A Convolution Estimate for a Measure on a Curve in $\mathbb{R}^4$, II

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Let γ be the curve in $\mathbb{R}^4$ defined by $\gamma(t) = (t, t^2, t^3, t^4)$, $t \in \mathbb{R}$. For an interval $I \subseteq \mathbb{R}$ we consider the convolution operator T on $\mathbb{R}^4$ defined by

$$Tf(x) = \int_I f(x - \gamma(t))dt, \qquad x \in \mathbb{R}^4.$$

Our interest is in determining the type-set $\mathcal{T}$ of T — the set of all $(\frac{1}{p}, \frac{1}{q}) \in [0,1] \times [0,1]$ such that $T(L^p(\mathbb{R}^4)) \subseteq L^q(\mathbb{R}^4)$. Let $A = (0,0)$, $B = (\frac{4}{10}, \frac{3}{10})$, $C = (\frac{5}{10}, \frac{4}{10})$, $D = (\frac{6}{10}, \frac{5}{10})$, $E = (\frac{7}{10}, \frac{6}{10})$, and $F = (1,1)$. Since T is a convolution operator, $\mathcal{T}$ lies on or below the closed segment $[A, F]$. Estimating Tf when f is the characteristic function of a small ball shows that $\mathcal{T}$ lies on or above the line through A and B and so, by duality, on or below the line through F and E. Estimating Tf when f is the characteristic function of

$$[0, \epsilon] \times [0, \epsilon^2] \times [0, \epsilon^3] \times [0, \epsilon^4]$$

shows that $\mathcal{T}$ lies on or above the line through B and E. Thus $\mathcal{T}$ lies inside the closed quadrilateral $ABEF$ (the closed convex hull of A, B, E, and F). If the interval of integration I is unbounded, then homogeneity considerations show that $\mathcal{T} \subseteq [B, E]$. It is conjectured that these necessary conditions for the boundedness of T are also sufficient. By duality and interpolation, it would suffice to show that $B \in \mathcal{T}$ if $I = (0, \infty)$. The best previous result [O1] is that $[C, D] \subseteq \mathcal{T}$ for any I. Here we establish an almost sharp result when I is bounded.

Theorem. *If I is bounded then, excepting possibly the segments $(A, B]$, $[B, C)$, $(D, E]$, and $[E, F)$, the closed quadrilateral $ABEF$ lies in $\mathcal{T}$.*

Proof. Let $G = (\frac{1}{2}, \frac{1}{2})$. By duality, interpolation, and the result of [O1], it is enough to show that $[G, B) \subseteq \mathcal{T}$. And a homogeneity argument shows that it is enough to do this for $I = [0, 1]$. To this end we will study an analytic family T_z of convolution operators given formally by

$$T_z f(x) = \frac{1}{\Gamma(\frac{(z+1)}{2})} \int_0^1 \int_{-1}^1 f(x - \gamma(t) - u\gamma''(t))|u|^z \, du \, dt.$$

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We will establish, with acceptable estimates on the growth of the operator norms, that

(1) T_z is bounded on $L^2(\mathbb{R}^4)$ when $\mathrm{Re}(z) = -\frac{3}{2}$

and, if $H = (\frac{1}{3}, \frac{1}{6})$, that

(2) T_z maps $L^p(\mathbb{R}^4)$ into $L^q(\mathbb{R}^4)$ if $(\frac{1}{p}, \frac{1}{q}) \in [G, H)$ and $\mathrm{Re}\ z = -\frac{2}{3}$.

Analytic interpolation and the fact that T_{-1} is a multiple of T then show that $[G, B) \subseteq \mathcal{T}$ as claimed.

To establish (1) we need to estimate the Fourier transform $\widehat{T}_{-\frac{3}{2}+is}$ given formally by

$$\widehat{T}_{-\frac{3}{2}+is}(y) = \frac{1}{\Gamma((-\frac{1}{2}+is)/2)} \int_0^1 \int_{-1}^1 e^{-iy\cdot(\gamma(t)+u\gamma''(t))} |u|^{-\frac{3}{2}+is} du\, dt.$$

Because the integral

$$\int_{-1}^{\infty} |u|^{-\frac{3}{2}} du$$

converges, it is enough to estimate the integral

$$\int_0^1 \int_{-\infty}^{\infty} e^{-iy\cdot(\gamma(t)+u\gamma''(t))} |u|^{-\frac{3}{2}+is} du\, dt. \tag{3}$$

If $y = (y_1, y_2, y_3, y_4)$, write $p(t) = \sum_{j=1}^{4} y_j t^j$. Since the Fourier transform of the one-dimensional distribution $|u|^{-\frac{3}{2}+is} du$ is a multiple of $|x|^{\frac{1}{2}-is} dx$, (3) is essentially

$$\int_0^1 e^{-ip(t)} |p''(t)|^{\frac{1}{2}-is} dt.$$

A result of [O3] bounds this integral by $C(1+|s|)^{\frac{1}{2}}$ independently of y. Thus

$$\| \widehat{T}_{-\frac{3}{2}+is} \|_\infty \le C(1+|s|)^{\frac{1}{2}},$$

yielding (1).

To prove (2) we begin by defining

$$S_1 f(x) = \int_0^\infty \int_{\frac{1}{2}}^1 f(x - \gamma(t) - u\gamma''(t)) du\, dt \doteq \mu_1 * f(x),$$

$$S_2 f(x) = \int_0^\infty \int_{-1}^{-\frac{1}{2}} f(x - \gamma(t) - u\gamma''(t)) du\, dt \doteq \mu_2 * f(x).$$

With $S = S_1 + S_2$, our immediate goal is to establish that

$$S \text{ maps } L^3(\mathbb{R}^4) \text{ into } L^6(\mathbb{R}^4). \tag{4}$$

We will do this by adapting an argument from [O2]. Now (4) follows from the two multilinear estimates

$$\left| \int_{\mathbf{R}^4} \prod_{j=1}^{6} \mu_\ell * f_j(x)\, dx \right| \le C \prod_{j=1}^{6} \|f_j\|_3, \quad \ell = 1, 2. \tag{5}$$

Utilizing Christ's remarkable observation about multilinear interpolation ([C], pp. 227-228), (5) will follow from the estimates

$$\left| \int_{\mathbf{R}^4} \prod_{j=1}^{6} \mu_\ell * f_j(x)\, dx \right| \le C \|f_1\|_1 \prod_{j=2}^{6} \|f_j\|_{5,1}, \quad \ell = 1, 2, \tag{6}$$

where the symbol $\|\cdot\|_{p,r}$ stands for a Lorentz space norm on $\mathbb{R}^4$. It is enough to prove (6) with f_1 replaced by the point mass at the origin in $\mathbb{R}^4$ and so (6) reduces to the estimate

$$\|S_\ell f\|_{L^5(\mu)} \le C \|f\|_{5,1}.$$

We will actually establish the dual of this estimate,

$$\|S_\ell^* g\|_{\frac{5}{4},\infty} \le C \|g\|_{L^{\frac{5}{4}}(\mu_\ell)}, \tag{7}$$

for functions g on the two-dimensional manifold that is the support of μ_ℓ. Let ϕ be the map of $\mathbb{R}^4$ into itself defined by

$$\phi(t, u, s, v) = \gamma(t) + u\gamma''(t) - \gamma(s) - v\gamma''(s).$$

Computations show that

$$\phi \text{ is a most three to one on the set } \{t \ne s\} \tag{8}$$

and that the absolute value J of the Jacobian of ϕ is given by

$$J(t, u, s, v) = 24(s-t)^2 |(s-t)^2 - 6(u+v)|. \tag{9}$$

Let m denote Lebesgue measure on $\mathbb{R}^4$. From (8) it follows that if $U \subseteq \mathbb{R}^4$ is open and if the measure λ_U is defined on $\phi(U)$ by

$$\int_{\phi(U)} h d\lambda_U = \int_U h \circ \phi(t, u, s, v) J(t, u, s, v) d(t, u, s, v)$$

then

$$m(E) \le \lambda_U(E) \le 3m(E), \qquad E \subseteq U. \tag{10}$$

Let $Q_1 = \{(s,v) : 0 < s < \infty, \frac{1}{2} < v < 1\}$, $Q_2 = \{(s,v) : 0 < s < \infty, -1 < v < -\frac{1}{2}\}$. If f is a function on $\mathbb{R}^4$ and g is defined on the support of μ_ℓ, then we have, writing $g(s,v)$ for $g(\gamma(s) + v\gamma''(s))$,

$$\int_{\mathbf{R}^4} f(x) S_\ell^* g(x) dx = \int_{\mathbf{R}^4} S_\ell f(y) g(y) d\mu_\ell(y)$$

$$= \int_{Q_\ell \times Q_\ell} f(\phi(t,u,s,v)) \frac{g(s,v)}{J(t,u,s,v)} J(t,u,s,v) d(t,u,s,v).$$

The last integral is bounded by the product of the two Lorentz norms

$$\|f \circ \phi\|_{L^{5,1}(Jdm)}, \qquad \|g/J\|_{L^{\frac{5}{4},\infty}(Jdm)}.$$

By (10) the first of these is bounded by $\|f\|_{5,1}$. We will establish (7) by observing that there is a constant C such that, for any $\alpha > 0$,

$$\int_{(Q_\ell \times Q_\ell) \cap \{|g(s,v)| > \alpha J(t,u,s,v)\}} J(t,u,s,v) d(t,u,s,v) \le \frac{C}{\alpha^{\frac{5}{4}}} \int_{Q_\ell} |g(s,v)|^{\frac{5}{4}} d(s,v).$$

And this will follow from the inequality, uniform in $A > 0$ and $(s,v) \in Q_\ell$,

$$\int_{Q_\ell \cap \{J(t,u,s,v) < A\}} J(t,u,s,v) d(t,u) \le C A^{\frac{5}{4}}.$$

Taking account of (9) it is enough to observe that there is C such that for $A > 0$ and $6 < |k| < 12$ we have

$$\int_{\{t^2|t^2-k| < A\}} t^2 |t^2 - k| dt \le C A^{\frac{5}{4}}. \tag{11}$$

But this inequality is easily checked by considering separately the cases of large and small A. The first case follows from the fact that if $|t|$ is large, then $t^2|t^2-k| \sim t^4$, leading to (11). In the case of small A, $t^2|t^2-k| < A$ implies either $t^2|t^2-k| \sim Ct^2$ or $t^2|t^2-k| \sim C|t \pm \sqrt{k}|$, leading to estimates of $CA^{\frac{3}{2}}$ or CA^2 respectively for the left-hand side of (11). These complete the proof of (4).

Now define operators V_k by

$$V_k f(x) = \int_0^1 \int_{2^{-k} < |u| < 2^{-k+1}} f(x - \gamma(t) - u\gamma''(t)) du\, dt.$$

Let $\|V_k\|_{pq}$ denote the norm of V_k from $L^p(\mathbb{R}^4)$ to $L^q(\mathbb{R}^4)$. Then, since

$$T_{-\frac{3}{2}} \sim \sum_{k=1}^{\infty} 2^{\frac{2k}{3}} V_k,$$

(2) will follow from the estimate

$$\|V_k\|_{pq} \leq C 2^{-k(1-2\alpha)} \tag{12}$$

for $(\frac{1}{p}, \frac{1}{q})$ on $[G, H)$ with $\frac{1}{p} - \frac{1}{q} = \alpha$ along with the fact that $\alpha < \frac{1}{6}$. Inequality (12) comes from interpolating the cases $\alpha = 0$ (corresponding to $G = (\frac{1}{2}, \frac{1}{2})$) and $\alpha = \frac{1}{6}$ (corresponding to $H = (\frac{1}{3}, \frac{1}{6})$). The first of these two estimates is clear. The second is a consequence of (4) and a homogeneity argument:

For $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $\lambda \in \mathbb{R}$, let $[\lambda]x$ be the nonisotropic dilate of x given by $(\lambda x_1, \lambda^2 x_2, \lambda^3 x_3, \lambda^4 x_4)$. For $\lambda > 0$, let $V_{[\lambda]}$ be the dilate of V given by

$$V_{[\lambda]} f(x) = \int_0^{\infty} \int_{\frac{1}{2} < |u| < 1} f(x - [\lambda](\gamma(t) + u\gamma''(t)))du\, dt.$$

Then it is easy to see that

$$\|V_{[\lambda]}\|_{pq} = \lambda^{-(\frac{10}{p} - \frac{10}{q})} \|S\|_{pq}$$

and that

$$V_{[\lambda]} f(x) = \lambda^{-3} \int_0^{\infty} \int_{\frac{\lambda^2}{2} < |u| < \lambda^2} f(x - \gamma(t) - u\gamma''(t))du\, dt.$$

Taking $\lambda = 2^{-\frac{k}{2}}$ and combining these two observations with (4) gives (12) for $\alpha = \frac{1}{6}$ (and $(\frac{1}{p}, \frac{1}{q}) = H = (\frac{1}{3}, \frac{1}{6})$).

References

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