by-two matrices cannot be upper triangular, since adjacent Seifert pieces in a (minimal) JSJ decomposition cannot have freely homotopic regular fibers.

The choice of vertical/horizontal homological basis in each boundary torus of $N(Z_i)$ naturally prescribes a choice of vertical/horizontal homological basis in each boundary torus of $P_i \approx N(Z_i)$.

Hence we have collected all the necessary combinatorial data necessary for the construction of a model flow Ψ on a manifold M_Ψ obtained by glueing the various pieces P_i. The resulting manifold M_Ψ is homeomorphic to M.

Now it should be clear to the reader that the (M, Φ) and (M_Ψ, Ψ) have the same combinatorial data, so that they are topologically equivalent by Theorem D. This completes the proof of the Main Theorem.

7. Concluding remarks

Throughout this section M is a graph manifold admitting a totally periodic pseudo-Anosov flow.

Number of topological equivalence classes and isotopic equivalence classes of pseudo-Anosov flows. The mapping class group of a given topological surface $S = \Sigma_i$ with more than one boundary component is infinite. This implies that the number of fat graphs on S up to isotopy is infinite, unless S is a surface of small complexity. One example of a surface of small complexity is the annulus. Here there is only one fat graph T which is a core circle. Notice that such surfaces also need to be considered for pseudo-Anosov flows. At first one may think that a fat graph T a circle as above will only yield one-prong pseudo-Anosov flows. However one can perform Dehn surgery on the vertical orbits to obtain true pseudo-Anosov flows without one-prong orbits. In this article we do not explicitly determine all the surfaces of small complexity with respect to this question.

In addition if a homeomorphism g of S maps a given fat graph T onto a fat graph isotopic to T, then g is isotopic to the identity, because T is a deformation retract of S. It follows from these facts, and from Theorem D', that the number of isotopic equivalence classes of pseudo-Anosov flows on the same graph manifold so that at least one of the corresponding fat graph surfaces does not have small complexity, is infinite.

Let us now discuss the number of topological equivalence classes. Waller [Wa] proved that in general the number of fat graph structures on a given topological surface $S = \Sigma_i$, up to homeomorphisms of S, even if always finite, can be quite big. These all will generate topologically inequivalent pseudo-Anosov flows. That is, different graphs X_i yield different flows. With the same fat graphs, with the careful choices of particular sections and Dehn filling coefficients, made above, then the following happens. The only choice left is the direction of the vertical periodic orbits. Once the direction of a single vertical periodic orbit in a fixed P_i is chosen, then all the directions in the other vertical orbits are determined, because the choice of orientations propagates along Birkhoff annuli, i.e. edges of the fat graph. But there are two choices here. So there are at least 2^k inequivalent such pseudo-Anosov flows if there are k pieces in the JSJ decomposition of M.

Therefore, we have proved that there is no upper bound on the number of topological equivalence classes of pseudo-Anosov flows on 3-manifolds.

Action of the mapping class group on the space of isotopic equivalence classes. The mapping class group of the graph manifold M preserves the JSJ decomposition; more precisely, induces a permutation on the space of Seifert pieces of the JSJ decomposition. In general, if the permutation is non trivial, then it will map a totally periodic pseudo-Anosov flow Φ in M to a flow non isotopically equivalent to Φ, except maybe in exceptional situations where M has some symmetries - for example, when M has exactly two pieces which are both homeomorphic to the product $S \times \mathbf{S}^1$ of a surface S with more than one boundary component with the circle.

Let us focus on non-trivial elements of the mapping class group preserving every JSJ component. As observed in the previous section, in general it will preserve no isotopic equivalence class of totally periodic pseudo-Anosov flow, simply because the induced map on the base surface S of some Seifert piece does not preserve the isotopy class of a fat graph in S.

However, there is an interesting phenomenom we want to discuss here: consider a torus T of the JSJ decomposition of M. Let P_i, P_j be the two adjacent Seifert pieces of M containing T in their boundary. Let $U(T)$ be a tubular neighborhood of T. Since M is orientable, $U(T)$ is diffeomorphic to the product of an annulus A by the circle $\mathbf{S}^1$. Let τ be a Dehn twist in the annulus A; then the map $\tau_T : A\times\mathbf{S}^1 \to A\times\mathbf{S}^1$ defined by $\tau_T(x,\theta) = (\tau(x),\theta)$ defines a homeomorphism of M, with support contained in $U(T)$, which is not homotopically trivial modulo the boundary. Therefore τ_T induces a homeomorphism of M which is not isotopic to the identity, because it induces a non trivial isomorphism of $\pi_1(M)$. Notice in addition that there are infinitely many inequivalent ways of expressing $U(T)$ as a product of an annulus and a circle: they are in one-to-one correspondance with the space of indivisible homology classes in $H^1(T,\mathbb{Z})$.

But the following remarkable property holds: Suppose that T is a torus of the JSJ decomposition and τ_T is a Dehn twist in a *vertical* direction of T, i.e. for which the annulus A contains a loop freely homotopic to the regular fibers of one of the two Seifert pieces P_i, P_j bounded by T. Then τ_T does not change the isotopy class of totally periodic pseudo-Anosov flows, i.e. the conjugate of the flow by any representant of τ_T is isotopically equivalent to the initial flow.

Indeed, let Φ be any such a flow. Up to isotopic equivalence, one can assume that T is contained in one neighborhood $N(Z_i)$ which is a representative of the Seifert piece P_i, and in the region between Z_i and a boundary component of $N(Z_i)$ isotopic to T. Then, we can select in the isotopy class of τ_T another homeomorphism f with support disjoint from Z_i, disjoint from all the $N(Z_k)$ with $k \neq i$, and which preserves all the vertical stable/unstable annuli in $N(Z_i)$. In addition f preserves the Z_i and the vertical annuli in $N(Z_i)$ for all i. Therefore, the conjugate of Φ by f has the same combinatorial/topological data than Φ. Thus, according to Theorem D', the conjugate of Φ by f is isotopically equivalent to Φ, since the identity map realizes the isotopy involved in the statement of Theorem D'.

Similarly, the same applies for Dehn twists along the vertical direction defined by the regular fibers of the adjoining Seifert piece P_j. In this way, we can obtain many other elements of the mapping class group in the stabilizer of the isotopic equivalence class by composing Dehn twists in the vertical direction for P_i with Dehn twists in the vertical direction for P_j. In particular, if the intersection number between the vertical direction defined by P_i and the one defined by P_j is ± 1, then *any* Dehn twist along *any* closed simple loop in T preserves the isotopy class of totally periodic pseudo-Anosov flow in M. In general since the fibers in P_j and P_i do not have common powers, they generate a finite index subgroup of $\pi_1(T)$. Hence all such Dehn twists produce only finitely many isotopic equivalence classes of pseudo-Anosov flows.

Topological transitivity of totally periodic pseudo-Anosov flows. It is easy to construct non transitive totally periodic pseudo-Anosov flows. For simplicity we do an example without Dehn surgery. Start with a surface S with high enough genus and three boundary components. Russ Waller [Wa] showed that S has a structure as a fat graph with the properties of section 5 and one can then easily construct a semiflow Ψ' in $S \times \mathbf{S}^1$, with one exiting boundary component and two entering components. Glue one entering boundary component with the exiting one by an admissible glueing. This is manifold M_1 with semiflow Ψ_1 with one entering component. Do a copy of M_1 with a flow reversal of Ψ_1 and then glue it to M_1 by an admissible map. The resulting manifold is a graph manifold with a model flow which is clearly not transitive: the boundary component of M_1 is transverse to the flow and any orbit intersecting this torus is trapped in M_1.

It is not very hard to characterize when exactly the model flow is transitive: we claim that it is equivalent to the following property: the oriented graph $\mathfrak{G}$, which is the quotient by $\pi_1(M)$ of the oriented graph $\widetilde{\mathfrak{G}}$ defined in section 4.3, is *strongly connected:* for any pair of vertex T and T' in $\mathfrak{G}$, there must be an oriented path going from T to T' and another oriented path going from T' to T.

This is done for a more general class of flows (at least the Anosov case) in a forthcoming article [BBB] by Béguin, Bonatti and Yu. Because of that we do not discuss further transitivity of model flows here.

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