Reduction numbers, Rees algebras, and Pfaffian ideals

by

Ian M. Aberbach[1]
Sam Huckaba[2]
Craig Huneke[3]

1. Introduction

This paper continues a search for practical methods (in terms of a presentation matrix of I) of testing the Cohen-Macaulay property of the Rees algebra $R[It] = \bigoplus_{n\geq 0} I^n t^n$, or the associated graded ring $G(I) = \bigoplus_{n\geq 0} (I^n/I^{n+1})$, of an ideal I in a Cohen-Macaulay (sometimes Gorenstein or regular) local ring R. It was inspired specifically by the recent work of Vasconcelos [Vas] and Aberbach-Huneke [AH], and more broadly by the work of several authors (e.g. [Ba], [Val], [Sa], [Hun1], [HSV1-2], and [HH1-2]). Vasconcelos searched for criteria which forced $R[Pt]$ to be Cohen-Macaulay where R is a 3-dimensional local ring and P is a 4-generated height 2 normal prime ideal of R. He was able to obtain results based on the nature of a presentation matrix φ of P (see [Vas] Theorem 3.1.3). Aberbach and Huneke followed with a generalized version in [AH] Theorem 8.2, one that applies to unmixed height 2 ideals of R that are generically complete intersections. Below we will recall their result and explain the terminology surrounding their approach. We have two goals in this paper. First we will establish bounds on the reduction numbers of certain ideals I of a Cohen-Macaulay local ring R, based on information from a presentation matrix φ of I (Theorem 2.2). Then we will prove a result analogous to [AH, Theorem 8.2] for certain height 3 ideals of a regular local ring R such that R/I is Gorenstein (Theorem 3.1). Before describing our results in more detail, let us record some basic concepts and terminology.

Let (R, m) be a local ring and I an ideal of R. The *analytic deviation* of I was defined in [HH1] to be the difference $ad(I) = \ell(I) - ht(I)$ where $\ell(I) = dim(\bigoplus_{n\geq 0}(I^n/mI^n))$ is the analytic spread of I (introduced in [NR]). An ideal J contained in I is called a reduction of I if $JI^n = I^{n+1}$ for some integer $n \geq 0$. If J is minimal with respect to being a reduction of I then it is called a minimal reduction of I. These concepts were also introduced in [NR]. In general minimal reductions always exist, and if R/m is infinite then every minimal reduction of I is generated by $\ell(I)$ elements. For J a reduction of I we define the *reduction number of I with respect to J* by $r_J(I) = \min\{n \geq 0 | JI^n = I^{n+1}\}$, and the *reduction*

[1] Partially supported by the NSF
[2] Partially supported by the NSA (MDA904-92-H-3040)
[3] Partially supported by the NSF

number of I by $r(I)$ =min$\{r_J(I)|J$ is a minimal reduction of $I\}$. This concept goes back to [Sa].

If (R,m) is Cohen-Macaulay and I is an ideal with "small" reduction number, then one can sometimes conclude that $G(I)$ or $R[It]$ is Cohen-Macaulay. For example, Sally showed that if $r(m) \leq 1$ and $dim(R) > 0$ then $G(m)$ is Cohen-Macaulay [Sa], and in [HH1-2] Huckaba and Huneke gave versions of this applicable to ideals of analytic deviation either 1 or 2 (which are of course not m-primary), showing additionally that $R[It]$ is Cohen-Macaulay. (The results on ideals of positive analytic deviation often carry the additional assumption that I is *generically a complete intersection*; i.e. that $\mu(I_P) = ht(I)$ for all $P \in min(R/I)$.) These results give practical conditions in the sense that one can often calculate bounds on the reduction number.

The 'equations' of the Rees algebra $R[It]$ play a crucial role in the study of $R[It]$. Suppose $I = (a_1, \ldots, a_n)$ is an ideal of R and consider the surjection $R[T_1, \ldots, T_n] \to R[It]$. We let Q denote the kernel of this homomorphism (a graded ideal), and refer to its elements as the equations of $R[It]$. Let Q_i be the ideal generated by $\{F \in Q | F$ is homogeneous and $deg(F) \leq i\}$. Then $Q_1 \subseteq Q_2 \subseteq \ldots$ stabilizes and the integer $N(I)$ such that $Q_{N(I)-1} \neq Q$ but $Q_{N(I)} = Q$ is called the *relation type* of I. The relation type is closely connected to the reduction number provided that the number of generators of I is small relative to its analytic spread (cf.[Huc],[T]). We note here the definition of syzygetic; I is said to be *syzygetic* if Q_1=Q_2. Any prime ideal of height 2 in a 3-dimensional regular local ring is syzygetic [SV] Theorem 2.2. In [Vas] Vasconcelos explicitly calculated the equations of $R[Pt]$ for P a 4-generated height 2 normal prime ideal of a 3-dimensional regular local ring R, such that P has presentation matrix satisfying certain properties. Using these equations he was able to establish that $R[Pt]$ is Cohen-Macaulay. Aberbach and Huneke extended and strengthened this result in [AH] Theorem 8.2. Before stating their result we recall a definition from [AH] that follows [Vas]. Let I be an ideal of a local ring R and let $R^m \xrightarrow{\varphi} R^n \to I$ be a presentation matrix of I.

Definition 1.0. We say that φ satisfies the *row condition* if by performing invertible row and column operations we can obtain a matrix ψ such that the ideal of entries $I_1(\psi)$ is generated by the entries of some row of ψ.

Theorem 1.1. [AH, Theorem 8.2] *Let (R,m) be a 3-dimensional regular local ring containing a field and having an infinite residue field and let I be an unmixed 4-generated height 2 ideal of R such that I is generically a complete intersection. Then the following are equivalent.*

(1) Any presentation matrix of I satisfies the row condition;

(2) $G(I)$ is Cohen-Macaulay;

(3) $r(I) \leq 2$;

(4) $N(I) \leq 3$;

(5) $R[It]$ *is Cohen-Macaulay;*

We note here an improvement to this theorem that will appear in section 2.

Theorem 2.12. *Let* (R, m) *be a Cohen-Macaulay local ring with* R/m *infinite and let* I *be a 4-generated height 2 ideal of* R *such that* R/I *is Cohen-Macaulay and has finite projective dimension. Assume that* I *is generically a complete intersection and has analytic deviation 1. Then the following are equivalent.*

(1) *Any presentation matrix of* I *satisfies the row condition;*

(2) $G(I)$ *is Cohen-Macaulay;*

(3) $r(I) \leq 2$;

(4) $N(I) \leq 3$;

(5) $R[It]$ *is Cohen-Macaulay;*

In section 3 we will produce an analogue of Theorem 2.12 for height 3 Gorenstein ideals of a Gorenstein local ring. Here is the statement:

Theorem 3.1. *Let* (R, m) *be a Gorenstein local ring with* R/m *infinite. Let* I *be a height 3 ideal of* R *such that* R/I *is Gorenstein and has finite projective dimension. Assume that* I *has analytic deviation one, is generically a complete intersection, and* $\mu(I) = 5$. *Then the following are equivalent:*

(1) *There exists a presentation matrix* φ *of* I *such that* φ *is an alternating matrix that satisfies the row condition;*

(2) $G(I)$ *is Cohen-Macaulay;*

(3) $r(I) \leq 2$;

(4) $N(I) \leq 3$;

(5) $R[It]$ *is Cohen-Macaulay.*

As with Theorem 1.1, Theorem 3.1 gives a usable method for testing whether $R[It]$ is Cohen-Macaulay (see Example 3.5). Note that by [BE] Theorem 2.1 any ideal satisfying the assumptions of Theorem 3.1 is generated by the 4^{th} order Pfaffians of a 5×5 alternating matrix.

Our work in section 2 also has a practical flavor, and it is needed for part of our proof of Theorem 3.1. We show in section 2 that if I "behaves well" with respect to the entries of a presentation matrix of I, then $r(I)$ is bounded in a predictable way. In particular we will prove the following result.

Theorem 2.2. *Let* (R, m) *be a Cohen-Macaulay local ring with* R/m *infinite and let* I

be an analytic deviation one ideal that is generically a complete intersection and unmixed. Let φ be a presentation matrix of I and assume that $G(I)$ is Cohen-Macaulay and $\mu(I) = \ell(I)+1$. If J is a minimal reduction of I and $I \subseteq I_1(\varphi)^j$ where j is a positive integer then $r_J(I) \leq \lceil \frac{r_J(I)}{j} \rceil + 1$ where $\lceil k \rceil$ denotes the smallest integer $\geq k$.

This theorem leads to small bounds on $r(I)$ if one knows that $j \geq 2$ (see Corollaries 2.3 and 2.4). By making use of the Hilbert-Burch theorem for height two perfect ideals, and the structure theorem of Buchsbaum and Eisenbud for height three Gorenstein ideals, we are able to apply these results to those classes of ideals (see Corollaries 2.9 and 2.10).

Throughout the paper the rings involved are assumed to be commutative and Noetherian (with an identity). We make one final remark on terminology; for I an ideal and $a \in I$, we let a' denote the leading form of a in $G(I)$; that is, a' is the equivalence class $a + I^{n+1}$ in I^n/I^{n+1} where $a \in I^n \backslash I^{n+1}$.

2. Bounds on reduction numbers

In this section we prove our main result about bounding the reduction number of an analytic deviation one ideal. Notationally we will follow the lead of Vasconcelos [Vas]. Let (R, m) be a local ring and I an ideal of R with $\mu(I) = d+1$. Let

$$R^n \xrightarrow{\varphi} R^{d+1} \to I \to 0$$

be a presentation of I. We denote by $I_1(\varphi)$ the ideal generated by the entries of φ. If ψ is obtained from φ by invertible row and column operations then ψ also presents I and corresponds to a change of basis in R^n and/or R^{d+1}. In this case we have that $I_1(\varphi) = I_1(\psi)$.

We start with a useful lemma.

Lemma 2.1. *Let (R, m) be a Cohen-Macaulay local ring and let I be an analytic deviation one ideal that is generically a complete intersection. Assume that $\mu(I) = \ell(I) + 1 = d + 1$ and grade $(G(I)^+) = \mathrm{ht}(I) = d - 1$. Let φ be a presentation matrix of I and let Q be the kernel of $R[T_1, \ldots, T_{d+1}] \to R[It]$. Then $I_1(\varphi)Q_n \subseteq Q_{n-1}$ $\forall\ n > 1$.*

Proof. First note that by passing to the extension $R[X]_{m[X]}$ we may assume that R/m is infinite. Write $I = (a_1, \ldots, a_{d+1})$. By using prime avoidance along with a general position argument we may assume that $(a_{i_1}, \ldots, a_{i_{d-1}})$ generates I generically and $\{a'_{i_1}, \ldots, a'_{i_{d-1}}\}$ is a $G(I)$-regular sequence for distinct integers $i_1, \ldots, i_{d-1}$ between 1 and $d+1$, and also that $(a_{i_1}, \ldots, a_{i_d})$ is a minimal reduction of I for distinct integers $i_1, \ldots, i_d$ between 1 and $d+1$.

Let $F \in Q_n$ be homogeneous and let $b \in I_1(\varphi)$. Since φ is a relations matrix for I we may assume that $b \in ((a_1, \ldots, a_d) : a_{d+1})$. Write $F = rT_{d+1}^n + G$ where $G \in$

$(T_1,\ldots,T_d)R[T_1,\ldots,T_{d+1}]$. Then $bF = rbT_{d+1}^n + bG$. There exists $r_i \in R$ such that $ba_{d+1} - r_1a_1 - \ldots - r_da_d = 0$, hence $bT_{d+1} - r_1T_1 - \ldots - r_dT_d \in Q_1$. Thus $bF = rT_{d+1}^{n-1}(bT_{d+1} - r_1T_1 - \ldots - r_dT_d) + rT_{d+1}^{n-1}(r_1T_1 + \ldots + r_dT_d) + bG \in Q_1 + ((T_1,\ldots,T_d)\cap Q_n)$. By [AH] Proposition 2.5 we have that $(T_1,\ldots,T_d)\cap Q_n \subseteq Q_{n-1}$, therefore $bF \in Q_{n-1}$. ∎

Theorem 2.2. *Let (R,m) be a Cohen-Macaulay local ring with R/m infinite and let I be an analytic deviation one ideal that is generically a complete intersection and unmixed. Let φ be a presentation matrix of I and assume that $G(I)$ is Cohen-Macaulay and $\mu(I) = \ell(I)+1$. If J is a minimal reduction of I and $I \subseteq I_1(\varphi)^j$ where j is a positive integer then $r_J(I) \leq \lceil \frac{r_J(I)}{j} \rceil + 1$ where $\lceil k \rceil$ denotes the smallest integer $\geq k$.*

This theorem leads to some rather nice bounds on reduction numbers. We state these as corollaries whose proofs we will record below (after the proof of Theorem 2.2).

Corollary 2.3. *If R, I, and J are as in Theorem 2.2 and if $j > 1$, then $r_J(I) \leq 3$.*

Corollary 2.4. *Let R, I, and J be as in Theorem 2.2. If either $j > 1$ and I is syzygetic, or if $j > 2$, then $r_J(I) \leq 2$.*

Proof of Theorem 2.2. Let $d = \ell(I)$ and $I = (a_1,\ldots,a_d,a_{d+1})$ where $J = (a_1,\ldots,a_d)$ is the given minimal reduction of I. We may assume that $(a_1,\ldots,a_{d-1})$ generates I generically and $\{a_1',\ldots,a_{d-1}'\}$ is a $G(I)$-regular sequence. Let $\psi : R[T_1,\ldots,T_{d+1}] \to R[It]$ by $T_i \mapsto a_it$ and let $Q = \ker(\psi)$. Set $n = r_J(I)$. Then there exists a homogeneous polynomial $F \in Q_{n+1}$ of the form $F = T_{d+1}^{n+1} + G$ where G contains no term having T_{d+1}^{n+1} as a factor (this holds simply because $JI^n = I^{n+1}$). As $G(I)$ is Cohen-Macaulay we have that grade $(G(I)^+) = \mathrm{ht}(I)$. Thus we obtain, by repeated use of Lemma 2.1, that $I_1(\varphi)^nF \subseteq Q_1$. If C denotes the ideal generated by the last row of φ then $C = (J : a_{d+1})$. We claim that $I_1(\varphi)^n \subseteq C$. Let $b \in I_1(\varphi)^n$. Then $bF \in Q_1$ from above, hence there exists $L_i \in Q_1$ and $H_i \in R[T_1,\ldots,T_{d+1}]$ such that $bT_{d+1}^{n+1} + bG = L_1H_1 + \ldots + L_mH_m$. By equating the coefficients of T_{d+1}^{n+1} on each side and using that $L_i \in Q_1$ it follows that $b \in C$, thus establishing the claim. By assumption $a_{d+1} \in I_1(\varphi)^j$, hence $a_{d+1}^{\lceil \frac{n}{j} \rceil + 1} \in a_{d+1}(I_1(\varphi)^j)^{\lceil \frac{n}{j} \rceil} \subseteq a_{d+1}C \subseteq J$. If we set $k = \lceil \frac{n}{j} \rceil + 1$ then this implies $I^k \subseteq J$, thus $I^{k+1} \subseteq JI$. Since $(a_1,\ldots,a_{d-1})$ generates I generically we have that $ht((a_1,\ldots,a_{d-1}) : I) + I) \geq d$, thus J is a "special reduction" of I (see [AH] Definition 5.1 and Lemma 6.1). Therefore we may apply the arguments in the proof of [AH] Theorem 5.2 to conclude that $I^{k+1} = JI^k$. For convenience we add a few of the details here. Let $f \in I^{k+1}$. If $f \in JI^m$ for some m, $1 \leq m < k$, then we may write $f = a_1u_1 + \ldots + a_du_d + w$ where $u_i \in I^m$ and $w \in JI^{m+1}$. We claim that in fact $f \in JI^{m+1}$. By interpreting the equation in $G(I)_{m+1}$ we have $0 = a_1'u_1' + \ldots + a_d'u_d'$, hence $u_d' \in ((a_1',\ldots,a_{d-1}') : a_d')_m$. By [AH, Proposition 5.10] it follows that $u_d' \in (a_1',\ldots,a_{d-1}')_m$, thus $a_1u_1 + \ldots + a_du_d \in (a_1,\ldots,a_{d-1})I^m + (a_d)I^{m+1}$. Therefore, in order to show $f \in JI^{m+1}$ it suffices to show that $(a_1,\ldots,a_{d-1})I^m \cap$

$I^{m+2} = (a_1, \ldots, a_{d-1})I^{m+1}$. But $(a_1, \ldots, a_{d-1})I^m \cap I^{m+2} \subseteq (a_1, \ldots, a_{d-1}) \cap I^{m+2} = (a_1, \ldots, a_{d-1})I^{m+1}$ because $\{a'_1, \ldots, a'_{d-1}\}$ is a $G(I)$-regular sequence. This completes the proof. ∎

Remark 2.5. If I is assumed to be syzygetic in Theorem 2.1 (i.e. that $Q_1 = Q_2$) then the conclusion can be strengthened to $r_J(I) \leq \lceil \frac{r_J(I)-1}{j} \rceil + 1$. This holds because in this case, with $n = r_J(I)$, we obtain $I_1(\varphi)^{n-1}F \subseteq Q_2 = Q_1$, hence $I_1(\varphi)^{n-1} \subseteq C$, thus $n \leq \lceil \frac{n-1}{j} \rceil + 1$.

Proof of Corollary 2.3. Set $n = r_J(I)$. Then $n \leq \lceil \frac{n}{j} \rceil + 1$ by Theorem 2.2. If $j \geq 2$ then $\lceil \frac{n}{j} \rceil \leq \lceil \frac{n}{2} \rceil$, hence $n \leq \lceil \frac{n}{2} \rceil + 1$. If $n = 2m$ is even then $2m \leq \lceil \frac{2m}{2} \rceil + 1 = m + 1$, thus $m \leq 1$. If $n = 2m + 1$ is odd then $2m + 1 \leq \lceil \frac{2m+1}{2} \rceil + 1 = m + 2$, thus $m \leq 1$. These inequalities show that $n \leq 3$ if $j \geq 2$. ∎

Proof of Corollary 2.4. Similar to the proof of 2.3, although use Remark 2.5. ∎

By applying [GH] Proposition 2.4 we are able to interpret these corollaries in terms of the a-invariant.

Corollary 2.6. *Let R, J, and I be as in Theorem 2.2 with $j > 1$. Then $a(G(I)) \leq 2 - \mathrm{ht}(I)$. If $j > 2$ or I is syzygetic then $a(G(I)) \leq 1 - \mathrm{ht}(I)$.*

Proof. By [GH] Proposition 2.4, $a(G(I)) = r - \mathrm{ht}(I) - 1$ where $r = \min\{n > 0 | JI^n = I^{n+1}\}$. Now use Corollaries 2.3 and 2.4. ∎

We can also obtain bounds on the relation type by invoking results from [Huc].

Corollary 2.7. *Let R and I be as in Theorem 2.2 with $j > 1$. Then $N(I) \leq 4$. If $j > 2$ or I is syzygetic then $N(I) \leq 3$.*

Proof. By [Huc] Corollary 1.6, $N(I) \leq r + 1$ where r is the reduction number of I. Thus the result follows from Corollaries 2.3 and 2.4. ∎

Remark 2.8. The ideals satisfying $\mu(I) = \ell(I) + 1$ (as do those in our results) are referred to as having "second deviation one" by Ulrich and Vasconcelos in [UV]. They define the second deviation of I to be the integer $\mu(I) - \ell(I)$.

Next we record two families of examples. The first of these was initially proved in [AH] for R of dimension 3.

Corollary 2.9. *Let (R, m) be a Cohen-Macaulay local ring with R/m infinite and let I be a height 2 ideal of R such that R/I is Cohen-Macaulay and has finite projective dimension. Assume that I has analytic deviation one, I is generically a complete intersection, and $\mu(I) = 4$. If $G(I)$ is Cohen-Macaulay and J is a minimal reduction of I then $r_J(I) \leq 2$.*

Consequently, $R[It]$ is Cohen-Macaulay if and only if $G(I)$ is Cohen-Macaulay.

Proof. By the Hilbert-Burch Theorem [Bu] I is generated by the 3×3 minors of a 3×4 presentation matrix φ. Thus $I \subseteq I_1(\varphi)^3$. Therefore $r_J(I) \leq 2$ by Corollary 2.4. The last statement follows from [Hun1, Proposition 1.1] and [GH] Theorem 2.1. ∎

Corollary 2.10. *Let (R, m) be a Gorenstein local ring with R/m infinite and let I be a height 3 ideal of R such that R/I is Gorenstein of finite projective dimension. Assume that I has analytic deviation one, I is generically a complete intersection, and $\mu(I) = 5$. If $G(I)$ is Cohen-Macaulay and J is a minimal reduction of I then $r_J(I) \leq 2$.*

Consequently, $R[It]$ is Cohen-Macaulay if and only if $G(I)$ is Cohen-Macaulay.

Proof. By the structure theorem of Buchsbaum and Eisenbud [BE] Theorem 2.1, I is generated by the 4^{th} order Pfaffians of some 5×5 skew-symmetric presentation matrix φ with zeros along the main diagonal. By definition these Pfaffians are contained in $I_1(\varphi)^2$, thus $I \subseteq I_1(\varphi)^2$. As I is in the linkage class of a complete intersection it holds that I is syzygetic (c.f.[Hun2] Proposition 1.19 and Example 2.2, [SV] Example). Therefore $r_J(I) \leq 2$ by Corollary 2.4. The last statement follows from [Hun1] Proposition 1.1 and [GH] Theorem 2.1. ∎

We end this section with an extension of [AH] Theorem 8.2 (appearing as Theorem 1.1 in the introduction). First we need to recall an exact sequence, discovered in [SV], involving the 1^{st} Koszul homology of I. If H_1 denotes the first homology module of the Koszul complex built on a generating set $a_1, ..., a_n$ of I, then there is an exact sequence

$$0 \to \delta(I) \to H_1 \to [R/I]^n \to I/I^2 \to 0$$

of R-modules where $\delta(I) = ker(Sym_2(I) \to I^2)$. Recall that I is syzygetic if and only if $\delta(I) = 0$, therefore if I is syzygetic then there is an exact sequence

$$0 \to H_1 \to [R/I]^n \to I/I^2 \to 0. \tag{2.11}$$

As in the introduction let Q denote the kernel of the homomorphism $R[T_1, ..., T_n] \to R[It]$ and define Q_i accordingly.

Theorem 2.12. *Let (R, m) be a Cohen-Macaulay local ring with R/m infinite and let I be a 4-generated height 2 ideal of R such that R/I is Cohen-Macaulay and has finite* *projective dimension. Assume that I is generically a complete intersection and has analytic deviation 1. Then the following are equivalent.*

(1) Any presentation matrix of I satisfies the row condition;

(2) $G(I)$ is Cohen-Macaulay;

(3) $r(I) \leq 2$;

(4) $N(I) \leq 3$;

(5) $R[It]$ is Cohen-Macaulay;

Proof. We note first that I is syzygetic [SV] and the koszul homology of I is Cohen-Macaulay [Hun2, Theorem 1.14]. (5) implies (2) by [Hun1] Proposition 1.1 and (2) implies (3) and (5) by Corollary 2.9. By applying the Depth Lemma (cf. [HH2] Remark 1.9) to the exact sequence (2.11) we obtain that $depth(I/I^2) \geq dim(R/I) - 1$, thus $depth(R/I^2) \geq dim(R/I) - 1$. Therefore (3) implies (2) by [GN] Theorem 1.5. Using [Huc] Corollary 1.6 we obtain that (3) implies (4). Assume that (4) holds and let J be a minimal reduction of I. By [NR] we may assume that $I = (a_1, a_2, a_3, a_4)$ and $J = (a_1, a_2, a_3)$. By the definition of reduction we know that Q contains, as a minimal generator, a homogeneous polynomial monic in T_4. As $N(I) \leq 3$ this polynomial has degree at most 3. Therefore we must have $a_4^3 \in JI^2$ proving that (3) holds. Therefore (2) through (5) are equivalent.

The argument given in [AH] Theorem 8.2 ((3) implies (1)) shows that (1) implies (3). Assume that (3) holds and let J be a minimal reduction of I satisfying $JI^2 = I^3$. Write $J = (a_1, a_2, a_3)$ and $I = (a_1, a_2, a_3, a_4)$ and let φ be a presentation matrix of I. By the Hilbert-Burch Theorem [Bu], I is generated by the 3×3 minors of φ. We may assume (by using invertible operations) that these minors are a_1, a_2, a_3, a_4 so that $[a_1 a_2 a_3 a_4]\varphi = 0$. By condition (4) we have $Q = Q_3$, hence by Lemma 2.1 (applicable because of (2)) $I_1(\varphi)Q \subseteq Q_2$. As I is syzygetic (cf. [SV] Example) we obtain that $I_1(\varphi)Q \subseteq Q_1$. Since $a_4^3 \in JI^2$ there exists $G = T_4^3 + H \in Q$ homogeneous where $H \in (T_1, T_2, T_3)R[T_1, T_2, T_3, T_4]$ and $deg(H) = 3$. Let $y \in I_1(\varphi)$ be arbitrary. Then $yG \in Q_1$, say $yG = L_1H_1 + L_2H_2 + L_3H_3$ where $Q_1 = (L_1, L_2, L_3)$ (the generators L_1, L_2, L_3 are given by the matrix equation $[L_1 L_2 L_3] = [T_1 \cdots T_4]\varphi$). If the last row of ϕ is $[x_1 x_2 x_3]$ then since φ presents I we have $(x_1, x_2, x_3) = (J : a_4)$. We claim that $y \in (J : a_4)$. Consider the equation $yG = L_1H_1 + L_2H_2 + L_3H_3$. The coefficient of T_4^3 on the left is y. On the right the coeffiecient of T_4^3 is in the ideal generated by the coefficients of T_4 appearing in L_1, L_2, L_3 because each L_i is homogeneous of degree 1. But this ideal is contained in $(J : a_4)$ because each L_i is in Q. Therefore $y \in (J : a_4)$. As y was chosen arbitrarily we have shown that $I_1(\varphi) \subseteq (J : a_4)$, hence $I_1(\varphi) \subseteq (x_1, x_2, x_3)$. Clearly the reverse containment holds, therefore φ satisfies the row condition. This completes the proof of Theorem 2.12. ∎

3. Pfaffian ideals

We now focus on height 3 Gorenstein ideals of a regular local ring. In particular we will prove that the equivalences of Theorem 1 also hold for height 3 Gorenstein ideals.

Let $I = (a_1, \ldots, a_n)$ and as in the introduction let Q be the kernel of the homomorphism $R[T_1, \ldots, T_n] \to R[It]$. Then Q_1 is the defining ideal of the symmetric algebra of I. If φ is a given presentation matrix of I then $Q_1 = (L_1, \ldots, L_n)$ where $[L_1 \cdots L_n] = [T_1 \cdots T_n]\,\varphi$. Vasconcelos in [Vas] first studied the effect on Q of imposing restrictions on φ along the lines of the above definition. He was able to both spell out the generators of Q and determine the Cohen-Macaulay property of $R[It]$ for certain choices of I. The next result was inspired by his work, and by the follow-up work in [AH]. A square matrix is said to be *alternating* if it is skew-symmetric and has zeroes along the main diagonal.

Theorem 3.1. *Let (R, m) be a regular local ring with R/m infinite. Let I be a height 3 ideal of R such that R/I is Gorenstein. Assume that I has analytic deviation one, is generically a complete intersection, and $\mu(I) = 5$. Then the following are equivalent:*

(1) There exists a presentation matrix φ of I such that φ is an alternating matrix that satisfies the row condition;

(2) $G(I)$ is Cohen-Macaulay;

(3) $r(I) \leq 2$;

(4) $N(I) \leq 3$;

(5) $R[It]$ is Cohen-Macaulay.

Remark 3.2. It is worth noting that the proof below will show that any height 3, 5-generated, Gorenstein ideal of R that satisfies condition (1) will necessarily have analytic deviation at least one. This is because the proof will show that such an ideal has a 4-generated reduction, and (under the assumption that R/m is infinite) this reduction will be a minimal reduction by [CN] since I is not a complete intersection.

Remark 3.3. Certain equivalences of Theorem 3.1 can be strengthened by making use of a result of Noh and Vasconcelos. It follows from [NV] Theorem 2.6 that, for I as in the theorem, if $G(I)$ is Cohen-Macaulay then I is generically a complete intersection.

Proof. (5) implies (2) by [Hun1] Proposition 1.1 and (2) implies (3) and (5) by Corollary 2.10. It follows from [BU] that I/I^2 is a Cohen-Macaulay R-module (see also [HU] Theorem 2.8), thus R/I^2 is Cohen-Macaulay by the 'Depth Lemma' (cf. [HH2] Remark 1.9]). Thus [GN] Theorem 1.5 implies that (3) implies (2). If $r(I) \leq 2$ then by [Huc] Corollary 1.6 it follows that (4) holds (because $G(I)$ is Cohen-Macaulay by (2)). Therefore (3) implies (4). The proof of (4) implies (3) is almost identical to the analogous part of the proof of Theorem 2.12, and we omit the details.

We now show that the equivalent conditions (2) through (4) imply condition (1). This argument also follows closely the argument in Theorem 2.12, but for convenience we will include most of the details. Let J be a minimal reduction of I satisfying $JI^2 = I^3$ and write $I = (a_1, \ldots, a_4, a_5)$ where $J = (a_1, \ldots, a_4)$. As above consider the homomorphism $R[T_1, \ldots, T_5] \to R[It]$. By condition (4) we have that $Q = Q_3$. The structure theorem of Buchsbaum and Eisenbud [BE] Theorem 2.1 implies that there exists an alternating 5×5 presentation matrix for I whose 4^{th} order Pfaffians are $a_1, \ldots, a_5$. Let φ be such a matrix. Then condition (2) allows us to apply Lemma 2.1 to conclude that $I_1(\varphi)Q \subseteq Q_2$. By [Hun2] Proposition 1.19 and Example 2.2 (see also [SV] Example) I is syzygetic, so $Q_1 = Q_2$. Hence $I_1(\varphi)Q \subseteq Q_1$. We have ${a_5}^3 \in JI^2$, thus there exists $G = {T_5}^3 + H \in Q$ where $H \in (T_1, \ldots, T_4)R[T_1, \ldots, T_5]$ and H has degree 3. Let $y \in I_1(\varphi)$ be arbitrary. Then $yG \in Q_1$, say $yG = L_1H_1 + \ldots + L_5H_5$ where $Q_1 = (L_1, \ldots, L_5)$. If the last row of φ is $[x_1 \cdots x_4 0]$ then since φ presents I we have $(x_1, \ldots, x_4) = ((a_1, \ldots, a_4) : a_5)$. We claim that $y \in ((a_1, \ldots, a_4) : a_5)$. This follows as in the proof of Theorem 2.12. As y was chosen arbitrarily we have that $I_1(\varphi) \subseteq ((a_1, \ldots, a_4) : a_5)$, hence $I_1(\varphi) \subseteq (x_1, \ldots, x_4)$. Clearly the reverse containment holds, therefore φ satisfies the row condition.

We now prove that (1) implies (3). Our approach to this is as follows: We let φ be a "generic" 5×5 alternating matrix that satisfies the row condition and show that the Pfaffian ideal associated with it has reduction number at most 2. It then follows that the Pfaffian ideal associated with any 5×5 alternating matrix satisfying the row condition has reduction number at most 2. By "generic" we mean that the first row consists of zero followed by 4 independent variables a, b, c and d, and the other entries are generic combinations of a, b, c, d whose coefficients are new variables.

At the outset this matrix will involve 28 variables, but after eliminating two of the variables through row and column operations we may assume that φ has the form

$$\begin{pmatrix} 0 & a & b & c & d \\ -a & 0 & ae+bf+cg+dh & ai+bj+ck+dl & bm+cn \\ -b & -ae-bf-cg-dh & 0 & ax_1+bp+cq+dr & as+ct+by+dz \\ -c & -ai-bj-ck-dl & -ax_1-bp-cq-dr & 0 & au+bv+cw+dx \\ -d & -bm-cn & -as-ct-by-dz & -au-bv-cw-dx & 0 \end{pmatrix}$$

We view this as being a matrix over the polynomial ring $k[a, \ldots, z]$ where $k = \mathbb{Z}/(31,991)\mathbb{Z}$ and where we denote the variable "o" by x_1 (so as not to confuse with zero). Let I be the ideal $(a_1, \ldots, a_5)$ and $J = (a_1, \ldots, a_4)$, where $a_1, \ldots, a_5$ are the 4^{th} order Pfaffians of φ and a_5 is the Pfaffian obtained by deleting the 1^{st} row and 1^{st} column. By using the computer algebra system `Macaulay` we obtain that $JI^2 = I^3$; below is an outline of the calculation. The reader will notice that the calculation can be verified using any computer system capable of multiplying together polynomials.

Let V denote the row vector $[{a_1}^3, {a_1}^2a_2, {a_1}^2a_3, \ldots, a_4{a_5}^2]$, which is obtained by listing the elements of I^3 under a lexicographic ordering but leaving off the last element ${a_5}^3$; in other words we are simply listing the generators of JI^2. The generators for the ideal I are as follows:

$a_1 = ace + bcf + c2g + cdh - abi - b^2j - bck - bdl + a^2x_1 + abp + acq + adr$

$a_2 = ade + bdf + cdg + d^2h - b^2m - bcn + a^2s + act + aby + adz$

$a_3 = adi + bdj + cdk + d^2l - bcm - c^2n + a^2u + abv + acw + adx$

$a_4 = adx_1 + bdp + cdq + d^2r - acs - c^2t + abu + b^2v + bcw + bdx - bcy - cdz$

$a_5 = abmx_1 + acnx_1 + b^2mp + bcnp + bcmq + c^2nq + bdmr + cdnr - a^2is - abjs - acks - adls - acit - bcjt - c^2kt - cdlt + a^2eu + abfu + acgu + adhu + abev + b^2fv + bcgv + bdhv + acew + bcfw + c^2gw + cdhw + adex + bdfx + cdgx + d^2hx - abiy - b^2jy - bcky - bdly - adiz - bdjz - cdkz - d^2lz$

Let H be the following 34×1 column vector. It consists of the coefficients which will establish that ${a_5}^3 \in JI^2$. These coefficients were discovered by using `Macaulay`.

(1,1) $mtu + nsv - msw - nuy$

(2,1) $npu - mqu - jtu - nx_1v - ksv + itv + mx_1w + jsw - msx + kuy - iwy + muz$

(3,1) $-nps + mqs - mx_1t + ftu + gsv - etv - fsw - nsx + nx_1y - guy + ewy + nuz$

(4,1) $-kms + jns + imt + gmu - fnu + env - emw + mtx - iny - nxy + nvz - mwz$

(5,1) $ms + nu + tv - wy$

(6,1) $-kpu + jqu - mru + kx_1v - iqv - lsv - jx_1w + ipw + mx_1x + jsx + luy - ixy - juz + ivz$

(7,1) $kps - jqs + mrs + jx_1t - ipt + gpu - fqu - nru + ltu - gx_1v + eqv + hsv + fx_1w - epw - lsw + nx_1x - fsx + ksx - itx - kx_1y + iqy - huy + exy - mx_1z + fuz - kuz - evz + iwz$

(8,1) $kmx_1 - jnx_1 + inp - imq - lms - gju + fku + hmu + giv - ekv - nrv + ltv - fiw + ejw + mrw - emx + npx - mqx - jtx - lwy + kxy + imz - kvz + jwz$

(9,1) $-mx_1 - js - ku - qv + iw + pw + iy - xy + vz$

(10,1) $-gps + fqs + nrs - fx_1t + ept - htu + hsw - gsx + etx + gx_1y - eqy - nx_1z + guz - ewz$

(11,1) $-gmx_1 + fnx_1 - enp + emq + gjs - fks - lns + fit - ejt - mrt + hnu - htv - enx + ftx - giy + eky + nry + hwy - gxy + inz - npz + mqz + gvz - fwz$

(12,1) $-nx_1 + fs - pt + gu - ew - tx - ey + qy + wz$

(13,1) $gim - ekm - fin + ejn + lmt + hnv - hmw + gmx - fnx - lny - kmz + jnz$

(14,1) $em + in + jt + gv - fw + nx - ky + mz$

(15,1) $w + y$

(16,1) $-lpu + jru + lx_1v - irv - jx_1x + ipx$

(17,1) $lps - jrs + hpu - lqu - fru + kru - hx_1v + erv + lx_1w - irw + fx_1x - kx_1x - epx + iqx - lx_1y + iry + jx_1z - ipz$

(18,1) $lmx_1 - imr - hju + flu + hiv - elv - lqv + krv + lpw - jrw - fix + ejx - kpx + jqx$

(19,1) $jx_1 - ip - lu - rv + ix + px$

(20,1) $-hps + lqs + frs - krs - lx_1t + irt + hqu - gru - hx_1w + erw + gx_1x - eqx + hx_1y - ery - fx_1z + kx_1z + epz - iqz$

(21,1) $-hmx_1 + lnx_1 + emr - inr + hjs - fls - lpt + jrt - hku + glu + hqv - grv + hiw - elw - hpw + frw - gix + ekx + gpx - fqx - hiy + ely + lqy - kry + fiz - ejz + kpz - jqz$

(22,1) $-fx_1 + kx_1 + ep - iq + ls + hu - rw - ex + qx + ry - iz - pz$

(23,1) $him - elm + lnp - lmq + kmr - jnr - hkv + glv + hjw - flw - gjx + fkx$

(24,1) $fi - ej + kp - jq - mr + hv + lw - fx - kx$

(25,1) $-i - p + x$

(26,1) $-hqs + grs + hx_1t - ert - gx_1z + eqz$

(27,1) $-hnx_1 + enr + hks - gls - hit + elt + hpt - frt - hqy + gry + giz - ekz - gpz + fqz$

(28,1) $-gx_1 + eq - hs + rt + ez - qz$

(29,1) $hin - eln - hnp + hmq - gmr + fnr - hjt + flt + hky - gly + gjz - fkz$

(30,1) $gi - ek - gp + fq - nr - lt - hy + fz + kz$

(31,1) $e - q - z$

(32,1) $-hkm + glm + hjn - fln$

(33,1) $gj - fk + hm + ln$

(34,1) $f + k$

A calculation using `Macaulay` shows that $VH = {a_5}^3$, thus establishing that ${a_5}^3 \in JI^2$. This calculation involves polynomials whose terms have coefficients equaling either 1 or -1, which in this case prevents the integer coefficients from reaching $31,991$. Therefore the `Macaulay` calculation is valid over the integers. This proves (modulo the validity of the `Macaulay` program) that the equation $JI^2 = I^3$ holds under any specialization of the ring $\mathbb{Z}[a, \ldots, z]$. Therefore condition (3) must hold. ∎

Corollary 3.4. *For R and I as in Theorem 3.1, if $I_1(\varphi)$ is not a complete intersection then neither $R[It]$ nor $G(I)$ is Cohen-Macaulay.*

Proof. We have $ht(I_1(\varphi)) = 4$ because I is generically a complete intersection, thus if $I_1(\varphi)$ is not a complete intersection then it cannot be generated by the four entries of any row of a 5×5 alternating matrix. ∎

We end by presenting the following family of examples, illustrative of Theorem 3.1.

Example 3.5. Let k be an infinite field and let $S = k[u, v, w, x]$. Let n be a positive integer and define I to be the kernel of the surjection

$$S \to k[s^{3n+2}, s^{3n+3}, s^{3n+4}, s^{3n+5}]$$

where $u \to s^{3n+2}$, $v \to s^{3n+3}$, $w \to s^{3n+4}$, and $x \to s^{3n+5}$. Let $m = (u, v, w, x)S$, set T equal to the localization of S at m, and denote IS by P. Then P satisfies the equivalent conditions of Theorem 3.1.

Proof. One can check that I, hence P, is generated by $a_1 = v^2 - uw$, $a_2 = ux - vw$, $a_3 = vx - w^2$, $a_4 = u^{n+2} - wx^n$, and $a_5 = x^{n+1} - u^{n+1}v$. These elements are the 4^{th} order Pfaffians of the alternating matrix

$$\begin{pmatrix} 0 & -u^{n+1} & 0 & -x & -w \\ u^{n+1} & 0 & -x^n & -w & -v \\ 0 & x^n & 0 & v & u \\ x & w & -v & 0 & 0 \\ w & v & -u & 0 & 0 \end{pmatrix}.$$

By adding the first row to the fifth row, and the first column to the fifth column, we obtain the matrix

$$\begin{pmatrix} 0 & -u^{n+1} & 0 & -x & -w \\ u^{n+1} & 0 & -x^n & -w & u^{n+1} - v \\ 0 & x^n & 0 & v & u \\ x & w & -v & 0 & x \\ w & -u^{n+1} + v & -u & -x & 0 \end{pmatrix}.$$

This matrix satisfies condition (1) of Theorem 3.1. ∎

Bibliography

[AH] I. M. Aberbach and C. Huneke, An improved Briançon-Skoda theorem with applications to the Cohen-Macaulayness of Rees algebras, to appear, Math. Ann.

[Ba] J. Barshay, Graded algebras of powers of ideals generated by A-sequences, J. Algebra 25(1973), 90-99.

[BE] D.A. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math., 99(1977), 447-485.

[BU] R. O. Buchweitz and B. Ulrich, Homological properties which are invariant under linkage, preprint.

[B] L. Burch, On ideals of finite homological dimension in local rings, Proc. of Camb. Phil. Soc. 64(1968), 941-948.

[CN] R. Cowsik and M.V. Nori, On the fibers of blowing up, J. Indian Math. Soc., 40(1976), 217-222.

[GH] S. Goto and S. Huckaba, On graded rings associated to analytic deviation one ideals, to appear, Amer. J. Math.

[GN] S. Goto and Y. Nakamura, On the Gorensteinness of graded rings associated to ideals of analytic deviation one, Contemporary Mathematics vol. 159.

[HSV1] J. Herzog, A. Simis, and W. V. Vasconcelos, Approximation complexes of blowing up rings, J. Algebra 74(1982), 466-493.

[HSV1] J. Herzog, A. Simis, and W. V. Vasconcelos, Approximation complexes of blowing up rings II, J. Algebra 82(1983), 53-83.

[Huc] S. Huckaba, On complete d-sequences and the defining ideals of Rees algebras, Math. Proc. Cambridge Phil. Soc., 106(1989), 445-458.

[HH1] S. Huckaba and C. Huneke, Powers of ideals having small analytic deviation, Amer. J. Math., 114(1992), 367-403.

[HH2] S. Huckaba and C. Huneke, Rees algebras of ideals having small analytic deviation, Trans. Amer. Math. Soc. 339(1993), 373-402.

[Hun1] C. Huneke, On the associated graded ring of an ideal, Illinois J. Math. 26(1982), 121-137.

[Hun2] C. Huneke, Linkage and Koszul homology of ideals, Amer. J. Math. 104(5)(1982), 1043-1062.

[HU] C. Huneke and B. Ulrich, Powers of licci ideals, Commutative Algebra, Springer-Verlag, New York, 1989, 339-346.

[Macaulay] D. Bayer and M. Stillman, MACAULAY, computer algebra program.

[NV] S. Noh and W. V. Vasconcelos, The S_2-closure of a Rees Algebra, Results in Mathematics 23 (1993), 149-162.

[NR] D.G. Northcott and D. Rees, Reductions of ideals in local rings, Math. Proc. Cambridge Phil. Soc. 50(1954), 145-158.

[PS] C. Peskine and L. Szpiro, Liasions des variétes algebriques, Invent. Math. 26(1974), 271-302.

[Sa] J. Sally, On the associated graded ring of a local Cohen-Macaulay ring, J. Math. Kyoto Univ. 17(1977), 19-21.

[SV] A. Simis and W.V. Vasconcelos, The syzygies of the conormal module, Amer. J. Math., 103(1981), 203-224.

[T] N. V. Trung, Reduction exponent and degree bound for the defining equations of graded rings, Proc. Amer. Math. Soc. 101(1987), 229-236.

[UV] B. Ulrich and W.V. Vasconcelos, The equations of Rees algebras of ideals with linear presentation, Math. Z. 214(1993), 79-92.

[Val] G.Valla, Certain graded algebras are always Cohen-Macaulay, J. Algebra 42(1976), 537-548.

[Vas] W.V. Vasconcelos, On the equations of Rees algebras, J. reine angew. Math., 418(1991), 189-218.

Department of Mathematics, Purdue University, West Lafayette, IN 47907.

Department of Mathematics, Florida State University, Tallahassee, FL 32306-3027.

Department of Mathematics, Purdue University, West Lafayette, IN 47907.