An Application of the Resultant to Oscillatory Integrals with Polynomial Phase

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Fix $n = 2, 3, \ldots$ and let $p(t) = \sum_{j=1}^{n} a_j t^j$ be a polynomial with real coefficients. Let $I \subseteq \mathbb{R}$ be an interval. This paper is concerned with estimates of the form

$$\left| \int_I e^{ip(t)}\, dt \right| \leq M(a_1, \ldots, a_n) \tag{1}$$

where $M(a_1, \ldots, a_n)$ is independent of I.

If $q(t) = p(st)$ for $s > 0$, then

$$\int_a^b e^{iq(t)}\, dt = s^{-1} \int_{as}^{bs} e^{ip(t)}\, dt.$$

Since we want (1) to be uniform in I, the homogeneity condition

$$M(sa_1, s^2 a_2, \ldots, s^n a_n) = s^{-1} M(a_1, \ldots, a_n), \qquad s > 0 \tag{2}$$

is a reasonable property to require. We would also like for (1) to give the correct rate of decay for

$$\left| \int_I e^{i\lambda p(t)}\, dt \right| \tag{3}$$

as $|\lambda| \to \infty$. More specifically, if $1 \leq j_1 < j_2 \leq n$ and the derivatives $p^{(j_1)}$ and $p^{(j_2)}$ have no common zero in $\mathbb{R}$, then van der Corput's lemma ([Z], p. 197) shows that (3) is $O(|\lambda|^{-\frac{1}{j_2}})$ as $|\lambda| \to \infty$. The best estimates (1) should recover this fact with a bound on the "big oh" constant as well. The paper [O2] contains such estimates in case either $n \leq 4$ or $j_2 = n-1$.

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(The case $j_2 = n$ is covered by van der Corput's lemma.) Here we obtain estimates in the same spirit for any n and $1 \leq j_1 < j_2 \leq n$. Unfortunately these estimates do not give the optimal rate of decay for (3) if $j_2 < n-1$. They do, however, yield Theorem 1 and Corollary 3 of [O2] as special cases and also provide an extension of that Corollary 3 to n dimensions. (Corollary 3 of [O2] was a key part of the proof of the theorem in [O1], a convolution estimate in $\mathbb{R}^4$. The extension of Corollary 3 to n dimensions is the first step in our attempt to extend the work of [O1] to the n-dimensional setting.)

We begin by recalling the definition of the resultant of two polynomials. If $q(t) = \sum_{j=0}^{k} b_j t^j$ and $r(t) = \sum_{j=0}^{\ell} c_j t^j$ are polynomials, their resultant $R = R(q,r)$ is the polynomial in the variables b_j and c_j defined by the determinant of the $(k+\ell)$ by $(k+\ell)$ matrix:

$$\begin{array}{c} \ell\left\{\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right. \\ k\left\{\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right. \end{array}
\left[\begin{array}{cccccccc}
b_k & b_{k-1} & \cdot & \cdot & \cdot & b_0 & & \\
 & b_k & b_{k-1} & \cdot & \cdot & b_1 & b_0 & \\
 & & b_k & b_{k-1} & \cdot & \cdot & b_1 & b_0 \\
 & & & \cdot & \cdot\ \ \cdot & \cdot & \cdot & \cdot \\
c_\ell & c_{\ell-1} & \cdot & \cdot & \cdot & c_0 & & \\
 & c_\ell & c_{\ell-1} & \cdot & \cdot & c_1 & c_0 & \\
 & & c_\ell & c_{\ell-1} & \cdot & \cdot & c_1 & c_0 \\
 & & & \cdot & \cdot\ \ \cdot & \cdot & \cdot & \cdot
\end{array}\right].$$

Thus if $n = 2, 3, \ldots$, if $1 \leq j_1 < j_2 \leq n$, and if $p(t) = \sum_{j=0}^{n} a_j t^j$, then $R(p^{(j_1)}, p^{(j_2)})$ is a homogeneous polynomial of degree $2n - j_1 - j_2$ in $a_1, \ldots, a_n$. It is clear that if

$$P = P_{j_1 j_2 n}(a_1, \ldots, a_n) = \frac{R(p^{(j_1)}, p^{(j_2)})}{a_n},$$

then P is a homogeneous polynomial of degree $2n - j_1 - j_2 - 1$ in $a_1, \ldots, a_n$. These polynomials P play a central role in our estimate. Specific examples are easily obtainable using computer algebra systems.

Theorem. *With n, j_1, and j_2 as above and $P = P_{j_1 j_2 n}$, there is a constant C, depending only on n, j_1, and j_2, such that if $p(t) = \sum_{j=1}^{n} a_j t^j$ (with $a_j \in \mathbb{R}$) then, for every interval $I \subseteq \mathbb{R}$,*

$$\left| \int_I e^{ip(t)}\, dt \right| \leq \frac{C}{|P(a_1, \ldots, a_n)|^{\frac{1}{n^2 - j_1 j_2 - n}}}. \tag{4}$$

As we will see, the polynomial P has the anisotropic homogeneity property

$$P(sa_1, s^2a_2, \ldots, s^n a_n) = s^{n^2-j_1j_2-n} P(a_1, \ldots, a_n) \tag{5}$$

so that (2) holds. Also, P is nonzero if and only if $p^{(j_1)}$ and $p^{(j_2)}$ have no common complex root. Then the (isotropic) homogeneity of P shows that (3) decays at least as fast as $|\lambda|^{-(2n-j_1-j_2-1)/(n^2-j_1j_2-n)}$ as $|\lambda| \to \infty$ if $a_n \neq 0$. The choices $(n, j_1, j_2) = (n, j, n-1)$ and $(n, j_1, j_2) = (4, 1, 2)$ recover, respectively, Theorem 1 and Corollary 3 of [O2].

Proof. Our proof hinges on the following fact about the resultant $R(q, r)$: with q and r as above, if $q_1, \ldots, a_k$ are the (complex) roots of q and $r_1, \ldots, r_\ell$ are the roots of r, then

$$R(q, r) = b_k^\ell c_\ell^k \prod_{i=1}^{k} \prod_{j=1}^{\ell} (q_i - r_j). \tag{6}$$

(See, e.g., p. 137 in [L].) Thus, for example, it is clear that P is nonzero if and only if $p^{(j_1)}$ and $p^{(j_2)}$ have no common root (assuming $a_n \neq 0$). We will now observe two more consequences of (6): the first is that (5) holds and the second is

$$\text{if } t_0 \in \mathbb{R} \text{ and } p(t + t_0) = \sum_{j=0}^{n} b_j t^j, \text{ then } P(b_1, \ldots, b_n) = P(a_1, \ldots, a_n). \tag{7}$$

To check (5), fix $p(t)(= \sum_{j=1}^{n} a_j t^j)$ and $s > 0$. Let $u(t) = p(st)$. Then

$$P(sa_1, s^2a_2, \ldots, s^n a_n) = \frac{R(u^{(j_1)}, u^{(j_2)})}{s^n a_n}. \tag{8}$$

Since the roots of $u^{(j)}$ are the multiples by s^{-1} of the roots of $p^{(j)}$, (6) shows that

$$R(u^{(j_1)}, u^{(j_2)}) = s^{n(n-j_1)+n(n-j_2)-(n-j_1)(n-j_2)} R(p^{(j_1)}, p^{(j_2)}).$$

With (8) this gives (5). The proof of (7) is somewhat similar: if we let $w(t) = p(t + t_0)$, then the roots of $w^{(j)}$ are the translates by $-t_0$ of the roots of $p^{(j)}$. Since $w^{(j)}$ and $p^{(j)}$ have the same leading coefficient, (7) follows from (6).

Our next goal is the inequality

$$|P(a_1,\dots,a_n)|^{\frac{1}{n^2-j_1j_2-n}} \le C \inf_{t\in\mathbb{R}} \sum_{j=1}^{n} \left|p^{(j)}(t)\right|^{\frac{1}{j}}. \tag{9}$$

Define C to be

$$\sup\left\{ |P(a_1,\dots,a_n)|^{\frac{1}{n^2-j_1j_2-n}} \ :\ \sum_{j=1}^{n} |a_j|^{\frac{1}{j}} = 1 \right\}.$$

Since for $s>0$

$$\left|P(sa_1, s^2a_2,\dots,s^na_n)\right|^{\frac{1}{n^2-j_1j_2-n}} = s\,|P(a_1,\dots,a_n)|^{\frac{1}{n^2-j_1j_2-n}}$$

by (5) and since

$$\sum_{j=1}^{n} \left|s^j a_j\right|^{\frac{1}{j}} = s\sum_{j=1}^{n} |a_j|^{\frac{1}{j}},$$

it follows from the definition of C that

$$|P(a_1,\dots,a_n)|^{\frac{1}{n^2-j_1j_2-n}} \le C\sum_{j=1}^{n} \left|p^{(j)}(0)\right|^{\frac{1}{j}}$$

for any $p(t)=\sum_{j=0}^{n} a_j t^j$. Replacing $p(t)$ by $p(t+t_0)$ and applying (7) gives (9).

Now let $N=N(n)$ be a positive integer so large that, for any $p(t)=\sum_{j=1}^{n} a_j t^j$, the cardinality of

$$\left\{ t\in\mathbb{R} : \left(p^{(k_1)}(t)\right)^{k_2} = \left(p^{(k_2)}(t)\right)^{k_1} \text{ for some } k_1,k_2 \text{ with } 1\le k_1<k_2\le n \right\}$$

is bounded by $N-1$. Then, given p, $\mathbb{R}$ can be written as a disjoint union of at most N intervals I_ℓ such that for each I_ℓ there is $j'=j'(\ell)\in\{1,\dots,n\}$ with

$$\left|p^{(j')}(t)\right|^{\frac{1}{j'}} = \sup_{1\le j\le n} \left|p^{(j)}(t)\right|^{\frac{1}{j}}, \quad t\in I_\ell.$$

Thus (9) and van der Corput's lemma give

$$\left|\int_{I\cap I_\ell} e^{ip(t)}\,dt\right| \le \frac{C}{|P(a_1,\dots,a_n)|^{\frac{1}{n^2-j_1j_2-n}}}.$$

Since

$$\left|\int_I e^{ip(t)}\,dt\right| \le \sum_{\ell=1}^{N(n)} \left|\int_{I\cap I_\ell} e^{ip(t)}\,dt\right|,$$

(4) follows.

References

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