Biextensions, bimonoidal functors, multilinear functor calculus, and categorical rings

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Abstract

We associate to a bimonoidal functor, i.e. a bifunctor which is monoidal in each variable, a nonabelian version of a biextension. We show that such a biextension satisfies additional triviality conditions which make it a bilinear analog of the kind of spans known as butterflies and, conversely, these data determine a bimonoidal functor. We extend this result to n-variables, and prove that, in a manner analogous to that of butterflies, these multi-extensions can be composed. This is phrased in terms of a multilinear functor calculus in a bicategory. As an application, we study a bimonoidal category or stack, treating the multiplicative structure as a bimonoidal functor with respect to the additive one. In the context of the multilinear functor calculus, we view the bimonoidal structure as an instance of the general notion of pseudo-monoid. We show that when the structure is ring-like, i.e. the pseudo-monoid is a stack whose fibers are categorical rings, we can recover the classification by the third Mac Lane cohomology of a ring with values in a bimodule.

Introduction

Let $\mathcal{H}$ and $\mathcal{G}$ be monoidal stacks in a topos $\mathbf{T}$ and F a monoidal functor $F\colon \mathcal{H} \to \mathcal{G}$. It is well known that F can be represented by a special kind of span. More precisely, if the monoidal laws are group-like, or if we restrict to the invertible objects (restricting to the invertible objects does not affect the characteristic class), we can find presentations $H_\bullet$ for $\mathcal{H}$ and $G_\bullet$ for $\mathcal{G}$ by crossed modules of $\mathbf{T}$ so that F is represented by a diagram of group-objects of the form

$$
\begin{array}{ccc}
H_1 & & G_1 \\
& \searrow \quad \swarrow & \\
& E & \\
& \swarrow \quad \searrow & \\
H_0 & & G_0
\end{array}
$$

where the salient feature is that the sequence $G_1 \to E \to H_0$ of objects of $\mathbf{T}$ is an extension of H_0 by $G_1 \to G_0$: an exact sequence in which the conjugation action of E on G_1 is compatible with that of G_0 [AN09]. Such an extension is conveniently described in geometric terms as a G_1-bitorsor E over H_0, with the property that one of the actions is obtained by way of the crossed module structure of $G_\bullet$ [Bre90]. A converse of this correspondence is also available, establishing an equivalence between the groupoid of monoidal functors from $\mathcal{H}$ to $\mathcal{G}$ and that of diagrams like the one above (a morphism between two diagrams with the same wings $H_\bullet$ and $G_\bullet$ is a homomorphism of $E \to E'$ compatible with all the maps).

One of the main results of this paper is an analogous result for *bi*monoidal functors $F\colon \mathcal{H} \times \mathcal{H} \to \mathcal{G}$, that is, bifunctors which are monoidal in each variable.

The concept of bimonoidal functor requires $\mathcal{G}$ to be braided. In addition, F must satisfy a compatibility condition requiring that applying the monoidal condition on both of its variables in the two possible orders lead to the same result. (We can think of F as an obvious generalization of a bilinear map, in which case this condition is trivially satisfied.) This condition is formally equal to the compatibility one between the two

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partial multiplication laws of a biextension [Mum69; Gro72]. Recall that for abelian groups A, B, C of $\mathbf{T}$, a biextension of B and C by A consists of an A-torsor E over $B \times C$ equipped with two partial commutative and associative product laws, making E, for each generalized point $b \in B$, an extension of C by A, and, similarly, for each $c \in C$, an extension of B by A. The coincidence between these conditions is not purely formal: the bilinear point of view is to regard a biextension as providing a bifunctor $\varphi_E \colon B \times C \to \mathrm{Tors}(A)$, which is then monoidal (i.e. a homomorphism) in each variable. Notable examples are certain duality pairings where, in particular, $C = B^\vee$ and $A = \mathbb{G}_m$ [see, e.g. Boy10; Dat10].

Our result begins from a generalization of these ideas, starting with that of biextension. Thus, for two groups H and K of $\mathbf{T}$, and a braided crossed module (G_1, G_0), we define a biextension of H, K by (G_1, G_0) as a G_1-bitorsor (or, more appropriately, using the terminology of [Bre90], a (G_1, G_0)-torsor) E over $H \times K$ such that, for each point $x \in H$ (resp. $y \in K$), E is an extension of K (resp. of H) by the crossed module $G_1 \to G_0$. The braiding is required by the compatibility between the two partial product laws of E. The relative diagram is formally the same as in the classical situation, with some differences due to the fact that none of the groups involved is assumed abelian—although the crossed module $G_1 \to G_0$ and its associated stack are braided.

Now, let F be a bimonoidal functor $F \colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$, where each of $\mathscr{K}, \mathscr{H}$, and $\mathscr{G}$ has a presentation by a crossed module. We show that F determines a biextension of H_0, K_0 by the crossed module $G_1 \to G_0$, equipped with a pair of compatible trivializations for the two pullbacks along the maps $\partial \times \mathrm{id} \colon H_1 \times K_0 \to H_0 \times K_0$ and $\mathrm{id} \times \partial \colon H_0 \times K_1 \to H_0 \times K_0$. This is what we call, in this context, a butterfly —the bilinear version of the notion introduced in [AN09; earlier Noo05, over a point]. Viceversa, given a biextension E of (H_0, K_0) by $G_1 \to G_0$, we obtain a bimonoidal functor

$$\varphi_E \colon H_0 \times K_0 \longrightarrow \mathrm{Tors}(G_1, G_0),$$

where the right hand side denotes the stack of (G_1, G_0)-torsors equivalent to $\mathscr{G}$, and where H_0 and K_0, are interpreted as discrete monoidal stacks. The additional data provided by the two compatible trivializations allow to conclude that φ_E is compatible with descent along the presentations $H_0 \ltimes H_1 \rightrightarrows H_0 \to \mathscr{H}$ and $K_0 \ltimes K_1 \rightrightarrows K_0 \to \mathscr{K}$ by the groupoids determined by their respective crossed modules, so it determines a bimonoidal functor $F_E \colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$. With the obvious notion of morphism, we obtain:

Theorem (Theorem 7.1 in the main text). *There is an equivalence of (pointed) groupoids*

$$u \colon \mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet) \xrightarrow{\sim} \mathbf{Hom}(\mathscr{H}, \mathscr{K}; \mathscr{G}),$$

where the right hand side denotes the groupoid of bimonoidal functors, and left hand side that of biextensions equipped with the aforementioned trivializations.

This result is actually valid over a variable object S of $\mathbf{T}$, hence we have a similar statement where the left and right side above are replaced by the corresponding stacks, which one obtains by letting $S \to \mathbf{Biext}(H_\bullet|_S, K_\bullet|_S; G_\bullet|_S)$, and similarly for the right hand side.

Bimonoidal categories or stacks provide examples of bimonoidal morphisms. If $\mathscr{R}$ is bimonoidal, it has two monoidal structures, say $\boxplus$ and $\boxtimes$, satisfying an appropriate set of axioms [Lap72b; Lap72a]; the distributivity one, in particular, says that $\boxtimes$ is a bimonoidal functor $\boxtimes \colon \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ with respect to the other structure, $\boxplus$. Now, assuming $\boxplus$ to be group-like (see [BDRR13] for passing from a merely additive monoidal to a full group-like one, i.e. from "rig" to "ring") and $\mathscr{R}$ to have a presentation of the form $R_0 \ltimes R_1 \rightrightarrows R_0 \to \mathscr{R}$ as above, the bifunctor $\boxtimes$ can be described by a biextension $E_\boxtimes$ of (R_0, R_0) by $R_1 \to R_0$. We are interested in extracting informations about $\mathscr{R}$, especially of a cohomological nature such as the characteristic class, from the biextension $E_\boxtimes$. Informations of this kind, which reduces to (usually complicated) cocycle calculations, ordinarily come from the coherence diagrams of $\boxtimes$, among others. These diagrams involve morphisms like $\boxtimes \circ (\boxtimes \times \mathrm{id})$ and $\boxtimes \circ (\mathrm{id} \times \boxtimes)$, which are monoidal in each of the three variables, and higher iterations of compositions involving $\boxtimes$ and id. Ideally, each of those multi-functors corresponds to some kind of iteration of $E_\boxtimes$. So we need to generalize the representation of bimonoidal functors by biextensions to an arbitrary number of variables.

Extending the concept of bimonoidal functor to n variables is immediate, the only difference being the same compatibility condition we have in the $n = 2$ case must hold for each pair of variables. Therefore, for the multivariable analog of the right hand side in the above statement, the only substantial difference is in the

bookeping aspect, and we immediately see that monoidal functors in n variables can be composed—provided we restrict ourselves to considering braided objects. While this is easy to verify, it has the far-reaching consequence that braided monoidal categories, or more generally stacks, comprise the 2-categorical analog of a multicategory [CS10; Shu10] we denote $\mathrm{M}\mathfrak{BGrSt}$, where $\mathfrak{BGrSt}$ is the 2-category they form relative to ordinary unary monoidal functors.

What about the left hand side of the equivalence in the theorem? For that we must extend the concept of biextension to n variables and investigate whether such objects admit a composition law. We define a multi-extension, or an n-extension, by $G_1 \to G_0$ as a (G_1, G_0)-torsor (same as for biextensions) over an n-fold product, this time equipped with n partial product laws which are required to be pairwise compatible in the same manner as those in a biextension. (The generalization of a biextension to n variables, in the fully abelian context, is outlined in a remark in [Gro72].) Given n crossed modules $H_{i,\bullet} = (\partial\colon H_{i,1} \to H_{i,0})$, $i = 1,\dots,n$, we are interested in the n-extensions of $H_{1,0} \times \cdots \times H_{n,0}$ by $G_1 \to G_0$ which are also equipped with n compatible trivializations of their pullbacks along each of the n morphisms $\partial_i = \mathrm{id} \times \cdots \times \partial \times \cdots \times \mathrm{id}$. We call them n-butterflies. Now, a direct extension of the above theorem (cf. Theorem 9.1 below) provides an equivalence

$$\mathbf{MExt}(H_{1,\bullet},\dots,H_{n,\bullet}; G_\bullet) \xrightarrow{\sim} \mathbf{Hom}(\mathscr{H}_1,\dots,\mathscr{H}_n; \mathscr{G})$$

of pointed groupoids, where each $H_{i,\bullet}$ is a presentation of the corresponding $\mathscr{H}_i$, and $G_\bullet$ of $\mathscr{G}$, the left hand side is the groupoid of n-extensions, and the right hand side denotes the groupoid of n-monoidal functors $\mathscr{H}_1 \times \cdots \times \mathscr{H}_n \to \mathscr{G}$.

The remarkable fact is that these multi-butterflies can be composed in a way reminiscent of the unary case discussed in [AN09; Noo05], that is by "wing juxtaposition," which turns out to be associative up to coherent isomorphism. Wing juxtaposition (we informally use this term thanks to the shape of the composition diagrams in the unary case) is an operation which allows to associate to butterflies $F_1,\dots,F_n, E$, where E is an n-butterfly, a new one, denoted $E(F_1,\dots,F_n)$, whose underlying bitorsor is a quotient of the pullback of E to the product $F_1 \times \cdots \times F_n$—each F_i is equipped with an equivariant section with values in the ith factor in the base of E; the pullback is along the product of these maps. While we will discuss this in more detail later, the upshot is that braided crossed modules of $\mathbf{T}$ now form a bi-multicategory (i.e. *bi*categorical analog of a multi-category) of their own if we use these multi-butterflies as morphisms. In addition, the composition is compatible with the previous equivalence. More precisely, we have:

Theorem (Theorem 10.1, Propositions 10.3 and 10.4, and Theorem 11.3). *Braided crossed modules of* $\mathbf{T}$, *equipped with the groupoids* $\mathbf{MExt}(H_{1,\bullet},\dots,H_{n,\bullet}; G_\bullet)$ *as* Hom-*categories, form a bi-multicategory* $\mathrm{M}\mathfrak{BXMod}$. *Further, there is an equivalence of bi-multicategories*

$$\mathrm{M}\mathfrak{BXMod} \xrightarrow{\sim} \mathrm{M}\mathfrak{BGrSt},$$

induced by the associated stack functor, carrying each $\mathbf{MExt}(H_{1,\bullet},\dots,H_{n,\bullet}; G_\bullet)$ *to* $\mathbf{Hom}(\mathscr{H}_1,\dots,\mathscr{H}_n; \mathscr{G})$.

This is the multi-categorical analog of the unary case treated in [AN09], which states the bicategory $\mathfrak{BXMod}$ of braided crossed modules, equipped with groupoids of spans as categories of morphisms, is equivalent (as a bicategory) to $\mathfrak{BGrSt}$. (In fact the unary version proved therein holds without any commutativity assumption.)

Returning to the bimonoidal categories mentioned earlier, we can now discuss them from the broader and more convenient perspective afforded by the idea of a bi-multicategory. If $\mathrm{M}\mathfrak{C}$ is a bi-multicategory, we informally look into monoids with respect to the multi-composition structure of $\mathrm{M}\mathfrak{C}$. Formally, we say that an object X of $\mathrm{M}\mathfrak{C}$ is a (pseudo-)monoid of $\mathfrak{C}$, the underlying bicategory of $\mathrm{M}\mathfrak{C}$, if it is a pseudo-algebra over the *club* $\mathbf{N}$, where $\mathbf{N}$, after [Kel72], denotes the natural numbers equipped with the category structure described in section A; a club is a multicategory with one object.[1] The monoidal structure carried by X is of the "unbiased" kind [Lei04], namely we have an n-ary operation $m_n\colon X \times \cdots \times X \to X$ for each object $n \in \mathbf{N}$, as opposed to a privileged binary one, with the various coherence conditions being taken care of by the pseudo-algebra structure.

We apply the previous observation to $\mathfrak{C} = \mathfrak{BGrSt}$ or $\mathfrak{C} = \mathfrak{BXMod}$. We say that $\mathscr{R}$ is weakly ring-like (for want of a better name) if it is a pseudo-monoid of $\mathfrak{BGrSt}$. The "weakly" adverb refers to the fact that

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[1] For the sake of simplicity we are ignoring the "extraordinary" structure given by the action of permutations.

the underlying categorical group structure of $\mathcal{R}$ is only braided commutative, whereas we usually define categorical rings as having underlying braided *symmetric* categorical groups [see e.g. JP07]. Thus "weak" does not mean "not strict," but it signals the commutative law of the additive structure of $\mathcal{R}$ is weaker than usually required.

We make a similar definition for a pseudo monoid in $\mathfrak{BXMod}$: this is a novel object comprised of a braided crossed module $R_\bullet$ and multi-extensions E_n of $(R_0, \ldots, R_0)$ (n factors) by $R_1 \to R_0$ for each $n \in \mathbf{N}$, subject to the conditions dictated by the pseudo-algebra structure. We show that even in the realm of a genuine bicategory such as $\mathfrak{BXMod}$, we obtain coherence conditions resembling a weak version of Mac Lane's pentagon (see sections B and C, in particular diagram (C.2)).

Since pseudo-monoids are transported across equivalences, pseudo-monoids in $\mathfrak{BGrSt}$ are presented by those in $\mathfrak{BXMod}$. We have:

Proposition (Proposition 12.5, Proposition 13.1, and Corollary 13.2)**.** *Let $\mathcal{R}$ be (weakly) ring-like. Let $R_0 \ltimes R_1 \rightrightarrows R_0 \to \mathcal{R}$ be a presentation by a braided crossed module. Then is a pseudo-monoid in $\mathfrak{BXMod}$. Furthermore, if $A = \pi_0(\mathcal{R}) \simeq \operatorname{coker} \partial$ and $M = \pi_1(\mathcal{R}) \simeq \ker \partial$, then A is a (possibly non-unital) ring of $\mathbf{T}$ and M is an A-bimodule. The converse also holds.*

The A-bimodule structure on M is of course a standard fact. Observe that the presenting crossed module is far from being internal to (non-unital) rings of $\mathbf{T}$. This in particular holds for ordinary categorical rings, which are resolved by symmetric crossed modules carrying multi-extensions encoding the ring structure.

We may ask whether the weakly ring-like structure is strong enough to force the underlying braiding to become symmetric. The answer is affirmative, at least in the more interesting (to us) case where the multiplicative structure has a unit object. As we shall see, this is tied to the seemingly unrelated problem of computing a characteristic class for weakly categorical rings.

Our procedure is a bit nonstandard in two aspects. The first is that to compute the class corresponding to $\mathcal{R}$ we only choose local data for the underlying braided group-like structure of $\mathcal{R}$, or which is the same thing, for the presentation. We compute the rest of the cocycle from the pseudo-monoidal structure carried by the presentation, by examining the biextension's behavior with respect to the chosen local data.

The second aspect is that we choose to initially dispense with the symmetry condition, therefore we start with a choice of local data for the underlying braided group-like structure of $\mathcal{R}$. This yields an invariant in the Eilenberg Mac Lane group $H^4(K(A, 2), M)$, lifting to $H^5(K(A, 3), M)$ if braiding is symmetric [JS93; Bre99].[2] In fact the initial choice for the local data steers the computation into being based on the Eilenberg Mac Lane iterated bar construction [EM53; EM54], in place of the more customary Mac Lane's Q-construction. (In the stack context, the iterated bar construction emerges as part of the decomposition of $\mathcal{R}$ as a gerbe over A, together with its additional braiding structure [Bre94; Bre99].)

Over a point, a standard categorical ring gives rise to an element in the third Mac Lane cohomology group $HML^3(A, M)$ [JP07; see also Lod98, chap. 13, for general definitions], so the question is whether our procedure yields anything standard, and in particular the expected invariant in $HML^3(A, M)$.

That is does, at least in the important unital case, is due to the fact that, by [Mac58, §11], the third Mac Lane cohomology can also be computed by employing the infinite bar construction $B^\infty(A) := B(A, 1) \subseteq B(A, 2) \subseteq \cdots$ instead of the cubical complex $Q(A)$.[3] One needs a product structure on $B^\infty(A)$, explicitly up to degrees corresponding to the subcomplex $B(A, 2)$, to use in the multiplicative bar construction. It follows that we can plug one of $B(A, 2)$, $B(A, 3)$, or $B^\infty(A)$ to calculate a cohomology of A with values in the bimodule M. In the unital case, all these choices give rise to isomorphic cohomologies. The product structure on $B(A, 2)$, which is unusual, is explicitly given in Mac Lane's work. We recover that product, and hence the rest of the structure, directly from the pseudo-monoidal structure attached to the presentation of $\mathcal{R}$, namely the biextension plus the various compatibility conditions it satisfies.

We do not obtain the representing cocycle for the class of $\mathcal{R}$ in the form corresponding to class in $HML^3(A, M)$ right away. Instead, we find that the class of $\mathcal{R}$ is represented by a twisted cocycle in the multiplicative bar resolution for $B(A, 2)$, but the twisting disappears in the unital case, which yields the desired result. In particular this implies that the braiding is necessarily symmetric. More precisely, we have:

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[2] In a general topos we should interpret them as hypercohomology groups. See the discussion in [Bre70, §2; Bre94, §7; Bre99, §6].

[3] Recall that classically MacL̄ane cohomology is defined as the Hochschild cohomology of the complex $Q(A)$. The latter computes the stable homology of the Eilenberg-Mac Lane spaces, that is, $H_q(Q(A)) \simeq H_{n+q}(K(A, n))$, $n > q$.

Theorem (Theorem 14.1). *There is a bijective correspondence between equivalence classes of weak ring-like stacks $\mathscr{R}$, with $A = \pi_0(\mathscr{R})$ and $M = \pi_1(\mathscr{R})$, and twisted classes in $\widetilde{H}_2^3(A, M)$ defined below (cf. Definition 14.6). In the unital case, i.e. when the external monoidal structure of $\mathscr{R}$ has a unit element, A is likewise unital and the underlying braiding of $\mathscr{R}$ is necessarily symmetric. Hence the weak ring-like structure of $\mathscr{R}$ is fully ring-like, and $[\mathscr{R}] \in \widetilde{H}_2^3(A, M) \simeq HML^3(A, M)$.*

Hence, unital categorical rings in the sense of the present paper *are* categorical rings tout-court and, by [Mac58], we re-obtain their classification in terms of $HML^3(A, M)$. (Had we chosen to work with braided symmetric objects from the beginning there is little doubt the same procedure—i.e. a decomposition of the underlying group-like stack followed by an analysis of the attached biextension—would have yielded a Mac Lane cocycle in the right group right from the start. However, the interest was to test to what extent our framework successfully captured the standard theory, not to just reproduce known facts.)

Organization of the material

We have collected various preliminary items in section 1, including a quick review of standard biextensions. The recapitulation of the relation between group-like stacks and their presentations by crossed modules, though standard, involved a contravariance issue which becomes relevant, therefore we provided a brief outline.

Aside from the preliminary section, we can divide the rest in roughly three parts, plus another containing some appendices.

We develop the generalization of biextensions and their symmetry properties in sections 2 to 5, and the butterfly special kind, including the representation of bimonoidal functors, from section 6 to 8.

The extension to an arbitrary number of variables takes sections 9 to 11. In particular, the composition of butterflies is discussed in section 10. The bi-multicategories comprised by braided group-like stacks on one side, and braided crossed modules of **T** on the other, are discussed in section 11.

The idea of categorical ring, or ring-like stack, and its presentation as a pseudo-monoids in their respective bicategories is expounded in sections 11 and 13, where we also discuss some specific facts about the presentations. In section 14 we discuss the cohomology of rings and the computation of the characteristic class.

To avoid unnecessary and possibly long detours, parts of the material have been placed in a number of appendices. Some of this material is necessary for self-consistency but known to the experts. Thus, bi-multicategories are discussed in appendices A and B, the analog of the pentagon in a bicategory in C, and finally appendix D contains just some technical lemmas pertaining to section 5.

What to read

One possibility is to only read Part I, possibly followed by Part III if the reader is willing to only skim through Part II. For the latter, it is possible to just read the statements, in particular for the n-fold composition of multi-extensions theorem proved in sect. 10. A detailed proof is included because the juxtaposition product has features similar to the unary case discussed in [AN09] and [Noo05], but its proof must rely on geometrical, rather than group-theoretical, techniques, which makes it not an obvious generalization of the one used in the references. It seems quite possible to just read Part III, in particular section 14, by only skimming through the previous two. Parts II and III depend the multivariable functor calculus in a bicategory, an account of which is contained in the first two appendices, which may only be referred to when needed. Most their content should be known to the experts. An account of what the pentagonal axiom would look like in a bicategory is contained in Appendix C, and in general a reader will need only equation (C.1) and diagram (C.2).

Notations and conventions

The convention we use is to equate "additive" with "monoidal," therefore the term "biadditive" means "bimonoidal" in the sense used above in the introduction. By extension, "n-additive" means "n-monoidal," i.e. monoidal in each of the n-variables.

A notation of the form $G_\bullet$ or G denotes a crossed module (G_1, G_0, ∂), where $\partial\colon G_1 \to G_0$ is the homomorphism, and the (right) action is denoted element-wise by $(x, g) \mapsto g^x$, but is not otherwise labeled. Very often we will simply write (G_1, G_0).

For ease of notation we will often use the convention: $(G_1, G_0) \cong (G, \Pi)$ and, later in the paper, $(R_1, R_0) \cong (R, \Lambda)$ when we discuss ring-like structures.

Up to section 12, all monoidal structures are notated multiplicatively, with I denoting the identity object. We switch to an additive notation for one of the two monoidal structures in a bimonoidal situation. In that case, the identity object is 0. If this is the structure for which we construct a presentation by a crossed module, then we use an additive notation for the groups in the crossed module, even though they are by no means assumed commutative.

Acknowledgements

Contents

Part I

1   Preliminaries

In the following, let $\mathbf{T}$ be a topos which is assumed to be $\mathrm{Sh}(\mathbf{C})$ for a site $\mathbf{C}$. For the reader's convenience we recall some well know facts on monoidal stacks and their relations with crossed modules of $\mathbf{T}$.

1.1   Group-like stacks, crossed modules, and bitorsors

We follow [Bre90; AN09], to which we refer for further details. A **crossed module** of $\mathbf{T}$ consists of a group homomorphism $\partial\colon G \to \Pi$ of $\mathbf{T}$ and a right action of Π on G such that: (1) ∂ is a morphism of right Π-objects, with Π acting on itself by conjugation; (2) the action of G on itself induced by the Π-action via ∂ coincides with conjugation. If g, h are (generalized) points of G and x of Π we have the familiar conditions:

$$\partial(g^x) = x^{-1}\,\partial(g)\,x$$
$$g^{\partial(h)} = h^{-1}\,g\,h,$$

where the exponent notation denotes the action.

The crossed module G gives rise to a groupoid $\Gamma\colon \Pi \ltimes G \rightrightarrows \Pi$, and hence, via the nerve construction, to a simplicial group $N\mathsf{G}$ whose object in degree p is $(\cdots(\Pi \ltimes G) \ltimes \cdots \ltimes G$ (p factors). The groups $\pi_1 = \ker \partial$ and $\pi_0 = \operatorname{coker} \partial$ are the homotopy (sheaves of) groups of $G_\bullet$ are in fact the only two nonzero homotopy sheaves of $N\mathsf{G}$. It is easily verified that π_1 is abelian and central in G and that π_0 is the sheaf of connected components of the groupoid corresponding to G (and hence of $N\mathsf{G}$).

The group laws of G and $\Pi \ltimes G$ equip the groupoid Γ with a structure of strict categorical group, that is, a morphism

$$m\colon \Gamma \times \Gamma \longrightarrow \Gamma$$

of groupoids satisfying the usual axioms of a group-like, strict monoidal category [see also JS93].

The associated stack construction performed on Γ yields a stack $\mathscr{G}$ of $\mathbf{T}$, which inherits the—now lax—group-like structure. Thus, $\mathscr{G}$ is a monoidal stack, in the sense that there exists a stack morphism

$$m\colon \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G},$$

and an associator $\mu\colon m \circ (m \times 1) \Rightarrow m \circ (1 \times m)$ satisfying the axioms of a monoidal category. We specify the rest of the group-like structure by requiring the existence of a unit object I, which we can identify with a morphism $I\colon e \to \mathscr{G}$ (where e is the terminal object of $\mathbf{T}$), and that left and right-multiplications by a fixed object x induce functorial equivalences

$$m(x, -)\colon \mathscr{G} \longrightarrow \mathscr{G} \quad m(-, x)\colon \mathscr{G} \longrightarrow \mathscr{G}.$$

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The group-like stack obtained in this way from a crossed module will be denoted $[\partial\colon G \to \Pi]^{\sim}$.

There is a compelling geometric picture for the associated stack obtained in this way from a crossed module. Recall the stack $\mathrm{Tors}(G,\Pi)$ is the stack of right G-torsors P whose structure group extension $P \wedge^G \Pi$ is isomorphic to the trivial torsor [see Del79]. Equivalently, an object of this stack is a pair (P,s) where P is a G-torsor and $s\colon P \to \Pi$ a G-right-equivariant morphism, where G acts on Π on the right via multiplication. Objects of $\mathrm{Tors}(G,\Pi)$ are called (G,Π)-torsors. The notion makes sense for each group homomorphism, but if $\partial\colon G \to \Pi$ is a crossed module, then $\mathrm{Tors}(G,\Pi)$ becomes group-like. Indeed, each torsor is also equipped with a *left G-action* defined by

$$gu = u g^{s(u)},$$

where $u \in P$ and $g \in G$, which makes it into a G-*bi*torsor. The monoidal structure is given by the contracted product

$$m((P,s),(Q,t)) = (P \wedge^G Q, st),$$

and it is easily proved to be group-like. Finally, the stack $\mathrm{Tors}(G,\Pi)$ is equivalent, as a group-like stack, to $[\partial\colon G \to \Pi]^{\sim}$, obtained via the associated stack construction described above.

Conversely, for any group-like stack $\mathscr{G}$ there exists a *presentation* by a crossed module, namely there exists a crossed module $\mathsf{G} = (G,\Pi,\partial)$ and an equivalence $\mathscr{G} \simeq [\partial\colon G \to \Pi]^{\sim}$. For any group-like stack $\mathscr{G}$ we can define $\pi_1 = \mathrm{Aut}_{\mathscr{G}}(I)$, and π_0 as the sheaf associated to the presheaf of connected components. Upon choosing a presentation, its homotopy sheaves are isomorphic to the ones of $\mathscr{G}$.

In view of the previous geometric interpretation, $\mathscr{G}$ is also equivalent to $\mathrm{Tors}(G,\Pi)$. In addition, the sequence

(1.1)
$$G \xrightarrow{\ \partial\ } \Pi \xrightarrow{\ \pi\ } \mathscr{G},$$

where each object is regarded as a stack, is exact in a homotopical sense. If $\mathscr{G}$ is identified with $\mathrm{Tors}(G,\Pi)$, the projection $\Pi \to \mathscr{G}$ sends $x \in \Pi$ to the bitorsor (G,x), that is the trivial right G-torsor whose left action on itself is given by $g \cdot u = g^x u$, with $u,g \in G$. (This follows from the fact that x is identified with the equivariant section that assigns x to the unit section e_G.) In particular, (G,e_{Π}) can be identified with the unit object of $\mathscr{G}$.

1.1 Remark. Since a morphism $\varphi\colon (P,s) \to (Q,t)$ in $\mathrm{Tors}(G,\Pi)$ has the form

$$
\begin{array}{ccc}
P & \xrightarrow{\ \varphi\ } & Q \\
& {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle t} & \\
& \Pi &
\end{array}
$$

it immediately seen the equivalence

$$[\partial\colon G \to \Pi]^{\sim} \longrightarrow \mathrm{Tors}(G,\Pi)$$

is *contravariant*. In particular, if $g\colon x \to x'$ is a morphism in the strict categorical group Γ, that is, $x' = x\partial g$, the corresponding morphism of (G,Π)-torsors is $(G,x') \to (G,x)$.[4]

A **morphism** of group-like stacks is a stack morphism $F\colon \mathscr{H} \to \mathscr{G}$ preserving the monoidal structure. A morphism of crossed modules determines one between the associated group-like stacks in the obvious way. In the converse direction, a morphism $F\colon \mathscr{H} \to \mathscr{G}$ only determines a morphism in the *homotopy category* between corresponding crossed modules. This morphism can be represented by a **butterfly,** namely a diagram of group objects of **T** of the form:

$$
\begin{array}{ccccc}
H & & & & G \\
\Big\downarrow & {\scriptstyle\sigma}\searrow & & \swarrow{\scriptstyle\imath} & \Big\downarrow \\
{\scriptstyle\partial} & & E & & {\scriptstyle\partial} \\
\Big\downarrow & {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\jmath} & \Big\downarrow \\
\Sigma & & & & \Pi
\end{array}
$$

(where the vertical arrows are crossed modules) from which a fraction representing the morphism $F\colon \mathscr{H} \to \mathscr{G}$ can be obtained (see loc. cit. and below for more details).

---

[4]There appears to be no good way to get around the issue. A "fix" is to replace the crossed module with a left one. This restores the expected direction of the arrows, at the cost of turning one of the monoidal laws into the opposite one [see e.g. Bre90, after Théorème 4.5].

1.2 Braidings

A **braided crossed module,** [see JS93; AN09], is a crossed module $\mathsf{G} = (G, \Pi, \partial)$ equipped with a bracket

$$\{-, -\} \colon \Pi \times \Pi \longrightarrow G$$

such that $\partial\{x, y\} = y^{-1} x^{-1} y x$, for all points x, y of Π. The bracket satisfies:

$$
(1.2) \qquad
\begin{aligned}
\{x, yz\} &= \{x, y\}^z \{x, z\}, & \{x, \partial h\} &= h^{-1} h^x, \\
\{xy, z\} &= \{y, z\}\{x, z\}^y, & \{\partial g, y\} &= g^{-y} g,
\end{aligned}
$$

for all $x, y, z \in \Pi$ and $g, h \in G$. These properties all arise from the observation that the bracket $\{-, -\}$ corresponds to a braiding in the usual sense for the categorical group determined by $G_\bullet$. Thus $c_{x,y} = \{x, y\}$ gives a family of functorial isomorphisms

$$c_{x,y} \colon xy \longrightarrow yx$$

for each pair (x, y) of objects in Γ, so that $c \colon m \circ \tau \Rightarrow m \colon \Gamma \times \Gamma \to \Gamma$, where τ is the interchange functor. The properties above can be derived from functoriality and Mac Lane's hexagon diagrams. Conversely, if Γ is a braided strict categorical group, setting $\{x, y\} = y^{-1} x^{-1} c_{x,y} \colon e \to y^{-1} x^{-1} y x$ defines a braiding on the corresponding crossed module.

The braiding is **symmetric** if it has the property that $\{y, x\} = \{x, y\}^{-1}$ for all $x, y \in \Pi$. A symmetric braiding is **Picard** if in addition it satisfies the condition $\{x, x\} = e$. These conditions match the corresponding ones for the categorical group Γ. As observed in Breen [Bre94, §1], for a Picard crossed module the bracket is a full lift of the commutator map.

The braided, symmetric, and Picard structures translate in the expected manner to the associated group-like stack $\mathscr{G} \simeq [\partial \colon G \to \Pi]^\sim$, and conversely, if $\mathscr{G}$ is a braided (resp. symmetric, Picard) stack with presentation given by $[\partial \colon G \to \Pi]$, then the latter acquires a braided (resp. symmetric, Picard) structure as above. Let us observe here that a braiding on $\mathscr{G} = \mathrm{Tors}(G, \Pi)$ induces one on the crossed module by way of the butterfly representing the morphism $m \colon \mathscr{G} \times \mathscr{G} \to \mathscr{G}$. In the Picard case this leads to two possible presentations: by a braided crossed module satisfying the Picard condition, or, according to Deligne [Del73], by a length-one complex of abelian sheaves. We refer the reader to [AN09, §7] for full details on this correspondence. Here we limit ourselves to observe that if (P, s) and (Q, t) are two (G, Π)-torsors, the choice of two sections $u \in P$ and $v \in Q$ allows us to write the braiding morphism $c_{P,Q} \colon P \wedge^G Q \to Q \wedge^G P$ as $c_{P,Q}(u \wedge v) = (v \wedge u) \chi_{u,v}$, where $\chi \colon P \wedge^G Q \to G$, is a coordinate representation of $c_{P,Q}$ subject to certain equivariance conditions dictated by the requirements:

$$
\begin{aligned}
c_{P,Q}(gu \wedge v) &= g\, c_{P,Q}(u \wedge v), \\
c_{P,Q}(ug \wedge v) &= c_{P,Q}(u \wedge g v), \\
c_{P,Q}(u \wedge vg) &= c_{P,Q}(u \wedge v) g.
\end{aligned}
$$

1.2 Lemma. *The coordinate $\chi \colon P \wedge^G Q \to G$ has the expression:*

$$
(1.3) \qquad \chi_{u,v} = \{s(u), t(v)\}^{-1}.
$$

Proof (Sketch). The relation (1.3) is immediate for $(P, s) = (G, x)$ and $(Q, t) = (G, y)$, where $x, y \in \Pi$, using Remark 1.1. The general case follows by descent by exploiting the equivariance conditions for χ and for $\{-, -\}$ recalled above. $\qquad\square$

1.3 Biextensions

Biextensions were introduced in Mumford [Mum69], and later reexamined by Grothendieck [Gro72]. We refer to the latter and the text by Breen [Bre83] for details on the standard concept of biextensions. We briefly recall the basic definitions, and later extend them to introduce crossed modules as coefficients.

Let H, K, G be groups of **T**, with G assumed to be abelian. A **biextension** of H, K by G is a $G_{H \times K}$-torsor E on $H \times K$ equipped with two partial composition laws $\times_1$ (resp. $\times_2$) making E a central extension of H_K (resp. K_H) by G_K (resp. by G_H). These composition laws are required to be compatible with one another. Usually one also requires $\times_1$ and $\times_2$ to be commutative, which make sense whenever H and K are abelian. Our definition, in which $\times_1$ and $\times_2$ are not required to be commutative is then referred to as a **weak biextension.** Even when H and K are commutative, this notion of biextension is weaker than that of loc. cit. in which the partial multiplication laws are commutative.

Analogously to extensions, the composition laws can be represented by morphisms of G-torsors. Let BH and BK be the standard classifying simplicial objects of **T**.[5] The bisimplicial object $BH \times BK$ has face maps d^h and d^v [see e.g. GJ99]. In particular, for $i = 0, 1, 2$ we consider $d_i^h \colon H \times H \times K \to H \times K$ and $d_i^v \colon H \times K \times K \to H \times K$. Then $\times_1$ and $\times_2$ correspond to morphisms of G-torsors

(1.4a) $$\gamma^1 \colon d_2^{h*} E \wedge^G d_0^{h*} E \longrightarrow d_1^{h*} E$$

and

(1.4b) $$\gamma^2 \colon d_2^{v*} E \wedge^G d_0^{v*} E \longrightarrow d_1^{v*} E.$$

The associativity and commutativity diagrams for γ^1 and γ^2 are the obvious ones. More interesting is the compatibility condition of the two structures, which can be expressed as follows.

Let $E_{h,k}$ be the fiber of E over a generalized point (h, k) of $H \times K$. Then the compatibility condition reads

(1.5)

where we have suppressed the torsor contraction symbol. The horizontal arrow is the canonical symmetry map swapping the factors in $\mathrm{Tors}(G)$, which exists whenever G is abelian. A more intrinsic way to express the same thing is to observe that the above compatibility condition amounts to the equality

(1.6) $$\gamma^2_{d_1^{h*} E} \circ (d_2^{v*} \gamma^1 d_0^{v*} \gamma^1) = (1 \times c \times 1) \circ \gamma^1_{d_1^{v*} E} \circ (d_2^{h*} \gamma^2 d_0^{h*} \gamma^2)$$

as morphisms from $(d_0^h d_2^v)^* E \wedge (d_2^h d_0^v)^* E \wedge (d_0^h d_0^v)^* E \wedge (d_1^h d_1^v)^* E$ to $(d_1^h d_1^v)^* E$ over $H \times H \times K \times K$. c is the symmetry morphism which swaps the two terms in the middle.

With the straightforward notion of morphism, when H, K are also abelian, biextensions of **T** form a Picard category **Biext**$(H, K; G)$. The relative version of it, where we consider the various **Biext**$(H_S, K_S; G_S)$ over a variable base S, is a Picard stack, denoted Вıєхт$(H, K; G)$. As observed in Grothendieck [Gro72] and Breen [Bre83], Вıєхт$(H, K; G)$ is biadditive in all three variables.

1.4   Schreier-Grothendieck-Breen theory of extensions

It is well-known that an extension E of K with a possibly nonabelian kernel G determines a morphism $\jmath \colon E \to \mathrm{Aut}(G)$ from the action of E on G by conjugation [see, e.g. Mac95]. A refinement of this situation is when G is part of a crossed module $\partial \colon G \to \Pi$. Since part of the crossed module data is precisely a homomorphism $\Pi \to \mathrm{Aut}(G)$, an extension of K by that crossed module, or by (G, Π) for short, is a commutative diagram of group objects of the form [Bre90, §8]

---

[5]That is, $BH = NH[1]$, where $H[1]$ is the groupoid with a single object e and $\mathrm{Aut}(e) = H$.

where the row is exact and $\iota(g)e = e\iota(g^{\jmath(e)})$, for points g of G and e of E. An equivalent characterization is that E is a (G,Π)-torsor over K such that the equivariant morphism structural morphism to Π is a homomorphism. By loc. cit. and [AN10], such an extension corresponds to a group-like stack morphism $F\colon K \to [G \to \Pi]^\sim$, where K is identified with the (discrete) group-like stack. In this sense the above diagram is a butterfly representing this morphism.

In all these characterizations the morphism $\jmath$ arises as the structural equivariant section of the (G,Π)-torsor, whereas ι is the identification $G \simeq E_1$, where the (G,Π)-torsor E_1, isomorphic to the unit one, is the fiber over the unit section of K. This identification is explicitly given as $\iota(g) = g\,e_0 = e_0\,g$, where e_0 is the central section of E_1 corresponding to the unit section of G. Since $\jmath$ is a homomorphism, we obtain the commutativity of the "wing" in the diagram above. The conjugation property satisfied by ι and $\jmath$ reflects the structure of (G,Π)-torsor of E. The product structure of E is encoded by a 1-cocycle of (G,Π)-torsor isomorphisms

$$\gamma_{k,k'}\colon E_k \wedge^G E_{k'} \xrightarrow{\ \sim\ } E_{kk'},$$

satisfying the standard associativity condition [Gro72; Bre90; AN10]. In particular, we have that $\iota(g)e$ (resp. $e\iota(g)$), product in E, agrees with the left (resp. right) G_1-action on E.

2 Biadditive morphisms

If H, K, G are groups of $\mathbf{T}$, a biadditive (or bimultiplicative) morphism is a map $f\colon H \times K \to G$ which is a homomorphism in each variable. For abelian groups this is none other than a $\mathbb{Z}$-bilinear morphism.

More generally, if $\mathbf{H}$, $\mathbf{K}$, and $\mathbf{G}$ are categories, which we assume to be group-like, we say that a bifunctor is **biadditive** (or **bimultiplicative**) if it is monoidal in both variables, in a compatible way.[6] For this, we must also assume $\mathbf{G}$ be endowed with a braiding c. More precisely, we have:

2.1 Definition. A bifunctor $F\colon \mathbf{H} \times \mathbf{K} \to \mathbf{G}$ is biadditive if:

1. it has the structure of additive functor with respect to each variable, namely there exist functorial (iso)morphisms

$$\lambda^1_{h,h';k}\colon F(h,k)F(h',k) \longrightarrow F(hh',k) \quad \text{and} \quad \lambda^2_{h;k,k'}\colon F(h,k)F(h,k') \longrightarrow F(h,kk')$$

satisfying the standard associativity conditions and compatibility with the braiding of $\mathbf{G}$;

2. for all objects h, h' of $\mathbf{H}$ and k, k' of $\mathbf{K}$ we have a functorial commutative diagram

(2.1)
$$
\begin{array}{ccc}
\big(F(h,k)F(h',k)\big)\big(F(h,k')F(h',k')\big) & \xrightarrow{\quad\hat{c}\quad} & \big(F(h,k)F(h,k')\big)\big(F(h',k)F(h',k')\big) \\
{\scriptstyle \lambda^1_{h,h';k}\lambda^1_{h,h';k'}}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda^2_{h;k,k'}\lambda^2_{h';k,k'}} \\
F(hh',k)F(hh',k') \xrightarrow[\ \lambda^2_{hh';k,k'}\ ]{} F(hh',kk') & \xleftarrow[\ \lambda^1_{h,h';kk'}\ ]{} & F(h,kk')F(h',kk')
\end{array}
$$

where the upper horizontal arrow is the "commuto-associativity" morphism

$$(x\,y)(z\,w) \longrightarrow (x\,z)(y\,w)$$

arising from the braiding of $\mathbf{G}$ (an explicit definition can be found in [JP07]);

3. the two morphisms $F(I_{\mathbf{H}}, I_{\mathbf{K}}) \to I_{\mathbf{G}}$ that can be deduced from the first condition coincide.

The second condition ensures that the two possible ways to compute $F(hh',kk')$ agree. The obvious similarity with Diagram (1.5) will be exploited below.

---

[6]A better choice would be to use "bi-monoidal" in place of biadditive. The latter is motivated by continuity with the naming convention in [AN09], where morphisms between gr-stacks are called additive—a convention derived from the particular but significant Picard case.

The definition can be extended to the case where all three categories are braided. In this case, we add to the biadditivity the condition that F be *braided* in each variable, namely that the following diagrams commute:

$$
\begin{array}{ccc}
F(h,k)F(h',k) & \xrightarrow{\ \lambda^1\ } & F(hh',k) \\
{\scriptstyle c_{F,F}}\downarrow & & \downarrow{\scriptstyle F(c,k)} \\
F(h',k)F(h,k) & \xrightarrow[\lambda^1]{} & F(h'h,k)
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
F(h,k)F(h,k') & \xrightarrow{\ \lambda^2\ } & F(h,kk') \\
{\scriptstyle c_{F,F}}\downarrow & & \downarrow{\scriptstyle F(h,c)} \\
F(h,k')F(h,k) & \xrightarrow[\lambda^2]{} & F(h,k'k)
\end{array}
$$

A natural transformation $\varphi\colon F \Rightarrow F'$ between two biadditive morphisms is a natural transformation of bifunctors such that in each variable the conditions to be a natural transformation of additive functors are satisfied. We denote by $\mathbf{Hom}(\mathbf{H},\mathbf{K};\mathbf{G})$ the resulting groupoid of biadditive morphisms.

It is clear that the same definitions can be stated in the case of group-like stacks as opposed to categories. Similarly to the pointwise case, we obtain a groupoid $\mathbf{Hom}(\mathscr{H},\mathscr{K};\mathscr{G})$ of biadditive morphisms. For the latter, by way of a standard process of restricting to variable base S, we obtain a stack $\textsc{Hom}(\mathscr{H},\mathscr{K};\mathscr{G})$ arising from the various fiber categories $S \to \mathbf{Hom}(\mathscr{H}|_S,\mathscr{K}|_S;\mathscr{G}|_S)$.

3 Biextensions by braided crossed modules

We extend the notion of biextension by adapting it to include braided crossed modules as coefficients.

Let $\mathsf{G} = (\partial\colon G \to \Pi, \{\cdot,\cdot\})$ be a braided crossed module of $\mathbf{T}$, and let H, K be groups of $\mathbf{T}$.

3.1 Definition. A biextension of H, K by (G,Π) is a $(G,\Pi)_{H\times K}$-torsor E over $H\times K$ equipped with two partial composition laws $\times_1$ (resp. $\times_2$) making E into an extension of H_K (resp. K_H) by $(G,\Pi)_K$ (resp. by $(G,\Pi)_H$)—cf. sect. 1.4. We require the composition laws to be compatible in the manner described by a diagram formally identical to (1.5) in sect. 1.3.

To be precise, in the last condition of the above definition we must specify that the horizontal arrow in diagram (1.5), adapted to the present case,

$$(3.1) \qquad \hat{c}\colon (E_{h,k} \wedge^G E_{h',k}) \wedge^G (E_{h,k'} \wedge^G E_{h',k'}) \longrightarrow (E_{h,k} \wedge^G E_{h,k'}) \wedge^G (E_{h',k} \wedge^G E_{h',k'})$$

is given by the braiding of the stack $\mathscr{G} = [\partial\colon G \to \Pi]^\sim$ in a manner analogous to the horizontal arrow of diagram (2.1) in Definition 2.1. A global version of the compatibility condition is given by equation (1.6).

The two partial composition laws give rise to morphisms of (G,Π)-torsors

$$\gamma^1_{h,h';k}\colon E_{h,k} \wedge^G E_{h',k} \xrightarrow{\ \sim\ } E_{hh',k}$$

and

$$\gamma^2_{h;k,k'}\colon E_{h,k} \wedge^G E_{h,k'} \xrightarrow{\ \sim\ } E_{h,kk'}$$

each satisfying the obvious associativity condition, relative to the relevant variables, which can be obtained, mutatis mutandis, from [refs. Gro72; Bre83]. Again, a global version is given by (1.4).

The morphism (3.1) can be explicitly computed through the braiding of G as follows.

3.2 Proposition. *Let u, u', v, v' be points of $E_{h,k}, E_{h',k}, E_{h,k'},$ and $E_{h',k'}$, respectively. Then we have*

$$(3.2) \qquad (u \times_1 u') \times_2 (v \times_1 v') = (u \times_2 v) \times_1 (u' \times_2 v')\{f(u'),f(v)\}^{-f(v')},$$

where f is the equivariant section into the trivial Π-torsor.

Proof. Use Lemma 1.2 to express the morphism $E_{h',k} \wedge^G E_{h,k'} \to E_{h,k} \wedge^G E_{h',k}$. The right action by $f(v')$ arises from the relation between the left and right G-torsor structures of E. $\qquad\square$

3.3 Example. A standard biextension E of H, K by G (all groups assumed commutative) is a biextension in the sense specified above with coefficients in the complex $G[1]: [G \to 0]$. The interchange law, when expressed in terms of points, takes the standard form

$$(u \times_1 u') \times_2 (v \times_1 v') = (u \times_2 v) \times_1 (u' \times_2 v').$$

Dropping the commutativity of the partial multiplication laws, E becomes a biextension of the sort mentioned in the "variants" in [Gro72]. For such a biextension each extension determined by the partial multiplication laws is a central extension.

3.4 Remark. We do not require the partial multiplication laws to be commutative, even if the groups H and K are. It is not difficult to verify that if we do require the multiplication laws to be commutative, G becomes abelian and $\partial: G \to \Pi$ factors through zero, hence the biextension reduces to one of the type mentioned in Example 3.3 above. See however below for a more appropriate notion of commutativity.

3.5 Definition. A morphism of biextensions of H, K by (G, Π) is a morphism of the underlying (G, Π)-torsors compatible with the partial composition laws.

The compatibility with the composition laws means that there are commutative diagrams of (G, Π)-torsors

$$
\begin{array}{ccc}
E_{h,k} \wedge^G E_{h',k} & \xrightarrow{\gamma_1} & E_{hh',k} \\
{\scriptstyle \varphi \wedge \varphi} \downarrow & & \downarrow {\scriptstyle \varphi} \\
E'_{h,k} \wedge^G E'_{h',k} & \xrightarrow{\gamma'_1} & E'_{hh',k}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
E_{h,k} \wedge^G E_{h,k'} & \xrightarrow{\gamma_2} & E_{h,kk'} \\
{\scriptstyle \varphi \wedge \varphi} \downarrow & & \downarrow {\scriptstyle \varphi} \\
E'_{h,k} \wedge^G E'_{h,k'} & \xrightarrow{\gamma'_2} & E'_{h,kk'}.
\end{array}
$$

The **trivial biextension** is $I = (G_{H \times K}, e_\Pi)$. More generally, a biextension will be considered trivial if it is isomorphic, in the sense specified above, to the trivial one.

With the above notion of morphism, biextensions of H, K by G form a *pointed* groupoid $\mathbf{Biext}(H, K; \mathsf{G})$. A standard argument based on descent shows that over a variable base S, the collection of all $\mathbf{Biext}(H_S, K_S; \mathsf{G}_S)$ forms a stack, equally pointed, denoted $\mathrm{Biext}(H, K; \mathsf{G})$. The question of whether either of $\mathbf{Biext}(H, K; \mathsf{G})$ and $\mathrm{Biext}(H, K; \mathsf{G})$ possess a monoidal structure is more delicate, and it will require additional commutativity properties from G, as discussed below.

The groupoid $\mathbf{Biext}(H, K; \mathsf{G})$ is evidently additive with respect to the last variable, namely

$$\mathbf{Biext}(H, K; \mathsf{G}_1 \times \mathsf{G}_2) \simeq \mathbf{Biext}(H, K; \mathsf{G}_1) \times \mathbf{Biext}(H, K; \mathsf{G}_2),$$

and the same holds for $\mathrm{Biext}(H, K; \mathsf{G})$. Additivity on the other two variables relies on the commutativity of the partial multiplications for the same reason as [Bre83, §1.2], hence it will not hold in our case.

Biextensions by (G, Π) have certain structural properties analogous to those of extensions reviewed in sect. 1.4, more precisely, we have:

3.6 Lemma. *If $E \to H \times K$ is a biextension of H, K by (G, Π), then E is equipped with a map $\jmath: E \to \Pi$ which is a homomorphism for each partial composition law. For all points $e \in E_{h,1}$, or $e \in E_{1,k}$, and $g \in G$ we have the relation*

$$(3.3) \qquad\qquad\qquad \imath(g)e = e\imath(g^{\jmath(e)}).$$

and the diagram

$$
(3.4) \qquad
\begin{array}{ccc}
G & \xrightarrow{\imath} E \xrightarrow{p} & H \times K \\
{\scriptstyle \partial} \downarrow & \swarrow {\scriptstyle \jmath} & \\
\Pi & &
\end{array}
$$

commutes, where $\imath$ is the identification $(G_1, e_\Pi) \xrightarrow{\sim} E_{1,1}$ as (G, Π)-torsors.

The identification of the fiber $E_{1,1}$ above the unit (e_H, e_K) of $H \times K$ with the unit torsor $I = (G, e_\Pi)$ amounts to the definition of exactness of the row in (3.4).

Proof. The arguments are virtually the same as those of [Gro72], except for the part concerning the morphism $\jmath$. Following loc. cit., we can use the facts about extensions recalled in sect. 1.4 by working over the terminal object K (resp. H) when we view E as an extension of H_K (resp. K_H) by $(G, \Pi)_H$ (resp. $(G, \Pi)_K$).

With this in mind, let $f = (\jmath, p) \colon E \to \Pi_{H \times K}$, with $\jmath \colon E \to \Pi$, be the equivariant morphism which is part of the $(G, \Pi)_{H \times K}$-torsor structure. As an extension of H_K by $(G, \Pi)_K$, E has a structure of $(G, \Pi)_K \times_K H_K$-torsor over H_K, hence $f \colon E \to \Pi_K \times_K H_K \simeq \Pi \times H \times K$ is written as $f = (\jmath_K, p)$, where $\jmath_K \colon E \to \Pi_K$. We have $\jmath_K = (\jmath, p_K)$, where p_K (resp. p_H) is the composite of p with the projection to K. A similar picture holds by exchanging H with K. Since $\jmath_K$ and $\jmath_H$ are homomorphisms for the corresponding product structure, $\jmath$ is a homomorphism for both.

Let $e_{H/K} = (e_H, \mathrm{id}_K)$ be the unit section of H_K and $e_{K/H} = (\mathrm{id}_H, e_K)$ that of K_H. Since the fibers $E_{1,k} = e^*_{H/K} E$ and $E_{h,1} = e^*_{K/H} E$ are respectively identified with G_K and G_H as (G, Π)-torsors, we have the unit section $e_{E/K}$ (resp. $e_{E/H}$) in $E_{1,k}$ (resp. $E_{h,1}$). Now let $E_{1,1} = (e_H, e_K)^* E$ be the fiber over the unit section (e_H, e_K) of $H \times K$. This is the common pullback of both $E_{1,k}$ and $E_{h,1}$. The same argument as in [Gro72, §2.2] shows that the restriction $e_{E/K}$ and $e_{E/H}$ to (e_H, e_K) agree, so let $e_E \in E_{1,1}$ be their common value. From [Gro72, §2.1] e_E is central, since both $e_{E/K}$ and $e_{E/H}$ are, and we define $\imath$ by $\imath(g) = g\, e_E = e_E\, g$. Note, $\imath$ is the common restriction of $\imath_K$ and $\imath_H$ respectively defined by $e_{E/K}$ and $e_{E/H}$.

Analogously to the case of extensions, the commutativity of the "wing" in (3.4) follows from the equivariance of f, and hence of $\jmath$, having observed that $\jmath(e_E)$ must be equal to the unit of Π. Finally, the relation (3.3) follows from the definition of the bitorsor structure underlying that of a (G, Π)-bitorsor (cf. the end of sect. 1.1). $\qquad\square$

3.7 Remark. In the converse direction to Lemma 3.6 a biextension of H, K by (G, Π) can be recovered from the datum $(E, \times_1, \times_2, \imath, \jmath)$, where $p \colon E \to H \times K$ is an object of $\mathbf{T}_{/H \times K}$ equipped with two partial composition laws satisfying the interchange law, and $\imath$ and $\jmath$ satisfy the conditions of the lemma.

If E' is a second biextension, with maps $\imath' \colon G \to E$ and $\jmath' \colon E \to \Pi$ satisfying the conditions of Lemma 3.6, a biextension morphism $\varphi \colon E \to E'$ makes the following diagram

commutative.

For a biextension which is globally of the form $(G_{H \times K}, x)$, where $x \colon H \times K \to \Pi$ characterizes the equivariant structural morphism, we obtain a decomposition in terms of a pair of nonabelian cocycles, which we briefly describe.

The partial multiplication laws are described by

$$(3.5a) \qquad (h, k, a)(h', k, a') = (hh', k, g_1(h, h'; k) a^{x(h', k)} a'),$$

$$(3.5b) \qquad (h, k, b)(h, k', b') = (h, kk', g_2(h; k, k') b^{x(h, k')} b').$$

The actions stem from the change from right to left action,

$$(h, k, a)(h', k, a') = ((h, k, e_G)a)(h', k, a') = (h, k, e_G)(a(h', k, a')) = (h, k, e_G)(h', k, a^{x(h', k)} a'),$$

and similarly for the other one. From the associativity property and the fact that the equivariant section $G_{H \times K} \to \Pi$ must be a homomorphism for both laws, we find that (g_1, g_2, x) must satisfy a pair of nonabelian cocycle conditions:

$$(3.6a) \qquad g_1(hh', h''; k) g_1(h, h'; k)^{x(h'', k)} = g(h, h'h''; k) g_1(h', h''; k)$$

$$(3.6b) \qquad x(h, k) x(h', k) = x(hh', k)\, \partial g_1(h, h'; k)$$

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and

(3.6c) $$g_2(h;kk',k'')g_2(h;k,k')^{x(h,k'')} = g_2(h;k,k'k'')g_2(h;k',k'')$$

(3.6d) $$x(h,k)x(h,k') = x(h,kk')\partial g_2(h;k,k')$$

The cocycles (g_1,x) and (g_2,x) are not independent: from the compatibility between the partial multiplication laws, using Proposition 3.2, we find:

(3.7) $$g_2(hh';k,k')g_1(h,h';k)^{x(hh',k')}g_1(h,h';k') =$$
$$g_1(h,h';kk')g_2(h,k,k')^{x(h',kk')}g_2(h';k,k')\{x(h',k),x(h,k')\}^{-x(h',k')}.$$

These cocycles are a coboundary if there exists $u\colon H\times K\to G$ such that:

(3.8a) $$u(hh',k)\,g_1(h,h';k) = u(h,k)u(h',k)$$

(3.8b) $$u(h,kk')\,g_2(h;k,k') = u(h,k)u(h,k')$$

(3.8c) $$x(h,k) = \partial u(h,k).$$

4 Biadditive morphisms and biextensions

Let E be a biextension of H,K by (G,Π). Then E defines a biadditive morphism

$$F_E\colon H\times K\longrightarrow \mathscr{G},$$

in the sense of Definition 2.1, where $\mathscr{G}$ is the group-like associated stack, by assigning to a point $(h,k)\in H\times K$ the (G,Π)-torsor $E_{h,k}$. Indeed, it is easily seen that the isomorphisms γ^1 and γ^2 plus the compatibility of the two composition laws expressed by (1.5) satisfy the required conditions. It is also easily verified that a morphism of biextensions $\varphi\colon E\to E'$ induces a natural transformation (denoted with the same symbol)

$$\varphi\colon F_E \Longrightarrow F_{E'}\colon H\times K\longrightarrow \mathscr{G}.$$

This establishes a functor

$$u\colon \mathbf{Biext}(H,K;\mathsf{G})\longrightarrow \mathbf{Hom}(H,K;\mathscr{G}),$$

and in fact one between stacks

$$u\colon \textsc{Biext}(H,K;\mathsf{G})\longrightarrow \textsc{Hom}(H,K;\mathscr{G}),$$

These functors are evidently fully faithful. In the converse direction, we have:

4.1 Proposition. *Let $F\colon H\times K\to \mathscr{G}$ be a biadditive morphism, and let $\mathscr{G}$ have a presentation by way of the braided crossed module $\mathsf{G}=(G,\Pi,\partial,\{\cdot,\cdot\})$. Then the pullback of the sequence (1.1) along F determines a biextension $E=E_F$ of H,K by G.*

Proof. Let $E=(H\times K)\times_{\mathscr{G}}\Pi$ be the pullback. Set $p\colon E\to H\times K$ (resp. $\jmath\colon E\to \Pi$) equal to the first (resp. second) projection. We claim that E is a biextension of H,K by G.

The sequence (1.1) can be seen as the universal extension by (G,Π). In particular, Π can be identified with the universal G,Π-torsor (the identity id_Π is tautologically the structural equivariant section). This makes apparent that the pullback to $H\times K$ is a (G,Π)-torsor with structural map $\jmath$. As for the partial multiplication laws, observe that, with the same notation as sect. 1.3, the biadditivity of F amounts to a pair of natural transformations

$$(F\circ d_2^h)(F\circ d_0^h) \Longrightarrow (F\circ d_1^h)\colon H\times H\times K\longrightarrow \mathscr{G}, \qquad (F\circ d_2^h)(F\circ d_0^h) \Longrightarrow (F\circ d_1^h)\colon H\times K\times K\longrightarrow \mathscr{G},$$

and the equality of two transformations (see eqns. (1.6) and (2.1))

$$(Fd_0^h d_2^v)(Fd_2^h d_0^v)(Fd_0^h d_0^v)(Fd_1^h d_1^v) \Longrightarrow (Fd_1^h d_1^v)\colon H\times H\times K\times K\longrightarrow \mathscr{G}.$$

This translates into the required properties for E once we observe that the universality of Π (as a (G,Π)-torsor over $\mathcal{G}$) implies that for any pair $f,g\colon S\to\mathcal{G}$ we have an isomorphism $f^*\Pi\wedge^G g^*\Pi\simeq(fg)^*\Pi$ of (G,Π)-torsors. Last, the morphism $\imath$ is the image, under a further pullback along the unit section $(e_H,e_K)\in H\times K$, of $\partial\colon G\to\Pi$. It is clear $\imath$ and $\jmath$ have the properties stated in Lemma 3.6.

In more down-to-earth terms, everything can be explicitly checked by writing out explicit expressions for the points of the pullback. Thus, a point $e\in E$ is a triple $e=((h,k),a,x)$, where $(h,k)\in H\times K$, $x\in\Pi$, and $a\colon F(h,k)\to(G,x)$. The G-action becomes evident by looking at two points e,e' in the same fiber. We must have a commutative diagram of (G,Π)-torsors

$$
\begin{array}{ccc}
F(h,k) & \xrightarrow{\ a'\ } & (G,x') \\
& {\scriptstyle a}\searrow & \downarrow{\scriptstyle g} \\
& & (G,x)
\end{array}
$$

where the vertical arrow, as a map of (G,Π)-torsors whose underlying G-torsor is trivial, is completely determined by a point $g\in G$ (the image of the unit section e_G) such that

$$x'=x\,\partial g.$$

The situation for the left action is completely analogous, with $x'=\partial g\,x$.

When $(h,k)=(e_H,e_K)$, the point $e_E=((e_H,e_K),l,e_\Pi)$, where l is the unique morphism coming from the third condition in Definition 2.1, is the central unit section of E. This gives the morphism $\imath\colon G\to E$. The partial multiplications are obtained as follows. Let $e=((h,k),a,x)$ and $e=((h',k),a',x')$. Then $ee'=((hh',k),a'',xx')$, where the morphism a'' is obtained from the diagram

$$
\begin{array}{ccc}
F(h,k)F(h',k) & \xrightarrow{\ aa'\ } & (G,xx') \\
{\scriptstyle \lambda^1_{hh',k}}\downarrow & \nearrow & \\
F(hh',k) & {\scriptstyle a''} &
\end{array}
$$

The second multiplication is defined in the same way. The interchange law immediately follows from the above and (2.1). $\qquad\square$

Since it is clear that if $\varphi\colon F\Rightarrow F'$ is a natural transformations of biadditive morphisms by pullback we obtain a corresponding isomorphim $E_F\to E_{F'}$ of biextensions, the construction of Proposition 4.1 provides functors

$$v\colon \mathbf{Hom}(H,K;\mathcal{G})\longrightarrow\mathbf{Biext}(H,K;\mathsf{G})$$

and

$$v\colon \textsc{Hom}(H,K;\mathcal{G})\longrightarrow\textsc{Biext}(H,K;\mathsf{G}).$$

4.2 Lemma. *The functor u is essentially surjective.*

Proof. For each F biadditive, we construct a morphism $F\to F'=F_E$, where $E=(H\times K)\times_\mathcal{G}\Pi$. The biadditive morphism determined by E assigns to each $(h,k)\in H\times K$ the (G,Π)-torsor $E_{h,k}$ consisting of pairs (f,x) such that

$$f\colon F(h,k)\xrightarrow{\ \sim\ }(G,x).$$

Having observed this, the proof proceeds in a manner identical to that of Proposition 4.2.2 in [AN09] (in particular, cf. 4.4.2.1 and 4.4.2.2). $\qquad\square$

Summarizing the previous discussion, we have

4.3 Proposition. *The functors u and v are quasi-inverses.* $\qquad\square$

In words, $\mathbf{Hom}(H,K;\mathcal{G})$ and $\mathbf{Biext}(H,K;\mathsf{G})$ are equivalent, *pointed* groupoids. The distinguished point is the trivial biextension, corresponding to the trivial biadditive morphism sending $(h,k)\in H\times K$ to the trivial (G,Π)-torsor. The same holds for $\textsc{Hom}(H,K;\mathcal{G})$ and $\textsc{Biext}(H,K;\mathsf{G})$.

5 Symmetry properties of biextensions

It is well-known (and see the discussion at the end of [AN10]) that the groupoid of extensions of K by (G, Π) admits a monoidal structure if and only if the crossed module $\partial \colon G \to \Pi$ (or equivalently the associated stack $\mathscr{G}$) is braided. This monoidal structure is itself equipped with a commutativity isomorphism if and only if the braiding carried by $\mathscr{G}$ is symmetric.

For biextensions the situation is more rigid. This is already apparent from the fact that in the very definition of a biextension a braided structure is required in order to be able to formulate the compatibility condition between the two multiplication laws. Carrying on this line of thought, we have

5.1 Proposition. *Let E and F be two biextensions of (H, K) by (G, Π). The contracted product (as (G, Π)-torsors) $E \wedge^G F$ carries a biextension structure if and only if the braiding on the crossed module $\mathsf{G} = (G, \Pi, \partial, \{-, -\})$ (or equivalently $\mathscr{G}$) is symmetric.*

Proof. If E and F are biextensions, we can use the braiding of $\mathscr{G}$ to construct the partial multiplication laws in the expected manner:

$$(E_{h,k} \wedge^G F_{h,k}) \wedge^G (E_{h',k} \wedge^G F_{h',k}) \xrightarrow{\hat{c}} (E_{h,k} \wedge^G E_{h',k}) \wedge^G (F_{h,k} \wedge^G F_{h',k}) \xrightarrow{\gamma^1 \wedge \mu^1} E_{hh',k} \wedge F_{hh',k}.$$

Let us simply denote the first multiplication morphism so obtained by $\gamma^1 \mu^1$, and let $\gamma^2 \mu^2$ denote the second multiplication structure obtained in exactly the same way.

For the compatibility condition between the two multiplications we must write diagram (1.5) for $\gamma^1 \mu^1$ and $\gamma^2 \mu^2$. To simplify notations and readability, let us suppress the symbol $\wedge^G$ and the associativity morphisms. Set: $a = E_{h,k}$, $x = F_{h,k}$, $y = E_{h',k}$, $z = F_{h',k}$, $u = E_{h,k'}$, $v = F_{h,k'}$, $w = E_{h',k'}$, $b = F_{h',k'}$. Also, let us use parentheses to denote the various codomains of γ, μ, namely $(ay) = E_{hh',k}$, $(xz) = F_{hh',k}$, etc. Consider the following diagram:
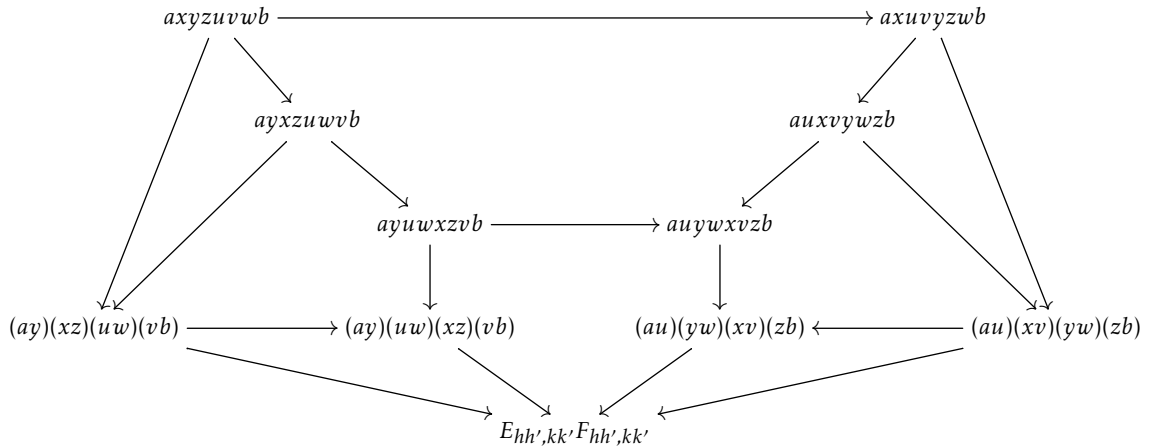
The outer pentagon corresponds to the sought-after compatibility for the partial multiplications of $E \wedge^G F$, whereas the little inner one is obtained by juxtaposing the compatibility conditions for E and F. Thus the inner pentagon commutes. The triangles commute by definition, since they just hold the definitions of $\gamma^1 \mu^1$ and $\gamma^2 \mu^2$. The quadrangles are commutative by inspection. This leaves the big hexagon in the middle. The position of the external variables is unchanged throughout, hence it reduces to the commutativity of the diagram in Lemma D.2. This proves the proposition. $\qquad\square$

Once the coefficient crossed module carries a symmetric braiding, we are provided with a product of biextensions. An easy argument, based on the definition of the partial multiplication laws in the proof of Proposition 5.1 and Lemma D.1, or alternatively [AN10, §8.2], shows that the braiding provides a morphism

$$c \colon E \wedge^G F \longrightarrow F \wedge^G E$$

of biextensions, which is evidently symmetric. We summarize the previous discussion as

5.2 Proposition. $\textbf{Biext}(H,K;\mathsf{G})$ *(resp.* $\textsc{Biext}(H,K;\mathsf{G})$*) is a braided symmetric group-like groupoid (resp. stack) if and only if* $\mathsf{G} = (G,\Pi,\partial,\{-,-\})$ *carries a symmetric braiding.* $\qquad\square$

By equivalence, the same statement holds for $\textbf{Hom}(H,K;\mathscr{G})$ and $\textsc{Hom}(H,K;\mathscr{G})$, respectively; the resulting symmetric monoidal structure is given by the pointwise product of functors (with respect to the monoidal structure of $\mathscr{G}$).

Let $\mathscr{G}$ be symmetric. If $F_1, F_2 \colon H \times K \to \mathscr{G}$ correspond, via Proposition 4.3, to the biextensions E_1 and E_2, then it is not difficult to see that $E_1 \wedge^G E_2$ is the biextension corresponding to $F_1 F_2 \colon H \times K \to \mathscr{G}$, where $F_1 F_2$ is computed pointwise, i.e. for all $h \in H$ and $k \in K$ we set $(F_1 F_2)(h,k) := F_1(h,k) F_2(h,k)$, where the juxtaposition on the right indicates the monoidal structure in $\mathscr{G}$. Thus, Proposition 4.3 is upgraded to an equivalence of symmetric group-like groupoids or stacks. More precisely, we get:

5.3 Proposition. *The assignment defined by u in Proposition 4.3 gives an equivalence*

$$u \colon \textbf{Biext}(H,K;\mathsf{G}) \xrightarrow{\sim} \textbf{Hom}(H,K;\mathscr{G})$$

(resp.

$$u \colon \textsc{Biext}(H,K;\mathsf{G}) \xrightarrow{\sim} \textsc{Hom}(H,K;\mathscr{G}))$$

of symmetric group-like categories (resp. stacks). $\qquad\square$

6 Butterflies

Let $H_\bullet \colon H_1 \xrightarrow{\partial} H_0$ and $K_\bullet \colon K_1 \xrightarrow{\partial} K_0$ be a pair of crossed modules, and let $\partial \colon G_1 \to G_0$ be a crossed module equipped with a braiding $\{-,-\}$. We denote by $\mathscr{H}$, $\mathscr{K}$, and $\mathscr{G}$ the corresponding associated stacks.

6.1 Definition. A **butterfly** from $H_\bullet \times K_\bullet$ to $G_\bullet$ is a biextension $E \in \textbf{Biext}(H_0, K_0; G_\bullet)$ equipped with trivializations of its pullbacks $(\partial,\mathrm{id})^* E$ and $(\mathrm{id},\partial)^* E$, i.e. maps $s_1 \colon H_1 \times K_0 \to E$ and $s_2 \colon H_0 \times K_1 \to E$ subject to the following conditions:

1. (Restriction) s_1 and s_2 agree on $H_1 \times K_1$: $(\mathrm{id},\partial)^* s_1 = (\partial,\mathrm{id})^* s_2$.

2. (Compatibility) For all $(h,z) \in H_1 \times K_0$ and $(y,k) \in H_0 \times K_1$, and $e \in E_{y,z}$, s_1 and s_2 satisfy:

$$(6.1) \qquad \begin{aligned} s_1(h,z) \times_1 e &= e \times_1 s_1(h^y,z), \\ s_2(y,k) \times_2 e &= e \times_2 s_2(y,k^z). \end{aligned}$$

A morphism $\varphi \colon (E, s_1, s_2) \to (E', s_1', s_2')$ is a morphism of the underlying biextensions preserving the trivializations.

Butterflies form a pointed groupoid, denoted $\textbf{Biext}(H_\bullet, K_\bullet; G_\bullet)$. The distinguished object is the trivial biextension.

If we identify the sections corresponding to the trivializations with $s_1 \colon H_1 \times K_0 \to E$ and $s_2 \colon H_0 \times K_1 \to E$, condition 1 in the Definition implies that

$$s_1(h,\partial k) = s_2(\partial h, k).$$

Let $s \colon H_1 \times K_1 \to E$ be the resulting morphism. The next lemmas characterize these objects in terms of explicit diagrams (see (6.2) and (6.6) below). Their shape justifies the name.

6.2 Lemma. *The trivializations s_1 and s_2, as maps $s_1 \colon H_1 \times K_0 \to E$ and $s_2 \colon H_0 \times K_1 \to E$, render the following diagrams commutative*

(6.2)

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with $\jmath \circ s_1$ and $\jmath \circ s_2$ equal to the trivial map (identically equal to the unit e_0 of G_0). In addition, s_1 and s_2 have the following properties:[7]

1. *(Multiplicative) s_1 and s_2 are multiplicative in each variable, namely*

$$(6.3) \qquad \begin{aligned} s_1(h,z) \times_1 s_1(h',z) = s_1(hh',z), &\quad s_1(h,z_1) \times_2 s_1(h,z') = s_1(h,zz'), \\ s_2(y,k) \times_1 s_2(y',k) = s_2(yy',k), &\quad s_2(y,k) \times_2 s_2(y,k') = s_2(y,kk'); \end{aligned}$$

2. *(Central) For all $g \in G_1$, and whenever ∂h (resp. ∂k) is equal to the unit section of H_0 (resp. K_0), we have:*

$$(6.4) \qquad \begin{aligned} s_1(h,z)\imath(g) &= \imath(g)s_1(h,z), \\ s_2(y,k)\imath(g) &= \imath(g)s_2(y,k). \end{aligned}$$

Proof. Since the pullbacks $(\partial,\mathrm{id})^* E$ and $(\mathrm{id},\partial)^* E$ split as extensions, we immediately get relations (6.3), as well as the fact that the compositions $\jmath \circ s_1$ and $\jmath \circ s_2$ must be equal to the unit element.

The trivializing section of a bitorsor must be central, hence, say

$$(6.5) \qquad s_1(h,z)\,g = g\,s_1(h,z)$$

where the juxtaposition indicates the G_1-action. Of course, a similar relations is valid for s_2, as well. Then the relations (6.4) follow from it and (3.3) of Lemma 3.6. $\qquad \square$

6.3 Remark. An alternative way to characterize a butterfly is to say that it consists of a biextension E of (H_0,K_0) by $G_\bullet$ such that s_1 and s_2 provide it with the structure of butterflies in the ordinary sense of [AN09] from $(H_\bullet)_{K_0}$ (resp. $(K_\bullet)_{H_0}$) to $G_\bullet$ in a compatible way.

It is clear that properties similar to those pertaining to s_1 and s_2 stated in Lemma 6.2 hold for their common restriction $s \colon H_1 \times K_1 \to E$. In particular, the following diagram

$$(6.6)$$

is commutative, with $\jmath \circ s$ equal to the trivial map (identically equal to the unit e_0 of G_0). In addition, s is multiplicative in each variable:

$$(6.7) \qquad \begin{aligned} s(h,k) \times_1 s(h',k) &= s(hh',k) \\ s(h,k) \times_2 s(h,k') &= s(h,kk'). \end{aligned}$$

Relations (6.4) (collapsed into one) and (6.1) also hold. As an easy consequence we have the following

6.4 Lemma. *Let $(E,s_1,s_2) \in \mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet)$. The pullback $(\partial,\partial)^* E$ is isomorphic (via s) to the trivial biextension in $\mathbf{Biext}(H_1, K_1; G_\bullet)$.* $\qquad \square$

6.5 Remark. The correspondence $(E,s_1,s_2) \mapsto (E,s)$ determines a morphism

$$\mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet) \longrightarrow \mathrm{H\,Ker}\Big(\mathbf{Biext}(H_0, K_0; G_\bullet) \xrightarrow{(\partial,\partial)^*} \mathbf{Biext}(H_1, K_1; G_\bullet) \Big)$$

where $\mathrm{H\,Ker}$ denotes the homotopy kernel (recall both groupoids are pointed) which is not an equivalence, in general. Requesting that the biextension E become trivial when pulled back to $H_1 \times K_1$ is a weaker condition for it does not provide for the two other pullbacks to be trivializable.

---

[7] In some of the following formulas we explicitly write the the product symbols $\times_1$ and $\times_2$ to avoid ambiguities.

6.6 Remark. A butterfly has a description in terms of cocycles if the underlying (G_1, G_0)-torsor E is globally trivial as a right G_1-torsor. As seen at the end of sect. 3, E has the form $(H_0 \times K_0 \times G_1, x)$, with $x \colon H_0 \times K_0 \to G_0$. Thus, a cocyclic description has formulas (3.5), (3.6), (3.7) (with $h, h' \in H_0$ and $k, k' \in K_0$). We must add formulas for the trivializations of the two pullbacks to $H_1 \times K_0$ and $H_0 \times K_1$. The required triviality conditions are the same as those for a nonabelian 1-cocycles of the sort that appears in the theory of extensions. Thus there must be $u_1 \colon H_1 \times K_0 \to G_1$ and $u_2 \colon H_0 \times K_1 \to G_1$ such that

$$(6.8a) \qquad u_1(hh', z)g_1(\partial h, \partial h'; z) = u_1(h, z)u_2(h', z)$$

$$(6.8b) \qquad x(\partial h, z) = \partial u_1(h, z)$$

and

$$(6.8c) \qquad u_2(y, kk')g_2(y; \partial k, \partial k') = u_2(y, k)u_2(y, k')$$

$$(6.8d) \qquad x(y, \partial k) = \partial u_2(h, z)$$

for all pairs $(h, z), (h', z) \in H_1 \times K_0$ and $(y, k), (y, k') \in H_0 \times K_1$. Moreover, since the two trivializations must agree when pulled back to $H_1 \times K_1$, we must have $u_1(h, \partial k) = u_2(\partial h, k)$. Denoting this common restriction by $u \colon H_1 \times K_1 \to G_1$, we must have

$$(6.9a) \qquad u(hh', k)g_1(\partial h, \partial h'; \partial k) = u(h, k)u(h', k)$$

$$(6.9b) \qquad u(h, kk')g_2(\partial h; \partial k, \partial k') = u(h, k)u(h, k')$$

$$(6.9c) \qquad x(\partial h, \partial k) = \partial u(h, k).$$

for all $h, h' \in H_1$ and $k, k' \in K_1$. Comparing with (3.8), we see that the latter set corresponds to a trivialization as a biextension, as implied by Lemma 6.4.

7 Biadditive morphisms and butterflies

We now consider biadditive morphisms

$$F \colon \mathscr{H} \times \mathscr{K} \longrightarrow \mathscr{G}$$

with $\mathscr{G}$ braided (cf. sect. 2). As before, we assume we have presentations by crossed modules $\mathscr{H} \simeq [H_1 \to H_0]^\sim$, $\mathscr{K} \simeq [K_1 \to K_0]^\sim$, and $\mathscr{G} \simeq [G_1 \to G_0]^\sim$, the latter equipped with a braiding structure $\{-, -\}$. Our purpose is to prove the following

7.1 Theorem. *There is an equivalence of (pointed) groupoids*

$$u \colon \mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet) \xrightarrow{\sim} \mathbf{Hom}(\mathscr{H}, \mathscr{K}; \mathscr{G}).$$

In essence, the theorem states that any biadditive functor $F \colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$ can be represented by a butterfly involving the presentations. This equivalence is compatible with restriction and base-change so that, via the usual mechanism of considering the above equivalence relative to a variable object S of $\mathbf{T}$, we obtain a corresponding equivalence

$$u \colon \mathrm{B\textsc{iext}}(H_\bullet, K_\bullet; G_\bullet) \xrightarrow{\sim} \mathrm{H\textsc{om}}(\mathscr{H}, \mathscr{K}; \mathscr{G}).$$

Following [AN09] and sect. 4, we exhibit a pair of quasi-inverse functors. We will essentially confine ourselves to just exhibit the relevant definitions, for the methods are quite similar to those in [AN09, §4], with the exception of biadditivity. The latter is discussed explicitly, where the need arises.

Let $\bar{F}$ be the composition of the projection $\pi \colon H_0 \times K_0 \to \mathscr{H} \times \mathscr{K}$ with F. It is evidently biadditive.

7.2 Proposition. *There exists a functor $v \colon \mathbf{Hom}(\mathscr{H}, \mathscr{K}; \mathscr{G}) \to \mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet)$ whose value at an object F is the pullback $E = (H_0 \times K_0) \times_{\bar{F}, \mathscr{G}, \pi} G_0$.*

7.3 Lemma. *Let E be as in proposition 7.2. The biextension E satisfies the conditions in the statement of lemma 6.2.*

Proof. We must verify the pullbacks of E to $H_1 \times K_0$ and $H_0 \times K_1$ are trivializable. This is a direct consequence of the fact that $H_1 \to H_0 \to \mathcal{H}$ and $K_1 \to K_0 \to \mathcal{K}$ are (homotopically) exact. Furthermore, the decomposition $(\partial, \partial) = (\mathrm{id}, \partial) \circ (\partial, \mathrm{id}) = (\partial, \mathrm{id}) \circ (\mathrm{id}, \partial)$ and condition 3 of definition 2.1 ensure that the restriction condition in definition 6.1 is satisfied.

In explicit terms, a point of the pullback $(\partial, \mathrm{id})^* E$ is given by a triple $((h,z), f, x)$ where $(h,z) \in H_1 \times K_0$, $x \in G_0$ and f is an isomorphism

$$F(\pi(\partial h), \pi(z)) \xrightarrow{\ f\ } (G_1, x).$$

We have $\pi(\partial h) = (H_1, \partial h)$, so there must be an isomorphism $I_{\mathcal{H}} = (H_1, e_{h_0}) \longrightarrow (H_1, \partial h)$. Thus, there is a chain of isomorphisms

$$F(I_{\mathcal{H}}, \pi(z)) \longrightarrow F(\pi(\partial h), \pi(z)) \xrightarrow{\ f\ } (G_1, x)$$
$$\downarrow$$
$$I_{\mathcal{G}}$$

so that we must have $x = \partial g$, $g \in G_1$. This provides an explicit trivializing section s_1 for $(\partial, \mathrm{id})^* E$. Similarly for the other one, s_2, and their common restriction s.

A computation based on the biadditivity of F and the same technique at the end of the proof of 4.1 shows that both s_1 and s_2 are multiplicative, i.e. each satisfies condition 1 of lemma 6.2, with respect to both variables. Condition 2 also follows from a direct calculation, as in [AN09, §4.3.6]. $\qquad\square$

The pullback biextension construction is functorial (cf. the end of section 4). In particular, if $\varphi \colon F \Rightarrow F'$ is a morphism of biadditive functors, the resulting morphism of biextensions is compatible with the trivializations. This observation proves Proposition 7.2. $\qquad\square$

For the converse direction, we proceed as in ref. [AN09, §§4.3.2–4.3.4]. Specifically, an object of $\mathcal{H} \times \mathcal{K}$ will be represented by a pair of torsors, as $((Y,y),(Z,z))$ where (Y,y) is an (H_1, H_0)-torsor and (Z,z) an (K_1, K_0)-torsor. Let (E,s) be an object of $\mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet)$. We define a $F_{E,s_1,s_2} \colon \mathcal{H} \times \mathcal{K} \to \mathcal{G}$ by assigning to the above pair the (G_1, G_0)-torsor (X,x) where:

$$X := \mathrm{Hom}_{H_1, K_1}(Y, Z; E)_{(y,z)}, \qquad x \colon X \longrightarrow G_0$$
$$e \longmapsto \jmath \circ e.$$

X consists of separately H_1 or K_1-equivariant local lifts of $(y,z) \colon Y \times Z \to H_0 \times K_0$ to E. By this we mean that the lift of $(y\,\partial h, z)$ is related to that of (y,z) by $e(y\,\partial h, z) = e(y,z) s_1(h,z)$. Similarly, $e(y, z\,\partial k) = e(y,z) s_2(y,k)$. The notion is consistent thanks to Conditions 1 and the multiplicativity of s_1 and s_2. This construction is obviously functorial with respect to each of its arguments. It gives the defining part of the

7.4 Proposition. *There exists a functor*

$$u \colon \mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet) \longrightarrow \mathbf{Hom}(\mathcal{H}, \mathcal{K}; \mathcal{G}),$$

whose value at (E, s_1, s_2) is the bifunctor F_{E,s_1,s_2} defined above.

Proof. We must verify the biadditivity property, namely that $F(Y_1, Z) \wedge^{G_1} F(Y_2, Z) \xrightarrow{\sim} F(Y_1 \wedge^{H_1} Y_2, Z)$, which follows from the diagram:

$$
\begin{array}{ccccc}
& & E \times_{K_0} E & \xrightarrow{\ \gamma^1\ } & E \\
& \nearrow^{(e_1,e_2)} & \downarrow{\scriptstyle(p,p)} & & \downarrow{\scriptstyle p} \\
Y_1 \times Y_2 \times Z \xrightarrow{\Delta_Z} Y_1 \times Z \times Y_2 \times Z \longrightarrow & & (H_0 \times K_0) \times_{K_0} (H_0 \times K_0) \longrightarrow & & H_0 \times K_0
\end{array}
$$

It is easy (and left to the reader) to check that the diagram is invariant under replacing $(e_1 g, e_2)$ by $(e_1, g e_2)$ and $(y_1 h, y_2)$ by $(y_1, h y_2)$. (The first arrow to the left is the diagonal of Z followed by the swap of the two inner factors.) $\qquad\square$

7.5 Remark. A coordinate version of the construction of the functor u above is as follows. According to the beginning of section 4, if E is the underlying biextension of an object in $\mathbf{Biext}(H_\bullet, K_\bullet; \mathscr{G}_\bullet)$, we obtain a biadditive morphism $H_0 \times K_0 \to \mathscr{G}$ by sending the pair (y, z) to the (G_1, G_0)-torsor $E_{y,z}$. Since E is part of a butterfly, this construction is compatible with morphisms in $\mathscr{H}$ and $\mathscr{K}$ (in fact, in the prestacks defined by the presentations) because, if say $y' = y \, \partial h$, we have

$$E_{y',z} \longleftarrow E_{y,z} \wedge^{G_1} E_{\partial h, z} \overset{\sim}{\longleftarrow} E_{y,z},$$

since from the definition the existence of s_1 implies $E_{\partial h, z}$ is a trivial (G_1, G_0)-torsor. (Similarly for $E_{y, \partial k}$.) We have a similar calculation whenever $z' = z \, \partial k$, and the properties of the butterfly (plus the compatibility of $\times_1$ and $\times_2$) ensure we obtain a unique morphism

$$E_{y,z} \longrightarrow E_{y',z'},$$

which allows us to define $u(E)$ on more general objects by descent. The connection with the global version above is of course that $\mathrm{Hom}_{H_1, K_1}(Y, Z; E)_{(y,z)}$ reduces to $E_{y,z}$ when $(Y, y) = (H_1, y)$ and $(Z, z) = (H_1, z)$.

Proof of Theorem 7.1. We prove that u and v are quasi-inverses. To this end, recall that for a G_1-torsor P we have the isomorphism

$$(7.1) \qquad\qquad \mathrm{Hom}_{G_1}(G_1, P) \overset{\sim}{\to} P, \qquad m \mapsto m(e_{G_1}),$$

[Gir71, Proposition III 1.2.7]. This extends to (G_1, G_0)-torsors by assigning to $m \colon G_1 \to P$ the element $s(m(e_{G_1})) \in G_0$ (see [AN09, n. 4.4.2]). We apply this observation to both $v(u(E))$ and $u(v(F))$, where (E, s_1, s_2) is a butterfly with underlying biextension E and $F \colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$ is biadditive. In the first case, a point of the biextension $v(u(E))$ is given by a tuple $((y, z), f, x)$ with $(y, z) \in H_0 \times K_0$ and $x \in G_0$, such that

$$f \colon E_{y,z} \overset{\sim}{\longrightarrow} (G_1, x).$$

Considering the fiber over (y, z) we have the following chain of morphisms of (G_1, G_0)-torsors:

$$(7.2) \qquad\qquad v(u(E))_{y,z} \overset{\sim}{\longrightarrow} \mathrm{Hom}_{G_1}(G_1, E_{y,z}) \overset{\sim}{\longrightarrow} E_{y,z},$$

where the first arrow sends (f, x) to f^{-1}, and the projection $(f, x) \to x$, namely the equivariant section, to $\jmath \circ f^{-1}$. Therefore we have obtained an isomorphism of biextensions $v(u(E)) \overset{\sim}{\to} E$, by virtue of the result quoted at the beginning. This is (tautologically) an isomorphism of butterflies. For this, consider the composite isomorphism $I_{\mathscr{G}} = (G_1, e) \overset{\sim}{\to} E_{\partial h, z} \overset{\sim}{\to} (G_1, x)$ (resp. $I_{\mathscr{G}} = (G_1, e) \overset{\sim}{\to} E_{y, \partial k} \overset{\sim}{\to} (G_1, x)$) which provides the trivialization of the pullback $v(u(E))$ to $H_1 \times K_0$ (resp. $H_0 \times K_1$), as in the proof of Lemma 7.3. The resulting identifications of the pair (f, x) with an element $g \in G_1$ is clearly compatible via the chain (7.2), with the trivialization $(G_1, e) \overset{\sim}{\to} E_{y,z}$.

In the second case, $v(F) = (H_0 \times K_0) \times_{\mathscr{G}} G_0$ and $u(v(F))$ is the biadditive morphism that assigns to $(y, z) \in H_0 \times K_0$ the (G_1, G_0)-torsor

$$v(F)_{y,z} = \{(f, x) | F(\pi(y), \pi(z)) \overset{f}{\longrightarrow} (G_1, x)\} \overset{\sim}{\longrightarrow} F(\pi(y), \pi(z)),$$

where the isomorphism on the right is by way of (7.1) and loc. cit. The pullback $v(F)$ has the required trivializations by Lemma 7.3, which are evidently compatible with $F(\pi(\partial h), \pi(z)) \overset{\sim}{\to} (G_1, e)$ and $F(\pi(y), \pi(\partial k)) \overset{\sim}{\to} (G_1, e)$, showing that the above isomorphism holds for general objects and is functorial, proving there is an isomorphism $u(v(F)) \simeq F$. $\qquad\square$

7.6 Theorem (Theorem 7.1, symmetric version). *Let the monoidal structure of $\mathscr{G}$ be symmetric. Then the equivalence in Theorem 7.1 extends to one of group-like groupoids or stacks.*

Proof (Sketch). We need only check the trivializations. Suppose F, F' and E, E' are the corresponding butterflies. In particular, let s_1, s_2 and s_1', s_2' be trivialization morphisms of E and E', respectively. We claim the pairs (s_1, s_1') and (s_2, s_2') give the two trivialization morphisms turning the biextension $E \wedge^G E'$ into a butterfly.

The restriction condition 1 in Definition 6.1 is immediate. The compatibility condition 2 can verified by a simple computation, by using the definition of the partial multiplications from the proof of Proposition 5.1 and Lemma 1.2. $\qquad\square$

8 Commutative structures

With the equivalence between $u\colon \mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet) \to \mathbf{Hom}(\mathcal{H}, \mathcal{K}; \mathcal{G})$ at our disposal, we can discuss commutativity conditions for biextensions (cf. Remark 3.4 above).

8.1 Definition. Let $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{G}$ all be equipped with a braiding structure. Then we say that $(E, s_1, s_2) \in \mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet)$ (or simply E by abuse of language) is **braided** if the biadditive morphism $u(E)$ is braided in the sense of sect. 2.

In order to turn the definition into actual diagrams, we use the explicit variance of the biextension E, Remark 7.5. Here the point of Remark 1.1 becomes relevant. Whereas the morphism $u(E)\colon \mathcal{H} \times \mathcal{K} \to \mathcal{G}$ is covariant in each variable, the biextension E itself is a (G_0, G_1)-*torsor* over $H_0 \times K_0$, and $\mathrm{Tors}(G_1, G_0)$ is anti-equivalent to $\mathcal{G}$.

Using the trivialization $s_1\colon H_1 \times K_0 \to E$ and $s_2\colon H_0 \times K_1 \to E$, and the braidings $\{-,-\}_H$ and $\{-,-\}_K$ for the presentations, we have morphisms η^1 and η^2:

$$\eta^1_{y,y';z}\colon E_{y'y,z} \longrightarrow E_{yy',z} \qquad\qquad \eta^2_{y;z,z'}\colon E_{y,z'z} \longrightarrow E_{y,zz'}$$
$$e \longmapsto e \times_1 s_1(\{y',y\}_H, z) \qquad\qquad e' \longmapsto e' \times_2 s_2(y, \{z',z\}_K).$$

Thus, for $y, y' \in H_0$ and $z, z' \in K_0$ the following diagrams must commute:

(8.1)

$$
\begin{array}{ccc}
E_{y,z} \wedge^{G_1} E_{y',z} & \xrightarrow{\gamma^1_{y,y';z}} & E_{yy',z} \\
{\scriptstyle c_{y,y';z}}\big\downarrow & & \big\uparrow{\scriptstyle \eta^1_{y,y';z}} \\
E_{y',z} \wedge^{G_1} E_{y,z} & \xrightarrow[\gamma^1_{y',y;z}]{} & E_{y'y,z}
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
E_{y,z} \wedge^{G_1} E_{y,z'} & \xrightarrow{\gamma^2_{y;z,z'}} & E_{y,zz'} \\
{\scriptstyle c_{y;z,z'}}\big\downarrow & & \big\uparrow{\scriptstyle \eta^2_{y;z,z'}} \\
E_{y,z'} \wedge^{G_1} E_{y,z} & \xrightarrow[\gamma^2_{y;z',z}]{} & E_{y,z'z}
\end{array}
$$

In both diagrams the vertical arrow to the left comes from the braiding in $\mathcal{G}$. The two vertical arrows on the right express the functoriality with respect to the braiding structures of $H_\bullet$ and $K_\bullet$.

Alternatively, the directions in the right vertical arrows of both diagrams in (8.1) can be restored if we interpret them as morphism in $\mathrm{Tors}(G_1, G_0)$ arising from the morphisms

$$(H_0, yy') \longrightarrow (H_0, y'y) \qquad\text{and}\qquad (K_0, zz') \longrightarrow (K_0, z'z)$$

in $\mathrm{Tors}(H_1, H_0)$ and $\mathrm{Tors}(K_1, K_0)$, respectively. This requires writing η^1 and η^2 (going in the opposite direction) in terms of

$$s_1(\{y',y\}_H^{-1}, z) \qquad\text{and}\qquad s_2(y, \{z',z\}_K^{-1}).$$

Using Lemma 1.2 to express the braiding morphisms $c_{y,y';z}$ and $c_{y;z,z'}$ we arrive at the expressions, valid for $e \in E_{y,z}$, $e' \in E_{y',z}$:

(8.2a)
$$e' \times_1 e = \left(e \times_1 e' \times_1 s_1(\{y',y\}_H^{-1}, z)\right)\{\jmath(e), \jmath(e')\}_G;$$

and for $e \in E_{y,z}$, $e' \in E_{y,z'}$:

(8.2b)
$$e' \times_2 e = \left(e \times_2 e' \times_2 s_2(y, \{z',z\}_K^{-1})\right)\{\jmath(e), \jmath(e')\}_G.$$

It follows that the (ordinary) butterfly corresponding to each variable must be braided in the sense of [AN09, §7.4.1].

Theorems 7.1 and 7.6 specialize to this situation.

8.2 Theorem (Theorems 7.1, 7.6 fully commutative case). *Let $\mathcal{K}$, $\mathcal{H}$, and $\mathcal{G}$ be all braided and have presentations by braided crossed modules. The equivalence u of Theorem 7.1 restricts to an equivalence*

$$u\colon \mathbf{Biext}^b(H_\bullet, K_\bullet; G_\bullet) \xrightarrow{\sim} \mathbf{Hom}^b(\mathcal{H}, \mathcal{K}; \mathcal{G})$$

of groupoids. Furthermore, if $\mathcal{G}$ is braided symmetric, u (from Theorem 7.6) becomes an equivalence of group-like groupoids. Similar statements hold for $\mathrm{Biext}^b(-,-;-)$ and $\mathrm{Hom}^b(-,-;-)$. $\qquad\square$

In the following we will restrict to this commutative situation, and consistently drop the superscript "b" from the notation.

As an application of these additional assumptions, we can discuss whether $\mathbf{Biext}(H_\bullet, K_\bullet; G_\bullet)$ and $\mathbf{Hom}(\mathscr{H}, \mathscr{K}; \mathscr{G})$ are biadditive in their variables. We have observed that regardless the commutativity assumptions (but keeping $\mathscr{G}$ braided) they always are biadditive in the third variable (the same conclusion holds for $\mathrm{Biext}(H_\bullet, K_\bullet; G_\bullet)$ and $\mathrm{Hom}(\mathscr{H}, \mathscr{K}; \mathscr{G})$). More precisely, we have

8.3 Proposition. *The groupoid* $\mathbf{Hom}(-, -; -)$ *is biadditive in all variables for symmetric group-like stacks.*

Proof. We can use the same argument as [Bre83, §1.2]. Specifically, consider the three (additive) functors $d_i \colon \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ (d_1 is additive since $\mathscr{H}$ is braided). Let $F \colon \mathscr{H} \times \mathscr{K} \to \mathscr{G}$ be a biadditive morphism, and let $E = v(F)$ be the corresponding butterfly. Then

$$\lambda^1 \colon d_2^* F d_0^* F \longrightarrow d_1^* F$$

is a morphism in $\mathbf{Hom}(\mathscr{H} \times \mathscr{H}, \mathscr{K}; \mathscr{G})$. As in loc. cit. this follows from the compatibility between λ^1 and λ^2 and the fact that λ^1 is an additive morphism thanks to the (symmetric) braiding. In view of the equivalence $\mathbf{Hom}(-, -; -) \simeq \mathbf{Biext}(-, -; -)$, the above morphism corresponds to a morphism of biextensions

$$\gamma^1 \colon d_2^* E \wedge^{G_1} d_0^* E \longrightarrow d_1^* E$$

which is in fact a morphism of butterflies in $\mathbf{Biext}(H_\bullet \times H_\bullet, K_\bullet; G_\bullet)$, after one checks the trivializations. From Proposition 5.1 we see that the braiding must be symmetric. Now, for additive morphisms $R, S \colon \mathscr{H}' \to \mathscr{H}$, we obtain a morphism

$$(R \times I)^* F (S \times I)^* F \longrightarrow ((RS) \times I)^* F$$

in $\mathbf{Hom}(\mathscr{H}', \mathscr{K}; \mathscr{G})$, showing biadditivity in the first variable. The second variable is treated analogously. $\square$

8.4 Remark. In the proof, if E_R and E_S are the butterflies corresponding to $R, S \colon \mathscr{H}' \to \mathscr{H}$, we obtain the morphism of butterflies in $\mathbf{Biext}(H_\bullet', K_\bullet; G_\bullet)$ is

$$\left((E_R \times I) \times_{H_0}^{H_1} E\right) \wedge^{G_1} \left((E_S \times I) \times_{H_0}^{H_1} E\right) \longrightarrow \left((E_R \wedge^{H_1} E_S) \times I\right) \times_{H_0}^{H_1} E$$

expressing the biadditivity in the first variable at the level of biextensions. Here I is a shorthand for the diagram corresponding to the identity morphism. (This kind of compositions is systematically studied in the next Part II.)

Part II

9   Multiextensions and compositions

The generalizations of the previous notions of biextension and biadditive morphism to the case of n-variables is straightforward [see Gro72, §2.10.2; Bre99, §7]. Let $(G, \Pi, \partial, \{-, -\})$ be a braided crossed module. A **multiextension,** or n-**extension,** of $(H_1, \ldots H_n)$ by (G, Π) is a $(G, \Pi)_{H_1 \times \cdots \times H_n}$-torsor E over $H_1 \times \cdots \times H_n$ equipped with n partial multiplication laws $\times_1, \ldots, \times_n$, plus a compatibility relation of the type (1.5) for each pair $(\times_i, \times_j)$.

Lemma 3.6 remains valid, as well as the analogs of Propositions 4.1 and 4.3. In fact, the notion of butterfly can be extended to this case. Let us consider $\mathscr{H}_1, \ldots, \mathscr{H}_n$ and $\mathscr{G}$, the latter equipped with a braiding as usual. There is an evident notion of n-additive functor $F \colon \mathscr{H}_1 \times \cdots \times \mathscr{H}_n \to \mathscr{G}$, which can be defined by an appropriate generalization of Definition 2.1. If we suppose that each $\mathscr{H}_i$ has a presentation $\mathscr{H}_i \simeq [H_{i,1} \to H_{i,0}]^\sim$, we can define a butterfly (or, more precisely, an n-butterfly) from $H_{1,\bullet} \times \cdots \times H_{n,\bullet}$ to $G_\bullet$ as an n-extension E of $(H_{1,0} \times \cdots \times H_{n,0})$ by $G_\bullet$ equipped with n trivializations $s_i \colon H_{1,0} \times \cdots \times H_{i,1} \times \cdots \times H_{n,0} \to E$, each satisfying the conditions of Definition 6.1, with the obvious modifications. Theorem 7.1 and its symmetric variant 7.6 generalize to this case. Let us state this independently for future reference. Denoting by $\mathbf{MExt}$ the groupoid of n-butterflies, we have

9.1 Theorem. *There is an equivalence of (pointed) groupoids:*

$$u : \mathbf{MExt}(H_{1,\bullet}, \ldots, H_{n,\bullet}; G_\bullet) \xrightarrow{\sim} \mathbf{Hom}(\mathcal{H}_1, \ldots, \mathcal{H}_n; \mathcal{G}),$$

sending a multi-extension E, to the n-additive functor $u(E)$ that to the object $(y_1, \ldots, y_n)$ of $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, where each y_i is in the essential image of $H_{i,0}$, assigns the (G, Π)-torsor $E_{y_1, \ldots, y_n}$. It is an equivalence of group-like groupoids whenever $\mathcal{G}$ is symmetric. □

Once again, the axioms of Definition 6.1 ensure this is compatible with the descent, ensuring $u(E)$ is a well defined morphism. Just like for biextensions, the equivalence is compatible with localization, with the stacks MExt and Hom in place of the global groupoids.

Multi-additive functors can be composed in the following way. Let $\mathcal{G}$, $\mathcal{H}_1, \ldots, \mathcal{H}_n$, $\mathcal{K}_{i,1}, \ldots, \mathcal{K}_{i,m_i}$, $i = 1, \ldots, n$, be group-like stacks, with $\mathcal{G}$ and $\mathcal{H}_1, \ldots, \mathcal{H}_n$ braided. Let $F \in \mathbf{Hom}(\mathcal{H}_1, \ldots, \mathcal{H}_n; \mathcal{G})$, and for $i = 1, \ldots, n$ let $G_i \in \mathbf{Hom}(\mathcal{K}_{i,1}, \ldots, \mathcal{K}_{i,m_i}; \mathcal{H}_i)$. Then if $x_{1,1}, \ldots, x_{n,m_n}$, collectively denoted $x_1, \ldots, x_m$, are objects of $\mathcal{K}_{1,1}, \ldots, \mathcal{K}_{n,m_n}$, define $F(G_1, \ldots, G_n)$ as usual by

$$(9.1) \qquad F(G_1, \ldots, G_n)(x_1 \ldots, x_m) := F(G_1(x_1, \ldots), \ldots, G_n(\ldots, x_m)).$$

9.2 Proposition. *The composition defined in (9.1) assigns to the tuple $(F, G_1, \ldots, G_n)$ a well defined object $F(G_1, \ldots, G_n)$ of $\mathbf{Hom}(\mathcal{K}_{1,1}, \ldots, \mathcal{K}_{n,m_n}; \mathcal{G})$. This composition is associative.*

An identical statement holds with $\mathbf{Hom}$ replaced by Hom.

Proof. The only thing to check is that the composition $F(G_1, \ldots, G_n)$ satisfy the conditions (2.1) for each pair (i, j) of indices within the list $\{1, \ldots, m_1 + m_2 + \cdots + m_n\}$, where m_k is the arity of G_k. There are two cases depending on whether the variables corresponding to the pair (i, j) belongs to the same "slot," say relative to G_k, or when i and j fall into two different slots, relative to G_k and G_l, with $k \neq l$. We now indicate the main points of the verification, leaving the easy task of writing the complete diagrams to the reader.

In the former case, the mechanics of the verification are completely captured by considering the values $n = 1$, $m_1 = 2$. First we write the pentagonal diagram corresponding to (2.1) for $F(G)$. In it, we use the functoriality of F to reduce the arrow

$$(F(G(x,y))F(G(x',y))) (F(G(x,y'))F(G(x',y'))) \longrightarrow (F(G(x,y))F(G(x,y'))) (F(G(x',y))F(G(x',y')))$$

to the arrow

$$F((G(x,y)G(x',y))(G(x,y')G(x',y'))) \longrightarrow F((G(x,y)G(x,y'))(G(x',y)G(x',y'))),$$

and then use the fact that G itself satisfies (2.1). For the latter case, the general situation is captured by considering $n = 2$, $m_1 = m_2 = 1$, so we need to write the diagram (2.1) for $F(G_1, G_2)$. For this, the interchange law for F gives us a unique morphism from

$$(F(G_1(x), G_2(y)) F(G_1(x'), G_2(y))) (F(G_1(x), G_2(y')) F(G_1(x'), G_2(y')))$$

to

$$F(G_1(x) G_1(x'), G_2(y) G_2(y')).$$

Now the functoriality of F gives the morphism

$$F(G_1(x) G_1(x'), G_2(y) G_2(y')) \longrightarrow F(G_1(xx'), G_2(yy'))$$

by way of the square

$$
\begin{array}{ccc}
F(G_1(x) G_1(x'), G_2(y) G_2(y')) & \longrightarrow & F(G_1(x) G_1(x'), G_2(yy')) \\
\downarrow & & \downarrow \\
F(G_1(xx'), G_2(y) G_2(y')) & \longrightarrow & F(G_1(xx'), G_2(yy'))
\end{array}
$$

The statement about associativity is immediate. □

10   Compositions of n-butterflies

From Theorem 9.1, the multilinear functors can be expressed in terms of multi-extensions, and from Proposition 9.2 multilinear functors can be composed. Hence we expect an analogous composition exists for multi-extensions. In this section we provide a construction of this composition, generalizing the composition for single butterflies found in [AN09, §5.1].

For $i = 1,\dots,n$, and integers $j_i = 1,\dots,m_i$, let $G_\bullet$, $H_{1,\bullet},\dots H_{n,\bullet}$ and $K_{i,1,\bullet},\dots,K_{i,m_i,\bullet}$ crossed modules, of which $G_\bullet$, $H_{1,\bullet},\dots H_{n,\bullet}$ are assumed to be braided. Let E be a braided n-butterfly from $H_{1,\bullet} \times \cdots \times H_{n,\bullet}$ to $G_\bullet$, and for $i = 1,\dots,n$ let F_i be an m_i-butterfly from $K_{i,1,\bullet} \times \cdots \times K_{i,m_i,\bullet}$ to $H_{i,\bullet}$. In the following we let

$$P = (F_1 \times \cdots \times F_n) \bigtimes_{H_{1,0}\times\cdots\times H_{n,0}} E,$$

be the $(H_{1,1} \times \cdots \times H_{n,1})$-equivariant (G_1, G_0)-torsor *over $F_1 \times \cdots \times F_n$*.

10.1 Theorem. *The (G_1, G_0)-torsor P has a well defined quotient by $H_{1,1} \times \cdots \times H_{n,1}$, denoted*

$$(F_1 \times \cdots \times F_n) \overset{H_{1,1}\times\cdots\times H_{n,1}}{\underset{H_{1,0}\times\cdots\times H_{n,0}}{\bigtimes}} E,$$

which carries a structure of $m_1 + \cdots + m_n$-extension (=butterfly) from $K_{1,1,\bullet} \times \cdots \times K_{n,m_n,\bullet}$ to $G_\bullet$.

10.2 Definition. The composition $E(F_1,\dots,F_n)$ of E with $F_1,\dots,F_n$ is

$$E(F_1,\dots,F_n) := (F_1 \times \cdots \times F_n) \overset{H_{1,1}\times\cdots\times H_{n,1}}{\underset{H_{1,0}\times\cdots\times H_{n,0}}{\bigtimes}} E.$$

We call this composition the *juxtaposition product*, since it entails placing the butterflies wing-by-wing next to one another.

Proof of Theorem 10.1–Construction of the juxtaposition product. We will use element notation throughout. The procedure consists of several steps. The first is to form the fiber product exactly as in loc. cit.

$$P = (F_1 \times \cdots \times F_n) \bigtimes_{H_{1,0}\times\cdots\times H_{n,0}} E,$$

which as noted is an $(H_{1,1} \times \cdots \times H_{n,1})$-equivariant (G_1, G_0)-torsor *over $F_1 \times \cdots \times F_n$*, and to mod out the left and right actions of $H_{1,1} \times \cdots \times H_{n,1}$. Let us use the notation

$$(y_1,\dots,y_n) := (j_1(v_1),\dots j_n(v_n)) = p(u)$$

for the tuple in $H_{1,0} \times \cdots \times H_{n,0}$, where $j_i \colon F_i \to H_{i,0}$ for each $(H_{i,1}, H_{i,0})$-torsor F_i, $i = 1,\dots,n$.

Consider first the right action of elements of the form $(1,\dots,h_i,\dots,1)$, where $h_i \in H_{i,1}$. The action on P is given by sending a point $(v_1,\dots,v_n,u)$ to

$$(v_1,\dots,v_i h_i,\dots v_n, u \times_i s_i(y_1,\dots,h_i,\dots,y_n)),$$

For the left action, we have

$$(v_1,\dots,h_i v_i,\dots v_n, s_i(y_1,\dots,h_i,\dots,y_n) \times_i u) = (v_1,\dots,v_i h_i^{y_i},\dots v_n, u \times_i s_i(y_1,\dots,h_i^{y_i},\dots,y_n)),$$

which follows from Definition 6.1, adapted to the multi-variable case. These actions are compatible for $i \neq j$. Acting with h_i and then with h_j, assuming for example that $i < j$, we have

$$\big(u \times_i s_i(y_1,\dots,h_i,\dots,y_n)\big) \times_j s_j(y_1,\dots,y_i \partial h_i,\dots,h_j,\dots,y_n)$$
$$= \big(u \times_i s_i(y_1,\dots,h_i,\dots,y_n)\big) \times_j \big(s_j(y_1,\dots,y_i\dots,h_j,\dots,y_n) \times_i s_j(y_1,\dots,\partial h_i,\dots,h_j,\dots,y_n)\big),$$

where we used the multiplicativity property of s_j. Now, using the interchange property for $\times_i$ and $\times_j$, and the compatibility between the trivializations s_i and s_j, the right hand side becomes

$$\left(u \times_j s_j(y_1,\ldots,h_j,\ldots,y_n)\right) \times_i \left(s_i(y_1,\ldots,h_i,\ldots,y_n) \times_j s_i(y_1,\ldots,h_i,\ldots,\partial h_j,\ldots,y_n)\right),$$

which coincides with the action of h_j first, followed by that of h_i. This ensures that following formula for the right action of a generic point $(h_1,\ldots,h_n)$ is well defined:

$$(10.1) \quad \left(v_1\, h_1,\ldots v_n\, h_n, \left(\cdots\left(u \times_1 s_1(h_1,y_2,\ldots,y_n)\right) \times_2 s_2(y_1 \partial h_1, h_2,\ldots,y_n) \cdots\right) \times_n s_n(y_1 \partial h_1,\ldots,y_{n-1}\partial h_{n-1}, h_n)\right).$$

The reader will have no difficulty in writing the corresponding formulas for the left action. Since each F_i is an $(H_{i,1}, H_{i,0})$-torsor, the action is free.

Note also that the G_1-actions on P (both left and right) are compatible with the action of $H_{1,1} \times \cdots \times H_{n,1}$ thanks to the relations (6.4) in Lemma 6.2 (property 2). In particular, the G_1-actions happen by way of those on the last element of the tuple; denoting the class of a tuple by brackets we have:

$$[v_1,\ldots,v_n,u]\, g = [v_1,\ldots,v_n,u\, g], \qquad g\,[v_1,\ldots,v_n,u] = [v_1,\ldots,v_n,g\, u] = [v_1,\ldots,v_n,u\, g^{ju}].$$

Finally, the equivariant section from the quotient of P to G_0 is defined to be

$$(v_1,\ldots,v_n,u) \longmapsto j(u).$$

Next, we define $m_1 + \cdots + m_n$ partial product structures $\times_{i,j}$ for $i = 1,\ldots,n$ and $j = 1,\ldots,m_i$ as follows. Using our established index convention, let us use the notation

$$(z_{1,1},\ldots,z_{i,j},\ldots,z_{n,m_n}) \in K_{1,1,0} \times \cdots \times K_{i,j,0} \times \cdots \times K_{n,m_n,0}$$

for a point of the base of $P/(H_{1,1} \times \cdots \times H_{n,1})$. To begin with, consider the special case of two points $(v_1,\ldots,v_j,\ldots,v_n,u)$ and $(v_1,\ldots,v_j',\ldots,v_n,u')$ of P such that

$$p_i(v_i) = (z_{i,1},\ldots,z_{i,j},\ldots,z_{i,m_i}), \quad p_i(v_i') = (z_{i,1},\ldots,z_{i,j}',\ldots,z_{i,m_i}),$$

in $K_{i,j,0}$, for fixed i and $j \in \{1,\ldots,m_i\}$, everything else being equal. Define:

$$(10.2) \qquad (v_1,\ldots,v_j,\ldots,v_n,u) \times_{i,j} (v_1,\ldots,v_j',\ldots,v_n,u) := (v_1,\ldots,v_j \times_{i,j} v_j',\ldots,v_n,u \times_i u').$$

The symbol $\times_{i,j}$ on the right hand side of (10.2) denotes the jth product structure (within $j = 1,\ldots,m_i$) of F_i, and the resulting point of P projects onto the point $(z_{1,1},\ldots,z_{i,j}z_{i,j}',\ldots,z_{n,m_n})$.

In general, let us consider points $e = [v_1,\ldots,v_n,u]$ and $e' = [v_1',\ldots,v_n',u']$ of $P/(H_{1,1} \times \cdots \times H_{n,1})$ above $(z_{1,1},\ldots,z_{i,j},\ldots,z_{n,m_n})$ and $(z_{1,1},\ldots,z_{i,j}',\ldots,z_{n,m_n})$, respectively. (As before, only the $z_{i,j}$ and $z_{i,j}'$ coordinates are different.) For $j = 1,\ldots,n$, $j \neq i$, there exist unique $h_j \in H_{j,1}$ such that $v_j' = v_j h_j$. As a result,

$$[v_1',\ldots,v_n',u'] = [v_1,\ldots,v_i',\ldots,v_n,u''],$$

where u'' is related to u' by an application of (10.1). Then we define $e \times_{i,j} e'$ as the class

$$[v_1,\ldots,v_j \times_{i,j} v_j',\ldots,v_n,u \times_i u''],$$

which is computed using (10.2) above. We must show that this is independent of the various choices involved through the use of (10.1). The computation is quite elaborate, but otherwise not illuminating nor eventful, therefore we omit it. We also omit the easy verification that each of these partial multiplication laws is associative.

On the other hand, to claim that we have constructed a genuine multi-extension, we must prove the partial multiplication laws just defined obey pairwise compatibility (interchange) laws. This we prove explicitly. Like in the proof of Proposition 9.2, there are two distinct cases to address, depending on whether

the two partial product laws have the same first index. The easiest is when they do not, so we treat it first. Thus, let $i \neq k \in \{1, \ldots n\}$. For brevity let $Q = P/(H_{1,1} \times \cdots \times H_{n,1})$ and consider:

$$e_{i,j} = [v_1, \ldots, v_i, \ldots, v_k, \ldots, v_n, u_i] \in Q_{z_{1,1}, \ldots, z_{i,j}, \ldots, z_{k,l}, \ldots, z_{n,m_n}}$$

$$e'_{i,j} = [v_1, \ldots, v'_i, \ldots, v_k, \ldots, v_n, u'_i] \in Q_{z_{1,1}, \ldots, z'_{i,j}, \ldots, z_{k,l}, \ldots, z_{n,m_n}}$$

$$e_{k,l} = [v_1, \ldots, v_i, \ldots, v'_k, \ldots, v_n, u_k] \in Q_{z_{1,1}, \ldots, z_{i,j}, \ldots, z'_{k,l}, \ldots, z_{n,m_n}}$$

$$e'_{k,l} = [v_1, \ldots, v'_i, \ldots, v'_k, \ldots, v_n, u'_k] \in Q_{z_{1,1}, \ldots, z'_{i,j}, \ldots, z'_{k,l}, \ldots, z_{n,m_n}}.$$

We first compute the products with $\times_{i,j}$:

$$e_{i,j} \times_{i,j} e'_{i,j} = [v_1, \ldots, v_i \times_{i,j} v'_i, \ldots, v_k, \ldots, v_n, u_i \times_i u'_i] \qquad e_{k,l} \times_{i,j} e'_{k,l} = [v_1, \ldots, v_i \times_{i,j} v'_i, \ldots, v'_k, \ldots, v_n, u_k \times_i u'_k].$$

Then

$$(e_{i,j} \times_{i,j} e'_{i,j}) \times_{k,l} (e_{k,l} \times_{i,j} e'_{k,l}) = [v_1, \ldots, v_i \times_{i,j} v'_i, \ldots, v_k \times_{k,l} v'_k, \ldots, v_n, (u_i \times_i u'_i) \times_k (u_k \times_i u'_k)]$$

$$= [v_1, \ldots, v_i \times_{i,j} v'_i, \ldots, v_k \times_{k,l} v'_k, \ldots, v_n, (u_i \times_k u_k) \times_i (u'_i \times_k u'_k)\{J(u'_i), J(u_k)\}^{-J(u'_k)}]$$

$$= (e_{i,j} \times_{k,l} e_{k,l}) \times_{i,j} (e'_{i,j} \times_{k,l} e'_{k,l})\{J(u'_i), J(u_k)\}^{-J(u'_k)},$$

where in the second line we have used the interchange law for the ith and k^{th} product laws in E, Proposition 3.2.

The other case is the one with the same first index, say $\times_{i,j}$ and $\times_{i,l}$, for $j \neq l$. In this situation we consider instead:

$$e_{i,j} = [v_1, \ldots, v_i, \ldots, v_n, u_i] \in Q_{z_{1,1}, \ldots, z_{i,j}, \ldots, z_{i,l}, \ldots, z_{n,m_n}}$$

$$e'_{i,j} = [v_1, \ldots, v'_i, \ldots, v_n, u'_i] \in Q_{z_{1,1}, \ldots, z'_{i,j}, \ldots, z_{i,l}, \ldots, z_{n,m_n}}$$

$$e_{i,l} = [v_1, \ldots, w_i, \ldots, v_n, t_i] \in Q_{z_{1,1}, \ldots, z_{i,j}, \ldots, z'_{i,l}, \ldots, z_{n,m_n}}$$

$$e'_{i,l} = [v_1, \ldots, w'_i, \ldots, v_n, t'_i] \in Q_{z_{1,1}, \ldots, z'_{i,j}, \ldots, z'_{i,l}, \ldots, z_{n,m_n}}.$$

Proceeding in an analogous way we first get:

$$e_{i,j} \times_{i,j} e'_{i,j} = [v_1, \ldots, v_i \times_{i,j} v'_i, \ldots, v_k, \ldots, v_n, u_i \times_i u'_i], \qquad e_{k,l} \times_{i,j} e'_{k,l} = [v_1, \ldots, w_i \times_{i,j} w'_i, \ldots, v_n, t_k \times_i t'_k].$$

And then:

$$(e_{i,j} \times_{i,j} e'_{i,j}) \times_{i,l} (e_{i,l} \times_{i,j} e'_{i,l}) = [v_1, \ldots, (v_i \times_{i,j} v'_i) \times_{i,l} (w_i \times_{i,j} w'_i), \ldots, v_n, (u_i \times_i u'_i) \times_i (t_i \times_i t'_i)]$$

$$= [v_1, \ldots, (v_i \times_{i,j} v'_i) \times_{i,l} (w_i \times_{i,j} w'_i)\{J_i(w_i), J_i(v'_i)\}_{H_i}^{-J_i(w'_i)}, \ldots, v_n,$$

$$(u_i \times_i u'_i) \times_i (t_i \times_i t'_i) \times_i s_i(y_1, \ldots, \{J_i(w_i), J_i(v'_i)\}_{H_i}^{-J_i(w'_i)}, \ldots, y_n)]$$

$$= [v_1, \ldots, (v_i \times_{i,l} w_i) \times_{i,j} (v'_i \times_{i,l} w'_i), \ldots, v_n, (u_i \times_i t_i) \times_i (u'_i \times_i t'_i)\{J(u'_i), J(t_i)\}_G^{-J(t'_i)}],$$

where in the intermediate equality we used the the action by an element of $H_{i,1}$, whereas in the last one the interchange law (within the multi-extension F_i) and (8.2) were used. But the last line equals

$$(e_{i,j} \times_{i,l} e_{i,l}) \times_{i,j} (e'_{i,j} \times_{i,l} e'_{i,l})\{J(u'_i), J(t_i)\}_G^{-J(t'_i)},$$

as wanted.

The final step is to show that Q, in addition to being a multi-extension of $K_{1,1,0} \times \cdots \times K_{n,m_n,0}$ by $G_\bullet$, is in fact a butterfly, that is, it carries a trivialization for each of its pullbacks to $K_{1,1,0} \times \cdots \times K_{i,j,1} \times \cdots \times K_{n,m_n,0}$. Such trivializations can be defined as follows.

For each $i = 1, \ldots, n$ and for each $j = 1, \ldots, m_i$, let $s_{i,j}$ be the trivialization $s_{i,j}: K_{i,1,0} \times \cdots \times K_{i,j,1} \times \cdots \times K_{i,m_i,0} \to F_i$. Observe that the fiber of P above $(v_1, \ldots, s_{i,j}(z_{i,1}, \ldots, k_{i,j}, \ldots, z_{i,m_i}), \ldots, v_n) \in F_1 \times \cdots \times F_i \times \cdots \times F_n$ (where

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v_k is a point of F_k, for $k \neq i$) can be identified with G_1. This is because that point maps via $(\jmath_1, \ldots, \jmath_n)$ to $(y_1, \ldots, 1, \ldots, y_n)$, and the fiber of E above it is identified with G_1. (In turn, this follows from the fact that biextensions in particular, and multi-extensions in general, are canonically trivialized over identity sections, see sect. 3.) It follows that the fiber of Q over $(z_{1,1}, \ldots, \partial k_{i,j}, \ldots, z_{n,m_n})$ is similarly identified with G_1. Thus, we obtain a trivializing section $\hat{s}_{i,j}$ of the pullback of Q to $K_{1,1,0} \times \cdots \times K_{i,j,1} \times \cdots \times K_{n,m_n,0}$ by sending the point $(z_{1,1}, \ldots, k_{i,j}, \ldots, z_{n,m_n})$ to $[v_1, \ldots, s_{i,j}(z_{i,1}, \ldots, k_{i,j}, \ldots, z_{i,m_i}), \ldots, v_n, e_i]$, where e_i is the unit (central) section of the restriction of E to $H_{1,0} \times \ldots \{1\} \times \cdots \times H_{n,0}$. Observe this does not depend on the choice of $v_j \in F_j$, for $j \neq i$, since F_j/H_j is just $(z_{j,1}, \ldots, z_{j,m_j})$. Note that $\jmath \colon Q \to G_0$ composed with any of $\hat{s}_{i,j}$ is identically equal to 1, since so is the result of applying $\jmath_E \colon E \to G_0$ to e_i. The conditions (Compatibility and Restriction) of Definition 6.1 are easy to verify and so this task is left to the reader.

This ends the proof. $\qquad\square$

10.3 Proposition. *The composition in Definition 10.2 gives rise to a well-defined composition functor*

$$\mathbf{MExt}(K_{1,1,\bullet}, \ldots; H_{1,\bullet}) \times \cdots \times \mathbf{MExt}(\ldots, K_{n,m_n,\bullet}; H_{n,\bullet}) \times \mathbf{MExt}(H_{1,\bullet}, \ldots, H_{n,\bullet}; G_\bullet) \longrightarrow \mathbf{MExt}(K_{1,1,\bullet}, \ldots, K_{n,m_n,\bullet}; G_\bullet),$$

where we are using the same convention for the indices as above and in sect. A. Same with MExt *in place of* **MExt**.

Proof. We need only check that if $f \colon E \to E'$ and $g_i \colon F_i \to F_i'$ are butterfly isomorphisms, then we get a well defined isomorphism of $k = m_1 + \cdots + m_n$-butterflies

$$f(g_1, \ldots, g_n) \colon E(F_1, \ldots, F_n) \longrightarrow E'(F_1', \ldots, F_n').$$

This is easy to check from the construction of the juxtaposition product provided above. $\qquad\square$

Associativity only holds up to isomorphism. This is due to the same phenomenon, ultimately a manifestation of the same lack of strict associativity for fiber products, observed in the unary case [see AN09; Noo05]. We omit the proof.

10.4 Proposition. *The composition in Definition 10.2 is associative up to coherent isomorphism.* $\qquad\square$

11 The bi-multicategories of group-like stacks and crossed modules

By using Proposition 9.2 we can rely on the framework recalled in the technical section A to promote the 2-category of braided group-like stacks to a 2-multicategory. In particular, by putting together all

$$\mathbf{Hom}(\mathscr{H}^n; \mathscr{G}) = \mathbf{Hom}(\underbrace{\mathscr{H}, \ldots, \mathscr{H}}_{n-\text{times}}; \mathscr{G}),$$

we can define categories $\{\mathscr{H}, \mathscr{G}\}$. We ambiguously use the same symbol to denote the stack obtained using Hom in place of **Hom**, as which version will be in use will be clear from the context. We immediately obtain:

11.1 Proposition. *For group-like stacks $\mathscr{K}$, $\mathscr{H}$, and $\mathscr{G}$, we have the assembly map*

$$(11.1) \qquad\qquad \bigcirc \colon \{\mathscr{H}, \mathscr{G}\} \wr \{\mathscr{K}, \mathscr{H}\} \longrightarrow \{\mathscr{K}, \mathscr{G}\}$$

which to the object $(F; G_1, \ldots, G_n)$ assigns the composition $F(G_1, \ldots, G_n)$.

In particular, $M\mathscr{G} \coloneqq \{\mathscr{G}, \mathscr{G}\}$ is a monoid for this composition, whenever $\mathscr{G}$ is braided, with identity object given by (id;) *(the empty string in the second slot).* $\qquad\square$

11.2 Remark. As alluded in sect. A, the formalism includes the symmetry structure, which makes it slightly more general than needed in the sequel. All objects of interest shall have augmentations factoring through the discrete subcategory **N** of **S**, which amounts to ignoring the permutation structure and hence symmetry conditions.

From Propositions 10.3 and 10.4 and the formalism of sect. A the bicategory of braided crossed modules of **T** equipped with butterflies as morphisms [see AN09, Theorem 5.3.6] is promoted to a genuine bi-multicategory. In particular, we obtain objects $\{H_\bullet, G_\bullet\}$ and an analog of Proposition 11.1 with assembly map

$$(11.2) \qquad \bigcirc \colon \{H_\bullet, G_\bullet\} \wr \{K_\bullet, H_\bullet\} \longrightarrow \{K_\bullet, G_\bullet\}$$

for braided crossed modules $K_\bullet$, $H_\bullet$, and $G_\bullet$.

The equivalence between the 2-category $\mathfrak{BGrSt}$ of (braided) group-like stacks and the bicategory $\mathfrak{BXMod}$ of (braided) crossed modules [cf. AN09, Theorem 5.3.6] lifts to the present case.

11.3 Theorem. *There is an equivalence of bi-multicategories*

$$\mathrm{M}a \colon \mathrm{M}\mathfrak{BXMod} \xrightarrow{\sim} \mathrm{M}\mathfrak{BGrSt}$$

induced by the associated stack functor.

Proof. Recall that a functor $F \colon M\mathbb{C} \to M\mathbb{D}$ between bi-multicategories is an equivalence if for each tuple $(x; y_1, \ldots, y_n)$ of objects of $M\mathbb{C}$ the functor $F_{(x; y_1, \ldots, y_n)} \colon \mathbf{Hom}_{\mathbb{C}}(y_1, \ldots, y_n; x) \to \mathbf{Hom}_{\mathbb{D}}(F(y_1), \ldots, F(y_n))$ is an equivalence of categories, and the underlying ordinary homomorphism $[F]_1 \colon \mathbb{C} \to \mathbb{D}$ is essentially surjective.[8]

The associate stack functor at the level of the underlying bicategories $a \colon \mathfrak{BXMod} \to \mathfrak{BGrSt}$ is an equivalence, hence, in particular, essentially surjective. The equivalence is proved in [AN09, Theorem 5.3.6] without the braiding hypothesis, however Theorem 4.3.1 in loc. cit. applies to the braided case as well by §7, ibid., and presentations by braided crossed modules are easily obtained by using Lemma 1.2. Thus essential surjectivity follows.

The equivalence at the level of the multi-arrow categories directly follows from Theorem 9.1. Finally we check that the functor u of Theorem 9.1 preserves the (horizontal) compositions up to coherent isomorphism. Like the single variable case, this follows from the fiber product construction of the quasi-inverse to u. □

11.4 Remark. The situation becomes decidedly more pleasant if we confine ourselves to symmetric objects. We can consider a symmetric variant of the above based on the bicategories $\mathfrak{SXMod}$ and $\mathfrak{SGrSt}$. The equivalence in Theorem 11.3 restricts to between these new entities:

$$\mathrm{M}a \colon \mathrm{M}\mathfrak{SXMod} \xrightarrow{\sim} \mathrm{M}\mathfrak{SGrSt},$$

which, by Theorem 9.1, both ought to be regarded as multicategories enriched over symmetric group-like groupoids. Furthermore, using the version of $\{-, -\}$ based on Hom, $\mathfrak{SGrSt}$ becomes closed. Indeed one has the analog of [Kel72, Theorem 2], in that

$$\mathrm{Hom}_{\mathfrak{C}}(\mathscr{K} \wr \mathscr{H}, \mathscr{G}) \simeq \mathrm{Hom}_{\mathfrak{C}}(\mathscr{K}, \{\mathscr{H}, \mathscr{G}\}),$$

where $\mathfrak{C} = \mathfrak{SGrSt}$ (or even $\mathfrak{C} = \mathfrak{SGrSt}/\mathbf{S}$ using extraordinary structure briefly discussed in sect. A—see [Kel72] for all the details). A proof can be obtained along the same lines, proceeding from Theorem 9.1 and sect. A. We will not pursue this further.

Part III

12 Bimonoidal structures and weakly categorical rings

For a braided stack $\mathscr{R}$, an additive bifunctor $m \colon \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ gives a binary law which is automatically distributive with respect to the monoidal structure of $\mathscr{R}$ by virtue of (2.1) in Definition 2.1. If m is itself part of a monoidal structure, that is, there exists $\mu \colon m(m, \mathrm{id}) \Rightarrow m(\mathrm{id}, m)$ satisfying the standard pentagon identity, then $\mathscr{R}$ is said to be bimonoidal. In fact it satisfies the axioms of a categorical ring in the usual sense [JP07], except possibly the requirement for the underlying group-like structure of $\mathscr{R}$ to be symmetric.

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[8] $[F]_1$ is the homomorphism we obtain by restricting F to arrows of arity equal to one.

From sections 9 and A, the binary operation m and id are objects of $\mathrm{M}\mathscr{R} = \{\mathscr{R},\mathscr{R}\}$, and $\mu\colon m(m,\mathrm{id}) \Rightarrow m(\mathrm{id},m)$ is a morphism of $\mathrm{M}\mathscr{R}$, with the pentagon identity expressing the equality between two morphisms thereof. Rather than using the classical, "biased," definition of the monoidal structure, it is more convenient to exploit the multi-categorical structure comprising all the multilinear functors. Now, biased and unbiased definitions yield equivalent structures [Lei04], hence we can simply define it in terms of the multicategory $\mathrm{M}\mathfrak{B}\mathfrak{Gr}\mathfrak{St}$.

12.1 Definition. Let $\mathscr{R}$ be a braided stack. A *(weak) categorical ring structure* on $\mathscr{R}$ is a lax monoidal functor $(t,\theta)\colon \mathbf{N} \to \mathrm{M}\mathscr{R}$. A *(weak) ring-like stack* is a braided stack $\mathscr{R}$ equipped with such a structure, namely a pseudo-monoid in the 2-multicategory $\mathrm{M}\mathfrak{B}\mathfrak{Gr}\mathfrak{St}$.[9]

12.2 Remark. The 2-multicategory $\mathrm{M}\mathfrak{B}\mathfrak{Gr}\mathfrak{St}$ has of course more structure than in the abstract situation discussed in sections A and B. In particular, the objects of $\mathfrak{B}\mathfrak{Gr}\mathfrak{St}$, being themselves categories, have an internal structure, so for any two objects $\mathscr{G}$ and $\mathscr{H}$ the wreath product $\mathscr{G} \wr \mathscr{H}$ is defined, say by using the same definition as in sect. A with $\mathscr{G}$ and $\mathscr{H}$ interpreted as objects over $\mathbf{1}$. It follows that by a special case of the adjunction discussed in [Kel72], which in the present case can be verified by hand, the lax monoidal functor providing the exterior monoidal structure on $\mathscr{R}$ corresponds to

$$(\tilde{t},\tilde{\theta})\colon \mathbf{N} \wr \mathscr{R} \longrightarrow \mathscr{R},$$

which is required to satisfy diagrams similar to (B.1) and (B.3). In this form we have that $\mathscr{R}$ is a *pseudo algebra* over $\mathbf{N}$.

We drop the "weak" attribute is the underlying braiding is symmetric. In such a case, we will see the fibers of $\mathscr{R}$ are categorical rings in the usual sense, so that $\mathscr{R}$ becomes the stack analog of the categorical rings described in e.g. [JP07].

12.3 Remark. For a symmetric $\mathscr{R}$, we can define the ring-like structure in the same way, but working in $\mathfrak{S}\mathfrak{Gr}\mathfrak{St}$. In such case the pseudo-algebra structure of $\mathscr{R}$ can be verified directly from the adjunction mentioned in Remark 11.4 above.

A pseudo monoid $\mathscr{R}$ in $\mathrm{M}\mathfrak{B}\mathfrak{Gr}\mathfrak{St}$ ought to correspond to one in $\mathrm{M}\mathfrak{B}\mathfrak{X}\mathfrak{Mod}$. More precisely, any pseudo monoid $\mathscr{R}$ ought to be equivalent to one whose underlying stack is the one associated to a presentation by a braided crossed module, i.e. an object of $\mathfrak{B}\mathfrak{X}\mathfrak{Mod}$. This will be made precise below. First, we can write the analog of Definition 12.1, namely:

12.4 Definition. A *(weakly) ring-like crossed module* is a pseudo monoid in the bi-multicategory $\mathrm{M}\mathfrak{B}\mathfrak{X}\mathfrak{Mod}$, that is, a braided crossed module $R_\bullet\colon R_1 \xrightarrow{\partial} R_0$ equipped with a pseudo monoidal functor

$$(t,\theta)\colon \mathbf{N} \longrightarrow \mathrm{M}R_\bullet.$$

We drop the "weak" attribute if the underlying braiding is symmetric.

An operation of arity n is realized in this case by an n-butterfly $E_n = t(n)$, an object of $\mathbf{Hom}(R_\bullet^n; R_\bullet)$, composed according to (B.2), realized by the juxtaposition product

$$(12.1) \qquad \theta_{n;m_1,\dots,m_n}\colon E_{m_1+\cdots+m_k} \longrightarrow E_n(E_{m_1},\dots,E_{m_n}),$$

plus the coherence condition expressed by the full diagram (B.4), which reads

$$(12.2)$$

$$
\begin{array}{ccc}
E_k(E_{l_1},\dots,E_{l_k}) & \longleftarrow E_h \longrightarrow & E_n(E_{j_1},\dots,E_{j_n}) \\
\downarrow & & \downarrow \\
E_n(E_{m_1},\dots,E_{m_n})(E_{l_1},\dots,E_{l_k}) & \longrightarrow & E_n(E_{m_1}(E_{l_1},\dots),\dots,E_{m_n}(\dots,E_{l_k}))
\end{array}
$$

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[9]In the sequel we shall use "weak", when referring to bimonoidal structures, in this sense; it will not refer to the laxness of the monoidal structures.

The bottom arrow is the association isomorphism in the butterfly juxtaposition. For $n \leq 4$, with all the sequences emanating from $n = 4$ as in sect. B, we obtain the "generalized pentagon" diagram (C.2).

Theorem 11.3 and the fact that pseudo monoids are preserved by pseudo functors (Definition B.4) guarantee that a weakly ring-like $R_\bullet$ gives rise, through the associated stack construction, to a weakly ring-like stack. More interesting is the converse direction, namely:

12.5 Proposition. *If $(\mathscr{R}, t, \theta)$ is a weakly categorical ring and $R_1 \to R_0 \to \mathscr{R}$ a presentation, then $R_\bullet : R_1 \to R_0$ is a pseudo monoid in* $\mathfrak{MBXMod}$.

Proof. From Theorem 11.3 we get an equivalence $\mathrm{M}a_R \colon \{R_\bullet, R_\bullet\} \overset{\sim}{\to} \{\mathscr{R}, \mathscr{R}\}$. □

Here is an explicit construction of a quasi-inverse to $\mathrm{M}a_R$. By Theorem 9.1, or rather the generalization of the proof of Theorem 7.1, the functor u has a quasi-inverse v computed by the fiber product construction which yields an n-butterfly $E_n = v(t(n))$; for a composition $t(n)(t(m_1), \ldots, t(m_n))$ we obtain a morphism

$$v(t(n)(t(m_1), \ldots, t(m_n))) \overset{\sim}{\longrightarrow} E_n(E_{m_1}, \ldots, E_{m_n}).$$

Thus, for any

$$\theta_{n;m_1,\ldots,m_n} \colon t(m_1 + \cdots + m_n) \to t(n)(t(m_1), \ldots, t(m_n))$$

in $\mathrm{M}\mathscr{R}$, we obtain

$$E_{m_1 + \cdots + m_n} \overset{\sim}{\longrightarrow} v(t(n)(t(m_1), \ldots, t(m_n))) \overset{\sim}{\longrightarrow} E_n(E_{m_1}, \ldots, E_{m_n}),$$

eventually arriving at (12.2).

The notion of **morphism** for both weakly categorical rings and crossed modules is straightforward. In each instance, the notion is a special case of that of morphism of pseudo-monoid examined in sect. B.

Notation. From now on it will be convenient to refer to the monoidal structure of $\mathscr{R}$ as a group-like object as the "intrinsic" or "internal" one, and denote it by a plus. Correspondingly, the relative unit object will be denoted by $0_{\mathscr{R}}$, or simply 0, if no confusion is bound to arise. (This retroactively justifies the choice of the attribute "biadditive" for the functor m.) The second monoidal structure (m, μ) will be referred to as the "extrinsic" or "external" one. The result of the application of m will be denoted by a juxtaposition. The corresponding unit object, if it exists, will be denoted by $I_{\mathscr{R}}$ or simply I.

13   The presentation of a categorical ring

In this section we show that the presentation of a (weak) categorical ring or ring-like stack is a crossed module with certain additional properties. In particular, its zeroth and first homotopy sheaves are, respectively, a ring and a bimodule over that ring, as in the standard case.

Let $\mathscr{R}$ be as above. Let $A = \pi_0(\mathscr{R})$ and $M = \pi_1(\mathscr{R})$ be the sheaves of connected components and of automorphisms of the (additive) identity object. Since the definition of π_i applies to the underlying (group-like) stack, by the usual arguments A and M are also zeroth and first homotopy sheaves of any crossed module $R_\bullet$ used to present $\mathscr{R}$. In fact the homotopy kernel of the projection $\varpi \colon \mathscr{R} \to A$ is identified with the Picard stack $\mathscr{M} = \mathrm{Tors}(M)$ of M-torsors. One has the familiar diagram

with homotopy-exact columns and where the top two levels of each column (the part within the box above) provide a presentation of the item immediately below. The colored sequence in it is the standard extension of A by M by way of the crossed module $R_\bullet$. Of course, the top two rows, considered as a sequence of

complexes of length one, is exact only in the sense that the group-like categories or stacks they determine form a short exact sequence.

For objects $X_1, X_2, \ldots, X_n$ of $\mathscr{R}$ define the result of applying the n-ary operation $t(n)$ simply by $X_1 X_2 \cdots X_n$. Then, by defining $[X_1][X_2]\cdots[X_n] := [X_1 X_2 \cdots X_n]$, we immediately get that $\varpi \colon \mathscr{R} \to A$ is a morphism of pseudo monoids, so that A is a ring—unital if $\mathscr{R}$ possesses a unit object $I_{\mathscr{R}}$ for the external monoidal structure. (It follows that $\mathscr{M}$ behaves like a bilateral ideal in $\mathscr{R}$, namely the monoidal structure on $\mathscr{R}$, and more precisely its binary operation, restricts to a pair of actions:

$$\mathscr{R} \times \mathscr{M} \longrightarrow \mathscr{M} \quad \text{and} \quad \mathscr{M} \times \mathscr{R} \longrightarrow \mathscr{M}.)$$

While there is no direct map $R_0 \times R_0 \to R_0$, the exterior monoidal structure $m_2 = t(2)\colon \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ is "covered" by the diagram

$$
\begin{array}{ccc}
& E & \\
{\scriptstyle p}\swarrow & & \searrow{\scriptstyle \jmath} \\
R_0 \times R_0 & & R_0
\end{array}
$$

(13.1)

which is part of the structure of the biextension. Given a pair $(x,y) \in R_0 \times R_0$, and the choice of a point in the fiber $e \in E_{x,y}$, we get a "value" $\jmath(e) \in R_0$. The latter is of course only defined up to shifting e by the action of R_1, namely $\jmath(e\,r) = \jmath(e) + \partial r$, where $r \in R_1$. The connected component $q(x) \in A$ is well-defined and independent of all choices. Moreover, by Theorem 7.1, or rather its proof, we have $E_{x,y} = m_2(\pi(x), \pi(y))$, and therefore $\varpi(E_{x,y}) = \varpi(\pi(x))\varpi(\pi(y)) = q(x)q(y)$. We conclude that, with $e \in E_{x,y}$ as above, $q(\jmath(e)) = q(x)q(y)$. Put differently, the diagram displayed above, if interpreted as a "virtual ring structure" for R_0, covers the multiplication map of A. This consideration extends to any number of variables.

Hence the following standard fact holds:

13.1 Proposition. *M is an A-bimodule.*

We recall the main idea of the proof, as it will be needed to translate the above fact in terms of the presentation of $\mathscr{R}$ and the multi-extensions associated to its second monoidal structure.

Proof of 13.1 (Sketch). Recall that the standard way to relate the automorphisms of $0_{\mathscr{R}}$ to those of any other object X of $\mathscr{R}$ is to use the diagram

$$
\begin{array}{ccc}
X + 0_{\mathscr{R}} & \xrightarrow{\mathrm{id}_X + \alpha} & X + 0_{\mathscr{R}} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\tilde{\alpha}} & X
\end{array}
$$

Left and right multiplications by an object Y (i.e. the additive functors $m_2(Y,-)$ and $m_2(-,Y)$) translate the above diagram into the corresponding ones for automorphisms $Y\alpha$ and αY. If $\beta \colon Y \to Z$, so that $[Y] = [Z] \in A$, then we arrive at the square

$$
\begin{array}{ccc}
XY + 0_{\mathscr{R}} & \xrightarrow{\mathrm{id} + \alpha Y} & XY + 0_{\mathscr{R}} \\
{\scriptstyle \mathrm{id}_X\,\beta + \mathrm{id}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X\,\beta + \mathrm{id}_0} \\
XZ + 0_{\mathscr{R}} & \xrightarrow[\mathrm{id} + \alpha Z]{} & XZ + 0_{\mathscr{R}}
\end{array}
$$

from which we conclude that $\alpha Y = \alpha Z$. The situations for $Y\alpha$, $Z\alpha$ is analogous. $\qquad\square$

Combining this with Proposition 12.5 we have:

13.2 Corollary. *Every categorical ring $\mathscr{R}$ has a presentation*

$$0 \longrightarrow M \longrightarrow R_1 \xrightarrow{\;\partial\;} R_0 \longrightarrow A \longrightarrow 0,$$

where $R_\bullet \colon R_1 \to R_0$ is a braided crossed module, $A = \pi_0(\mathscr{R})$ a ring and $M = \pi_1(\mathscr{R})$ an A-bimodule. $\qquad\square$

In fact, something more can be said about the presentation, despite R_0 having the weaker structure given by (13.1). In addition, we will need to describe the A-bimodule structure of M in terms of (13.1) and the biextension E.

Consider the morphism examined in Remark 7.5: for an arrow $x \to x' = x + \partial r$ of $[R_1 \to R_0]^\sim$, with $x \in R_0$ and $r \in R_1$, we have the morphism of (R_1, R_0)-torsors $\rho_y \colon E_{x,y} \to E_{x',y}$, $e \mapsto e \times_1 s_1(r,y)$. If $\partial r = 0$, so $r \in \pi_1(\mathscr{R}) = M$, then ρ_y is an automorphism of $E_{x,y}$, hence $\rho \in M$ as well (depending on x, y), since in general $\mathrm{Aut}_{\mathrm{Tors}(R_1,R_0)}(E) \simeq M_{R_0 \times R_0}$. Similarly, we construct an automorphism $\lambda_x \colon E_{x,y} \to E_{x,y}$ corresponding to the automorphism of $y \in R_0$ given by $r \in R_1$.

13.3 Lemma-Definition. *The automorphism ρ_y (resp. λ_x) is identified with a section $-ry$ (resp. $-xr$) of M such that if $y, y' \in R_0$ (resp. $x, x' \in R_0$) and $q(y) = q(y') \in A$ (resp. $q(x) = q(x') \in A$), then $ry = ry'$ (resp. $xr = x'r$).*

With the above notations, define $ar := xr$ and $rb := ry$, where $a = q(x), b = q(y) \in A$.

13.4 Remark. The apparently bizarre choice of inserting an inverse (denoted additively) is due to the point raised in Remark 1.1.

Proof of the Lemma. The identification between the automorphisms of E and M is standard: for any section $e \in E_{x,y}$ we have $\rho_y(e) = e\, r_{\rho_y}(e)$, with $r_{\rho_y}(e) \in M$. On the other hand, this must be equal to $e \times_1 s_1(r,y)$, the latter coming from the composition

$$E_{x,y} \longrightarrow E_{x,y} \wedge^{R_1} E_{0,y} \longrightarrow E_{x,y}.$$

Recall from sect. 3 that, analogously to the standard case of abelian biextensions [Gro72], for $x, y \in R_0$, the fibers $E_{x,0}$ and $E_{0,y}$ are canonically identified with the unit (R_1, R_0)-torsor: the identification

$$R_1 \xrightarrow{\ \sim\ } E_{0,y}, \quad 0_{R_1} \longmapsto e_y,$$

sends the unit section of R_1 (which we write as 0 according to the current additive convention) to the central section $e_y \in E_{0,y}$. Note that as a result $\jmath(e_y) = 0_{R_0}$. By the above, we have

(13.2) $$s_1(r,y) = e_y(-ry),$$

where $ry := -r_{\rho_y}(e_y) \in R_1$, and the juxtaposition stands for the right R_1-action. In fact, we have $ry \in M$, since $\jmath \circ s_1$ is trivial (cf. sect. 6).

Now, assume we have $y \to y' = y + \partial u$, $u \in R_1$. We have

Using the identities (6.3) and the restriction condition 1 in Definition 6.1 we calculate:

$$
\begin{aligned}
s_1(r,y) \times_2 s_2(0,u) &= s_1(r,y) \times_2 s_2(\partial r, u) \\
&= s_1(r,y) \times_2 s_1(r, \partial u) \\
&= s_1(r, y + \partial u) \\
&= s_2(r, y').
\end{aligned}
$$

Using (13.2) the above calculation gives

$$
\begin{aligned}
e_y(ry) \times_2 s_2(0,u) &= e_y \times_2 (ry) s_2(0,u) \\
&= e_y \times_2 s_2(0,u)(ry) \\
&= e_{y'}(ry').
\end{aligned}
$$

The fact that the identification is canonical gives $e_y \times_2 s_2(0,u) = e_{y'}$ and hence $ry = ry'$.

The situation with $e_x \in E_{x,0}$, xr, and $x'r$ is entirely analogous. $\qquad\square$

13.5 Remark. In the extension of A by M above, R_1 and R_0 are only groups, in general, even though the crossed module $R_\bullet$ they form comes equipped with a braiding. In particular, unless certain strong triviality conditions on the biextension $E_2 = t(2) \in \mathbf{Biext}(R_\bullet, R_\bullet; R_\bullet)$ hold, R_0 is *not* a ring, in general. This remark applies to stock categorical rings as well, which therefore admit a presentation of the above type with $R_\bullet$ a braided symmetric crossed module. This is in sharp contrast with the so-called *Ann-Categories* [see e.g. Qua03; QHT08], whose underlying categories are Picard groupoids. In a companion paper [Ald15] we prove that Picard stacks with a ring-like structure of this type admit presentations given by crossed bimodules, namely crossed modules such as in the presentation above where R_0 is a ring and R_1 is an R_0-bimodule, with ∂ a bimodule morphism satisfying an appropriate version of the Pfeiffer identity. The general comparison between these structures is intricate and it will be addressed elsewhere.

14 Decomposition of categorical rings and the cohomology of rings

Let $\mathscr{R}$, A, and M be as above. In general, a **decomposition** [Bre94] of $\mathscr{R}$ consists in the choice of local data (objects, morphisms) subordinate to a hypercovering of A leading to the calculation of cohomological invariants of $\mathscr{R}$. We use the pseudo-monoidal structure carried by a presentation of $\mathscr{R}$ for this purpose.

We let $L_i^\bullet(A)$, $i = 2, 3$ be the sharp truncations of the iterated bar complexes $B(A, i)$, $i = 2, 3$ of A as an abelian group. By [Mac58, §11] it carries a product structure, defined explicitly up to cells in $B(A, 2)$. For $i = 2, 3$ we let $H_i^3(A, M)$ denote the (hyper)cohomology groups computed using the multiplicative bar construction over $L_\bullet^i(A)$. From the previous reference we have the $H_3^3(A, M) \simeq HML^3(A, M)$, the third Mac Lane cohomology of A with values in M. Furthermore, we let $\widetilde{H}_2^3(A, M)$ denote the group of twisted classes satisfying (14.2) below.

14.1 Theorem. *There is a bijective correspondence between equivalence classes of weak ring-like stacks $\mathscr{R}$, with $A = \pi_0(\mathscr{R})$ and $M = \pi_1(\mathscr{R})$, and twisted classes in $\widetilde{H}_2^3(A, M)$ defined below (cf. Definition 14.6). In the unital case, i.e. when the external monoidal structure of $\widetilde{\mathscr{R}}$ has a unit element, A is likewise unital and the underlying braiding of $\mathscr{R}$ is necessarily symmetric. Hence the weak ring-like structure of $\mathscr{R}$ is fully ring-like, and $[\mathscr{R}] \in \widetilde{H}_2^3(A, M) \simeq HML^3(A, M)$.*

The statement follows from Proposition 14.8, Corollary 14.9, and Proposition 14.10 below. The rest of this section is devoted to the details of the proof of the theorem.

14.1 The bar complex

We establish the notation for the iterated bar complex of the abelian group A. We quote, with minor changes in the notation, from [Dug14, §7]. The part of $B^\infty(A)$ of interest is the following complex, written homologically:

$$L_\bullet^3(A) : \mathbb{Z}[A] \xleftarrow{\;\partial_1\;} \mathbb{Z}[A^2] \xleftarrow{\;\partial_2\;} \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \xleftarrow{\;\partial_3\;} \mathbb{Z}[A^4] \oplus \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2]$$

We place the complex in homological degrees $[0, 3]$, with generators:

Degree	Generators			
0	$[a]$			
1	$[a \,|_1\, b]$			
2	$[a \,|_1\, b \,|_1\, c]$	$[a \,|_2\, b]$		
3	$[a \,|_1\, b \,|_1\, c \,|_1\, d]$	$[a \,|_1\, b \,|_2\, c]$	$[a \,|_2\, b \,|_1\, c]$	$[a \,|_3\, b]$

The differential is given by:

$$\partial_1[a\,|_1\,b] = [b] - [a+b] + [a]$$
$$\partial_2[a\,|_1\,b\,|_1\,c] = [b\,|_1\,c] - [a+b\,|_1\,c] + [a\,|_1\,b+c] - [a\,|_1\,b]$$
$$\partial_2[a\,|_2\,b] = [a\,|_1\,b] - [b\,|_1\,a]$$
$$\partial_3[a\,|_1\,b\,|_1\,c\,|_1\,d] = [b\,|_1\,c\,|_1\,d] - [a+b\,|_1\,c\,|_1\,d] + [a\,|_1\,b+c\,|_1\,d] - [a\,|_1\,b\,|_1\,c+d] + [a\,|_1\,b\,|_1\,c]$$
$$\partial_3[a\,|_1\,b\,|_2\,c] = [a\,|_1\,b\,|_1\,c] - [a\,|_1\,c\,|_1\,b] + [c\,|_1\,a\,|_1\,b] - [b\,|_2\,c] + [a+b\,|_2\,c] - [a\,|_2\,c]$$
$$\partial_3[a\,|_2\,b\,|_1\,c] = [a\,|_1\,b\,|_1\,c] - [b\,|_1\,a\,|_1\,c] + [b\,|_1\,c\,|_1\,a] + [a\,|_2\,c] - [a\,|_2\,b+c] + [a\,|_2\,b]$$
$$\partial_3[a\,|_3\,b] = [a\,|_2\,b] + [b\,|_2\,a].$$

The subcomplex $L^2_\bullet(A)$ is the one obtained by dropping the component $\mathbb{Z}[A^2]$ of $L^2_3(A)$, namely the one generated by symbols $[a\,|_3\,b]$. The degrees are shifted in a way compatible with the reset needed to form the infinite bar construction, so in effect we have $L^i_\bullet(A) = B(A,i)[-i]$, for $i = 2, 3$, and therefore

$$H_2(L^2_\bullet(A)) \simeq H_4(K(A,2)), \qquad H_2(L^3_\bullet(A)) \simeq H_5(K(A,3)).$$

In the stable situation, this shift is compatible with the degrees in the Q construction.

14.2   Product structure

Let now A be considered with its ring structure. We describe the product structure on $L^2_\bullet(A)$ and $L^3_\bullet(A)$ [see Mac58, §11]. The non zero products among the generators are the following:

$$[a][b] = [ab]$$
$$[a][b\,|_1\,c] = [ab\,|_1\,ac] \qquad\qquad [a\,|_1\,b][c] = [ac\,|_1\,bc]$$
$$[a][b\,|_1\,c\,|_1\,d] = [ab\,|_1\,ac\,|_1\,ad] \qquad [a\,|_1\,b\,|_1\,c][d] = [ad\,|_1\,bd\,|_1\,cd]$$
$$[a][b\,|_2\,c] = [ab\,|_2\,ac] \qquad\qquad [a\,|_2\,b][c] = [ac\,|_2\,bc]$$

and the most interesting one[10]:

$$[a\,|_1\,b][c\,|_1\,d] = [ac\,|_1\,bc\,|_1\,ad+bd] - [ac\,|_1\,ad\,|_1\,bc+bd] + [ad\,|_1\,bc\,|_1\,bd] - [bc\,|_1\,ad\,|_1\,bd] - [bc\,|_2\,ad].$$

With this product $L^2_\bullet(A)$ and $L^3_\bullet(A)$ become DGAs. Furthermore, both are equipped with the augmentation $\eta\colon L^k_\bullet(A) \to A$ given by $\eta([a]) = a$, for $x \in A$, and zero in all other degrees. Thus, they become augmented DGAs to which we can apply the (reduced) bar construction $\bar{B}_{i,\bullet}(A) := \bar{B}_N(L^i_\bullet(A), \eta)$ of loc. cit. (more details below). We are interested in the resulting cohomology groups $H^3(\mathrm{Hom}(\bar{B}_{i,\bullet}(A), M))$, where M is an A-bimodule as above. As remarked, for $i = 3$ we have $H^3(\mathrm{Hom}(\bar{B}_{3,\bullet}(A), M)) \simeq HML^3(A, M)$.

14.3   The multiplicative bar construction and third cohomology of rings

To avoid typographical clutter, the generators of $\bar{B}_{i,\bullet}(A)$ are denoted $[\![u_1, \ldots, u_n]\!]$, where the u_ks are homogeneous elements of $L^i_\bullet(A)$. The degree is

$$\deg[\![u_1, \ldots, u_n]\!] = n + |u_1| + \cdots + |u_n|,$$

where we let $|u_k| = \deg u_k$. We quote [Mac58, p. 323] the expression for the differential in the bar complex; it is $\partial_{\mathrm{tot}} = \partial' + \partial''$, where

$$\partial'[\![u_1, \ldots, u_n]\!] = -\sum_{i=1}^{n} (-1)^{\varepsilon_{i-1}} [\![u_1, \ldots, \partial u_i, \ldots, u_n]\!]$$

$$\partial''[\![u_1, \ldots, u_n]\!] = \eta(u_1)[\![u_2, \ldots, u_n]\!] + \sum_{i=1}^{n-1} (-1)^{\varepsilon_i} [\![u_1, \ldots, u_i u_{i+1}, \ldots, u_n]\!] + (-1)^{\varepsilon_n} [\![u_1, \ldots, u_{n-1}]\!]\eta(u_n),$$

---

[10]There appears to be a discrepancy in the second slot of the second term of the right hand side, presumably a misprint in [Mac58].

where we have set $\varepsilon_i = \deg[\![u_1,\ldots,u_i]\!]$. On the right hand side of the expression for ∂' the inner ∂u_i indicates the differential in the complex $L_\bullet^i(A)$.

The cells of total dimension (=degree) 3 are:

n	Generators	
3	$[\![[a],[b],[c]]\!]$	
2	$[\![[a\,|_1\,b],[c]]\!]$	$[\![[a],[b\,|_1\,c]]\!]$
1	$[\![[a\,|_1\,b\,|_1\,c]]\!]$	$[\![[a\,|_2\,b]]\!]$

Therefore a 3-dimensional cochain over $\bar{B}_{i,\bullet}(A)$ with values in M is a 5-tuple $\xi = (f,\alpha_1,\alpha_2,f_+,g_+)$, where

$$f\colon A^3 \longrightarrow M$$

$$\alpha_1\colon A^2 \times A \longrightarrow M, \quad \alpha_2\colon A \times A^2 \longrightarrow M$$

$$f_+\colon A^3 \longrightarrow M, \quad g_+\colon A^2 \longrightarrow M.$$

To write the cocycle condition $\delta\xi = \xi \circ (\partial' + \partial'') = 0$ in explicit form we need the cells of dimension four:

n	Generators				
4	$[\![[a],[b],[c],[d]]\!]$				
3	$[\![[a\,|_1\,b],[c],[d]]\!]$	$[\![[a],[b\,|_1\,c],[d]]\!]$	$[\![[a],[b],[c\,|_1\,d]]\!]$		
2	$[\![[a\,|_1\,b],[c\,|_1\,d]]\!]$	$[\![[a\,|_1\,b\,|_1\,c],[d]]\!]$	$[\![[a\,|_2\,b],[c]]\!]$	$[\![[a],[b\,|_1\,c\,|_1\,d]]\!]$	$[\![[a],[b\,|_2\,c]]\!]$
1	$[\![[a\,|_1\,b\,|_1\,c\,|_1\,d]]\!]$	$[\![[a\,|_1\,b\,|_2\,c]]\!]$	$[\![[a\,|_2\,b\,|_1\,c]]\!]$	$[\![[a\,|_3\,b]]\!]\dagger$	

The element marked with a (†) would only figure in the bar complex $\bar{B}_{3,\bullet}(A)$. In explicit form, the cocycle condition on the five components of ξ consists of several equations, which unfortunately makes it hard to take in. Arranging the various equations in different groups we have (see below for specific comments):

(14.1a)
$$a\,f(b,c,d) - f(ab,c,d) + f(a,bc,d) - f(a,b,cd) + f(a,b,c)\,d = 0$$

(14.1b)
$$f(b,c,d) - f(a+b,c,d) + f(a,c,d) = \alpha_1(ac,bc;d) - \alpha_1(a,b;cd) + \alpha_1(a,b;c)\,d$$
$$-f(a,c,d) + f(a,b+c,d) - f(a,b,d) = a\,\alpha_1(b,c;d) - \alpha_1(ab,ac;d) - \alpha_2(a;bd,cd) + \alpha_2(a;b,c)\,d$$
$$f(a,b,d) - f(a,b,c+d) + f(a,b,c) = a\,\alpha_2(b;c,d) - \alpha_2(ab;c,d) + \alpha_2(a;bc,bd)$$

(14.1c) $\quad f_+(ac,bc,ad+bd) - f_+(ac,ad,bc+bd) + f_+(ad,bc,bd) - f_+(bc,ad,bd) - g_+(bc,ad) =$
$$\alpha_1(a,b;d) - \alpha_1(a,b;c+d) + \alpha_1(a,b;c) + \alpha_2(b;c,d) - \alpha_2(a+b;c,d) + \alpha_2(a;c,d)$$

(14.1d)
$$f_+(ad,bd,cd) - f_+(a,b,c)\,d = -\alpha_1(b,c;d) + \alpha_1(a+b,c;d) - \alpha_1(a,b+c;d) + \alpha_1(a,b;d)$$
$$f_+(ab,ac,ad) - a\,f_+(b,c,d) = \alpha_2(a;c,d) - \alpha_2(a;b+c,d) + \alpha_2(a;b,c+d) - \alpha_1(a;b,c)$$
$$g_+(ac,bc) - g_+(a,b)\,c = -\alpha_1(a,b;c) + \alpha_1(b,a;c)$$
$$g_+(ab,ac) - a\,g_+(b,c) = \alpha_2(a;b,c) - \alpha_2(a;c,b)$$

(14.1e)
$$f_+(b,c,d) - f_+(a+b,c,d) + f_+(a,b+c,d) - f_+(a,b,c+d) + f_+(a,b,c) = 0$$
$$f_+(a,b,c) - f_+(a,c,b) + f_+(c,a,b) = g_+(b,c) - g_+(a+b,c) + g_+(a,c)$$
$$f_+(a,b,c) - f_+(b,a,c) + f_+(b,c,a) = -g_+(a,c) + g_+(a,b+c) - g_+(a,b)$$
$$g_+(a,b) + g_+(b,a) = 0.$$

14.2 Remarks.

1. The last group of equations (14.1e) is closed and it is the condition for the pair (f_+, g_+) to be an Eilenberg Mac Lane cocycle. Thus (f_+, g_+) represents a class in $H^4(K(A,2),M)$ or $H^5(K(A,3),M)$, depending on whether the last term is included.

2. The block (14.1d) can be re-written as follows. Let λ_a and ρ_a denote the left and right multiplications by $a \in A$. Also, write $a f_+$ and $f_+ a$ for the action of $a \in A$ on the values of f_+. Same for g_+. Then we can rewrite the block as

$$\rho_d^* f_+ - f_+ d = -\alpha_1(-,-;d) \circ \partial_3$$
$$\lambda_a^* f_+ - a f_+ = \alpha_2(a;-,-) \circ \partial_3$$
$$\rho_c^* g_+ - g_+ c = -\alpha_1(-,-;c) \circ \partial_2$$
$$\lambda_a^* g_+ - a g_+ = \alpha_2(a;-,-) \circ \partial_2$$

   The block (14.1d) expresses the invariance of the *class* of (f_+, g_+) under left and right multiplication in A.

3. Equation (14.1a) has the familiar form of a Hochschild cocycle. (As it might be expected, it arises from the associativity constraint on the exterior monoidal structure of $\mathscr{R}$, as it will be shown below.) The behavior of f with respect to the additivity, i.e. its failure to be multilinear, is expressed by the equation block (14.1b).

4. As it will be explained below, the meaning of the somewhat obscure relation (14.1c), is that it expresses the compatibility (interchange law) between the partial composition laws of the exterior monoidal structure of $\mathscr{R}$.

It is well known that $H_5(K(A,3)) \simeq A/2A$ and that $H^5(K(A,3),M) \simeq \mathrm{Hom}(A/2A,M) \simeq \mathrm{Hom}(A, {}_2M)$ [EM54, §23; see also Dug14, §7.3]. The former isomorphism is given by $a \mapsto [a \,|_2\, a]$, the latter sends the class of $[(f_+, g_+)]$ to the (linear) map $a \mapsto g_+(a,a)$. In fact, dropping the last of equations (14.1e), the same assignment gives a quadratic map from A to M. (Recall that $q \colon A \to M$ is quadratic if $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$, and if the associated symmetric function $\Delta q(a,b) = q(a+b) - q(a) - q(b)$, $a, b \in A$, is bilinear.) Thus $H^5(K(A,3),M) \simeq \mathrm{Hom}(\Gamma_2(A),M)$, where $\Gamma_2(A)$ is the Whitehead functor, i.e. the degree four component of the divided power algebra $\Gamma_\bullet(A)$ over A.

Recall that $m \in M$ is *central* if $am = ma$ for all $a \in A$. Let M^A be the A-module of central elements of M. Assume that ξ represents a class in $H^3(\mathrm{Hom}(\bar{B}_{2,\bullet}(A),M))$, that is, the pair (f_+, g_+) satisfies, as part of the full set $\xi = (f, \alpha_1, \alpha_2, f_+, g_+)$, equations (14.1d) and (14.1e), *except the last one*. We have the following observation.

14.3 Proposition. *Let A be unital. We have an isomorphism*

$$H^3(\mathrm{Hom}(\bar{B}_{2,\bullet}(A),M)) \simeq H^3(\mathrm{Hom}(\bar{B}_{3,\bullet}(A),M)) \simeq HML^3(A,M).$$

Moreover, the map $HML^3(A,M) \to \mathrm{Hom}(A, {}_2M)$ [Mac58, §11] factors through ${}_2M^A \simeq \mathrm{Hom}_A(A, {}_2M^A)$.

Proof. Let q be the map $a \mapsto g_+(a,a)$. The last two equations of (14.1d) imply that q is an A-bimodule homomorphism. As such, it is determined by a central element in M. This proves the first isomorphism. The second follows from [Mac58]. □

14.4 Remark. Because the complexes $B_{2,\bullet}(A)$ and $B_{2,\bullet}$ are equal below degree 3, their cohomologies coincide for $n \leq 2$. Combined with [Bre99, §6], we have that the statement of Proposition 14.3 holds in general when $H^\bullet$ is interpreted as hypercohomology.

The following variant of the previous constructions will be useful below. Given a 5-tuple ξ as above, consider the map $\beta_\xi \colon B_{2,4}(A)_2 \to M$ given by (the subscript 2 refers to the elements in $B_{2,4}(A)$ with $n = 2$):

$$\beta_\xi(\llbracket u_1, u_2 \rrbracket) = \begin{cases} g_+(ab, ac) + g_+(ac, ab) & \llbracket u_1, u_2 \rrbracket = \llbracket [a], [b \,|_2\, c] \rrbracket, \\ g_+(ac, bc) + g_+(bc, ac) & \llbracket u_1, u_2 \rrbracket = \llbracket [a \,|_2\, b], [c] \rrbracket, \\ 0 & \text{all other cases.} \end{cases}$$

Further, we define β_ξ to be zero on the other components of $B_{2,4}(A)$. Consider the twisted cocycle equation condition:

$$(14.2) \qquad\qquad D\xi := \delta\xi + \beta_\xi = 0.$$

This replaces the last two equations in the block (14.1d) with

$$(14.3) \qquad \begin{aligned} -g_+(bc,ac) - g_+(a,b)\,c &= -\alpha_1(a,b;c) + \alpha_1(b,a;c) \\ -g_+(ac,ab) - a\,g_+(b,c) &= \alpha_2(a;b,c) - \alpha_2(a;c,b) \end{aligned}$$

and it still drops the last one from (14.1e). Observe that for $i = 3$ equation (14.2) is vacuously equal to the set (14.1).

14.5 Remark. Note that replacing ξ with $\xi' = \xi + \delta\nu$ has the effect of adding to g_+ the alternation of a map $h_+ : A \times A \to M$, so that β_ξ does not depend on the particular choice of g_+ within its class modulo coboundaries. Hence we get a well defined class of solutions of equation (14.2) modulo adding coboundaries.

14.6 Definition. Let us denote by $\widetilde{H}_2^3(A, M)$ the hypercohomology group of classes satisfying (14.2) in degree three, relative to A (cf. Remark 14.4).

There is an evident map $HML^3(A, M) \to \widetilde{H}_2^3(A, M)$, which, at least in the cases we like to consider, is an isomorphism. Indeed we have

14.7 Lemma. *Let A be unital. Then (14.2) and (14.1) are equivalent. Therefore $\widetilde{H}_2^3(A, M) \simeq HML^3(A, M)$.*

Proof. Choosing a or $c = 1$ in (14.3) shows that the last equation in in (14.1e) holds. (Again, use Remark 14.4 for the general situation.) $\qquad\square$

14.4 Decomposition of $\mathscr{R}$

For convenience of notation, let us write $[\partial : R_1 \to R_0] \equiv [\partial : R \to \Lambda]$. Recall that since we denote the interior monoidal structure of $\mathscr{R}$ additively, we do the same for R and Λ in the presentation.

For a point $a \in A$, we let $\Lambda_a = a^*\Lambda$ be the fiber. For any multi-extension E_n in the weak ring-like structure of $R \to \Lambda$ denote by $E_{a_1,\ldots,a_n}$ the pullback of E to $\Lambda_{a_1} \times \cdots \times \Lambda_{a_n}$, where $a_1,\ldots,a_n \in A$.

There is an obvious morphism $\Lambda_{a_1} \times \cdots \times \Lambda_{a_n} \to \Lambda_{a_1+\cdots+a_n}$ covering the n-fold iteration of $+ : A \times A \to A$. If we assume a choice for a point $x_a \in \Lambda_a$ has been made for all points $a \in A$, i.e. we have a section x of $q : \Lambda \to A$, then $x_a + x_b$ (the image of (x_a, x_b) under $\Lambda_a \times \Lambda_b \to \Lambda_{a+b}$) and x_{a+b} will in general be different. This gives rise to $\sigma : A \times A \to R$ by way of

$$x_{a+b} = x_a + x_b + \partial\sigma_{a,b}.$$

In fact the objects $X_a := \pi(x_a)$ of $\mathscr{R}$ provide a decomposition of the sequence $\mathscr{M} \to \mathscr{R} \to A$, where $\mathscr{R}$ is considered as a gerbe over A [Bre94, §7]. Note that the above corresponds to the morphism

$$\sigma_{a,b} : X_{a+b} \longrightarrow X_a + X_b$$

of (R, Λ)-torsors. Therefore we have the classical fact that the difference between the two possible comparisons between $X_a + X_b + X_c$ and X_{a+b+c}, and the application of equations (1.2) to x_a, x_b, and x_c determines a pair (f_+, g_+) satisfying the cocycle equations (14.1e) above, hence a class in $H^4(K(A, 2), M)$ or $H^5(K(A, 3), M)$ [see, e.g. JS93; Bre99]. For future reference, the relevant relations are [cf. Bre94]:

$$(14.4) \qquad \begin{aligned} \sigma_{b,c} + \sigma_{a,b+c} - f_+(a,b,c) &= \sigma_{a,b}{}^{x_c} + \sigma_{a+b,c}, \\ -g_+(a,b) + \sigma_{a,b} &= \{x_a, x_b\} + \sigma_{b,a}. \end{aligned}$$

(There is an obvious generalization to n variables, but the classical situation is sufficient to describe the decomposition with respect to the "+" operation of $\mathscr{R}$.)

For the multiplication, i.e. the exterior monoidal structure, the reasoning at the end of sect. 12 implies we have instead

(14.5)

$$
\begin{array}{ccc}
 & E_{a_1,\ldots,a_n} & \\
 & {}^{p}\swarrow \quad \searrow^{\jmath} & \\
\Lambda_{a_1} \times \cdots \times \Lambda_{a_n} & & \Lambda_{a_1 \cdots a_n}
\end{array}
$$

covering the multiplication in A. Using the choice of a section of the fibers Λ_a as above, we have the isomorphism of (R,Λ)-torsors

(14.6)
$$
e_{a_1,\ldots,a_n} \colon E_{a_1,\ldots,a_n} \xrightarrow{\sim} X_{a_1 \cdots a_n} = (R, x_{a_1 \cdots a_n})
$$

which follows from (14.5) and again the end of sect. 12. We can assume this isomorphism is realized by the choice of a point $e_{a_1,\ldots,a_n} \in E_{a_1,\ldots,a_n}$ such that $\jmath(e_{a_1,\ldots,a_n}) = x_{a_1 \cdots a_n}$.

We can write the morphism (12.1) with respect to this choice of local data. According to the proof of Theorem 10.1, let $a_{1,1},\ldots,a_{1,m_1},\ldots,a_{n,1},\ldots,a_{n,m_n} \in A$. For $i = 1,\ldots,n$ define $b_i = a_{i,1} \cdots a_{i,m_i}$. Consider the points $e_i = e_{a_{i,1},\ldots,a_{i,m_i}} \in E_{a_{i,1},\ldots,a_{i,m_i}}$ and $e_{b_1,\ldots,b_n} \in E_{b_1,\ldots,b_n}$. Then $[e_1,\ldots,e_n, e_{b_1,\ldots,b_n}]$ is a point of the composition $E_n(E_{m_1},\ldots,E_{m_n})$ over $\Lambda_{a_{1,1}} \times \cdots \times \Lambda_{a_{n,a_n}}$, and $\jmath([e_1,\ldots,e_n,e_{b_1\ldots,b_n}]) = \jmath(e_{b_1,\ldots,b_n}) = b_1 \cdots b_n$. In other words, we have an isomorphism of (R,Λ)-torsors

$$
E_n(E_{m_1},\ldots,E_{m_n})_{a_{1,1},\ldots,a_{n,m_n}} \xrightarrow{\sim} (R, x_{b_1 \cdots b_n}).
$$

On the other hand, since $E_{m_1+\cdots+m_n}$ covers the multiplication $(a_{1,1},\ldots,a_{n,m_n}) \to a_{1,1} \cdots a_{n,m_n} = b_1 \cdots b_n$, we have another isomorphism

$$
\left(E_{m_1+\cdots+m_n} \right)_{a_{1,1},\ldots,a_{n,m_n}} \xrightarrow{\sim} (R, x_{b_1 \cdots b_n})
$$

determined by a chosen section of $E_{m_1+\cdots+m_n}$. Thus, the morphism (12.1) amounts to an automorphism

(14.7)
$$
f_{n;m_1,\ldots,m_n}(a_{1,1},\ldots,a_{n,m_n}) \colon (R, x_{b_1 \cdots b_n}) \xrightarrow{\sim} (R, x_{b_1 \cdots b_n}),
$$

which, using standard facts about (R,Λ)-torsors, we identify with an element of M.

14.5 Cocycle computations

We compute the full class determined by $\mathscr{R}$ from the multi-extension structure of the presentation $R \to \Lambda \to \mathscr{R}$ and prove that it satisfies the full set of equations (14.1).

14.5.1 Setup

First, consider the following (not necessarily commutative) diagrams of (R,Λ)-torsor morphisms

(14.8a)

$$
\begin{array}{ccccc}
E_{a,c} \wedge E_{b,c} & \longrightarrow & E_{x_a + x_b, x_c} & \longrightarrow & E_{a+b,c} \\
\downarrow & & & & \downarrow \\
(R, x_{ac}) \wedge (R, x_{bc}) & =\!=\!= & (R, x_{ac} + x_{bc}) & \xrightarrow{\sigma_{ac,bc}^{-1}} & (R, x_{(a+b)c})
\end{array}
$$

and

(14.8b)

$$
\begin{array}{ccccc}
E_{a,b} \wedge E_{a,c} & \longrightarrow & E_{x_a, x_b + x_c} & \longrightarrow & E_{a,b+c} \\
\downarrow & & & & \downarrow \\
(R, x_{ab}) \wedge (R, x_{ac}) & =\!=\!= & (R, x_{ab} + x_{ac}) & \xrightarrow{\sigma_{ab,ac}^{-1}} & (R, x_{a(b+c)})
\end{array}
$$

arising from the two partial product laws of $E = E_2$. The vertical arrows are the identification (14.6). Both diagrams in (14.8) can be construed as defining an automorphism of their respective lower right corners, which can be identified with an element of M: let them be $-\alpha_1(a, b; c)$ and $\alpha_2(a; b, c)$ respectively.

On the one hand the top rows can be written as

$$(14.9) \qquad e_{a,c} \wedge e_{b,c} \mapsto e_{a+b,c}\, g_1(a, b; c), \qquad e_{a,b} \wedge e_{a,c} \mapsto e_{a,b+c}\, g_2(a; b; c),$$

by invoking (3.5). On the other hand, following the bottom part, we have

$$e_{a,c} \wedge e_{b,c} \mapsto \sigma_{ac,bc}^{-1}, \qquad e_{a,b} \wedge e_{a,c} \mapsto \sigma_{ab,ac}^{-1}.$$

Comparing the two we get the relations

$$(14.10) \qquad -\alpha_1(a, b; c) = g_1(a, b; c) + \sigma_{ac,bc}, \qquad \alpha_2(a; b, c) = g_2(a; b; c) + \sigma_{ab,ac}.$$

14.5.2 The relations (14.1d)

Consider the diagrams:

(14.11a)

$$
\begin{array}{ccc}
E_{x_{a+b}+x_c, x_d} & \xrightarrow{\ \sigma_{a+b,c}\ } & E_{a+b+c,d} \\
\ \ \big\uparrow{\sigma_{a,b}^{x_c}} & & \big\downarrow \\
E_{x_a+x_b+x_c, x_d} & & \\
\ \ \big\downarrow{\sigma_{b,c}} & & \big\downarrow \\
E_{x_a+x_{b+c}, x_d} & \xrightarrow{\ \sigma_{a,b+c}\ } & E_{a+b+c,d}
\end{array}
\qquad
\begin{array}{ccc}
E_{x_a, x_{b+c}+x_d} & \xrightarrow{\ \sigma_{b+c,d}\ } & E_{a,b+c+d} \\
\ \ \big\uparrow{\sigma_{b,c}^{x_d}} & & \big\downarrow \\
E_{x_a, x_b+x_c+x_d} & & \\
\ \ \big\downarrow{\sigma_{c,d}} & & \big\downarrow \\
E_{x_a, x_b+x_{c+d}} & \xrightarrow{\ \sigma_{b,c+d}\ } & E_{a,b+c+d}
\end{array}
$$

and

(14.11b)

$$
\begin{array}{ccc}
E_{x_a+x_b, x_c} & \xrightarrow{\ \sigma_{a,b}\ } & E_{a+b,c} \\
\big\uparrow{\eta^1} & & \big\uparrow \\
E_{x_b+x_a, x_c} & \xrightarrow{\ \sigma_{b,a}\ } & E_{a+b,c}
\end{array}
\qquad
\begin{array}{ccc}
E_{x_a, x_b+x_c} & \xrightarrow{\ \sigma_{b,c}\ } & E_{a,b+c} \\
\big\uparrow{\eta^2} & & \big\uparrow \\
E_{x_a, x_c+x_b} & \xrightarrow{\ \sigma_{c,b}\ } & E_{a,b+c}
\end{array}
$$

Both diagrams (14.11) are parts of more extended ones, giving rise to relations linking f_+ and g_+ to the other quantities comprising a 5-tuple satisfying relations (14.1) as follows.

First observe that by applying Lemma 13.3, the right vertical give the automorphisms corresponding to $f_+(a, b, c)\, d$ (resp. $a\, f_+(b, c, d)$) for (14.11a), and $g_+(a, b)\, c$ (resp. $a\, g_+(b, c)$) for (14.11b).

Then from (14.8) and (14.11a) form the obvious associativity diagrams for the morphisms $E_{a,d} \wedge E_{b,d} \wedge E_{c,d} \to E_{a+b+c,d}$ and $E_{a,b} \wedge E_{a,c} \wedge E_{a,d} \to E_{a,b+c+d}$. Using the cocycle decomposition (3.6) (and Lemma 13.3 for the right vertical arrows of (14.11)) we arrive at:

$$(14.12) \qquad
\begin{aligned}
g_1(a+b, c; d) + g_1(a, b; d)^{x_{cd}} &= g_1(a, b+c; d) + g_1(b, c; d) + f_+(a, b, c)\, d, \\
g_2(a; b+c, d) + g_2(a; b, c)^{x_{ad}} &= g_2(a; b, c+d) + g_2(a; c, d) + a\, f_+(b, c, d),
\end{aligned}
$$

(The difference with equations (3.6a) and (3.6c) arises because the top rows of (14.8), contrary to the actual partial multiplication morphisms, lack associativity.) Using (14.10), (14.12), and the first of (14.4), we obtain the first two of the cocycle relations (14.1d).

The commutativity diagrams obtained from (14.11b) and (14.8) can be analyzed in an analogous manner, utilizing the second of (14.4). However, we do not directly obtain the other two equations in the block (14.1d); instead, we arrive at their "flipped" counterpart (14.3).

14.5.3 The interchange relation (14.1c)

We check the compatibility law (1.5) after pulling back to $\Lambda_a \times \Lambda_b \times \Lambda_c \times \Lambda_d$. Use the diagram

(14.13)

where the triangles commute by definition of the morphisms determined by the top rows of (14.8); the pentagon (marked I) commutes by the compatibility law; the squares II and the square III are obviously commutative by functoriality (an explicit calculation uses Proposition 3.2 and the equations (6.3)). Thus the diagram commutes, and we can use the cocycle equation (3.7) directly, written additively, which gives:

$$g_2(a+b;c,d) + g_1(a,b;c)^{x_{(a+b)d}} + g_1(a,b;d) = g_1(a,b;c+d) + g_2(a;c,d)^{x_{b(c+d)}} + g_2(b;c,d) - \{x_{bc}, x_{ad}\}^{x_{bd}}.$$

Inserting equations (14.10) and using (14.4) finally gives the third block (14.1c) of the cocycle relation (14.1).

14.5.4 The relations (14.1b) and (14.1a)

We specialize the expression (14.7) for the morphism (12.1) to the tuples $(2; 2, 1)$ and $(2; 1, 2)$. The morphism μ given by (C.1) corresponds to the element

$$f(a,b,c) = -f_{(2;1,2)}(a,b,c) + f_{(2;2,1)}(a,b,c) \in M.$$

More precisely, using the isomorphism with (R, x_{abc}) as a reference trivialization, we can identify $f(a,b,c)$ with an automorphism of E_3 pulled back to $\Lambda_a \times \Lambda_b \times \Lambda_c$, hence with a section of M over it. Explicitly, locally on $\Lambda_a \times \Lambda_b \times \Lambda_c$, we have the morphism of (R, Λ)-torsors

(14.14)
$$\mu_{a,b,c} \colon E_2(E_2, I)_{a,b,c} \longrightarrow E_2(I, E_2)_{a,b,c},$$

which, using the composition of multi-extensions given in section 10, we can write as

(14.15)
$$\mu([e_{a,b}, 0_c, e_{ab,c}]) = [0_a, e_{b,c}, e_{a,bc}] - f(a,b,c),$$

where $[e_{a,b}, e_{ab,c}] \in E_2(E_2, I)_{a,b,c}$ and $[e_{b,c}, e_{a,bc}] \in E_2(I, E_2)_{a,b,c}$. (Recall the additivity in the notation; we extend it to the action of R on torsors. The brackets denote the class under the action of R. The identity map is represented by the trivial butterfly, and here 0_a represents the unit section of the underlying trivial torsor $\Lambda \times R$ at the point x_a, say.) We have $\jmath([e_{a,b}, e_{ab,c}]) = \jmath([e_{b,c}, e_{a,bc}]) = x_{abc}$, so we can see directly that $f(a,b,c) \in M$.

As an isomorphism of tri-extensions, μ is a homomorphism for each of the three partial laws. Writing these conditions for (14.14), we must compute the maps along the following diagram

and the other two expressing the compatibility (or lack thereof) with the second and third partial laws. Because there are some new elements compared to the calculations which have appeared thus far, we sketch

some of the details. In particular, to compute the two horizontal maps in the second square above we need the explicit form of the trivializations s_1 for both tri-extensions. Similarly for the other two diagrams. According to the last part of section 10, the form of $s_1^{2,1}$ for $E_2(E_2, I)$ is:

$$s_1^{2,1}(r, x_c, x_d) = [s_1(r, x_c), 0_{x_d}, 1_{x_d}], \quad r = \sigma_{a,b},$$

where 0_{x_d} denotes the unit section of $\Lambda \times R$ computed at $x_d \in \Lambda$, and 1_{x_d} denotes the unit section of $E_2|_{\{0\} \times \Lambda} \simeq \Lambda \times R$ similarly computed at x_d. Thus, the map denoted by $\sigma_{a,b}$ in the top row is given by

$$e \longmapsto e \times_1 s_1(\sigma_{a,b}, x_c, x_d).$$

Similarly, for the analogous map in the bottom row we must use the expression

$$s_1^{1,2}(r, x_c, x_d) = [0_{\partial r}, e_{c,d}, s_1(r, x_c)], \quad r = \sigma_{a,b},$$

with a similar interpretation of the notation.

Thus, the upper path to the lower right corner gives

$$
\begin{aligned}
\mu\big([e_{a,c}, e_{ac,d}] \times_1 [e_{b,c}, e_{bc,d}] \times_1 [s_1(\sigma_{a,b}, x_c), 0_d, 1_{x_d}]\big) &= \mu[e_{a,c} \times_1 e_{b,c} \times_1 s_1(\sigma_{a,b}, x_c), 0_d, e_{ac,d} \times_1 e_{bc,d}] \\
&= \mu[e_{a+b,c} + g_1(a, b; c), 0_d, e_{ac,d} \times_1 e_{bc,d}] \\
&= \mu[e_{a+b,c}, 0_d, e_{ac,d} \times_1 e_{bc,d} \times_1 s_1(-g_1(a, b; c), x_d)] \\
&= \mu[e_{a+b,c}, 0_d, e_{ac,d} \times_1 e_{bc,d} \times_1 s_1(\sigma_{ac,bc} + \alpha_1(a, b; c), x_d)] \\
&= \mu[e_{a+b,c}, 0_d, e_{ac+bc,d}] + g_1(ac, bc, d) - \alpha_1(a, b; c)d \\
&= [0_{a+b}, e_{c,d}, e_{a+b,cd}] + g_1(ac, bc; d) - f(a + b, c, d) - \alpha_1(a, b; c)d
\end{aligned}
$$

where in the next to last we have used (6.3), the first of (14.9), and Lemma 13.3; to obtain the last we have used (14.15).

On the other hand, the lower path to the lower right corner gives, with similar calculations

$$\mu[e_{a,c}, e_{ac,d}] \times_1 \mu[e_{b,c}, e_{bc,d}] \times_1 [0_{\partial \sigma_{a,b}}, e_{c,d}, s_1(\sigma_{a,b}, x_{cd})] = [0_{a+b}, e_{c,d}, e_{a+b,cd}] + g_1(a, b; cd) - f(a, c, d) - f(b, c, d).$$

Comparing the two expressions and using (14.10) yield the first equation of block (14.1b). The others are obtained via identical means.

The last equation (14.1a) becomes now the easiest to obtain, as a condition satisfied by the automorphism μ upon considering the five possible pullbacks to the quadri-extension E_4, as per the pentagon diagram (C.2). We leave the details to the reader.

14.6 The class of a ring-like stack

Assembling the steps in sect. 14.5, we have the following

14.8 Proposition. *Let $\mathscr{R}$ be a (weakly) ring-like stack with $\pi_0(\mathscr{R}) = A$ and $\pi_1(\mathscr{R}) = M$. A decomposition of $\mathscr{R}$ determines a twisted cocycle $\xi = \xi_{\mathscr{R}}$ satisfying (14.2) with the same A-bimodule structure. An equivalence $\mathscr{R} \to \mathscr{R}'$ gives rise to two twisted cocycles $\xi_{\mathscr{R}}$ and $\xi_{\mathscr{R}'}$, differing by a coboundary, hence equivalence classes are in one-to-one correspondence with elements of $\widetilde{H}_2^3(A, M)$.*

Proof. The first statement is a consequence of the preceding calculations. The statement about the equivalence follows from the definition of morphism of pseudo-monoid and the fact that the structure is preserved across pseudo-functors between multi-bicategories (cf. B.3 and B.4), in particular between 12.5

Since the only difference between all these complexes occurs in degrees 3 and 4, the statements carry over to the hypercohomology situation. $\qquad \square$

Let us use the notation $[\mathscr{R}]$ for the class determined by $\mathscr{R}$. Thus of $[\mathscr{R}] = [\xi_{\mathscr{R}}]$. As we have seen, a consequence of Proposition 14.3 and Lemma 14.7 is that in the unital case the classes in $\widetilde{H}_2^3(A, M)$ lift to $HML^3(A, M)$.

14.9 Corollary. *Let $\mathscr{R}$ be as above, with in addition a unit object for the exterior monoidal structure. Then A is unital and there exists an equivalence $\mathscr{R} \xrightarrow{\sim} \mathscr{R}'$ where the underlying categorical group structure of $\mathscr{R}'$ is braided symmetric. Hence $[\mathscr{R}] = [\mathscr{R}'] \in HML^3(A, M)$.* $\qquad\square$

We briefly address the question of recovering $\mathscr{R}$ (up to equivalence) from $[\mathscr{R}]$. Consider a class $[\xi] \in \widetilde{H}_2^3(A, M)$. A portion of ξ will represent a class in $H^4(K(A, 2), M)$, possibly lifting into the stable range. Let ξ_+ denote this projection. Standard techniques [Bre94, §7.6–7] allow to reconstruct a braided (possibly symmetric) stack $\mathscr{R} = \mathscr{R}_\xi$ from ξ_+, equipped with $\varpi \colon \mathscr{R} \to A$ fitting into the standard short exact sequence $\mathscr{M} \to \mathscr{R} \to A$, with $\mathscr{M} = \mathrm{Tors}(M)$. Briefly, ξ_+ determines a 2-gerbe over a simplicial model of $K(A, 2)$ or $K(A, 3)$, suitably re-scaled so that the relevant class appears in degree three. $\mathscr{R}$ is obtained by gluing trivial gerbes with band $M \to 0$ over A along ξ_+. (We must supplement the cocycles in loc. cit. with those parts pertaining to the braiding structure.) The class of $\mathscr{R}$ is that of ξ_+, and therefore it is equipped with a decomposition (14.4).

14.10 Proposition. *Let $\mathscr{R}$, ξ, and ξ_+ be as above. Let $R \to \Lambda \to \mathscr{R}$ be a presentation by a braided crossed module. Then $R \to \Lambda$ carries a bi-extension whose class is ξ.*

Proof (Sketch). The main point is to reverse the computation of the cohomology class, using the equations for the cocycle ξ (either (14.1) or their twisted form (14.2)) to obtain a well-defined biextension $E \to \Lambda \times \Lambda$ providing (R, Λ) with a pseudo-monoid structure.

Starting with the implementation of (14.5), for all $a, b \in A$ define $E_{a,b} \to \Lambda_a \times \Lambda_b$ as $E_{a,b} = \Lambda_a \times \Lambda_b \times R$ equipped with $\jmath \colon E_{a,b} \to \Lambda$ given by $0_R \mapsto x_{ab}$. From (14.10), given ξ and the decomposition (14.4), we compute the nonabelian cocycles g_1 and g_2 we can use to define partial laws

$$\times_1 \colon E_{a,c} \wedge^R E_{b,c} \longrightarrow E_{a+b,c}, \qquad \times_2 \colon E_{a,b} \wedge^R E_{a,c} \longrightarrow E_{a,b+c},$$

which are well defined by equations (14.12) and (14.13).

Note that, unless $a = 0$, the pullback of $\partial \colon R \to \Lambda$ along $\Lambda_a \to \Lambda$ is trivial, whereas it is isomorphic to $\mathrm{Im}\,\partial$ for $a = 0$. This ensures (again from equation (14.5) and choosing a normalized section x) the triviality conditions required by Definition 6.1 to have a full-fledged butterfly are satisfied.

Finally, we use (14.14) and (14.15) to define $\mu \colon E_2(E_2, I) \to E_2(I, E_2)$; the identity

$$g_1(ac, bc; d) - f(a + b, c, d) - \alpha_1(a, b; c)d = g_1(a, b; cd) - f(a, c, d) - f(b, c, d),$$

and its companions found above, together with (14.1a) and (14.1b), ensure E satisfies the required pentagonal structure. $\qquad\square$

Part IV

A Multi-variable compositions

In this technical addendum we give a brief treatment of multivariable functor calculus in a bicategory. Our approach is descriptive and explicit, and it is aimed at a definition of pseudo-monoid suitable for the applications in the text to multi-additive functors and multi-linear butterflies (cf. sects. 9 and following).

We resort to multi categorical-based ideas, in fact we borrow some of kelly's clubs formalism [see Kel72; Kel74], which is convenient in the present context.[11] We include permutations, even though this is slightly more general than needed in the main part of the text. Permutations can be included at nearly no additional cost, covering the symmetric monoidal case, which is what the formalism was originally designed to do. As a result, the formalism can still be used to symmetrize the (external) monoidal structures described in the text. Unlike [Kel72], we need our objects to inhabit a bicategory, as opposed to a 2-category, due to the fact that crossed modules equipped with butterflies as morphisms form a genuine bicategory equivalent to the 2-category of group-like stacks.

---

[11]See also [Tho95] for further applications to (symmetric) monoidal categories.

Henceforth, we shall fix an ambient *groupoidal* bicategory $\mathfrak{C}$ of objects $x, y, \ldots$ Groupoidal means that its $\mathbf{Hom}_{\mathfrak{C}}$ categories are all groupoids. We shall require that for all tuples $(y_1, \ldots, y_n; x)$ of objects of $\mathfrak{C}$ there exists a groupoid $\mathbf{Hom}_{\mathfrak{C}}(y_1, \ldots, y_n; x)$ with compositions, formalized below, that follow the same pattern as in multi-categories (see e.g. [Lei04]). The examples occurring in the main text are the groupoids $\mathbf{MExt}(H_{1,\bullet}, \ldots, H_{n,\bullet}; G_\bullet)$ and $\mathbf{Hom}(\mathscr{H}_1, \ldots, \mathscr{H}_n; \mathscr{G})$ introduced in sect. 9. We also want to have, on each $\mathbf{Hom}_{\mathfrak{C}}(y_1, \ldots, y_n; x)$, a permutation action compatible with the composition.

Thus, our first step is to let $\mathrm{M}\mathfrak{C}$ (M is for "multi") be an **extraordinary (bi-)multicategory** with the same objects as $\mathfrak{C}$.[12] Explicitly, $\mathrm{M}\mathfrak{C}$ is given by:

- A class of objects $x, y, \ldots$ (those of $\mathfrak{C}$).

- For each tuple $(y_1, \ldots, y_n; x)$ of objects, a groupoid of arrows $\mathbf{Hom}_{\mathfrak{C}}(y_1, \ldots, y_n; x)$. The cells are denoted

$$(y_1, \ldots, y_n) \underset{g}{\overset{f}{\Longrightarrow}}^{\Downarrow \alpha} x.$$

- For each object x, a functor $\iota_x \colon \mathbf{1} \to \mathbf{Hom}_{\mathfrak{C}}(x; x)$, where $\mathbf{1}$ is the singleton category. The resulting distinguished object is the identity arrow $\mathrm{id}_x \colon (x) \to x$.

- Compositions functors

$$\mathbf{Hom}(y_1, \ldots, y_n; x) \times \mathbf{Hom}(z_{1,1}, \ldots, z_{1,m_1}; y_1) \times \cdots \times \mathbf{Hom}(z_{n,1} \ldots, z_{n,m_n}; y_n) \longrightarrow \mathbf{Hom}(z_{1,1}, \ldots, z_{n,m_n}; x)$$
$$(f; g_1, \ldots, g_n) \longrightarrow f(g_1, \ldots, g_n).$$

- Associativity data for the composition and the identities as in a bicategory.

- For each n and for each tuple $(y_1, \ldots, y_n; x)$, an action by Σ_n, that is, a functor

$$\xi^* \colon \mathbf{Hom}_{\mathfrak{C}}(y_{\xi(1)}, \ldots, y_{\xi(n)}; x) \longrightarrow \mathbf{Hom}_{\mathfrak{C}}(y_1, \ldots, y_n; x)$$

such that $(\xi \eta)^* \simeq \eta^* \xi^*$ and the composition functors are equivariant for this action.

The definition of pseudo- (or lax-)functor $F \colon \mathrm{M}\mathfrak{C} \to \mathrm{M}\mathfrak{B}$ between multi-bicategories is *mutatis mutandis* the same as for bicategories. For the symmetric structure, we add the condition that the functors $F_{y_1, \ldots, y_n; x} \colon \mathbf{Hom}_{\mathrm{M}\mathfrak{C}}(y_1, \ldots, y_n; x) \to \mathbf{Hom}_{\mathrm{M}\mathfrak{B}}(F(y_1), \ldots, F(y_n); F(x))$ preserve the Σ_n-action, for all n.

We identify the bicategory $\mathfrak{C}$ with the one obtained from $\mathrm{M}\mathfrak{C}$ by dropping all $\mathbf{Hom}$-groupoids except those of the form $\mathbf{Hom}_{\mathfrak{C}}(y; x)$.

Without the last requirement in the list defining $\mathrm{M}\mathfrak{C}$, the data define what ought to be indeed called a bi-multicategory. The last item intuitively affords for cells of the form

(A.1)

$$
\begin{array}{ccc}
(y_1, \ldots, y_n) & \longrightarrow & \\
\downarrow & \nearrow\!\!\!\nearrow & x \\
(y_{\xi(1)}, \ldots y_{\xi(n)}) & \longrightarrow &
\end{array}
$$

A.1 Remark. The above diagram becomes precise if we use the global definition of $\mathrm{M}\mathfrak{C}$ in terms of **spans** of categories [see Shu10; and CS10, for instance], namely as a diagram

$$
\begin{array}{ccc}
 & \mathbf{C}_1 & \\
{}^t \swarrow & & \searrow {}^s \\
C_0 & & \mathbf{T}C_0
\end{array}
$$

---

[12]Without the extraordinary structure given by permutations, the name "bi-multicategory" appears in [Pis14] to denote the bicategory-analog of a multicategory: a $\mathfrak{C}$at-enriched multicategory with weakly associative composition. For enrichment over simplicial sets, see also [Rob11], which also incorporates permutations.

where C_0 is a discrete category (corresponding to the class of objects of $\mathfrak{C}$) and $\mathbf{T}C_0$ is the strict free symmetric monoidal category on C_0— its objects are finite sequences of elements of C_0 with permutations as morphisms. Thus an object of $\mathbf{C}_1$ is a multi-arrow in M$\mathfrak{C}$, and an arrow in $\mathbf{C}_1$ gives rise to a diagram like eq (A.1). To recall the complete picture, the composition arises from the pullback diagram:
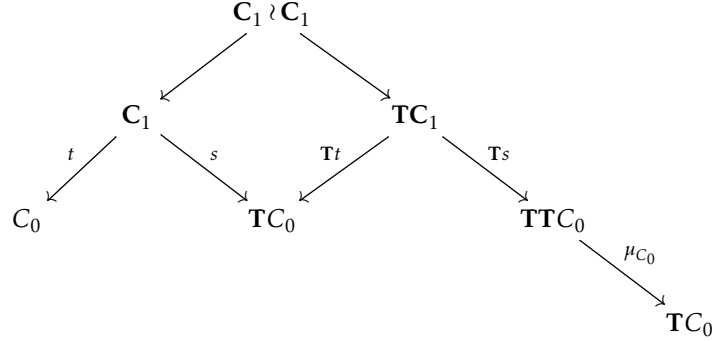
where $\mu\colon \mathbf{TT} \to \mathbf{T}$ is structure of the strict symmetric monoidal monad. On the discrete category C_0 it "flattens" the lists of objects as:

$$((z_{1,1},\ldots,z_{1,m_1}),\ldots,(z_{n,1},\ldots,z_{n,m_n})) \longrightarrow (z_{1,1},\ldots,z_{n,m_n})$$

whereas at the level of morphisms it is the standard map $\Sigma_{m_1} \times \ldots \Sigma_{m_n} \hookrightarrow \Sigma_{m_1+\cdots+m_n}$ of permutation groups sending the tuple $(\xi_1,\ldots,\xi_n)$ to $\xi_1 + \cdots + \xi_n$. The summit of the diagram being defined by the pullback, the composition amounts to a morphism $\bigcirc\colon \mathbf{C}_1 \wr \mathbf{C}_1 \to \mathbf{C}_1$. We shall refer to this map as the **assembly map**.[13] $\square$

We can recover the span version of M$\mathfrak{C}$ starting from the item-by-item description given above by applying the Grothendieck construction on the presheaf defined as follows. First, by calling C_0 the class of objects of M$\mathfrak{C}$, consider the category $C_0 \times \mathbf{T}C_0$ as in Remark A.1. The only possible morphisms are of the form $\tilde{\xi}\colon (y_1,\ldots,y_n;x) \to (y_{\xi(1)},\ldots,y_{\xi(n)};x)$ with $\xi \in \Sigma_n$. Then, the various $\mathbf{Hom}_{\mathfrak{C}}(y_1,\ldots,y_n;x)$ define a presheaf of groupoids on $C_0 \times \mathbf{T}C_0$ with pullback functors given by the various ξ^*. Let $\mathbf{C}_1$ be the Grothendieck construction performed on this presheaf. As a result, we obtain a fibration $\mathbf{C}_1 \to C_0 \times \mathbf{T}C_0$ for which a morphism, by construction, has precisely the form in eq. (A.1). Moreover, composing the fibration with the projections to C_0 and $\mathbf{T}C_0$ gives a span as in Remark A.1.

A.2 Definition (Kelly [Kel72; Kel74]). A **Club** is an extraordinary multicategory with one object.

We have $\mathbf{T}(\mathbf{1}) = \mathbf{S}$, the skeletal category of finite sets with permutations as morphisms. ($\mathbf{S}$ can be obtained as the core, i.e. the largest subgroupoid, of the Segal category Γ of finite sets.) As a result, the span corresponding to a club $\mathbf{k}$ has the form

$$\mathbf{k} \longrightarrow (\mathbf{1} \times \mathbf{S}) \simeq \mathbf{k} \longrightarrow \mathbf{S},$$

and so is an object of $\mathfrak{Cat}/\mathbf{S}$ (or actually $\mathfrak{Grpds}/\mathbf{S}$ in our case), as in Kelly's original definition. It is easy to see that the composite category $\mathbf{k} \wr \mathbf{k}$ is a wreath product [see e.g. Ber07], hence the choice of the symbol.

Examples of clubs are:

- $\mathbf{S}$ itself, equipped with id$\colon \mathbf{S} \to \mathbf{S}$.

- The natural numbers identified with a discrete subcategory $\mathbf{N}$ of $\mathbf{S}$.

In both cases $\mathbf{k} = \mathbf{S}$ and $\mathbf{k} = \mathbf{N}$, the composition morphism $\mathbf{k} \wr \mathbf{k} \to \mathbf{k}$ sends the object $(n;m_1,\ldots,m_n)$ to $m_1 + \cdots + m_n$.

---

[13]The choice of the symbol $\wr$ will be explained below.

A.3 Definition (Kelly [Kel72])**.** Let x, y be two objects of $\mathfrak{C}$. Let ι_x and ι_y be the inclusions of the singletons on x and y into the class of objects of $\mathfrak{C}$. Then $\{y, x\}$ denotes the category which is part of the span
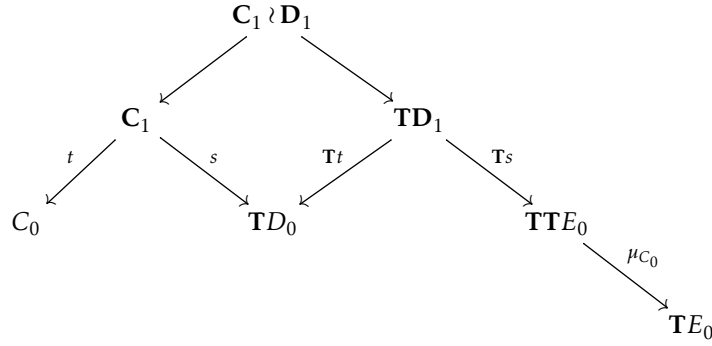
$$
\begin{array}{ccc}
 & \{y,x\} & \\
 {}^{t}\swarrow & & \searrow^{s} \\
\{x\} & & \mathbf{T}\{y\}
\end{array}
$$
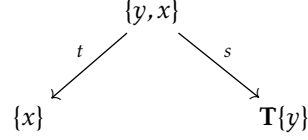
obtained by pulling back the span corresponding to $\mathrm{M}\mathfrak{C}$ along the functor $(\iota_x, \mathbf{T}\iota_y)\colon \{x\} \times \mathbf{T}\{y\} \to C_0 \times \mathbf{T}C_0$.

Observe that $\mathbf{T}\{y\} \simeq \mathbf{S} = \mathbf{T}(\mathbf{1})$, hence $\{y, x\}$ is again a category over $\mathbf{S}$. It is the category of all possible multi-arrows from any n-tuple $(y, \ldots, y)$ (n-copies) to x, for arbitrary values of the natural number n.

General spans can be composed by a procedure identical to that outlined in Remark A.1. If $C_0 \leftarrow \mathbf{C}_1 \to \mathbf{T}D_0$ and $D_0 \leftarrow \mathbf{D}_1 \to \mathbf{T}E_0$ are two spans, the composite is again[14]

$$
\begin{array}{ccccccc}
 & & & \mathbf{C}_1 \wr \mathbf{D}_1 & & & \\
 & & \swarrow & & \searrow & & \\
 & \mathbf{C}_1 & & & & \mathbf{T}\mathbf{D}_1 & \\
 {}^{t}\swarrow & & \searrow^{s} & & {}^{\mathbf{T}t}\swarrow & & \searrow^{\mathbf{T}s} \\
C_0 & & & \mathbf{T}D_0 & & & \mathbf{T}\mathbf{T}E_0 \\
 & & & & & & \searrow^{\mu_{C_0}} \\
 & & & & & & \mathbf{T}E_0
\end{array}
$$

By applying this construction to objects x, y, z and the categories $\{y, x\}$ and $\{z, y\}$ we immediately obtain

A.4 Proposition (Kelly [Kel72])**.** *There is an assembly map*

$$\{y, x\} \wr \{z, y\} \longrightarrow \{z, x\},$$

sending the object $(f; g_1, \ldots, g_n)$ *to the composition* $f(g_1, \ldots, g_n)$ *and the morphism* $(\alpha; \beta_1, \ldots, \beta_n)$ *to* $(\alpha^\xi; \beta_1^{\eta_1}, \ldots, \beta_n^{\eta_n})$, *where* $\xi \in \Sigma_n$, $\eta_i \in \Sigma_{m_i}$ *are images of* α, β_i *under the structure map to* $\mathbf{S}$.

Associativity for this composition is inherited from the ambient multicategory $\mathrm{M}\mathfrak{C}$, hence, for objects x, y, z, w of $\mathrm{M}\mathfrak{C}$ we have:

(A.2)

$$
\begin{array}{ccc}
\big(\{y,x\} \wr \{z,y\}\big) \wr \{w,z\} & \xrightarrow{\;\;ass.\;\;} & \{y,x\} \wr \big(\{z,y\} \wr \{w,z\}\big) \\
{\scriptstyle \circ \wr id}\downarrow & \quad \nearrow\!\!\!\!/ & \downarrow {\scriptstyle id \wr \circ} \\
\{z,x\} \wr \{w,z\} \longrightarrow \{w,x\} & \longleftarrow & \{y,x\} \wr \{w,y\}
\end{array}
$$

As a corollary we have that for any object x, $\{x, x\}$ is a (pseudo-)monoid for the assembly map, with identity given by (id;) (the empty string in the second slot).

B (Unbiased) Monoids and monoidal structures

We keep the notations and environment of section A. $\mathfrak{C}$ is a bicategory identified with the underlying bicategory of a bi-multicategory $\mathrm{M}\mathfrak{C}$. We add another piece of notation: for an object x, let $\mathrm{M}x = \{x, x\}$. Having observed that this is a monoid for the composition, a monoid object in $\mathfrak{C}$ is defined in the expected way. Recall that $\mathbf{N}$ is a club, as discussed in the previous section.

---

[14]Comparing with the span definition of $\mathrm{M}\mathfrak{C}$, this explains why the bi-multicategory can also be defined as a *monad* in spans.

B.1 Definition. A **(pseudo-)monoid** in $\mathfrak{C}$ is an object x of $M\mathfrak{C}$ equipped with a lax monoidal functor $(t,\theta)\colon \mathbf{N}\to Mx$.

Thus, a (pseudo-)monoid structure on x amounts to that of a **pseudo-algebra** over the club $\mathbf{N}$. Similarly, a symmetric monoid would be a pseudo-algebra over the club $\mathbf{S}$, instead.

Explicitly, the lax monoidal functor is given by a diagram

(B.1)
$$
\begin{array}{ccc}
\mathbf{N}\wr\mathbf{N} & \xrightarrow{\ \ \circ\ \ } & \mathbf{N} \\
{\scriptstyle t\wr t}\big\downarrow & \Swarrow_{\theta} & \big\downarrow{\scriptstyle t} \\
Mx\wr Mx & \xrightarrow[\ \ \circ\ \ ]{} & Mx
\end{array}
$$

From it we have the coordinate expression for the natural transformation θ, which assigns to the object $(n;m_1,\dots,m_n)\in\mathbf{N}\wr\mathbf{N}$ the morphism

(B.2)
$$\theta_{n;m_1,\dots,m_n}\colon t(m_1+\cdots+m_k)\longrightarrow t(n)(t(m_1),\dots,t(m_n))$$

in Mx. These data are subject to be compatible with the associativity conditions for the assembly maps of both $\mathbf{N}$ and Mx, namely they must satisfy the following commutative diagram of natural transformations:

(B.3)

In (B.3) the top quadrangle and the small pentagon are strictly commutative. For the quadrangle, it follows from the functoriality of the associator, whereas for the small pentagon the associativity morphism reads

$$((n;m_1,\dots,m_n);l_1,\dots,l_k)\longmapsto (n;(m_1;l_1,\dots),\dots,(m_n;\dots,l_k)),$$

where $k=m_1+\cdots+m_n$; both paths evaluate to $h=l_1+\cdots+l_k=j_1+\cdots+j_n$, where for $i=1,\dots,n$ we have $j_i=l_{m_{i-1}+1}+\cdots+l_{m_{i-1}+m_i}$. Lastly, the back face of the diagram, that is the large pentagon, is just (A.2) specialized to the same object. Thus, the commutativity in diagram (B.3), when expressed in coordinates, reads as follows:

(B.4)
$$
\begin{array}{ccc}
t(k)(t(l_1),\dots,t(l_k)) \longleftarrow\!\!\!\!\!\!\!\! & t(h) & \!\!\!\!\!\!\!\!\longrightarrow t(n)(t(j_1),\dots,t(j_n)) \\
\big\downarrow & & \big\downarrow \\
t(n)(t(m_1),\dots,t(m_n))(t(l_1),\dots,t(l_k)) & \longrightarrow & t(n)(t(m_1)(t(l_1),\dots),\dots,t(m_n)(\dots,t(l_k)))
\end{array}
$$

B.2 Remark. By construction, the pseudo-monoid structure on x given by the club morphism $t\colon \mathbf{N}\to Mx$ possesses n-ary operations $t(n)$ for arbitrary values of n. If x has an internal structure (namely objects, as in the main text), then this implements a monoidal structure in "unbiased" form [see e.g. Lei04], with the n-ary unprivileged operations of all degrees. In this case diagram (B.3) expresses the coherence condition in unbiased form, namely, that the two possible ways to remove parentheses from an expression must coincide as it is apparent from the coordinate version (B.4).

One can consider morphisms of monoids as follows. Let $f\colon y\to x$ be a (unary) arrow. It determines two functors

$$f_*\{y,y\}\longrightarrow\{y,x\} \qquad \mathbf{T}f^*\{x,x\}\longrightarrow\{y,x\}$$

by post-composing with f or pre-composing with $\mathbf{T}f$, respectively.

B.3 Definition. Let x, y be (pseudo-)monoids in $\mathcal{C}$. A **morphism of monoids** is a pair $(f\colon y \to x, \lambda)$ fitting the diagram

$$
\begin{array}{ccc}
\mathbf{N} & \xrightarrow{\ t_y\ } & \{y, y\} \\
{\scriptstyle t_x}\big\downarrow & \mathllap{\nearrow}_{\lambda} & \big\downarrow{\scriptstyle f_*} \\
\{x, x\} & \xrightarrow[\ \mathbf{T}f^*\ ]{} & \{y, x\}
\end{array}
$$

and compatible with the pseudo-algebra condition (B.1).

The compatibility between the above diagram and (B.1) means that in the coordinates of (B.2) we have

$$
\begin{array}{ccc}
f \circ t_y(m_1 + \cdots + m_k) \xrightarrow{\ f\theta_y\ } f \circ (t_y(n)(t_y(m_1), \ldots, t_y(m_n))) \xrightarrow{\ \textcircled{1}\ } (f \circ t_y(n))(t_y(m_1), \ldots, t_y(m_n)) \\
{\scriptstyle \lambda_{m_1 + \cdots + m_n}}\big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big\downarrow{\scriptstyle \lambda_n \circ \mathrm{Id}} \\
t_x(m_1 + \cdots + m_k) \circ \mathbf{T}f \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t_x(n) \circ \mathbf{T}f)(t_y(m_1), \ldots, t_y(m_n)) \\
{\scriptstyle \theta_x \mathbf{T}f}\big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big\downarrow{\scriptstyle \textcircled{2}} \\
(t_x(n)(t_x(m_1), \ldots, t_x(m_n))) \circ \mathbf{T}f \xrightarrow[\textcircled{3}]{} t_x(n)((t_x(m_1), \ldots, t_x(m_n)) \circ \mathbf{T}f) \xleftarrow[\mathrm{Id} \circ \lambda_{m_1, \ldots, m_n}]{} t_x(n)(f \circ t_y(m_1), \ldots, f \circ t_y(m_n))
\end{array}
$$

where the numbered arrows result from associativity isomorphism for composition.

Pseudo-monoids behave in the expected manner with respect to pseudo-functors. Specifically, let $F\colon \mathrm{M}\mathcal{C} \to \mathrm{M}\mathcal{B}$ be a pseudo-functor, which by virtue of our identification, we think of as coming from an pseudo-functor $F\colon \mathcal{C} \to \mathcal{B}$ between bicategories. It is clear that for any objects x, y of $\mathcal{C}$, F induces a functor

$$
F_{(y);x}\colon \{y, x\} \longrightarrow \{F(y), F(x)\}.
$$

B.4 Definition. Let x be a pseudo-monoid in $\mathcal{C}$. Then $F(x)$ acquires the structure of a pseudo-monoid in $\mathcal{B}$ by virtue of the composition:

$$
\mathbf{N} \xrightarrow{\ t\ } \mathrm{M}x \xrightarrow{\ F_\bullet\ } \mathrm{M}F(x),
$$

with the full pseudo-monoid structure for $\mathrm{M}F(x)$ results from the composition

$$
\begin{array}{ccccc}
\mathbf{N} \wr \mathbf{N} & \xrightarrow{\ t \wr t\ } & \mathrm{M}x \wr \mathrm{M}x & \xrightarrow{\ F_\bullet \wr F_\bullet\ } & \mathrm{M}F(x) \wr \mathrm{M}F(x) \\
{\scriptstyle \circ}\big\downarrow & {\scriptstyle \theta}\mathrlap{\nearrow} & \big\downarrow{\scriptstyle \circ} & {\scriptstyle \varepsilon}\mathrlap{\nearrow} & \big\downarrow{\scriptstyle \circ} \\
\mathbf{N} & \xrightarrow{\ t\ } & \mathrm{M}x & \xrightarrow{\ F_\bullet\ } & \mathrm{M}F(x)
\end{array}
$$

C Pentagons

Pentagon diagrams express the coherence condition in a monoidal category. This condition is replaced by a diagram of the form (B.3) or (B.4) if the monoidal structure is given in unbiased form [Lei04]. The actual pentagon can be recovered if these diagrams are specialized to arities equal to 4 and the monoidal category comes from a monoid inhabiting a 2-category. This shows the equivalence of the biased and unbiased definitions. A slight generalization of the pentagon occurs if the monoid inhabits a bicategory, and both these situations arise in the main text. In addition, conditions arising from pentagons are cocycle conditions in appropriate cohomology theories. It is useful to compute them once and for all in the general setting of a monoid object in a bicategory.

To begin with, observe that by applying (B.1) and (B.2) to the objects $(2; 2, 1)$ and $(2; 1, 2) \in \mathbf{N} \wr \mathbf{N}$ we obtain morphisms $\theta_{2;2,1}\colon t(3) \to t(2)(t(2), t(1))$, $\theta_{2;1,2}\colon t(3) \to t(2)(t(1), t(2))$, and combining these two we obtain

(C.1) $$\mu\colon t(2)(t(2), t(1)) \longrightarrow t(2)(t(1), t(2)).$$

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The pentagons (or the diagrams related to them) arise from the decomposition of the operation $t(4)$, by way of (B.4), down to terms only involving the binary and unary operations $t(2)$ and $t(1)$, respectively. We can assume that the unary operation $t(1)\colon x \to x$ coincides with the identity id_x. These decompositions can be encoded by working our way along the small pentagon in the diagram (B.3), so for instance one of them corresponds to the sequences

$$
\begin{array}{ccc}
((2;2,1);2,1,1) & \longrightarrow & (2;(2;2,1),1) \\
\downarrow & & \downarrow \\
(3;2,1,1) \longrightarrow & 4 & \longleftarrow (2;3,1)
\end{array}
$$

(We have simply written 1 in place of the more accurate but cumbersome expression $(1;1)$.)

There are six distinct sequences including the one above. Their starting points, counted from the upper left corner of the small pentagon in the diagram (B.3), are:

1. $((2;2,1);2,1,1)$,

2. $((2;2,1);1,2,1)$,

3. $((2;1,2);1,2,1)$,

4. $((2;1,2);1,1,2)$,

5. $((2;1,2);2,1,1)$,

6. $((2;2,1);1,1,2)$.

Let us also use the notations $m_i = t(i)$, $i = 1,\ldots,4$, with the special provision $m_1 = \mathrm{id}_x = \mathrm{id}$. With these, applying (B.4) and (B.3), we obtain the diagram on page 51:

(C.2)

$$m_2(\text{id}(m_2), m_2(\text{id}, \text{id})) \longrightarrow m_2(\text{id}, m_2)(m_2, \text{id}, \text{id})$$

$$m_2(m_2(\text{id}, \text{id}), \text{id}(m2))$$

⑤

$$m_2(m_2, \text{id})(m_2, \text{id}, \text{id})$$

μ

$$m_2(m_2, m_2) \longleftarrow m_3(m_2, \text{id}, \text{id})$$

$$m_2(m_2, \text{id})(\text{id}, \text{id}, m_2)$$

⑥

①

$$m_2(m_2(m_2, \text{id}), \text{id})$$

μ

$$m_3(\text{id}, \text{id}, m_2) \longleftarrow m_4 \longrightarrow m_2(m_3, \text{id})$$

μ

$$m_2(\text{id}, m_2)(\text{id}, \text{id}, m_2)$$

④

②

$$m_2(m_2(\text{id}, m_2), \text{id})$$

$$m_2(\text{id}, m_3) \qquad m_3(\text{id}, m_2, \text{id})$$

$$m_2(\text{id}, m_2(\text{id}, m_2))$$

③

$$m_2(m_2, \text{id})(\text{id}, m_2, \text{id})$$

μ

μ

$$m_2(\text{id}, m_2(m_2, \text{id})) \longrightarrow m_2(\text{id}, m_2)(\text{id}, m_2, \text{id})$$

The six petals correspond to the indicated sequences and are numbered accordingly. We have marked the (magenta) arrows resulting from the morphism (C.1). Note that there are five of them. In particular:

- If $M\mathfrak{C}$ is *unital*, the petals 5 and 6 collapse to the red arrows.

- If furthermore $M\mathfrak{C}$ is in fact a 2-multicategory, i.e. the 2-categorical analog of a multicategory, so that $\mathfrak{C}$ itself is a genuine 2-category, the associativity morphisms at the tops of all petals reduce to identities. As a result, the outer perimeter reduces to a standard pentagon

(C.3)
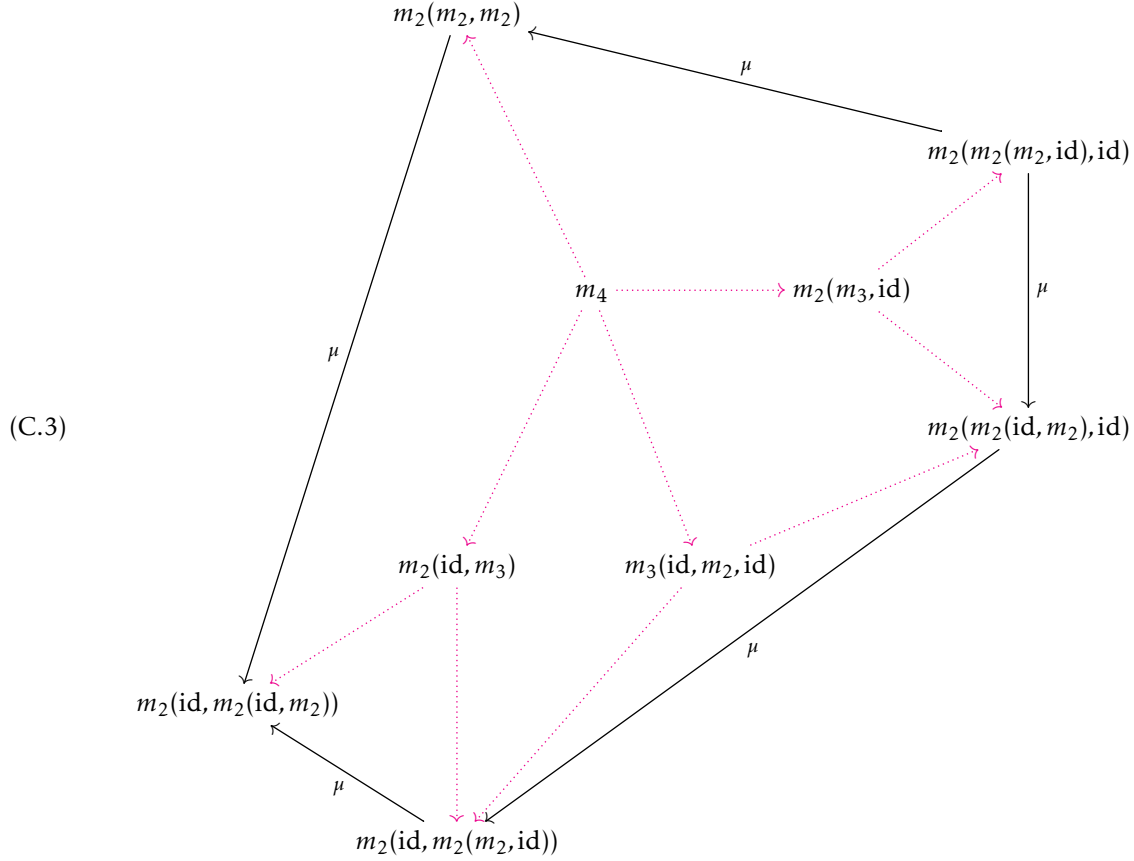
D   Some lemmas on symmetric braidings

D.1 Lemma. *Let* $(\mathbf{C}, \otimes, c)$ *be a strictly associative braided monoidal category. The diagram*

$$
\begin{array}{ccc}
x \otimes y \otimes z \otimes w & \xrightarrow{1 \otimes c_{y,z} \otimes 1} & x \otimes z \otimes y \otimes w \\
{\scriptstyle c_{x,y} \otimes c_{z,w}} \downarrow & & \downarrow {\scriptstyle c_{x \otimes z, y \otimes w}} \\
y \otimes x \otimes w \otimes z & \xrightarrow{1 \otimes c_{x,w} \otimes 1} & y \otimes w \otimes x \otimes z
\end{array}
$$

commutes if and only if the braiding c is symmetric.

Proof. The naturally occurring diagram relating $c_{x \otimes z, y \otimes w}$ to $c_{x,y} \otimes c_{z,w}$ is

$$
\begin{array}{ccc}
x \otimes y \otimes z \otimes w & \xleftarrow{1 \otimes c_{z,y} \otimes 1} & x \otimes z \otimes y \otimes w \\
{\scriptstyle c_{x,y} \otimes c_{z,w}} \downarrow & & \downarrow {\scriptstyle c_{x \otimes z, y \otimes w}} \\
y \otimes x \otimes w \otimes z & \xrightarrow{1 \otimes c_{x,w} \otimes 1} & y \otimes w \otimes x \otimes z
\end{array}
$$

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which is valid with respect to any braiding. This is equal to the diagram in the statement if and only if $c_{y,z}$ is the inverse of $c_{z,y}$. □

D.2 Lemma. *Let* $(\mathbf{C}, \otimes, c)$ *be a strictly associative braided monoidal category. The following diagram*

$$
\begin{array}{ccc}
x \otimes y \otimes z \otimes u \otimes v \otimes w & \xrightarrow{\ 1 \otimes c_{yz,uv} \otimes 1\ } & x \otimes u \otimes v \otimes y \otimes z \otimes w \\
\downarrow{\scriptstyle c_{x,y} \otimes 1 \otimes 1 \otimes c_{v,w}} & & \downarrow{\scriptstyle c_{x.u} \otimes 1 \otimes 1 \otimes c_{z,w}} \\
y \otimes x \otimes z \otimes u \otimes w \otimes v & & u \otimes x \otimes v \otimes y \otimes w \otimes z \\
\downarrow{\scriptstyle 1 \otimes c_{xz,uw} \otimes 1} & & \downarrow{\scriptstyle 1 \otimes c_{xv,yw} \otimes 1} \\
y \otimes u \otimes w \otimes x \otimes z \otimes v & \xrightarrow{\ c_{y.u} \otimes 1 \otimes 1 \otimes c_{z,v}\ } & u \otimes y \otimes w \otimes x \otimes v \otimes z
\end{array}
$$

commutes if and only if the braiding c *is symmetric.*

Proof. Use Lemma D.1 to replace $c_{yz,uv}$ with $c_{y,u} \otimes c_{z,v}$ and similarly for $c_{xz,uw}$ and $c_{xv,yw}$. This converts the diagram in the statement into the following one

$$
\begin{array}{ccc}
x \otimes y \otimes u \otimes z \otimes v \otimes w & \xrightarrow{\ 1 \otimes c_{y,u} \otimes c_{z,v} \otimes 1\ } & x \otimes u \otimes y \otimes v \otimes z \otimes w \\
\downarrow{\scriptstyle c_{x,y} \otimes 1 \otimes 1 \otimes c_{v,w}} & & \downarrow{\scriptstyle c_{x.u} \otimes 1 \otimes 1 \otimes c_{z,w}} \\
y \otimes x \otimes u \otimes z \otimes w \otimes v & & u \otimes x \otimes y \otimes v \otimes w \otimes z \\
\downarrow{\scriptstyle 1 \otimes c_{x,u} \otimes c_{z,w} \otimes 1} & & \downarrow{\scriptstyle 1 \otimes c_{x,y} \otimes c_{v,w} \otimes 1} \\
y \otimes u \otimes x \otimes w \otimes z \otimes v & \xrightarrow{\ c_{y.u} \otimes 1 \otimes 1 \otimes c_{z,v}\ } & u \otimes y \otimes x \otimes w \otimes v \otimes z
\end{array}
$$

which consists of two juxtaposed copies of the same kind of hexagon:

$$
\begin{array}{ccc}
x \otimes y \otimes u & \xrightarrow{\ 1 \otimes c_{y,u}\ } & x \otimes u \otimes y \\
\downarrow{\scriptstyle c_{x,y} \otimes 1} & & \downarrow{\scriptstyle c_{x.u} \otimes 1} \\
y \otimes x \otimes u & & u \otimes x \otimes y \\
\downarrow{\scriptstyle 1 \otimes c_{x,u}} & & \downarrow{\scriptstyle 1 \otimes c_{x,y}} \\
y \otimes u \otimes x & \xrightarrow{\ c_{y.u} \otimes 1\ } & u \otimes y \otimes x
\end{array}
$$

The latter commutes in any braided monoidal (strictly associative) category. Thus in effect the statement of the Lemma holds if and only if Lemma D.1 holds. □

A **permutative** category is a symmetric monoidal category $(\mathbf{C}, \otimes, c)$ in which associativity and unitality hold strictly. Thus the diagrams in Lemmas D.1 and D.2 commute if and only if the category $(\mathbf{C}, \otimes, c)$ is permutative. On the other hand, the statements of Lemma D.1 and Lemma D.2 hold in general. Requiring that $\mathbf{C}$ be strictly associative merely simplifies the diagrams and overall legibility. For example, in the

general case the diagram in the statement of Lemma D.1 would need replacing with:

$$
\begin{array}{ccccccc}
x\otimes(z\otimes(y\otimes w)) & \longleftarrow & (x\otimes z)\otimes(y\otimes w) & \xrightarrow{\ c_{x\otimes z,y\otimes w}\ } & (y\otimes w)\otimes(x\otimes z) & \longrightarrow & y\otimes(w\otimes(x\otimes z)) \\
\uparrow & & & & & & \uparrow \\
x\otimes((z\otimes y)\otimes w) & & & & & & y\otimes((w\otimes x)\otimes z) \\
{\scriptstyle 1\otimes c_{y,z}\otimes 1}\uparrow & & & & & & \uparrow{\scriptstyle 1\otimes c_{x,w}\otimes 1} \\
x\otimes((y\otimes z)\otimes w) & & & & & & y\otimes((x\otimes w)\otimes z) \\
\downarrow & & & & & & \downarrow \\
x\otimes(y\otimes(z\otimes w)) & \longleftarrow & (x\otimes y)\otimes(z\otimes w) & \xrightarrow[\ c_{x,y}\otimes c_{z,w}\ ]{} & (y\otimes x)\otimes(w\otimes z) & \longrightarrow & y\otimes(x\otimes(w\otimes z))
\end{array}
$$

with a similarly modified proof. A similar, but more complicated modifications apply to the diagram in the proof of Lemma D.2.

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