Initial-boundary layer associated with the nonlinear Darcy-Brinkman-Oberbeck-Boussinesq system

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Abstract

In this paper, we study the vanishing Darcy number limit of the nonlinear Darcy-Brinkman-Oberbeck-Boussinesq system (DBOB). This singular perturbation problem involves singular structures both in time and in space giving rise to initial layers, boundary layers and initial-boundary layers. We construct an approximate solution to the DBOB system by the method of multiple scale expansions. The convergence with optimal convergence rates in certain Sobolev norms is established rigorously via the energy method.

Keywords: initial layer, boundary layer, initial-boundary layer, Darcy-Brinkman-Oberbeck-Boussinesq system, Darcy-Oberbeck-Boussinesq system, Darcy number, vanishing Darcy number limit, Darcy equation

1. Introduction

In this article, we study a singular perturbation problem arising from the convection phenomena in porous media which are relevant to a variety of science and engineering problems [45]. On the physical side, the set-up of the problem is similar to the Rayleigh-Bénard convection in porous media. We consider a d-dimensional channel $\tilde{\Omega} = (0, 2\pi h)^{d-1} \times (0, h)$, $d = 2, 3$, periodic in the x- or x- and y-directions, bounded by two parallel planes in the z direction and saturated with fluids. The bottom plate is kept at temperature T_2 and the top plate is kept at temperature T_1 with $T_2 > T_1$. The governing equations are the following *Darcy-Brinkman-Oberbeck-Boussinesq system* in the non-dimensional form [45, 31]:

$$\gamma_a \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) + \mathbf{v} - \tilde{D}a\Delta\mathbf{v} + \nabla p = Ra_D\,\mathbf{k}T,$$
$$\operatorname{div}\mathbf{v} = 0,$$
$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \Delta T,$$

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where $\mathbf{k}$ is the unit normal vector directed upward (the positive z direction), $\mathbf{v}$ is the non-dimensional seepage velocity, p is the modified non-dimensional kinematic pressure, T is the non-dimensional temperature. Here $\gamma_a = \frac{Da}{Pr}$ is the inverse of the Prandtl-Darcy number with Da the Darcy number and Pr the Prandtl number; $\tilde{Da} = \lambda Da$ is the Brinkman-Darcy number with λ the ratio of effective viscosity to viscosity; Ra_D is the Rayleigh-Darcy number. We refer the interested readers to [45, 31] for the detailed definitions of these dimensionless parameters.

We note that the classical Darcy number Da is defined as the ratio of permeability over the cross-sectional area of the porous media. Thus, the Darcy number is relatively small in many physically interesting settings. In this work, we consider the vanishing Darcy number limit of the DBOB system and the validity of the resulting simplified model. For the convenience of the mathematical analysis, we rewrite the Darcy-Brinkman-Oberbeck-Boussinesq system (DBOB) in a 2D periodic channel $\Omega = [0,1] \times [0,1]$ as follows

$$\begin{cases} \epsilon\big(\frac{\partial \mathbf{v}^\epsilon}{\partial t} + (\mathbf{v}^\epsilon \cdot \nabla)\mathbf{v}^\epsilon\big) + \mathbf{v}^\epsilon - \epsilon\lambda\Delta\mathbf{v}^\epsilon + \nabla p^\epsilon = Ra_D\mathbf{k}T^\epsilon, \\ \frac{\partial T^\epsilon}{\partial t} + \mathbf{v}^\epsilon \cdot \nabla T^\epsilon = \Delta T^\epsilon, \\ \text{div } \mathbf{v}^\epsilon = 0, \\ \mathbf{v}^\epsilon|_{t=0} = \mathbf{v}_0, \quad T^\epsilon|_{t=0} = T_0, \\ \mathbf{v}^\epsilon|_{z=0,1} = 0, \quad T^\epsilon|_{z=0} = 1, \quad T^\epsilon|_{z=1} = 0, \\ \mathbf{v}^\epsilon, p^\epsilon, T^\epsilon \text{ are periodic in } x\text{-direction}, \end{cases} \tag{1.1}$$

where we use (x,z)-coordinates, ϵ is the small dimensionless parameter in our problem. $\mathbf{v}^\epsilon = (v_1^\epsilon, v_2^\epsilon)$ is the velocity field, p^ϵ is the pressure, and T^ϵ is the temperature. We point out that most of our analysis is valid in three dimensions, though we restrict ourselves to the two dimensions in the present work.

Formally setting ϵ to zero in the DBOB system (1.1), we arrive at the following *Darcy-Oberbeck-Boussinesq system* (DOB), see for instance [9]:

$$\begin{cases} \mathbf{v}^0 + \nabla p^0 = Ra_D\,\mathbf{k}T^0, \quad \text{div}\,\mathbf{v}^0 = 0, \\ \frac{\partial T^0}{\partial t} + \mathbf{v}^0 \cdot \nabla T^0 = \Delta T^0, \\ v_2^0\big|_{z=0,1} = 0, \quad T^0\big|_{z=0} = 1, \quad T^0\big|_{z=1} = 0. \\ T^0\big|_{t=0} = T_0 \end{cases} \tag{1.2}$$

Periodicity in the horizontal directions is assumed again. The system (1.2) is much simplified than the original DBOB system (1.1), and it is a commonly adopted model for heat transfer in porous media. Of particular physical interests in the context of porous media are the pressure distribution (or hydraulic head) and the rate of vertical heat transport (or Nusselt number). We note that these quantities are defined in terms of the uniform norm of the pressure and H^1 norm of the temperature, cf. [9, 45, 67, 75] and references therein.

Our aim in this article is to show the validity of the simplified DOB model (1.2) as an approximation of the DBOB system (1.1) in the vanishing Darcy number limit $\epsilon \to 0$. Note that the initial condition and no-slip boundary condition for the velocity fields are dropped in the DOB system (1.1). The discrepancy in initial and boundary conditions between the system (1.1) and (1.2) leads to the emergence of temporal and spatial singular structures in the solutions. As a result, we have to study a singular perturbation problem involving both an initial layer (multiple time scales) and a boundary layer (and hence multiple spatial scales). On the one hand, the boundary layer analysis of the DBOB system is very similar to that of the classical boundary layer problem for incompressible viscous fluids at small viscosity [52, 46, 56, 60, 61]. Indeed, following the original work of Prandtl [48], we derive a Prandtl type equation for this DBOB model which indicates the existence of a boundary layer of thickness proportional to $\sqrt{\epsilon}$ in the velocity field and no boundary layer in the temperature field or pressure field (in the leading order). On the other hand, the problem involves an initial layer as well. In this connection, a similar problem has been studied by the third author in the context of Rayleigh-Bénard convection [64, 65, 66], see also [47]. As we show below, the presence of both the boundary layer and the initial layer will incur another singular structure of corner layer type (initial-boundary layer) for the velocity near the intersection of $t = 0$ and the physical boundary. The study of the initial-boundary layer in the context of volume-averaged Navier-Stokes equation (i.e. without the temperature field) is completed in [22]. The inclusion of the heat convection process further complicates the analysis. Indeed, the leading order initial layer, boundary layer and initial boundary layer in the velocity fields introduce temporal and spatial singular structures in the temperature field through the convection mechanism. Though the singular structures in the temperature appears as low order terms, they have to be included in the asymptotic analysis for a robust convergence analysis.

Another singular perturbation problem related to the Darcy-Brinkman-Oberbeck-Boussinesq system is studied in [31] where the single vanishing Darcy number limit is considered by first neglecting the nonlinear advection in the velocity equation (i.e., $\gamma_a = 0$). As a result, the authors prove the existence of boundary layer in the velocity field but no leading order boundary layer for pressure and temperature field. There is an abundant literature on boundary layer associated with incompressible flows and the related question of vanishing viscosity (see for instance [3, 7, 50, 51, 12, 44, 17, 27, 49, 24, 25, 10, 32, 74, 35, 4, 5, 36, 34, 71, 1, 72, 26, 58, 59, 55, 57, 63, 28, 11, 29, 23, 30, 23, 30, 42, 6, 18, 16, 63, 43, 40, 39, 14, 15, 69, 70] among many others) among many others). We will refrain from surveying the literature here, but emphasize that the boundary layer problem associated with the Navier-Stokes equation is still open and that there is a need to develop tools and methods to tackle it, despite that much progress has been made in recent years [20, 2, 41, 8, 13, 38].

The definitions of all of our function spaces reflect the fact that we are working in a domain that is periodic in the horizontal direction(s). Thus, for instance, $H^m = H^m_{\overline{per}}(\Omega)$, m a nonnegative integer, is the Sobolev space consisting of all functions on Ω whose derivatives up to order m are square integrable and whose derivatives up to order $m-1$ are periodic in the horizontal direction(s), with the usual norm. Equivalently, we can view such functions as being defined on $\mathcal{R}^{d-1} \times (0,1)$ with period 2π in the horizontal direction(s). Similarly, $H^1_{0,per}(\Omega)$ is the subspace of functions in $H^1_{\overline{per}}(\Omega)$ that vanish on $z = 0, 1$. We will use the classical function spaces of fluid mechanics,

$$
\begin{aligned}
H = H(\Omega) &= \left\{ \mathbf{v} \in (L^2_{\overline{per}}(\Omega))^d \colon \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } z = 0, 1 \right\}, \\
V = V(\Omega) &= \left\{ \mathbf{v} \in (H^1_{0,per}(\Omega))^d \colon \operatorname{div} \mathbf{v} = 0 \right\},
\end{aligned}
$$

where $\mathbf{n}$ denotes the unit outer normal to $\partial\Omega$. We put the L^2-norm on H and the H^1-norm on V. Because of the Poincaré's inequality, we can equivalently use $||u||_V = ||\nabla u||_{L^2}$. We follow the convention that $||\cdot||$ is the L^2-norm. We seek pressure in $L^2_0(\Omega)$, the subspace of $L^2(\Omega)$ with mean zero. Since the parameter λ does not enter the analysis in an essential way, we will set $\lambda = 1$ from now on.

For the system (1.1), we work with weak solutions whose existence can be proved in a similar fashion as the classical theory of Navier-Stokes equation, cf. [53, 54]. The well-posedness of the system (1.2) can be found in [37] and references therein. The main result in this paper is summarized in the following theorem.

Theorem 1.1. *Assume* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $T_0 \in H^m(\Omega)$ *with* $m \geq 5$. *Then there exists an approximate solution* $(\tilde{\mathbf{v}}^{app}, T^0, p^0)$ *to the DBOB system* (1.1) *such that the following optimal convergence rates hold*

$$||\mathbf{v}^\epsilon - \tilde{\mathbf{v}}^{app}||_{L^\infty(0,T,L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{1.3a}$$
$$||\mathbf{v}^\epsilon - \tilde{\mathbf{v}}^{app}||_{L^\infty(0,T,H^1(\Omega))} \leq C\epsilon^{\frac{1}{4}}, \tag{1.3b}$$
$$||\mathbf{v}^\epsilon - \tilde{\mathbf{v}}^{app}||_{L^\infty(0,T,L^\infty(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{1.3c}$$
$$||T^\epsilon - T^0||_{L^\infty(0,T,L^\infty(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{1.3d}$$
$$||T^\epsilon - T^0||_{L^\infty(0,T,H^1(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{1.3e}$$
$$||\frac{\partial(T^\epsilon - T^0)}{\partial t}||_{L^2(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{1.3f}$$
$$||p^\epsilon - p^0||_{L^\infty(0,T;L^\infty(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{1.3g}$$
$$||\nabla(p^\epsilon - p^0)||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}. \tag{1.3h}$$

Here C *is a generic constant independent of* ϵ; $\tilde{\mathbf{v}}^{app}$ *is defined as the sum of the solution to the DOB system* (1.2), *an explicit initial layer, an explicit boundary layer and an initial-boundary layer (see* (2.27)*);* p^0 *and* T^0 *are the pressure and temperature fields to the DOB system* (1.2).

As we will show below, the boundary layer thickness is of the order of $\epsilon^{\frac{1}{2}}$ and the thickness of the initial layer is of the order of ϵ. The convergence rate estimates in Theorem 1.1 then reveal the singular structures of the solution $\{\mathbf{v}^\epsilon, p^\epsilon, T^\epsilon\}$ in the sense of asymptotic expansion: (1.3a) proves existence of an initial layer (initial layer is of O(1) in $L^\infty(L^2)$); both (1.3b) and (1.3c) show the presence of an initial layer, a boundary layer and an initial-boundary layer (boundary layer and initial-boundary layer are of $\mathrm{O}(\epsilon^{-\alpha}), \alpha > 0$ in $L^\infty(H^1)$ and of O(1) in $L^\infty(L^\infty)$); (1.3e) (1.3d) demonstrates that there are no spatial nor temporal singular structures of leading order in the temperature field; (1.3e) rules out the presence of order $\epsilon^{\frac{1}{2}}$ boundary layer or initial-boundary layer in T^ϵ; (1.3f) likewise says the order $\epsilon^{\frac{1}{2}}$ initial layer do not exist in T^ϵ; (1.3g) shows that pressure has no singular structure to the leading order; finally (1.3h) verifies that the first order boundary layer of pressure does not exist either.

We follow the classical Prandtl-type approach in establishing Theorem 1.1, cf. [62, 68, 21, 31]. Specifically, we derive the Prandtl-type effective equations for the velocity correctors in the region of initial layer, boundary layer and corner layer, respectively, that approximate $\mathbf{v}^\epsilon - \mathbf{v}^0$. Since the singular structures in the temperature field emerge only in the high order expansion, we employ simplified effective equations for temperature that contain singular structures, see sec. 2.2 for details. The high order expansion seems essential for proving the convergence theorem, in particular for establishing optimal convergence rates. A natural candidate for the approximate solution is the sum of the solution to the DOB system (1.2) and the correctors. The analysis of the initial boundary layer problem then consists of the study of the Prandtl-type equations, and proof of convergence of the approximate solution to the solution of the DBOB system (1.1). The key to our success here is a mild nonlinear term in the sense that the convection term $\mathbf{v}^\epsilon \cdot \nabla \mathbf{v}^\epsilon$ has a small coefficient ϵ. Because of this, the Prandtl type equations for the boundary layer and initial-boundary layer (to the leading order) are all linear though the DBOB model (1.1) itself is nonlinear. This is similar to the case of boundary layer for the incompressible Navier-Stokes flows with non-characteristic boundary conditions [61, 60] as well as secondly boundary layer associated with the Navier-Stokes equations under Navier-type slip boundary conditions [71, 69, 70, 25]. The main difficulty for us is the existence of an initial-boundary layer which necessitates the simultaneous treatment of multiple scales in space and in time.

The paper is organized as follows. In section 2, we construct two approximate solutions to the DBOD system in view of the asymptotic analysis given in the Appendix. The equations satisfied by the approximate solution are derived along with estimates of extra forcing terms. In section 3, we prove the main convergence results through a series of steps. Some concluding remarks are given in Section 4. We include the detailed asymptotic analysis of the DBOB system in an appendix.

2. The Construction of Approximate Solutions

In this section, we construct an approximate solution to the DBOB system by taking into account the asymptotic analysis given in the appendix. We note that the leading order boundary layer functions near the boundary $z = 0$ of the velocity fields do not vanish at the other boundary $z = 1$. Likewise, the boundary layer functions at $z = 1$ are not zero at the boundary $z = 0$. Although the differences are exponentially small, we explore a truncation technique to ensure the overall boundary layer profile satisfies the respective boundary conditions exactly. To maintain the divergence free condition, The truncation is done at the stream function level so as to maintain the divergence-free condition. Such a truncation procedure is common in the study of boundary layer problems for incompressible flow, cf. [56, 60, 61, 31, 22] for instance.

We start by recalling that the leading order outer solution satisfies the following system

$$\begin{cases} \mathbf{v}^{O,0} + \nabla p^{O,0} = Ra_D \mathbf{k} T^{O,0}, \\ \frac{\partial T^{O,0}}{\partial t} + \mathbf{v}^{O,0} \cdot \nabla T^{O,0} = \Delta T^{O,0}, \\ \operatorname{div} \mathbf{v}^{O,0} = 0, \\ v_2^{O,0}\big|_{z=0,1} = 0, \quad T^{O,0}\big|_{z=0} = 1, \quad T^{O,0}\big|_{z=1} = 0, \\ T^{O,0}\big|_{t=0} = T_0. \end{cases} \tag{2.1}$$

Here the capital letter O is appended to the variables in the DOB system (1.2) so signify the outer expansion. We point out that periodic boundary conditions are assumed in the x direction wherever is necessary. The leading order initial layer satisfies

$$\begin{cases} \epsilon \dfrac{\partial \mathbf{v}^{I,0}}{\partial t} + \mathbf{v}^{I,0} = 0, \\ \mathbf{v}^I|_{t=0} = \mathbf{v}_0(x, z) - \mathbf{v}^{O,0}(0, x, z). \end{cases} \tag{2.2}$$

It follows that

$$\mathbf{v}^{I,0} = \big(\mathbf{v}_0(x,z) - \mathbf{v}^{O,0}(0,x,z)\big)e^{-\frac{t}{\epsilon}}, \tag{2.3}$$

where one may recall that $\mathbf{v}_0$ is the initial condition in the original DBOB system. Notice that div $\mathbf{v}^{I,0} = 0$.

For the sake of convenience, the following notations will be assumed throughout

$$\begin{aligned} &a(t,x) = v_1^{O,0}(t,x,0), \quad b(t,x) = v_1^{O,0}(t,x,1), \quad c(x) = v_1^{O,0}(0,x,0), \\ &d(x) = v_1^{O,0}(0,x,1), \quad e(x) = v_1^{O,0}(0,x,1), \quad \Omega_\infty = \{(x,Z) | x \in [0,1], Z \in (0,\infty)\}. \end{aligned} \tag{2.4}$$

In addition, we shall adopt the stretched variables $\tau = \frac{t}{\epsilon}$ and $Z = \frac{z}{\sqrt{\epsilon}}$. The layer functions at respective boundaries will be denoted differently via subscripts at B and C, e.g. $\mathbf{v}^{B_0,0}$ indicates the leading order (zeroth order) boundary layer for velocity at $z=0$. We introduce a cut-off function $\rho_0 \in C^\infty[0,1]$ supported near $z=0$ such that

$$\begin{cases} \rho_0 = 1, & z \in [0, \dfrac{1}{4}], \\ \rho_0 = 0, & z \in [\dfrac{1}{2}, 1]. \end{cases} \tag{2.5}$$

Similarly, one can define the cut-off function ρ_1 supported near $z=1$.

2.1. The truncation of the velocity fields

We note that the leading order boundary layers $\mathbf{v}^{B_0,0}$ and initial-boundary layers $\mathbf{v}^{C_0,0}$ satisfy the same equations as those in the case of Navier-Stokes equations studied in [22] (cf. Eqs. 3.6 and 3.16, respectively). Thus, the truncated boundary layers and initial-boundary layers are exactly the same. For completeness, we reproduce the truncation procedure for the boundary layer $\mathbf{v}^{B_0,0}$ and give the equations satisfied by the truncated profiles. The interested readers are referred to [22] for details.

The leading order boundary layer at $z=0$ for the velocity fields $\mathbf{v}^{B_0,0}$ can be found explicitly

$$v_1^{B_0,0} = -a(t,x)e^{-\frac{z}{\sqrt{\epsilon}}}, \quad v_2^{B_0,0} = \sqrt{\epsilon}\frac{\partial a}{\partial x}(1 - e^{-\frac{z}{\sqrt{\epsilon}}}). \tag{2.6}$$

In view of the solution formula (2.6), we define the truncated stream function

$$\psi^{B_0,0} = \sqrt{\epsilon}a(t,x)\big(1 - e^{-z/\sqrt{\epsilon}}\big)\rho_0(z).$$

Then the truncated boundary layer profile at $z=0$ is given by

$$\begin{aligned} \tilde{v}_1^{B_0,0} &:= -\frac{\partial \psi^{B_0,0}}{\partial z} = -ae^{-Z}\rho - \sqrt{\epsilon}a(1-e^{-Z})\rho_0', \\ \tilde{v}_2^{B_0,0} &:= \frac{\partial \psi^{B_0,0}}{\partial x} = \sqrt{\epsilon}\frac{\partial a}{\partial x}(1-e^{-Z})\rho_0. \end{aligned} \tag{2.7}$$

One can obtain the truncated boundary layer profile $\tilde{\mathbf{v}}^{B_1,0}$ at $z=1$ in a similar fashion. We then define an overall truncated boundary layer $\tilde{\mathbf{v}}^{B_0}$ as

$$\tilde{\mathbf{v}}^{B,0} = \tilde{\mathbf{v}}^{B_0,0} + \tilde{\mathbf{v}}^{B_1,0}, \tag{2.8}$$

It follows that the truncated boundary layer $\tilde{\mathbf{v}}^{B,0}$ satisfy

$$\begin{cases} \tilde{\mathbf{v}}^{B,0} - \epsilon\Delta\tilde{\mathbf{v}}^{B,0} = \mathbf{f}^B, \\ \nabla\cdot\tilde{\mathbf{v}}^{B,0} = 0, \\ \tilde{\mathbf{v}}^{B,0}\big|_{z=0,1} = -\mathbf{v}^0\big|_{z=0,1}. \end{cases} \tag{2.9}$$

where $\mathbf{f}^B = \mathbf{f}^{B_0,0} + \mathbf{f}^{B_1,0}$ with $\mathbf{f}^{B_0,0} = (f_1^{B_0,0}, f_2^{B_0,0})$ defined as follows

$$\begin{aligned} f_1^{B_0,0} &= \epsilon^{\frac{3}{2}}\Delta(a\rho_0')(1-e^{-\frac{z}{\sqrt{\epsilon}}}) + \epsilon\big(\tfrac{\partial^2 a}{\partial x^2}\rho_0 + 3a\rho_0''\big)e^{-\frac{z}{\sqrt{\epsilon}}} \\ &\qquad -2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho_0' - \sqrt{\epsilon}a\rho_0', \end{aligned} \tag{2.10a}$$

$$f_2^{B_0,0} = -\epsilon^{\frac{3}{2}}\Delta(\tfrac{\partial a}{\partial x}\rho_0)(1-e^{-\frac{z}{\sqrt{\epsilon}}}) - 2\epsilon\tfrac{\partial a}{\partial x}e^{-\frac{z}{\sqrt{\epsilon}}}\rho_0' + \sqrt{\epsilon}\tfrac{\partial a}{\partial x}\rho_0. \tag{2.10b}$$

It is readily seen that the forcing terms have the estimate

$$||\partial_x^j\mathbf{f}^B||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{2}}, \quad j=0,1. \tag{2.11}$$

The stream function for the initial-boundary layer at $z=0$ is $\psi^{C_0,0} = \psi_1^0 + \psi_2^0$ with

$$\begin{aligned} \psi_1^0 = -\sqrt{\epsilon}c(x)e^{-\tau}\Bigg\{ &\frac{1}{\sqrt{4\pi\tau}}\int_0^{+\infty}(1-e^{-Z_0})\big(e^{-\frac{(Z-Z_0)^2}{4\tau}} + e^{-\frac{(Z+Z_0)^2}{4\tau}}\big)\,dZ_0 \\ &-2\int_0^\tau \frac{1}{\sqrt{4\pi s}}e^{-\frac{Z^2}{4s}}\,ds\Bigg\}, \end{aligned} \tag{2.12}$$

$$\psi_2^0 = 2\sqrt{\epsilon}c(x)e^{-\tau}\left\{\frac{1}{2} - \frac{e^\tau}{\sqrt{\pi}}\int_{\sqrt{\tau}}^{+\infty} e^{-z^2}\,dz - \int_0^\tau \frac{1}{\sqrt{4\pi s}}\,ds\right\}. \tag{2.13}$$

Note that $\psi_1 \to 0$ as $Z\to+\infty$ and ψ_2 is of initial layer type with exponential decay in time. The modified initial-boundary layer is given by

$$\begin{aligned} \tilde{v}_1^{C_0,0} &= v_1^{C,0}\rho_0(z) - \psi^{C_0,0}\rho_0'(z), \\ \tilde{v}_2^{C_0,0} &= v_2^{C_0,0}\rho_0(z). \end{aligned} \tag{2.14}$$

Define

$$\tilde{\mathbf{v}}^{C,0} = \tilde{\mathbf{v}}^{C_0,0} + \tilde{\mathbf{v}}^{C_1,0}. \tag{2.15}$$

One can verify that the truncated initial-boundary layer $\tilde{\mathbf{v}}^{C,0}$ satisfy

$$\begin{cases} \epsilon\dfrac{\partial\tilde{\mathbf{v}}^{C,0}}{\partial t} + \tilde{\mathbf{v}}^{C,0} - \epsilon\Delta\tilde{\mathbf{v}}^{C,0} = \mathbf{f}^C, \quad t\in(0,T),(x,z)\in\Omega, \\ \nabla\cdot\tilde{\mathbf{v}}^{C,0} = 0, \\ \tilde{\mathbf{v}}^{C,0}\big|_{t=0} = -\tilde{\mathbf{v}}^{B,0}\big|_{t=0}, \\ \tilde{\mathbf{v}}^{C,0}\big|_{z=0,1} = -\mathbf{v}^{I,0}\big|_{z=0,1}, \end{cases} \tag{2.16}$$

where $\mathbf{f}^C = \mathbf{f}_1^{C_0,0} + \mathbf{f}_2^{C_1,0}$, and $\mathbf{f}_1^{C_0,0} = (f_1^{C_0,0}, f_2^{C_0,0})$ with

$$
\begin{aligned}
f_1^{C_0,0} &= -f^0\rho_0' - 2\epsilon\frac{\partial v_1^{C_0,0}}{\partial z}\rho_0' - 3\epsilon v_1^{C_0,0}\rho_0'' \\
&\quad + \epsilon\psi^{C_0,0}\rho_0''' - \epsilon\frac{\partial^2 v_1^{C_0,0}}{\partial x^2}\rho_0 + \epsilon\frac{\partial v_2^{C_0,0}}{\partial x}\rho_0', \qquad (2.17a) \\
f_2^{C_0,0} &= \frac{\partial f^0}{\partial x}\rho_0 + \sqrt{\epsilon}\big(2\frac{\partial v_1^{C_0,0}}{\partial x}\rho_0' - v_2^{C_0,0}\rho_0'' - \frac{\partial^2 v_2^{C_0,0}}{\partial x^2}\rho_0\big). \qquad (2.17b)
\end{aligned}
$$

Here

$$
f^0(\tau, x) = -\frac{2\sqrt{\epsilon}c(x)}{\sqrt{\pi}}\int_{\sqrt{\tau}}^{+\infty} e^{-z^2}\,dz = -\frac{2\sqrt{\epsilon}c(x)e^{-\tau}}{\sqrt{4\pi\tau}}\int_0^{+\infty} e^{-\frac{z^2}{4\tau}-z}\,dz. \qquad (2.18)
$$

The forcing term has the following estimate

$$
\begin{aligned}
&||\partial_x^j \mathbf{f}^C||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{2}} \quad j = 0, 1, && (2.19) \\
&||\partial_x^j \mathbf{f}^C||_{L^2(0,T;L^2(\Omega))} \le C\epsilon \quad j = 0, 1, && (2.20) \\
&|f^0(\tau, x)| \le \sqrt{\epsilon}|c(x)|e^{-\tau}. && (2.21)
\end{aligned}
$$

The basic estimates for $\mathbf{v}^{B_0,0}, \mathbf{v}^{C_0,0}$ and $\tilde{\mathbf{v}}^{B,0}, \tilde{\mathbf{v}}^{C,0}$ are gathered in Lemma 2.1, cf. [22] for details of the proof.

Lemma 2.1. *Assume $\mathbf{v}_0 \in \mathbf{V} \cap H^4(\Omega)$ and $T_0 \in H^4(\Omega)$. The following estimates hold*

$$
\begin{aligned}
&||\tilde{\mathbf{v}}^{B,0}||_{L^\infty(0,T;L^\infty(\Omega))} \le C, \quad ||\tilde{\mathbf{v}}^{B,0}||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{4}}, \quad ||\tilde{\mathbf{v}}^{B,0}||_{L^\infty(0,T;H^1(\Omega))} \le C\epsilon^{-\frac{1}{4}}, && (2.22) \\
&||\tilde{\mathbf{v}}^{C,0}||_{L^\infty(0,T;L^\infty(\Omega))} \le C, \quad ||\tilde{\mathbf{v}}^{C,0}||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{4}}, \quad ||\tilde{\mathbf{v}}^{C,0}||_{L^\infty(0,T;H^1(\Omega))} \le C\epsilon^{-\frac{1}{4}}, && (2.23) \\
&||v_2^{C_0,0}, \tilde{v_2}^{C_0,0}||_{L^2(0,T;L^2(\Omega))} \le C\epsilon, \quad ||\psi^{C_0,0}, \psi_1^0, \psi_2^0||_{L^2(0,T;L^2(\Omega))} \le C\epsilon. && (2.24)
\end{aligned}
$$

In addition, $v_1^{B_0,0}$ and $v_1^{C_0,0}$ enjoy exponential decay

$$
|v_1^{B_0,0}| \le |a(t,x)|e^{-\frac{z}{\sqrt{\epsilon}}}, \quad |v_1^{C_0,0}| \le |c(x)|e^{-\frac{t}{2\epsilon}-\frac{z}{\sqrt{2\epsilon}}}, \qquad (2.25)
$$

Finally, the second component of the velocity has the following estimates

$$
||\partial_z v_2^{B_0,0}, \partial_z v_2^{C_0,0}|| \le C\epsilon^{\frac{1}{4}}, \quad ||\partial_{zz} v_2^{B_0,0}, \partial_{zz} v_2^{C_0,0}|| \le C\epsilon^{-\frac{1}{4}}. \qquad (2.26)
$$

Remark 2.1. *The point-wise estimates for $v_2^{B_0,0}$ and $v_2^{C_0,0}$ can be derived in view of estimates* (2.25) *and the divergence-free condition.*

Now we define an approximate solution $(\tilde{\mathbf{v}}^{app}, \tilde{p}^{app}, \tilde{T}^{app})$ to the DBOB system (1.1) as

$$
\begin{aligned}
&\tilde{\mathbf{v}}^{app} = \mathbf{v}^{O,0} + \mathbf{v}^{I,0} + \tilde{\mathbf{v}}^{B,0} + \tilde{\mathbf{v}}^{C,0}, && (2.27a) \\
&\tilde{p}^{app} = p^{O,0}, && (2.27b) \\
&\tilde{T}^{app} = T^{O,0}. && (2.27c)
\end{aligned}
$$

We recall that $\mathbf{v}^{O,0}, p^{O,0}, T^{O,0}$ are the outer solutions, $\mathbf{v}^{I,0}$ is the leading order initial layer function, $\tilde{\mathbf{v}}^{B,0}$ and $\tilde{\mathbf{v}}^{C,0}$ are the truncated boundary layer and initial-boundary layer of leading order, respectively. We remark that a convergence result (e.g. in $L^\infty L^2$ norm) can be obtained for $(\tilde{\mathbf{v}}^{app}, \tilde{p}^{app}, \tilde{T}^{app})$ by following the energy method in [22] and [31]. However, the convergence rates would not be optimal, since the truncation procedure incurs extra error of order $\epsilon^{\frac{1}{2}}$ into the system, cf. the last term in Eq. (2.10a) and the first term in Eq. (2.17a). Moreover, the convergence in physically interesting norms (e.g. $L^\infty H^1$ and $L^\infty L^\infty$) is difficult to establish by directly working with the approximate solution (2.27). These obstacles motivate us to construct another approximate solution by including higher order terms in the asymptotic expansion.

The equations satisfied by the first order terms ($\mathrm{O}(\sqrt{\epsilon})$) are given as follows, incorporating errors from the truncation of the leading order profiles.

- First order outer solution $(\mathbf{v}^{O,1}, p^{O,1}, T^{O,1})$

$$\begin{cases} \mathbf{v}^{O,1} + \nabla p^{O,1} = \mathbf{f}^O + Ra_D T^{O,1}\mathbf{k}, \\ \frac{\partial T^{O,1}}{\partial t} + \mathbf{v}^{O,0} \cdot \nabla T^{O,1} = \Delta T^{O,1} - f^T, \\ \text{div } \mathbf{v}^{O,1} = 0, \\ T^{O,1}(0,x,z) = 0, \\ \mathbf{v}^{O,1} \cdot \mathbf{n}|_{z=0,1} = 0, \ T^{O,1}(t,x,z)|_{z=0,1} = 0, \end{cases} \tag{2.28}$$

where

$$\mathbf{f}^O = \left(\rho_0' a + \rho_1' b, -\frac{\partial a}{\partial x}\rho_0 - \frac{\partial b}{\partial x}\rho_1\right), \tag{2.29}$$

$$f^T = (v_1^{O,1} - a\rho_0' - b\rho_1')\partial_x T^{O,0} + (v_2^{O,1} + \frac{\partial a}{\partial x}\rho_0 + \frac{\partial b}{\partial x}\rho_1)\partial_z T^{O,0}. \tag{2.30}$$

We note that $\lim_{Z\to\infty} \frac{\tilde{v}_2^{B,0}}{\sqrt{\epsilon}} = \frac{\partial a}{\partial x}\rho_0$, cf. Eq. (2.7). The readers are referred to Eq. (2.4) for the definitions of the functions a, b, c, d.

- First order initial layer $\mathbf{v}^{I,1}$

$$\begin{cases} \frac{\partial \mathbf{v}^{I,1}}{\partial \tau} + \mathbf{v}^{I,1} = \mathbf{f}^I, \\ \mathbf{v}^{I,1}(0,x,z) = -\mathbf{v}^{O,1}(0,x,z), \end{cases} \tag{2.31}$$

where $\mathbf{f}^I = \left(-(\rho_0' f^0 + \rho_1' f^1), (\rho_0 \frac{\partial f^0}{\partial x} + \rho_1 \frac{\partial f^1}{\partial x})\right)$. One may recall the definition of f^0 from Eq. (2.18). f^1 is the counterpart of f^0 at $z = 1$ (replacing $c(x)$ with $e(x)$). We note that $\nabla \cdot \mathbf{f}^I = 0$. It follows readily from the ODE (2.31) that $\nabla \cdot \mathbf{v}^{I,1} = 0$. We remark, however, that $v_2^{I,1}|_{z=0,1} \neq 0$.

- First order boundary layer $\mathbf{v}^{B_0,1}$ at $z = 0$

$$\begin{cases} v_1^{B_0,1} - \partial_{ZZ} v_1^{B_0,1} = 0, \quad Z \in (0,\infty), \\ \partial_x v_1^{B_0,1} + \frac{1}{\sqrt{\epsilon}}\partial_Z v_2^{B_0,1} = 0, \quad Z \in (0,\infty), \\ v_1^{B_0,1}(t,x,0) = -v_1^{O,1}(t,x,0), \quad v_2^{B_0,1}(t,x,0) = 0, \\ v_1^{B_0,1}(t,x,Z) \to 0, \quad Z \to \infty. \end{cases} \tag{2.32}$$

- First order initial-boundary layer $\mathbf{v}^{C_0,1}$ at $z=0$

$$\left\{\begin{array}{l}\frac{\partial v_1^{C_0,1}}{\partial\tau}+v_1^{C_0,1}-\partial_{ZZ}v_1^{C_0,1}=0,\quad Z\in(0,\infty),\\ \partial_x v_1^{C_0,1}+\frac{1}{\sqrt{\epsilon}}\partial_Z v_2^{C_0,1}=0,\quad Z\in(0,\infty),\\ v_1^{C_0,1}(0,x,Z)=-v_1^{B_0,1}(0,x,Z),\\ v_1^{C_0,1}(\tau,x,0)=-v_1^{I,1}(\tau,x,0),\ \ v_2^{C_0,1}(t,x,0)=-v_2^{I,1}(\tau,x,0),\\ v_1^{C_0,1}(t,x,Z)\to 0,\ Z\to\infty.\end{array}\right. \tag{2.33}$$

Note that we intentionally neglect the terms in the right-hand side of Eq. (4.12) and Eq. (4.13) in deriving the first order boundary layer system (2.32) and first order initial-boundary layer system (2.33), respectively. In the sequel, we will show that these terms can be controlled by $C\epsilon$ in the $L^\infty(L^2)$ norm. In doing so, we see that the first order boundary layer and initial boundary layer satisfy the same type of equations as their leading order counterparts. Thus, the truncation procedure and the estimates of the truncated profiles would be similar to the case of leading order expansion. Hereafter, we denote by $\tilde{\mathbf{v}}^{B,1}$ and $\tilde{\mathbf{v}}^{C,1}$ the first order truncated boundary layer and initial-boundary layer, respectively. The estimates given in Lemma 2.1 are valid for $\tilde{\mathbf{v}}^{B,1}$ and $\tilde{\mathbf{v}}^{C,1}$, as well. Moreover, thanks to the estimate (2.21), one finds that $\mathbf{v}^{I,1}$ has exponential decay in time.

2.2. Singular structures in the temperature field

We recall that the temporal and spatial singular structures in the velocity fields introduce singular structures to the temperature filed via advection mechanism. Formally, the singular structures in the temperature field start to appear in the order ϵ expansion. We modify the expansions according to the truncation in the velocity fields. Since our aim is to establish a convergence result, we will omit the detailed asymptotic analysis for each expansion. We will rather focus on the estimates of the functions in the original variable.

- Initial layer in the second order ($\mathrm{O}(\epsilon)$) expansion

$$\left\{\begin{array}{l}\frac{\partial T^{I,2}}{\partial\tau}=-[v_1^{I,0}-(\rho_0'\psi_2^0+\rho_1'\psi_2^1)]\frac{\partial T^{O,0}}{\partial x}-[v_2^{I,0}+\rho_0\frac{\partial\psi_2^0}{\partial x}+\rho_1\frac{\partial\psi_2^1}{\partial x})]\frac{\partial T^{O,0}}{\partial z},\\ T^{I,2}|_{\tau=0}=0.\end{array}\right. \tag{2.34}$$

Here ψ_2^0 is defined in Eq. (2.13), and ψ_2^1 is the counterpart of ψ_2^0 at $z=1$ with $c(x)$ replaced by $e(x)$. We note that the forcing terms enjoy exponential decay in time, cf. Eq. (2.3) and Eq. (2.13). We remark that a formal expansion of the forcing term would reveal the appearance of initial layers of $\mathrm{O}(\epsilon)$ contained in $T^{I,2}$.

- Boundary layer at $z=0$ in the second order ($\mathrm{O}(\epsilon)$) expansion

$$\left\{\begin{array}{l}\epsilon\frac{\partial^2 T^{B_0,2}}{\partial z^2}=(\tilde{v}_1^{B_0,0}+\sqrt{\epsilon}a\rho_0')\partial_x T^{O,0}+(\tilde{v}_2^{B_0,0}-\sqrt{\epsilon}\frac{\partial a}{\partial x}\rho_0)\partial_z T^{O,0},\\ T^{B,2}|_{z=0,1}=0.\end{array}\right. \tag{2.35}$$

The modification of the terms in the right-hand side is due to the fact that the non-boundary layer type functions are already taken care of in the first order outer expansion, cf. Eq. (2.28). Note that the equation is written in the original variable. To be precise, $T^{B_0,2}$ contains functions of boundary layer type if one formally performs an asymptotic analysis of Eq. (2.35). Indeed, the boundary layer in $T^{B_0,2}$ appears in the order $\mathrm{O}(\sqrt{\epsilon})$ expansion.

- Initial-boundary layer at $z=0$ in the second order ($\mathrm{O}(\epsilon)$) expansion

$$\begin{cases} \epsilon(\frac{\partial T^{C_0,2}}{\partial t} - \frac{\partial^2 T^{C_0,2}}{\partial z^2}) = -v_1^{C_0,0}\rho_0\partial_x T^{O,0} - \psi_1^0\rho_0'\partial_x T^{O,0} - \frac{\partial \psi_1^0}{\partial x}\rho_0\partial_z T^{O,0}, \\ T^{C,2}(0,x,z) = -T^{B_0,2}(0,x,z), \\ T^{C,2}(t,x,0) = -T^{I,2}(t,x,0), \quad T^{C,2}(t,x,1) = 0. \end{cases} \tag{2.36}$$

  As the functions of pure initial layer type in $\tilde{\mathbf{v}}^{C_0,0}$ are included in the Eq. (2.34), the forcing terms in Eq. (2.36) have exponential decay both in time and in space in terms of the stretched variables τ, Z.

The modified boundary layer Eq. (2.35) and modified initial-boundary layer Eq. (2.36) are both written in original variable. The boundary conditions are imposed such that no truncation is needed in defining the overall boundary layer $\tilde{T}^{B,2}$ and initial-boundary layer $\tilde{T}^{C,2}$, i.e.,

$$\tilde{T}^{B,2} = T^{B_0,2} + T^{B_1,2}, \quad \tilde{T}^{C,2} = T^{C_0,2} + T^{C_1,2}, \tag{2.37}$$

where $T^{B_1,2}$ and $T^{C_1,2}$ are the modified boundary layer and modified initial-boundary layer for temperature at $z=1$, respectively. One can readily check that $T^{I,2} + \tilde{T}^{B,2} + \tilde{T}^{C,2}$ satisfies homogeneous initial and boundary conditions.

In an entirely similar fashion, one can derive the equations satisfied by the third order expansion $T^{I,3}, \tilde{T}^{B,3}$ and $\tilde{T}^{C,3}$ that balance the $\epsilon^{\frac{1}{2}}$ terms in the temperature equation. The interested readers are referred to Eq. (4.19)–(4.20) for the original expansion. We gather the basic estimates for the temperature in the following lemma.

Lemma 2.2. *Assume $\mathbf{v}_0 \in \mathbf{V} \cap H^4(\Omega)$ and $T_0 \in H^4(\Omega)$. The following estimates hold*

$$||T^{I,2}||_{L^\infty(0,T;L^\infty(\Omega))} \le C, \quad ||\nabla T^{I,3}||_{L^\infty(0,T;L^\infty(\Omega))} \le C, \tag{2.38}$$
$$||\tilde{T}^{B,2}||_{L^\infty(0,T;L^\infty(\Omega))} \le C, \quad ||\nabla \tilde{T}^{B,2}||_{L^\infty(0,T;L^\infty(\Omega))} \le C, \tag{2.39}$$
$$||\tilde{T}^{C,2}||_{L^\infty(0,T;L^\infty(\Omega))} \le C, \quad ||\tilde{T}^{C,2}||_{L^\infty(0,T;H^1(\Omega))} \le C, \tag{2.40}$$
$$||\tilde{T}^{B,3}||_{L^\infty(0,T;L^\infty(\Omega))} \le C\epsilon^{-\frac{1}{2}}, \quad ||\tilde{T}^{B,3}||_{L^\infty(0,T;H^1(\Omega))} \le C\epsilon^{-\frac{1}{2}}, \tag{2.41}$$
$$||\tilde{T}^{C,3}||_{L^\infty(0,T;L^\infty(\Omega))} \le C\epsilon^{-\frac{1}{2}}, \quad ||\tilde{T}^{C,3}||_{L^\infty(0,T;H^1(\Omega))} \le C\epsilon^{-\frac{1}{2}}. \tag{2.42}$$

Remark 2.2. *Since there are no singular structures in the x variable, the derivatives of the layer functions in the x direction will satisfy the same estimates as the layer functions themselves, provided that the functions are as regular as needed.*

Proof. Thanks to the exponential decay in time for the forcing terms in Eq. (2.34) (i.e. $e^{-\frac{t}{\epsilon}}$), one readily obtains the estimate (2.38). In view of the definition of $\tilde{T}^{B,2}$ and $\tilde{T}^{C,2}$ from Eqs. (2.37), we only need to work on $T^{B_0,2}$ and $T^{C_0,2}$.

Recall the definition of $\tilde{\mathbf{v}}^{B_0,0}$ from Eqs. (2.7). One sees that the two typical terms in the right-hand side of Eq. (2.35) are $a\rho_0 e^{-\frac{z}{\sqrt{\epsilon}}}\partial_x T^{O,0}$ and $\epsilon^{\frac{1}{2}}\frac{\partial a}{\partial x}\rho_0 e^{-\frac{z}{\sqrt{\epsilon}}}\partial_z T^{O,0}$. It follows that

$$\frac{\partial T^{B_0,2}}{\partial z} = C(t,x) + \frac{a}{\epsilon}\int_0^z \rho_0(l)e^{-\frac{l}{\sqrt{\epsilon}}}\partial_x T^{O,0}dl + \frac{\partial_x a}{\sqrt{\epsilon}}\int_0^z \rho_0(l)e^{-\frac{l}{\sqrt{\epsilon}}}\partial_z T^{O,0}dl, \tag{2.43}$$

where the function $C(t,x)$ can be found by imposing the boundary conditions after integrating the above equations

$$C(t,x) = \frac{a}{\epsilon}\int_0^1\int_0^z \rho_0(l)e^{-\frac{l}{\sqrt{\epsilon}}}\partial_x T^{O,0}dldz + \frac{\partial_x a}{\sqrt{\epsilon}}\int_0^1\int_0^z \rho_0(l)e^{-\frac{l}{\sqrt{\epsilon}}}\partial_z T^{O,0}dldz. \tag{2.44}$$

The last term in Eq. (2.43) is bounded by $C||\partial_x a\partial_z T^{O,0}||_{L^\infty(0,T;L^\infty(\Omega))}$ by utilizing the exponential decay in z. For the second term in Eq. (2.43), we note that $\partial_x T^{O,0}|_{z=0} = 0$. One has, by Hardy's inequality

$$\begin{aligned}
|\frac{a}{\epsilon}\int_0^z \rho_0(l)e^{-\frac{l}{\sqrt{\epsilon}}}\partial_x T^{O,0}dl| &\leq ||a\rho_0||_{L^\infty(0,T;L^\infty(\Omega))}\int_0^z \frac{l}{\epsilon}e^{-\frac{l}{\sqrt{\epsilon}}}|\frac{\partial_x T^{O,0}}{l}|dl \\
&\leq ||a\rho_0\partial_{xz}T^{O,0}||_{L^\infty(0,T;L^\infty(\Omega))}\int_0^z \frac{l}{\epsilon}e^{-\frac{l}{\sqrt{\epsilon}}}dl \\
&\leq C||a\rho_0\partial_{xz}T^{O,0}||_{L^\infty(0,T;L^\infty(\Omega))}.
\end{aligned}$$

The function $C(t,x)$ in Eq. (2.44) has similar estimate. The estimates (2.39) follow immediately.

For the estimates of $T^{C_0,2}$, we first recall the definition of $\tilde{\mathbf{v}}^{C,0}$ from Eqs. (2.14) and the definitions of ψ_1^0 and ψ_2^0 from Eqs. (2.12) and (2.13), respectively. We introduce an auxiliary function $A(t,x,z) = T^{I,2}(t,x,0)(z-1)$. In view of Eq. (2.34), one finds that

$$\epsilon\frac{\partial A}{\partial t} = -\frac{\partial \psi_2^0}{\partial x}\partial_z T^{O,0}|_{z=0}(z-1).$$

where one has utilized the conditions $\partial_x T^{O,0}|_{z=0} = 0$ and $v_2^{I,0}|_{z=0} = 0$. It is clear that the difference $T^d = T^{C_0,2} - A$ would satisfy

$$\begin{cases} \frac{\partial T^d}{\partial t} - \partial_{zz}T^d = \frac{1}{\epsilon}f^d, \\ T^d|_{t=0} = -T^{B_0,2}(0,x,z), \quad T^d|_{z=0,1} = 0. \end{cases} \tag{2.45}$$

where

$$f^d = -v_1^{C_0,0}\rho_0\partial_x T^{O,0} - \psi_1^0\rho_0'\partial_x T^{O,0} - \frac{\partial \psi_1^0}{\partial x}\rho_0\partial_z T^{O,0} - \frac{\partial \psi_2^0}{\partial x}\partial_z T^{O,0}|_{z=0}(z-1). \tag{2.46}$$

One may apply the standard energy method to Eq. (2.45), i.e. testing the equation with T^d and $\partial_t T^d$ respectively. In view of the estimate (2.45), one only needs to control the first term in Eq. (2.46). One has

$$
\begin{aligned}
&\Big|\int_0^T\int_0^1 \frac{1}{\epsilon^2}(v_1^{C_0,0})^2\rho_0^2(\partial_x T^{O,0})^2 dzdt\Big| \\
&\leq ||c\rho_0||^2_{L^\infty(L^\infty(\Omega))}\int_0^T\int_0^1 \frac{1}{\epsilon^2}e^{-\frac{t}{\epsilon}-\frac{\sqrt{2}z}{\sqrt{\epsilon}}}(\partial_x T^{O,0})^2 dzdt \quad \text{thanks to the estimate (2.25)} \\
&= ||c\rho_0||^2_{L^\infty(0,T;L^\infty(\Omega))}\int_0^T\int_0^1 \frac{1}{\epsilon}e^{-\frac{t}{\epsilon}}\frac{z^2}{\epsilon}e^{-\frac{\sqrt{2}z}{\sqrt{\epsilon}}}(\partial_x T^{O,0}/z)^2 dzdt \\
&\leq ||c\rho_0\partial_{xz}T^{O,0}||^2_{L^\infty(0,T;L^\infty(\Omega))}\int_0^T(\frac{1}{\epsilon}e^{-\frac{t}{\epsilon}})dt\int_0^1\frac{z^2}{\epsilon}e^{-\frac{\sqrt{2}z}{\sqrt{\epsilon}}}dz \quad \text{thanks to Hardy's inequality} \\
&\leq ||c\rho_0\partial_{xz}T^{O,0}||^2_{L^\infty(0,T;L^\infty(\Omega))}\epsilon^{\frac{1}{2}}.
\end{aligned}
$$

One can then derive the second estimate in (2.40), by virtue of estimate (2.39). In fact, the estimate derived is $\int_0^1|\frac{\partial T^{C_0,2}}{\partial z}|^2dz \leq C$ with a constant C, as the equation (2.45) is a heat equation involving spatial derivatives only in the z direction. Then the first estimate in (2.40) follows from the 1D Sobolev embedding. We point out that the uniform estimate can also be derived by the anisotropic Sobolev embedding, cf. [22].

The estimates (2.42) are easy to derive by energy methods, cf. Eq. (4.19)–(4.20). This concludes our proof. □

Remark 2.3. *We note that the estimates* (2.39)– (2.42) *are by no means optimal in terms of the parameter ϵ. For instance, the estimates* (2.39) *imply that the boundary layer for temperature $T^{B_0,0}$ would be of order $\sqrt{\epsilon}$. An asymptotic expansion of the forcing terms in Eq.* (2.35) *will confirm this conclusion. We also remark that the estimates* (2.41) *and* (2.42) *are sufficient for the convergence analysis below, as $T^{B_0,3}$ and $T^{C_0,3}$ appear as order $\epsilon^{\frac{3}{2}}$ in the definition of the approximate solution, cf. the definition* (2.47).

2.3. The approximate solution

We define another approximate solution $(\mathbf{v}^{app}, p^{app}, T^{app})$ as follows

$$\mathbf{v}^{app} = \mathbf{v}^{O,0} + \mathbf{v}^{I,0} + \tilde{\mathbf{v}}^{B,0} + \tilde{\mathbf{v}}^{C,0} + \sqrt{\epsilon}(\mathbf{v}^{O,1} + \mathbf{v}^{I,1} + \tilde{\mathbf{v}}^{B,1} + \tilde{\mathbf{v}}^{C,1}), \tag{2.47a}$$

$$p^{app} = p^{O,0} + \sqrt{\epsilon}p^{O,1}, \tag{2.47b}$$

$$T^{app} = T^{O,0} + \sqrt{\epsilon}T^{O,1} + \epsilon(T^{I,2} + \tilde{T}^{B,2} + \tilde{T}^{C,2}) + \epsilon^{\frac{3}{2}}(T^{I,3} + \tilde{T}^{B,3} + \tilde{T}^{C,3}). \tag{2.47c}$$

By direct calculation, one can verify that the approximate solution $(\mathbf{v}^{app}, p^{app}, T^{app})$ satisfies the following system

$$
\begin{cases}
\epsilon\big(\frac{\partial \mathbf{v}^{app}}{\partial t} + (\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{app}\big) + \mathbf{v}^{app} - \epsilon\lambda\Delta\mathbf{v}^{app} + \nabla p^{app} = Ra_D\mathbf{k}T^{app} + \mathbf{f}^{err}, \\
\frac{\partial T^{app}}{\partial t} + \mathbf{v}^{app}\cdot\nabla T^{app} = \Delta T^{app} + g^{err}, \\
\operatorname{div}\mathbf{v}^{app} = 0, \\
\mathbf{v}^{app}|_{t=0} = \mathbf{v}_0, \quad T^{app}|_{t=0} = T_0, \\
\mathbf{v}^{app}|_{z=0,1} = 0, \quad T^{app}|_{z=0} = 1, \quad T^{app}|_{z=1} = 0, \\
\mathbf{v}^{app}, p^{app}, T^{app} \text{ are periodic in } x\text{-direction.}
\end{cases} \tag{2.48}
$$

The error functions take the form

$$
\begin{aligned}
\mathbf{f}^{err} &= \epsilon\big(\frac{\partial \mathbf{v}^{O,0}}{\partial t} + \frac{\partial \tilde{\mathbf{v}}^{B,0}}{\partial t}\big) + \epsilon(\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{app} - \epsilon\big(\Delta\mathbf{v}^{O,0} + \Delta\mathbf{v}^{I,0}\big)\\
&\quad + \epsilon^{\frac{3}{2}}\big(\frac{\partial \mathbf{v}^{O,1}}{\partial t} + \frac{\partial \tilde{\mathbf{v}}^{B,1}}{\partial t}\big) - \epsilon^{\frac{3}{2}}\big(\Delta\mathbf{v}^{O,1} + \Delta\mathbf{v}^{I,1}\big) + \tilde{\mathbf{f}}^{B} + \tilde{\mathbf{f}}^{C}\\
&\quad - Ra_D\mathbf{k}\big[\epsilon(T^{I,2} + \tilde{T}^{B,2} + \tilde{T}^{C,2}) + \epsilon^{\frac{3}{2}}(T^{I,3} + \tilde{T}^{B,3} + \tilde{T}^{C,3})\big], \qquad (2.49)\\
g^{err} &= -\epsilon(\frac{\partial \tilde{T}^{B,2}}{\partial t} + \mathbf{v}^{app}\cdot\nabla T^{I,2} + \Delta T^{I,2}) - \epsilon^{\frac{3}{2}}\big(\frac{\partial \tilde{T}^{B,3}}{\partial t} + \mathbf{v}^{app}\cdot\nabla(T^{I,3} + \tilde{T}^{B,3} + \tilde{T}^{C,3}) + \Delta T^{I,3}\big)\\
&\quad - \epsilon(\mathbf{v}^{O,1} + \mathbf{v}^{I,1} + \tilde{\mathbf{v}}^{B,1} + \tilde{\mathbf{v}}^{C,1})\cdot\nabla T^{O,1} - \epsilon^{\frac{3}{2}}(\mathbf{v}^{O,1} + \mathbf{v}^{I,1} + \tilde{\mathbf{v}}^{B,1} + \tilde{\mathbf{v}}^{C,1})\cdot\nabla(\tilde{T}^{B,2} + \tilde{T}^{C,2})\\
&\quad - \epsilon(\partial_{xx}\tilde{T}^{B,2} + \partial_{xx}\tilde{T}^{C,2}) - \epsilon^{\frac{3}{2}}(\partial_{xx}\tilde{T}^{B,3} + \partial_{xx}\tilde{T}^{C,3}). \qquad (2.50)
\end{aligned}
$$

Here $\tilde{\mathbf{f}}^B$ and $\tilde{\mathbf{f}}^C$ have similar terms as $\mathbf{f}^B$ and $\mathbf{f}^C$ except those $O(\sqrt{\epsilon})$ terms. The explicit formulation of $\tilde{\mathbf{f}}^B = \tilde{\mathbf{f}}^{B,0} + \tilde{\mathbf{f}}^{B,1}$ is illustrated as follows, cf. [22]:

$$
\begin{aligned}
\tilde{f}_1^{B,0} =& \epsilon^{\frac{3}{2}}\Delta(a\rho_0')(1 - e^{-\frac{z}{\sqrt{\epsilon}}}) + \epsilon\big(\frac{\partial^2 a}{\partial x^2}\rho_0 + 3a\rho''\big)e^{-\frac{z}{\sqrt{\epsilon}}}\\
&- 2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho_0' - \epsilon^{\frac{3}{2}}\Delta u_1^O + \epsilon^2\Delta(\bar{a}\rho_0')(1 - e^{-\frac{z}{\sqrt{\epsilon}}})\\
&+ \epsilon^{\frac{3}{2}}\big(\frac{\partial^2 \bar{a}}{\partial x^2}\rho_0 + 3\bar{a}\rho_0''\big)e^{-\frac{z}{\sqrt{\epsilon}}} - 2\epsilon\bar{a}e^{-\frac{z}{\sqrt{\epsilon}}}\rho_0' - \epsilon\bar{a}\rho_0',\\
\tilde{f}_2^{B,0} =& -\epsilon^{\frac{3}{2}}\Delta(\frac{\partial a}{\partial x}\rho_0)(1 - e^{-\frac{z}{\sqrt{\epsilon}}}) - 2\epsilon\frac{\partial a}{\partial x}e^{-\frac{z}{\sqrt{\epsilon}}}\rho_0' - \epsilon^{\frac{3}{2}}\Delta u_2^O\\
&- \epsilon^2\Delta(\frac{\partial \bar{a}}{\partial x}\rho_0)(1 - e^{-\frac{z}{\sqrt{\epsilon}}}) - 2\epsilon^{\frac{3}{2}}\frac{\partial \bar{a}}{\partial x}e^{-\frac{z}{\sqrt{\epsilon}}}\rho_0' + \epsilon\frac{\partial \bar{a}}{\partial x}\rho_0,
\end{aligned}
$$

with $\bar{a} = v_1^{O,1}(t,x,0)$.

For the error terms $\mathbf{f}^{err}$ and g^{err}, we have the following estimates.

Lemma 2.3. *Assume $\mathbf{v}_0 \in \mathbf{V}\cap H^4(\Omega)$ and $T_0 \in H^4(\Omega)$. The following estimates hold*

$$
||\partial_x^j\mathbf{f}^{err}||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon, \quad ||\partial_x^j g^{err}||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon, \quad j = 0, 1, \qquad (2.52)
$$

$$
||\nabla\cdot\mathbf{f}^{err}||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{3}{4}}, \quad ||f_2^{err}||_{L^\infty(0,T;H^1(\Omega))} \le C\epsilon^{\frac{3}{4}}, \qquad (2.53)
$$

where the constant C is independent of ϵ.

Proof. For the estimates (2.52), in view of the estimates from Lemma 2.1 and Lemma 2.2, one only needs control typical terms like $-2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho_0'$ in $\tilde{\mathbf{f}}^B$ and $\epsilon(\tilde{\mathbf{v}}^{app}\cdot\nabla)\tilde{\mathbf{v}}^{app}$. These are essentially done in our previous work [22]. Parts of the argument are given below for completeness. First of all, by the definition of the cut-off function, one has

$$
\begin{aligned}
||-2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho_0'||_{L^2}^2 &\le C\epsilon\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\frac{2z}{\sqrt{\epsilon}}}{\rho_0'}^2 dz\\
&\le C\epsilon^2\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{z^2}{\epsilon}e^{-\frac{2z}{\sqrt{\epsilon}}}{\rho_0'}^2 dz\\
&\le C\epsilon^{\frac{5}{2}}||Ze^{-Z}||_{L^2(0,\infty)}^2\\
&\le C\epsilon^{\frac{5}{2}}. \qquad (2.54)
\end{aligned}
$$

For the estimate of $\epsilon(\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{app}$, we only need to control terms like $v_2^{O,0}\frac{\partial\tilde{v}_1^{B,0}}{\partial z}$ or $v_2^{I,0}\frac{\partial\tilde{v}_1^{C_0,0}}{\partial z}$. Since $v_2^0\big|_{z=0,1}=0$, a direct application of Hardy's inequality yields

$$\begin{aligned}
||v_2^{O,0}\frac{\partial\tilde{v}_1^{B,0}}{\partial z}||_{L^2} &\leq ||v_2^{O,0}\frac{\partial\tilde{v}_1^{B_0,0}}{\partial z}||_{L^2}+||v_2^{O,0}\frac{\partial\tilde{v}_1^{B_1,0}}{\partial z}||_{L^2}\\
&\leq ||\partial_z v_2^{O,0}||_{L^\infty}\big(||z\frac{\partial\tilde{v}_1^{B_0,0}}{\partial z}||_{L^2}+||(1-z)\frac{\partial\tilde{v}_1^{B_1,0}}{\partial z}||_{L^2}\big)\\
&\leq C\epsilon^{\frac{1}{4}}||Ze^{-Z}||_{L^2(0,\infty)}. \qquad (2.55)
\end{aligned}$$

For the estimates (2.53), in light of the divergence-free conditions, special care needs to be taken for $\partial_z v_1^{app}\partial_x v_2^{app}$ and $v_2^{app}\partial_{zz}v_2^{app}$. According to the inequalities (2.26), these two terms are bounded above by $C\epsilon^{-\frac{1}{4}}$. This concludes the proof.

□

3. Error Estimates

Define the error functions

$$\mathbf{v}^{err}=\mathbf{v}^\epsilon-\mathbf{v}^{app},\quad p^{err}=p^\epsilon-p^{app},\quad T^{err}=T^\epsilon-T^{app},$$

then combining (1.1) and (2.48) we get

$$\begin{cases}
\epsilon\big(\frac{\partial\mathbf{v}^{err}}{\partial t}+(\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err}+(\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{app}+(\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err}\big)\\
\qquad\qquad +\mathbf{v}^{err}-\epsilon\Delta\mathbf{v}^{err}+\nabla p^{err}=Ra_D\mathbf{k}T^{err}-\mathbf{f}^{err},\\
\frac{\partial T^{err}}{\partial t}+\mathbf{v}^{err}\cdot\nabla T^{err}+\mathbf{v}^{app}\cdot\nabla T^{err}+\mathbf{v}^{err}\cdot\nabla T^{app}\\
\qquad\qquad =\Delta T^{err}-g^{err},\\
\text{div }\mathbf{v}^{err}=0,\\
\mathbf{v}^{err}\big|_{t=0}=0,\quad T^{err}\big|_{t=0}=0,\\
\mathbf{v}^{err}\big|_{z=0,1}=0,\quad T^{err}\big|_{z=0}=0,\quad T^{err}\big|_{z=1}=0,\\
\mathbf{v}^{err},p^{err},T^{err}\text{ are periodic in }x\text{-direction.}
\end{cases} \qquad (3.1)$$

Theorem 3.1. *Assume $\mathbf{v}_0\in V\cap H^m(\Omega)$ and $T_0\in H^m(\Omega)$ with $m\geq 5$ and that they satisfy certain compatibility conditions. Then the following convergence rates hold for small ϵ*

$$||\mathbf{v}^{err}||_{L^\infty(0,T;L^2(\Omega))}\leq C\epsilon, \qquad (3.2a)$$

$$||\mathbf{v}^{err}||_{L^\infty(0,T;H^1(\Omega))}\leq C\epsilon^{\frac{1}{2}}, \qquad (3.2b)$$

$$||\mathbf{v}^{err}||_{L^\infty(0,T;L^\infty(\Omega))}\leq C\epsilon^{\frac{3}{4}}, \qquad (3.2c)$$

$$||T^{err}||_{L^\infty(0,T;L^2(\Omega))}\leq C\epsilon, \qquad (3.2d)$$

$$||T^{err}||_{L^\infty(0,T;H^1(\Omega))}\leq C\epsilon, \qquad (3.2e)$$

$$||T^{err}||_{L^\infty(0,T;L^\infty(\Omega))}\leq C\epsilon, \qquad (3.2f)$$

$$||\tfrac{\partial T^{err}}{\partial t}||_{L^2(0,T;L^2(\Omega))}\leq C\epsilon, \qquad (3.2g)$$

$$||\nabla p^{err}||_{L^\infty(0,T;L^2(\Omega))}\leq C\epsilon, \qquad (3.2h)$$

$$||p^{err}||_{L^\infty(0,T;L^\infty(\Omega))}\leq C\epsilon^{\frac{3}{4}}, \qquad (3.2i)$$

where C is a generic constant independent of ϵ and $\overline{p^{err}}$ is the average of p^{err} over Ω.

In view of the estimates given in Lemma 2.1 and Lemma 2.2, one obtains

$$
\begin{aligned}
&\|\sqrt{\epsilon}(\mathbf{v}^{O,1}+\mathbf{v}^{I,1}+\tilde{\mathbf{v}}^{B,1}+\tilde{\mathbf{v}}^{C,1})\|_{L^\infty(0,T;L^2(\Omega))}\le C\epsilon^{\frac{1}{2}},\\
&\|\sqrt{\epsilon}(\mathbf{v}^{O,1}+\mathbf{v}^{I,1}+\tilde{\mathbf{v}}^{B,1}+\tilde{\mathbf{v}}^{C,1})\|_{L^\infty(0,T;H^1(\Omega))}\le C\epsilon^{\frac{1}{4}},\\
&\|\sqrt{\epsilon}(\mathbf{v}^{O,1}+\mathbf{v}^{I,1}+\tilde{\mathbf{v}}^{B,1}+\tilde{\mathbf{v}}^{C,1})\|_{L^\infty(0,T;L^\infty(\Omega))}\le C\epsilon^{\frac{1}{2}},\\
&\|\sqrt{\epsilon}p^{O,1}\|_{L^\infty(0,T;L^\infty(\Omega))}\le C\epsilon^{\frac{1}{2}},\\
&\|\sqrt{\epsilon}T^{O,1}+\epsilon(T^{I,2}+\tilde{T}^{B,2}+\tilde{T}^{C,2})+\epsilon^{\frac{3}{2}}(T^{I,3}+\tilde{T}^{B,3}+\tilde{T}^{C,3})\|_{L^\infty(0,T;L^2(\Omega))}\le C\epsilon^{\frac{1}{2}},\\
&\|\sqrt{\epsilon}T^{O,1}+\epsilon(T^{I,2}+\tilde{T}^{B,2}+\tilde{T}^{C,2})+\epsilon^{\frac{3}{2}}(T^{I,3}+\tilde{T}^{B,3}+\tilde{T}^{C,3})\|_{L^\infty(0,T;H^1(\Omega))}\le C\epsilon^{\frac{1}{2}},\\
&\|\sqrt{\epsilon}T^{O,1}+\epsilon(T^{I,2}+\tilde{T}^{B,2}+\tilde{T}^{C,2})+\epsilon^{\frac{3}{2}}(T^{I,3}+\tilde{T}^{B,3}+\tilde{T}^{C,3})\|_{L^\infty(0,T;L^\infty(\Omega))}\le C\epsilon^{\frac{1}{2}}.
\end{aligned}
$$

One can immediately derive Theorem 1.1 as a corollary of Theorem 3.1.

Corollary 3.1. *The assumptions are the same as in Theorem 3.1. Recall the definition of $\tilde{\mathbf{v}}^{app}$ from Eq. (2.27). Then we have the following convergence estimates*

$$||\mathbf{v}^\epsilon-\tilde{\mathbf{v}}^{app}||_{L^\infty(0,T,L^2(\Omega))}\le C\epsilon^{\frac{1}{2}}, \tag{3.3a}$$

$$||\mathbf{v}^\epsilon-\tilde{\mathbf{v}}^{app}||_{L^\infty(0,T,H^1(\Omega))}\le C\epsilon^{\frac{1}{4}}, \tag{3.3b}$$

$$||\mathbf{v}^\epsilon-\tilde{\mathbf{v}}^{app}||_{L^\infty(0,T,L^\infty(\Omega))}\le C\epsilon^{\frac{1}{2}}, \tag{3.3c}$$

$$||T^\epsilon-T^{O,0}||_{L^\infty(0,T,L^\infty(\Omega))}\le C\epsilon^{\frac{1}{2}}, \tag{3.3d}$$

$$||T^\epsilon-T^{O,0}||_{L^\infty(0,T,H^1(\Omega))}\le C\epsilon^{\frac{1}{2}}, \tag{3.3e}$$

$$||\frac{\partial(T^\epsilon-T^{O,0})}{\partial t}||_{L^2(0,T;L^2(\Omega))}\le C\epsilon^{\frac{1}{2}}, \tag{3.3f}$$

$$||\nabla(p^\epsilon-p^{O,0})||_{L^\infty(0,T;L^2(\Omega))}\le C\epsilon^{\frac{1}{2}}, \tag{3.3g}$$

$$||p^\epsilon-p^{O,0}||_{L^\infty(0,T;L^\infty(\Omega))}\le C\epsilon^{\frac{1}{2}}, \tag{3.3h}$$

where C is a generic constant independent of ϵ; $\tilde{\mathbf{v}}^{app}$ is defined as the sum of the solution to the DOB system (1.2)*, an explicit initial layer, an explicit boundary layer and an initial-boundary layer (see* (2.27)*); p^0 and T^0 are the pressure and temperature fields to the VDDB system* (1.2).

To prove Theorem 3.1, we firstly give the following two lemmas.

Lemma 3.1. *(Inequality for the trilinear form[54][19][22]) let $b(\mathbf{u},\mathbf{v},\mathbf{w})$ be the trilinear form on $V\times V\times V$ defined by*

$$b(\mathbf{u},\mathbf{v},\mathbf{w})=\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}dx.$$

Then b has the following propertites

$$
\begin{aligned}
&b(\mathbf{u},\mathbf{v},\mathbf{v})=0,\quad b(\mathbf{u},\mathbf{v},\mathbf{w})=-b(\mathbf{u},\mathbf{w},\mathbf{v}),\\
&|b(\mathbf{u},\mathbf{v},\mathbf{w})|\le C\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{v}\|_{L^2}\|\mathbf{w}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{w}\|_{L^2}^{\frac{1}{2}},\\
&|b(\mathbf{u},\mathbf{v},\mathbf{w})|\le C\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\Delta\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|\mathbf{w}\|_{L^2},\ \textit{provided}\\
&\qquad(\mathbf{u},\mathbf{v},\mathbf{w})\in V\times(V\cap H^2(\Omega))\times V.
\end{aligned}
$$

Lemma 3.2. *(Anisotropic Sobolev embedding[31][56][73][22]) There exists a positive constant C such that for any $u \in H^1_{0,per}(\Omega)$*

$$\|u\|_{L^\infty(\Omega)} \le C\big(\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|\partial_z u\|^{\frac{1}{2}}_{L^2(\Omega)} + \|\partial_x u\|^{\frac{1}{2}}_{L^2(\Omega)}\|\partial_z u\|^{\frac{1}{2}}_{L^2(\Omega)} + \|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|\partial_x\partial_z u\|^{\frac{1}{2}}_{L^2(\Omega)}\big),$$

where one or both sides of the inequality could be infinite.

Now we prove the Theorem 3.1 in six steps.

Proof.

3.1. $L^\infty(L^2)$ estimate of $\mathbf{v}^{err}$ and T^{err}

Multiplying the velocity error equations in (3.1) by $\mathbf{v}^{err}$ and integrating over Ω lead to, for small ϵ,

$$\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega |\mathbf{v}^{err}|^2 + \int_\Omega |\mathbf{v}^{err}|^2 + \epsilon\int_\Omega |\nabla \mathbf{v}^{err}|^2 \\
&= \epsilon\int_\Omega \big((\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err}\big)\cdot\mathbf{v}^{app} + Ra_D\int_\Omega \mathbf{k}\cdot\mathbf{v}^{err}T^{err} - \int_\Omega \mathbf{f}^{err}\cdot\mathbf{v}^{err} \\
&\le C\epsilon\|\mathbf{v}^{app}\|^2_{L^\infty(0,T;L^\infty(\Omega))}\int_\Omega |\mathbf{v}^{err}|^2 + \frac{1}{8}\int_\Omega |\mathbf{v}^{err}|^2 + 2(Ra_D)^2\int_\Omega |T^{err}|^2 \\
&\quad + \frac{\epsilon}{2}\int_\Omega |\nabla\mathbf{v}^{err}|^2 + \frac{1}{8}\int_\Omega |\mathbf{v}^{err}|^2 + 2\int_\Omega |\mathbf{f}^{err}|^2 \\
&\le \frac{3}{8}\int_\Omega |\mathbf{v}^{err}|^2 + \frac{\epsilon}{2}\int_\Omega |\nabla\mathbf{v}^{err}|^2 + 2(Ra_D)^2\int_\Omega |T^{err}|^2 + 2\int_\Omega |\mathbf{f}^{err}|^2. \qquad (3.4)
\end{aligned}$$

Here and in what follows the uniform estimate $\|\mathbf{v}^{app}\|_{L^\infty(0,T;L^\infty(\Omega))}$ follows from Lemma 2.1.

Multiplying the temperature error equations in (3.1) by T^{err} and integrating over Ω lead to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |T^{err}|^2 + \int_\Omega |\nabla T^{err}|^2 \\
&= -\int_\Omega (\mathbf{v}^{err}\cdot\nabla T^{O,0})T^{err} + \int_\Omega (\mathbf{v}^{err}\cdot\nabla T^{err})(T^{app} - T^{O,0}) - \int_\Omega g^{err}\cdot T^{err} \\
&\le \frac{1}{16}\int_\Omega |\mathbf{v}^{err}|^2 + C\|\nabla T^{O,0}\|^2_{L^\infty(0,T;L^\infty(\Omega))}\int_\Omega |T^{err}|^2 + C||T^{app} - T^{O,0}||^2_{L^\infty(0,T;L^\infty(\Omega))}\int_\Omega |\mathbf{v}^{err}|^2 \\
&\quad + \frac{1}{2}\int_\Omega |\nabla T^{err}|^2 + \frac{1}{4}\int_\Omega |T^{err}|^2 + \int_\Omega |g^{err}|^2 \\
&\le \frac{1}{8}\int_\Omega |\mathbf{v}^{err}|^2 + C\int_\Omega |T^{err}|^2 + \frac{1}{2}\int_\Omega |\nabla T^{err}|^2 + \int_\Omega |g^{err}|^2, \qquad (3.5)
\end{aligned}$$

where the estimate $||T^{app} - T^{O,0}||^2_{L^\infty(0,T;L^\infty(\Omega))} \le C\epsilon$ follows from Lemma 2.2. See Eq. (2.47) for the definition of T^{app}.

Combining (3.4) and (3.5), one concludes that there holds for small ϵ

$$\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega |\mathbf{v}^{err}|^2 + \frac{1}{2}\int_\Omega |\mathbf{v}^{err}|^2 + \frac{\epsilon}{2}\int_\Omega |\nabla \mathbf{v}^{err}|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |T^{err}|^2 + \frac{1}{2}\int_\Omega |\nabla T^{err}|^2$$
$$\leq C\int_\Omega |T^{err}|^2 + 2\int_\Omega |\mathbf{f}^{err}|^2 + \int_\Omega |g^{err}|^2, \tag{3.6}$$

upon which the application of Gronwall's inequality leads to for any $t \in [0, T]$

$$\int_\Omega \left(\epsilon|\mathbf{v}^{err}|^2 + |T^{err}|^2\right)(t) + \int_0^t \int_\Omega \left(\epsilon|\nabla \mathbf{v}^{err}|^2 + |\nabla T^{err}|^2\right) \leq C\epsilon^2.$$

Consequently, one has

$$\|\mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \quad \|T^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon, \tag{3.7}$$
$$\|\nabla \mathbf{v}^{err}\|_{L^2(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \quad \|\nabla T^{err}\|_{L^2(0,T;L^2(\Omega))} \leq C\epsilon. \tag{3.8}$$

The error estimates of velocity can be improved. Substituting (3.7) into (3.4) leads to for small ϵ

$$\epsilon\frac{d}{dt}\int_\Omega |\mathbf{v}^{err}|^2 + \int_\Omega |\mathbf{v}^{err}|^2 \leq C\epsilon^2.$$

Let $E_0(t) = \|\mathbf{v}^{err}\|^2_{L^2(\Omega)}$, then we have

$$E_0'(t) + \frac{1}{\epsilon}E_0(t) \leq C\epsilon \Rightarrow \frac{d}{dt}(e^{\frac{t}{\epsilon}}E_0(t)) \leq C\epsilon e^{\frac{t}{\epsilon}}$$
$$\Rightarrow E_0(t) \leq C\epsilon e^{-\frac{t}{\epsilon}}\int_0^t e^{\frac{\tau}{\epsilon}}d\tau = C\epsilon^2(1 - e^{-\frac{t}{\epsilon}}).$$

Consequently one has

$$\|\mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon. \tag{3.9}$$

The estimates (3.2a) and (3.2d) in Theorem 3.1 are hence established.

Remark 3.1. *Without including higher order expansions in the construction of approximate solution (i.e., $\tilde{\mathbf{v}}^{app}, p^{O,0}, T^{O,0}$), the error in the forcing terms ($\mathbf{f}^{err}, g^{err}$) is of order $\sqrt{\epsilon}$. The same energy method would still give the estimate* (3.3a) *in the Corollary 3.1. However, the error estimates in H^1 norm and uniform norm will not be optimal, cf. the arguments below (see also [31]).*

3.2. $L^\infty(L^\infty)$ estimate of the second component of velocity error v_2^{err}

Applying the operator ∂_x to the system (3.1) one has

$$
\begin{cases}
\epsilon\Big(\frac{\partial(\partial_x \mathbf{v}^{err})}{\partial t}+(\partial_x\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err}+(\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{err}+(\partial_x\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{app} \\
\qquad +(\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{app}+(\partial_x\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err}+(\mathbf{v}^{app}\cdot\nabla)\partial_x\mathbf{v}^{err}\Big) \\
\qquad +\partial_x\mathbf{v}^{err}-\epsilon\Delta\partial_x\mathbf{v}^{err}+\nabla\partial_x p^{err}=Ra_D\mathbf{k}\partial_x T^{err}-\partial_x\mathbf{f}^{err}, \\
\frac{\partial(\partial_x T^{err})}{\partial t}+\partial_x\mathbf{v}^{err}\cdot\nabla T^{err}+\mathbf{v}^{err}\cdot\nabla\partial_x T^{err}+\partial_x\mathbf{v}^{app}\cdot\nabla T^{err}+\mathbf{v}^{app}\cdot\nabla\partial_x T^{err} \\
\qquad +\partial_x\mathbf{v}^{err}\cdot\nabla T^{app}+\mathbf{v}^{err}\cdot\nabla\partial_x T^{app}=\Delta\partial_x T^{err}-\partial_x g^{err}, \\
\mathrm{div}\ \partial_x\mathbf{v}^{err}=0, \\
\partial_x\mathbf{v}^{err}|_{t=0}=0,\quad \partial_x T^{err}|_{t=0}=0, \\
\partial_x\mathbf{v}^{err}|_{z=0,1}=0,\quad \partial_x T^{err}|_{z=0}=0,\quad \partial_x T^{err}|_{z=1}=0, \\
\partial_x\mathbf{v}^{err},\partial_x p^{err},\partial_x T^{err} \text{ are periodic in } x\text{-direction.}
\end{cases} \tag{3.10}
$$

Multiplying the velocity error equations in (3.10) by $\partial_x\mathbf{v}^{err}$ and integrating over Ω lead to

$$
\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega|\partial_x\mathbf{v}^{err}|^2+\int_\Omega|\partial_x\mathbf{v}^{err}|^2+\epsilon\int_\Omega|\nabla\partial_x\mathbf{v}^{err}|^2 \\
&=-\epsilon\int_\Omega\big((\partial_x\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err}\big)\partial_x\mathbf{v}^{err}+\epsilon\int_\Omega\big((\partial_x\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{err}\big)\mathbf{v}^{app} \\
&\quad+\epsilon\int_\Omega\big((\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{err}\big)\partial_x\mathbf{v}^{app}-\epsilon\int_\Omega\big((\partial_x\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err}\big)\partial_x\mathbf{v}^{err} \\
&\quad+Ra_D\int_\Omega\mathbf{k}\cdot\partial_x\mathbf{v}^{err}\partial_x T^{err}-\int_\Omega\partial_x\mathbf{f}^{err}\cdot\partial_x\mathbf{v}^{err}.
\end{aligned} \tag{3.11}
$$

We control the right-hand side of Eq. (3.11) as follows.

$$
\begin{aligned}
-\epsilon\int_\Omega\big((\partial_x\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err}\big)\partial_x\mathbf{v}^{err}&\le C\epsilon\|\mathbf{v}^{err}\|^{\frac12}_{L^2(\Omega)}\|\partial_x\mathbf{v}^{err}\|^{\frac12}_{L^2(\Omega)}\|\nabla\mathbf{v}^{err}\|^{\frac12}_{L^2(\Omega)}\|\nabla\partial_x\mathbf{v}^{err}\|^{\frac32}_{L^2(\Omega)} \\
&\le C\epsilon^2\|\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)}\|\nabla\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\frac18\epsilon\|\nabla\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
\epsilon\int_\Omega\big((\partial_x\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{err}\big)\mathbf{v}^{app}&\le C\epsilon\|\partial_x\mathbf{v}^{err}\|_{L^2(\Omega)}\|\nabla\partial_x\mathbf{v}^{err}\|_{L^2(\Omega)}\|\mathbf{v}^{app}\|_{L^\infty(0,T;L^\infty(\Omega))} \\
&\le C\epsilon\|\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\frac18\epsilon\|\nabla\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)} \\
&\le \frac14\|\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)}+\frac18\epsilon\|\nabla\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
\epsilon\int_\Omega\big((\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{err}\big)\partial_x\mathbf{v}^{app}&\le C\epsilon\|\mathbf{v}^{err}\|^2_{L^\infty(0,T;L^2(\Omega))}\|\partial_x\mathbf{v}^{app}\|^2_{L^\infty(0,T;L^\infty(\Omega))}+\frac{1}{10}\epsilon\|\nabla\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)} \\
&\le C\epsilon^3+\frac18\epsilon\|\nabla\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
\epsilon\int_\Omega \big((\partial_x \mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err}\big)\partial_x\mathbf{v}^{err} &= -\epsilon\int_\Omega \big((\partial_x \mathbf{v}^{app}\cdot\nabla)\partial_x\mathbf{v}^{err}\big)\mathbf{v}^{err} \\
&\le C\epsilon\|\mathbf{v}^{err}\|^2_{L^\infty(0,T;L^2(\Omega))}\|\partial_x\mathbf{v}^{app}\|^2_{L^\infty(0,T;L^\infty(\Omega))} + \frac{1}{10}\epsilon\|\nabla\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)} \\
&\le C\epsilon^3 + \frac{1}{8}\epsilon\|\nabla\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)},
\end{aligned}
$$

$$
Ra_D\int_\Omega \mathbf{k}\cdot\partial_x\mathbf{v}^{err}\partial_x T^{err} - \int_\Omega \partial_x\mathbf{f}^{err}\cdot\partial_x\mathbf{v}^{err} \le \frac{1}{4}\|\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)} + C\|\partial_x T^{err}\|^2_{L^2(\Omega)} + C\epsilon^2.
$$

Here the uniform estimate $\|\partial_x\mathbf{v}^{app}\|_{L^\infty(0,T;L^\infty(\Omega))}$ follows from Lemma 2.1 and Remark 2.2.

The Eq. (3.11) then becomes

$$
\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega|\partial_x\mathbf{v}^{err}|^2 + \frac{1}{2}\int_\Omega|\partial_x\mathbf{v}^{err}|^2 + \frac{1}{2}\epsilon\int_\Omega|\nabla\partial_x\mathbf{v}^{err}|^2 \\
&\le C\epsilon^2\|\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)}\|\nabla\mathbf{v}^{err}\|^2_{L^2(\Omega)} + C\|\partial_x T^{err}\|^2_{L^2(\Omega)} + C\epsilon^2 + C\epsilon^3.
\end{aligned} \tag{3.12}
$$

In view of the estimates (3.8), one has by Gronwall's inequality that

$$
\|\partial_x\mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{2}}, \quad \|\nabla\partial_x\mathbf{v}^{err}\|_{L^2(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{2}}.
$$

Testing the temperature error equation in (3.10) by $\partial_x T^{err}$ gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\partial_x T^{err}|^2 + \int_\Omega|\nabla\partial_x T^{err}|^2 \\
&= -\int_\Omega(\partial_x\mathbf{v}^{err}\cdot\nabla T^{err})\partial_x T^{err} - \int_\Omega(\partial_x\mathbf{v}^{app}\cdot\nabla T^{err})\partial_x T^{err} \\
&\quad - \int_\Omega(\partial_x\mathbf{v}^{err}\cdot\nabla T^{app})\partial_x T^{err} - \int_\Omega(\mathbf{v}^{err}\cdot\nabla\partial_x T^{app})\partial_x T^{err} \\
&\quad - \int_\Omega \partial_x g^{err}\partial_x T^{err}.
\end{aligned} \tag{3.13}
$$

We estimate the right-side of of Eq. (3.13) as follows.

$$
\begin{aligned}
&-\int_\Omega(\partial_x\mathbf{v}^{err}\cdot\nabla T^{err})\partial_x T^{err} = \int_\Omega(\partial_x\mathbf{v}^{err}\cdot\nabla\partial_x T^{err})T^{err} \\
&\le \|\partial_x\mathbf{v}^{err}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla\partial_x\mathbf{v}^{err}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla\partial_x T^{err}\|_{L^2(\Omega)}\|T^{err}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla T^{err}\|^{\frac{1}{2}}_{L^2(\Omega)} \\
&\le C\|\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)}\|\nabla\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)} + \frac{1}{4}\|\nabla\partial_x T^{err}\|^2_{L^2(\Omega)} + \|T^{err}\|^2_{L^2(\Omega)}\|\nabla T^{err}\|^2_{L^2(\Omega)} \\
&\le C\epsilon\|\nabla\partial_x\mathbf{v}^{err}\|^2_{L^2(\Omega)} + \frac{1}{4}\|\nabla\partial_x T^{err}\|^2_{L^2(\Omega)} + C\epsilon^2\|\nabla T^{err}\|^2_{L^2(\Omega)},
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_\Omega(\partial_x\mathbf{v}^{app}\cdot\nabla T^{err})\partial_x T^{err} &= \int_\Omega(\partial_x\mathbf{v}^{app}\cdot\nabla\partial_x T^{err})T^{err} \\
&\le \|\partial_x\mathbf{v}^{app}\|_{L^\infty(0,T;L^\infty(\Omega))}\|\nabla\partial_x T^{err}\|_{L^2(\Omega)}\|T^{err}\|_{L^\infty(0,T;L^2(\Omega))} \\
&\le C\|\partial_x\mathbf{v}^{app}\|^2_{L^\infty(0,T;L^\infty(\Omega))}\|T^{err}\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{1}{8}\|\nabla\partial_x T^{err}\|^2_{L^2(\Omega)} \\
&\le C\epsilon^2 + \frac{1}{8}\|\nabla\partial_x T^{err}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Similarly, one has

$$
\begin{aligned}
&-\int_\Omega (\partial_x \mathbf{v}^{err} \cdot \nabla T^{app})\partial_x T^{err} - \int_\Omega (\mathbf{v}^{err} \cdot \nabla \partial_x T^{app})\partial_x T^{err} \\
&= \int_\Omega (\partial_x \mathbf{v}^{err} \cdot \nabla \partial_x T^{err})(T^{app} - T^{O,0}) - \int_\Omega (\partial_x \mathbf{v}^{err} \cdot \nabla T^{O,0})\partial_x T^{err} \\
&\quad + \int_\Omega (\partial_x \mathbf{v}^{err} \cdot \nabla \partial_x T^{err})\partial_x (T^{app} - T^{O,0}) - \int_\Omega (\mathbf{v}^{err} \cdot \nabla \partial_x T^{O,0})\partial_x T^{err} \\
&\leq \frac{1}{2}\|\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)} + \frac{1}{8}\|\nabla \partial_x T^{err}\|^2_{L^2(\Omega)} + C\|\partial_x T^{err}\|^2_{L^2(\Omega)} + C\epsilon^2 + C\epsilon^3.
\end{aligned}
$$

Here we use the estimate $||\partial_x(T^{app} - T^{O,0})||^2_{L^\infty(0,T;L^\infty(\Omega))} \leq C\epsilon$ from Lemma 2.2 and Remark 2.2. Finally,

$$
-\int_\Omega \partial_x g^{err} \partial_x T^{err} \leq 2\|\partial_x T^{err}\|^2_{L^2(\Omega)} + 2\|\partial_x g^{err}\|^2_{L^\infty(0,T;L^2(\Omega))} \leq 2\|\partial_x T^{err}\|^2_{L^2(\Omega)} + C\epsilon^2.
$$

Collecting the above estimate of the right-side of Eq. (3.13) into Eq. (3.13) we have for small ϵ

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_x T^{err}|^2 + \frac{1}{2}\int_\Omega |\nabla \partial_x T^{err}|^2 \leq C\|\partial_x T^{err}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)} \\
+ \epsilon\|\nabla \partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)} + C\epsilon^2\|\nabla T^{err}\|^2_{L^2(\Omega)} + C\epsilon^2,
\end{aligned}
\tag{3.14}
$$

Combine inequality (3.12) and (3.14) and apply Gronwall's inequality leads to

$$
||\partial_x T^{err}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon, \quad ||\nabla \partial_x T^{err}||_{L^2(0,T;L^2(\Omega))} \leq C\epsilon. \tag{3.15}
$$

Thanks to the estimate (3.15), the estimate on $\partial_x \mathbf{v}^{err}$ derived from inequality (3.12) can be improved as

$$
\|\partial_x \mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon, \quad \|\nabla \partial_x \mathbf{v}^{err}\|_{L^2(0,T;L^2(\Omega))} \leq C\epsilon. \tag{3.16}
$$

Indeed, a bootstrapping argument could yield that

$$
\|\partial_x^j \mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon,
$$

for any integer $j > 0$ permitted by the regularity of the data. Hereafter, we assume $j \geq 3$. In light of the divergence-free condition, one obtains

$$
||\partial_z v_2^{err}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon, \quad ||\partial_{xz} v_2^{err}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon, \quad ||\partial_{xxz} v_2^{err}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon.
$$

It follows from the Anisotropic Sobolev embedding Lemma that

$$
\|v_2^{err}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C\epsilon, \quad \|\partial_x v_2^{err}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C\epsilon. \tag{3.17}
$$

3.3. $L^\infty(H^1)$ estimates of $\mathbf{v}^{err}$

Multiplying the velocity error equations in (3.1) by $-\Delta\mathbf{v}^{err}$ and integrating over Ω lead to

$$\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega |\nabla\mathbf{v}^{err}|^2 + \int_\Omega |\nabla\mathbf{v}^{err}|^2 + \epsilon\int_\Omega |\Delta\mathbf{v}^{err}|^2 \\
&= \epsilon\int_\Omega \big((\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err}\big)\Delta\mathbf{v}^{err} + \epsilon\int_\Omega \big((\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{app}\big)\Delta\mathbf{v}^{err} \\
&\quad + \epsilon\int_\Omega \big((\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err}\big)\Delta\mathbf{v}^{err} - Ra_D\int_\Omega \Delta v_2^{err}T^{err} + \int_\Omega \mathbf{f}^{err}\Delta\mathbf{v}^{err}. \qquad (3.18)
\end{aligned}$$

The right-hand side of Eq. (3.18) can be controlled as follows.

$$\begin{aligned}
\epsilon\int_\Omega \big((\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err}\big)\Delta\mathbf{v}^{err} &= C\epsilon\|\mathbf{v}^{err}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\mathbf{v}^{err}\|_{L^2(\Omega)}\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^{\frac{3}{2}} \\
&\le C\epsilon\|\mathbf{v}^{err}\|_{L^\infty L^2}^2\|\nabla\mathbf{v}^{err}\|_{L^2(\Omega)}^4 + \frac{1}{10}\epsilon\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2 \\
&\le C\epsilon^3\|\nabla\mathbf{v}^{err}\|_{L^2(\Omega)}^4 + \frac{1}{10}\epsilon\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2,
\end{aligned}$$

and, by (3.7), (3.17) and Lemma 2.1,

$$\begin{aligned}
\epsilon\int_\Omega \big((\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{app}\big)\Delta\mathbf{v}^{err} &= \epsilon\int_\Omega \big(v_1^{err}\partial_x\mathbf{v}^{app}\big)\Delta\mathbf{v}^{err} + \epsilon\int_\Omega \big(v_2^{err}\partial_z\mathbf{v}^{app}\big)\Delta\mathbf{v}^{err} \\
&\le C\epsilon\|\mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))}^2\|\partial_x\mathbf{v}^{app}\|_{L^\infty(0,T;L^\infty(\Omega))}^2 \\
&\quad + C\epsilon\|v_2^{err}\|_{L^\infty(0,T;L^\infty(\Omega))}^2\|\nabla\mathbf{v}^{app}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \frac{1}{10}\epsilon\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2 \\
&\le C\epsilon^2 + \frac{1}{10}\epsilon\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2, \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
\epsilon\int_\Omega \big((\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err}\big)\Delta\mathbf{v}^{err} &\le C\epsilon\|\nabla\mathbf{v}^{err}\|_{L^2(\Omega)}^2\|\mathbf{v}^{app}\|_{L^\infty(0,T;L^\infty(\Omega))}^2 + \frac{1}{10}\epsilon\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2 \\
&\le C\epsilon\|\nabla\mathbf{v}^{err}\|_{L^2(\Omega)}^2 + \frac{1}{10}\epsilon\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2,
\end{aligned}$$

and

$$-Ra_D\int_\Omega \Delta v_2^{err}T^{err} \le \frac{\epsilon}{10}\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon}\|T^{err}\|_{L^2(\Omega)}^2 \le C\epsilon + \frac{\epsilon}{10}\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2,$$

and

$$\begin{aligned}
\int_\Omega \mathbf{f}^{err}\Delta\mathbf{v}^{err} \le \|\mathbf{f}^{err}\|_{L^2(\Omega)}\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)} &\le C\epsilon^{-1}\|\mathbf{f}^{err}\|_{L^2(\Omega)}^2 + \frac{1}{10}\epsilon\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2 \\
&\le C\epsilon + \frac{1}{10}\epsilon\|\Delta\mathbf{v}^{err}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Eq. (3.18) becomes, for small ϵ

$$\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega |\nabla \mathbf{v}^{err}|^2 + \frac{5}{8}\int_\Omega |\nabla \mathbf{v}^{err}|^2 + \frac{\epsilon}{2}\int_\Omega |\Delta \mathbf{v}^{err}|^2 \le C\epsilon^3 \|\nabla \mathbf{v}^{err}\|^4_{L^2(\Omega)} + C\epsilon. \tag{3.20}$$

It follows as usual that

$$\|\Delta \mathbf{v}^{err}\|_{L^2(0,T;L^2(\Omega))} \le C, \quad \|\nabla \mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{2}}. \tag{3.21}$$

This completes the proof of the estimate (3.2b). We remark that the estimate (2.52) from Lemma 2.3 is critical in deriving the inequality (3.21).

3.4. $L^\infty(L^\infty)$ estimate of $\mathbf{v}^{err}$

For the estimate of $\|\mathbf{v}^{err}\|_{L^\infty(0,T;L^\infty(\Omega))}$, we only need to estimate $\|\partial_x\partial_z \mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))}$ in view of the anisotropic Sobolev embedding.

Multiplying the velocity error equations in (3.10) by $-\Delta\partial_x \mathbf{v}^{err}$ and integrating over Ω lead to

$$\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega |\nabla\partial_x \mathbf{v}^{err}|^2 + \int_\Omega |\nabla\partial_x \mathbf{v}^{err}|^2 + \epsilon\int_\Omega |\Delta\partial_x \mathbf{v}^{err}|^2 \\
&= \epsilon\int_\Omega \big((\partial_x \mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err}\big)\Delta\partial_x \mathbf{v}^{err} + \epsilon\int_\Omega \big((\mathbf{v}^{err}\cdot\nabla)\partial_x \mathbf{v}^{err}\big)\Delta\partial_x \mathbf{v}^{err} \\
&\quad + \epsilon\int_\Omega \big((\partial_x \mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{app}\big)\Delta\partial_x \mathbf{v}^{err} - \epsilon\int_\Omega \big((\mathbf{v}^{err}\cdot\nabla)\partial_x \mathbf{v}^{app}\big)\Delta\partial_x \mathbf{v}^{err} \\
&\quad - \epsilon\int_\Omega \big((\partial_x \mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err}\big)\Delta\partial_x \mathbf{v}^{err} - \epsilon\int_\Omega \big((\mathbf{v}^{app}\cdot\nabla)\partial_x \mathbf{v}^{err}\big)\Delta\partial_x \mathbf{v}^{err} \\
&\quad - Ra_D\int_\Omega \mathbf{k}\cdot\Delta\partial_x \mathbf{v}^{err}\partial_x T^{err} + \int_\Omega \partial_x \mathbf{f}^{err}\cdot\Delta\partial_x \mathbf{v}^{err}.
\end{aligned} \tag{3.22}$$

With the estimate (3.21), the right-hand side of Eq. (3.22) can be controlled as following:

$$\begin{aligned}
&\epsilon\int_\Omega \big((\partial_x \mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err}\big)\Delta\partial_x \mathbf{v}^{err} \\
&\le C\epsilon\|\nabla\partial_x \mathbf{v}^{err}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla \mathbf{v}^{err}\|_{L^2(\Omega)}\|\Delta \mathbf{v}^{err}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\Delta\partial_x \mathbf{v}^{err}\|_{L^2(\Omega)} \\
&\le C\epsilon\|\nabla \mathbf{v}^{err}\|^2_{L^2(\Omega)} + C\epsilon\|\nabla \mathbf{v}^{err}\|^2_{L^2(\Omega)}\|\nabla\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)}\|\Delta \mathbf{v}^{err}\|^2_{L^2(\Omega)} + \frac{1}{10}\epsilon\|\Delta\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)} \\
&\le C\epsilon^2 + C\epsilon^2\|\nabla\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)}\|\Delta \mathbf{v}^{err}\|^2_{L^2(\Omega)} + \frac{1}{10}\epsilon\|\Delta\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
&\epsilon\int_\Omega \big((\mathbf{v}^{err}\cdot\nabla)\partial_x \mathbf{v}^{err}\big)\Delta\partial_x \mathbf{v}^{err} \\
&\le C\epsilon\|\mathbf{v}^{err}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla \mathbf{v}^{err}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla\partial_x \mathbf{v}^{err}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\Delta\partial_x \mathbf{v}^{err}\|^{\frac{3}{2}}_{L^2(\Omega)} \\
&\le C\epsilon^4\|\nabla\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)} + \frac{1}{10}\epsilon\|\Delta\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)},
\end{aligned}$$

and, with the help of the similar argument in (3.19),

$$
\begin{aligned}
&\epsilon \int_\Omega \big((\partial_x \mathbf{v}^{err} \cdot \nabla)\mathbf{v}^{app}\big)\Delta\partial_x \mathbf{v}^{err} - \epsilon \int_\Omega \big((\mathbf{v}^{err} \cdot \nabla)\partial_x \mathbf{v}^{app}\big)\Delta\partial_x \mathbf{v}^{err} \\
&\le C\epsilon^{\frac{3}{2}} + C\epsilon^2 + \frac{1}{20}\epsilon\|\Delta\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)},
\end{aligned}
$$

and

$$
\begin{aligned}
&- \epsilon \int_\Omega \big((\partial_x \mathbf{v}^{app} \cdot \nabla)\mathbf{v}^{err}\big)\Delta\partial_x \mathbf{v}^{err} - \epsilon \int_\Omega \big((\mathbf{v}^{app} \cdot \nabla)\partial_x \mathbf{v}^{err}\big)\Delta\partial_x \mathbf{v}^{err} \\
&\le C\epsilon^2 + C\epsilon\|\nabla\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)} + \frac{1}{20}\epsilon\|\Delta\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)},
\end{aligned}
$$

and by (3.15)

$$
\int_\Omega \partial_x \mathbf{f}^{err}\Delta\partial_x \mathbf{v}^{err} - Ra_D \int_\Omega \mathbf{k}\Delta\partial_x \mathbf{v}^{err} \cdot \partial_x T^{err} \le C\epsilon + \frac{1}{10}\epsilon\|\Delta\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)}.
$$

The Eq. (3.22) can be written as

$$
\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}\int_\Omega |\nabla\partial_x \mathbf{v}^{err}|^2 + \frac{5}{8}\int_\Omega |\nabla\partial_x \mathbf{v}^{err}|^2 + \frac{3}{5}\epsilon\int_\Omega |\Delta\partial_x \mathbf{v}^{err}|^2 \\
&\le C\epsilon + C\epsilon^2\|\Delta \mathbf{v}^{err}\|^2_{L^2(\Omega)} \cdot \|\nabla\partial_x \mathbf{v}^{err}\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.23}
$$

which leads to

$$
\|\nabla\partial_x \mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{2}}, \quad \|\Delta\partial_x \mathbf{v}^{err}\|_{L^2(0,T;L^2(\Omega))} \le C. \tag{3.24}
$$

In light of anasotropic Sobolev embedding, the estimates (3.7), (3.9), (3.26), (3.21) and (3.24), we obtain

$$
\begin{aligned}
\|\mathbf{v}^{err}\|_{L^\infty(0,T;L^\infty(\Omega))} \le C\Big(&\|\mathbf{v}^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\|\partial_z \mathbf{v}^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))} \\
&+ \|\partial_x \mathbf{v}^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\|\partial_z \mathbf{v}^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))} \\
&+ \|\mathbf{v}^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\|\partial_x\partial_z \mathbf{v}^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\Big) \\
\le C\epsilon^{\frac{3}{4}},&
\end{aligned}
\tag{3.25}
$$

This proves the estimate (3.2c).

3.5. $L^\infty(H^1)$ and $L^\infty(L^\infty)$ estimates of T^{err}

By the uniform estimates of velocity (3.17) and (3.25), one can easily establish the following estimate of T^{err}

$$
\|\nabla T^{err}\|_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon, \tag{3.26}
$$

$$
\|\Delta T^{err}\|_{L^2(0,T;L^2(\Omega))} \le C\epsilon, \tag{3.27}
$$

$$
\|\nabla\partial_x T^{err}\|_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon. \tag{3.28}
$$

In view of the anisotropic Sobolev embedding, one gets

$$\begin{aligned}\|T^{err}\|_{L^\infty(0,T;L^\infty(\Omega))} &\leq C\Big(\|T^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\|\partial_z T^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))} \\ &\qquad + \|\partial_x T^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\|\partial_z T^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))} \\ &\qquad + \|T^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\|\partial_x\partial_z T^{err}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))}\Big) \\ &\leq C\epsilon. \end{aligned} \tag{3.29}$$

Hence we arrive at (3.2f). Moreover, by applying (2.52), (3.2c), (3.2e) and (3.27) to the second equation in (3.1), we easily find (3.2g).

3.6. $L^\infty(L^\infty)$ estimate of p^{err}

We note that one can not apply the anisotropic Sobolev embedding for the uniform estimate of the pressure, due to the Neumnn boundary conditions. In order to estimate $\|p^{err}\|_{L^\infty(0,T;L^\infty(\Omega))}$, we rewrite the velocity error equation in (3.1) as follows

$$\begin{cases} \epsilon\frac{\partial \mathbf{v}^{err}}{\partial t} - \epsilon\Delta\mathbf{v}^{err} + \nabla p^{err} = \widetilde{\mathbf{f}}^{err}, \\ \text{div } \mathbf{v}^{err} = 0, \\ \mathbf{v}^{err}|_{t=0} = 0, \quad \mathbf{v}^{err}|_{z=0,1} = 0, \\ \mathbf{v}^{err}, \ p^{err} \text{ are periodic in } x\text{-direction.} \end{cases} \tag{3.30}$$

Here

$$\begin{aligned}\widetilde{\mathbf{f}}^{err} = &-\epsilon\Big((\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err} + (\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{app} + (\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err}\Big) \\ &- \mathbf{v}^{err} + Ra_D\mathbf{k}T^{err} - \mathbf{f}^{err},\end{aligned}$$

which satisfies $\|\widetilde{\mathbf{f}}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon$. Applying the regularity theory of Stokes system [54] to (3.30), we have

$$\|\nabla p^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon, \quad \|\Delta\mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \tag{3.31}$$

Moreover, according to (3.10) we have

$$\begin{cases} \epsilon\frac{\partial(\partial_x\mathbf{v}^{err})}{\partial t} - \epsilon\Delta\partial_x\mathbf{v}^{err} + \nabla\partial_x p^{err} = \mathbf{h}^{err}, \\ \text{div } \partial_x\mathbf{v}^{err} = 0, \\ \partial_x\mathbf{v}^{err}|_{t=0} = 0, \\ \partial_x\mathbf{v}^{err}|_{z=0,1} = 0, \\ \partial_x\mathbf{v}^{err}, \partial_x p^{err} \text{ are periodic in } x\text{-direction.} \end{cases} \tag{3.32}$$

where

$$\begin{aligned}\mathbf{h}^{err} = &-\epsilon\Big((\partial_x\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err} + (\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{err} + (\partial_x\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{app} \\ &+ (\mathbf{v}^{err}\cdot\nabla)\partial_x\mathbf{v}^{app} + (\partial_x\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{err} + (\mathbf{v}^{app}\cdot\nabla)\partial_x\mathbf{v}^{err}\Big) \\ &- \partial_x\mathbf{v}^{err} + Ra_D\mathbf{k}\partial_x T^{err} - \partial_x\mathbf{f}^{err}.\end{aligned}$$

It is easy to deduce from (3.9), (3.21), (3.24) and (3.25) that $\|\mathbf{h}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\varepsilon$. Applying the regularity theory of Stokes system [54] to (3.32), we have

$$\|\Delta\partial_x \mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C.$$

The divergence-free condition then implies

$$\|\Delta\partial_z v_2^{err}\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \tag{3.33}$$

Taking the divergence of the velocity error equation in (3.1), we get

$$\begin{cases} \Delta p^{err} = \bar{f}^{err}, \\ \partial_z p^{err}|_{z=0,1} = (\epsilon\Delta v_2^{err} - f_2^{err})|_{z=0,1}, \\ p^{err} \text{ is periodic in } x\text{-direction.} \end{cases} \tag{3.34}$$

Here

$$\bar{f}^{err} = -\varepsilon \sum_{i,j=1}^{2} \left(\partial_i \mathbf{v}_j^{err} \partial_j \mathbf{v}_i^{err} + \partial_i \mathbf{v}_j^{err} \partial_j \mathbf{v}_i^{app} + \partial_i \mathbf{v}_j^{app} \partial_j \mathbf{v}_i^{err}\right) + Ra_D \partial_z T^{err} - \mathrm{div}\mathbf{f}^{err}.$$

with $\mathbf{v}^{err} = (v_1^{err}, v_2^{err}), \mathbf{f}^{err} = (f_1^{err}, f_2^{err}),\ \partial_1 \triangleq \partial_x$ and $\partial_2 \triangleq \partial_z$.

The first term in the summation can be estimated by the interpolation inequality and regularity theory of Stokes system

$$\begin{aligned}\left\| \sum_{i,j=1}^{2} \left(\partial_i \mathbf{v}_j^{err} \partial_j \mathbf{v}_i^{err}\right) \right\|_{L^\infty(0,T;L^2(\Omega))} &\leq C\|\nabla \mathbf{v}^{err}\|^2_{L^\infty(0,T;L^4(\Omega))} \\ &\leq C\|\nabla \mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))}\|\Delta \mathbf{v}^{err}\|_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C\epsilon^{\frac{1}{2}}.\end{aligned}$$

The other terms in the summation can be estimated easily, thanks to the divergence-free condition, estimates (3.17), (3.21) and Lemma 2.1. One concludes by the inequalities (2.53) and (3.26) that $||\bar{f}^{err}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{3}{4}}$.

The regularity theory of elliptic equations [33] implies that

$$\begin{aligned}\|p^{err}\|_{L^\infty(0,T;H^2(\Omega))} &\leq C\big(\|\bar{f}^{err}\|_{L^\infty(0,T;L^2(\Omega))} + \|\epsilon\Delta \mathbf{v}_2^{err} - f_2^{err}\|_{L^\infty(0,T;H^1(\Omega))}\big) \\ &\leq C\epsilon^{\frac{3}{4}},\end{aligned} \tag{3.35}$$

where we have utilized the estimate (3.33) and (2.53). Recall also that we are restricting $p^{err} \in L_0^2(\Omega)$ with zero mean. By the Sobolev embedding, one concludes that

$$\|p^{err}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C\epsilon^{\frac{3}{4}}.$$

The proof of Theorem 3.1 is now complete.

$\square$

4. Conclusion

In this article, we have provided a rigorous asymptotic analysis of the nonlinear Darcy-Brinkman-Oberbeck-Boussinesq system in the vanishing Darcy number limit, which involves boundary layer, initial layer and their interaction initial-boundary layer. The optimal convergence rates in Sobolev norms including the physically interesting uniform norm are proved rigorously via a cascade of careful energy estimates. We remark that the analysis of the initial-boundary layer is novel, involving simultaneous two scale expansion in space and in time. The rigorous convergence result demonstrates detailed singular structures in the solution of the DBOB system. It further validates the applicability of the Darcy-Oberbeck-Boussinesq model for heat transport phenomenon in porous media if we view the nonlinear Darcy-Brinkman-Oberbeck-Boussinesq model as the "true" model.

The convergence results are derived under the zeroth order compatibility assumption $\mathbf{v}_0\big|_{z=0,1} = 0$. Additional singular structures will emerge without this compatibility condition. In [43, 16], the authors used semiclassical techniques and layer potentials to study the boundary layer. This approach does not rely on the Prandtl theory and does not require any type of compatibility conditions between the initial and boundary data. However, it yields only convergence in $L^\infty(L^p)$ with $p \in [1, +\infty]$ and does not provide any estimate on normal gradients at the boundary.

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Appendix: Asymptotic Analysis

In this section we mainly consider the system near $z = 0$. For that we introduce the stretched variables $\tau = \frac{t}{\epsilon} \in (0, +\infty)$ and $Z = \frac{z}{\sqrt{\epsilon}} \in (0, +\infty)$. We formally assume the solutions of the DBOB system have an asymptotic expansion of the form, taking the first component of the velocity for instance

$$
\begin{aligned}
v_1^\epsilon &= v_1^O(t,x,z) + v_1^I(\tau,x,z) + v_1^B(t,x,Z) + v_1^C(\tau,x,Z) \\
&= v_1^{O,0}(t,x,z) + \sqrt{\epsilon} v_1^{O,1}(t,x,z) + \cdots \\
&\quad + v_1^{I,0}(\tau,x,z) + \sqrt{\epsilon} v_1^{I,1}(\tau,x,z) + \cdots \\
&\quad + v_1^{B,0}(t,x,Z) + \sqrt{\epsilon} v_1^{B,1}(t,x,Z) + \cdots \\
&\quad + v_1^{C,0}(\tau,x,Z) + \sqrt{\epsilon} v_1^{C,1}(\tau,x,Z) + \cdots ,
\end{aligned} \tag{4.1}
$$

where the superscripts O, I, B, C denoting the outer term, initial layer, boundary layer and initial-boundary layer near $z = 0$, respectively.

We use the matched asymptotic expansion method to get the equations in Ω satisfied by the outer term, initial layer, boundary layer and initial-boundary layer. The following matching conditions are imposed: for all $0 \le k \le 1, 0 \le l \le 1, 0 \le m \le 1, 0 \le n \le 2, 1 \le i \le 2$,

$$\partial_\tau^k \partial_x^l \partial_z^m \big(\mathbf{v}^I,\ p^I,\ T^I\big) \to 0, \quad \partial_\tau^k \partial_x^l \partial_Z^n \big(v^C,\ p^C,\ T^C\big) \to 0, \quad \text{as} \quad \tau \to +\infty, \tag{4.2}$$

$$\partial_x^l \partial_Z^n \big(v_1^B,\ p^B,\ T^B\big) \to 0, \quad \partial_x^l \partial_Z^i v_2^B \to 0, \quad \text{as} \quad Z \to +\infty, \tag{4.3}$$

$$\partial_\tau^k \partial_x^l \partial_Z^n \big(v_1^C,\ p^C,\ T^C\big) \to 0, \ \partial_\tau^k \partial_x^l \partial_Z^i v_2^C \to 0, \ \text{as} \ \ Z \to +\infty. \tag{4.4}$$

The equations near $z = 1$ can be established similarly with the stretched variables $\tau = \frac{t}{\epsilon} \in (0, +\infty)$ and $Z = \frac{1-z}{\sqrt{\epsilon}} \in (-\infty, 0)$.

4.1. Leading order equations $O(1)$

The leading order outer system:

$$\begin{cases} \mathbf{v}^{O,0} + \nabla p^{O,0} = Ra_D \mathbf{k} T^{O,0}, \\ \frac{\partial T^{O,0}}{\partial t} + \mathbf{v}^{O,0} \cdot \nabla T^{O,0} = \Delta T^{O,0}, \\ \text{div } \mathbf{v}^{O,0} = 0, \\ T^{O,0}(0, x, z) = T_0, \\ \mathbf{v}^{O,0} \cdot \mathbf{n}|_{z=0,1} = 0, \ T^{O,0}(t, x, 0) = 1, \ T^{O,0}(t, x, 1) = 0. \end{cases} \tag{4.5}$$

The leading order initial layer system of velocity:

$$\begin{cases} \frac{\partial \mathbf{v}^{I,0}}{\partial \tau} + \mathbf{v}^{I,0} = 0, \\ \mathbf{v}^{I,0}(0, x, z) = \mathbf{v}_0(x, z) - \mathbf{v}^{O,0}(0, x, z). \end{cases} \tag{4.6}$$

The leading order boundary layer system of velocity:

$$\begin{cases} v_1^{B,0} - \partial_{ZZ} v_1^{B,0} = 0, \\ \partial_x v_1^{B,0} + \partial_Z v_2^{B,0} = 0, \\ v_1^{B,0}(t, x, 0) = -v_1^{O,0}(t, x, 0), \quad v_2^{B,0}(t, x, 0) = 0, \\ v_1^{B,0}(t, x, Z) \to 0, \quad Z \to +\infty. \end{cases} \tag{4.7}$$

The leading order initial-boundary layer system of velocity:

$$\begin{cases} \frac{\partial v_1^{C,0}}{\partial \tau} + v_1^{C,0} - \partial_{ZZ} v_1^{C,0} = 0, \\ \partial_x v_1^{C,0} + \partial_Z v_2^{C,0} = 0, \\ v_1^{C,0}(0, x, Z) = -v_1^{B,0}(0, x, Z), \\ v_1^{C,0}(\tau, x, 0) = -v_1^{I,0}(\tau, x, 0), \quad v_2^{C,0}(t, x, 0) = 0, \\ v_1^{B,0}(t, x, Z) \to 0, \ Z \to +\infty. \end{cases} \tag{4.8}$$

Here we impose that $v_2^{B,0}(t, x, 0) = 0$ and $v_2^{C,0}(t, x, 0) = 0$ since $\mathbf{v}^{O,0}$ and $\mathbf{v}^{I,0}$ satisfy no penetration boundary condition.

We also have

$$p^{I,0} = p^{B,0} = p^{C,0} = 0, \quad T^{I,0} = T^{B,0} = T^{C,0} = 0. \tag{4.9}$$

4.2. 1st-order equations $O(\sqrt{\epsilon})$

1st-order outer system:

$$\begin{cases} \mathbf{v}^{O,1} + \nabla p^{O,1} = \mathbf{k}(Ra_D T^{O,1} - \lim\limits_{Z\to+\infty} v_2^{B,0}), \\ \frac{\partial T^{O,1}}{\partial t} + \mathbf{v}^{O,0}\cdot\nabla T^{O,1} + v_1^{O,1}\partial_x T^{O,0} + (v_2^{O,1} + \lim\limits_{Z\to+\infty} v_2^{B,0})\partial_z T^{O,0} = \Delta T^{O,1}, \\ \mathrm{div}\ \mathbf{v}^{O,1} = 0, \\ T^{O,1}(0,x,z) = 0, \\ \mathbf{v}^{O,1}\cdot\mathbf{n}|_{z=0,1} = 0,\ T^{O,1}(t,x,0) = 0,\ T^{O,1}(t,x,1) = 0. \end{cases} \tag{4.10}$$

1st-order initial layer system of velocity:

$$\begin{cases} \frac{\partial \mathbf{v}^{I,1}}{\partial \tau} + \mathbf{v}^{I,1} = -\mathbf{k}\lim\limits_{Z\to+\infty}(v_2^{C,0} + \frac{\partial v_2^{C,0}}{\partial \tau}), \\ \mathrm{div}\ \mathbf{v}^{I,1} = 0, \\ \mathbf{v}^{I,1}\cdot\mathbf{n}|_{z=0,1} = 0, \\ v_1^{I,1}(0,x,z) = -v_1^{O,1}(0,x,z), \\ v_2^{I,1}(0,x,z) = -v_2^{O,1}(0,x,z) - \lim\limits_{Z\to+\infty}(v_2^{B,0}(0,x,Z) + v_2^{C,0}(0,x,Z)). \end{cases} \tag{4.11}$$

1st-order boundary layer system of velocity:

$$\begin{cases} v_1^{B,1} - \partial_{ZZ} v_1^{B,1} = -v_2^{O,0}\partial_Z v_1^{B,0}, \\ \partial_x v_1^{B,1} + \partial_Z v_2^{B,1} = 0, \\ v_1^{B,1}(t,x,0) = -v_1^{O,1}(t,x,0), \quad v_2^{B,1}(t,x,0) = 0, \\ v_1^{B,1}(t,x,Z) \to 0, \quad Z\to+\infty, \end{cases} \tag{4.12}$$

(iv) 1st-order initial-boundary layer system of velocity:

$$\begin{cases} \frac{\partial v_1^{C,1}}{\partial \tau} + v_1^{C,1} - \partial_{ZZ} v_1^{C,1} = v_2^{I,0}\big(\partial_Z v_1^{B,0} + \partial_Z v_1^{C,0}\big) + v_2^{O,0}\partial_Z v_1^{C,0}, \\ \partial_x v_1^{C,1} + \partial_Z v_2^{C,1} = 0, \\ v_1^{C,1}(0,x,Z) = -v_1^{B,1}(0,x,Z), \\ v_1^{C,1}(\tau,x,0) = -v_1^{I,1}(\tau,x,0), \quad v_2^{C,1}(t,x,0) = 0, \\ v_1^{C,1}(t,x,Z) \to 0,\ Z\to+\infty. \end{cases} \tag{4.13}$$

In addition, one has

$$p^{I,1} = p^{B,1} = p^{C,1} = 0, \quad T^{I,1} = T^{B,1} = T^{C,1} = 0. \tag{4.14}$$

4.3. 2nd-order equations for the temperature $O(\epsilon)$

Here we only derive the 2nd-order equations for $(T^{I,2}, T^{B,2}, T^{C,2})$. This is sufficient for our convergence analysis.

The initial layer equation:

$$\begin{cases} \frac{\partial T^{I,2}}{\partial \tau} + v_1^{I,0} \cdot \partial_x T^{O,0} + v_2^{I,0} \cdot \partial_z T^{O,0} = 0, \\ T^{I,2}(\tau, x, z) \to 0, \ \tau \to +\infty. \end{cases} \tag{4.15}$$

The boundary layer equation:

$$\begin{cases} \partial_{ZZ} T^{B,2} = v_1^{B,0} \cdot \partial_x T^{O,0}, \\ \partial_Z T^{B,2}(t, x, Z), T^{B,2}(t, x, Z) \to 0, \ Z \to +\infty. \end{cases} \tag{4.16}$$

The initial-boundary layer equation:

$$\begin{cases} \frac{\partial T^{C,2}}{\partial \tau} - \partial_{ZZ} T^{C,2} = -v_1^{C,0} \cdot \partial_x T^{O,0}, \\ T^{C,2}(0, x, Z) = -T^{B,2}(0, x, Z), \\ T^{C,2}(\tau, x, 0) = -T^{I,2}(\tau, x, 0), \quad T^{C,2}(t, x, Z) \to 0, \ Z \to +\infty. \end{cases} \tag{4.17}$$

4.4. 3rd-order equations for the temperature $O(\epsilon^{3/2})$

The initial layer equation:

$$\begin{cases} \frac{\partial T^{I,3}}{\partial \tau} + v_1^{I,0} \cdot \partial_x T^{O,1} + v_1^{I,1} \cdot \partial_x T^{O,0} + v_2^{I,0} \cdot \partial_z T^{O,1} \\ \qquad\qquad + (v_2^{I,1} + \lim\limits_{Z \to +\infty} v_2^{C,0}) \cdot \partial_z T^{O,0} = 0, \\ T^{I,3}(\tau, x, z) \to 0, \ \tau \to +\infty. \end{cases} \tag{4.18}$$

The boundary layer equation:

$$\begin{cases} \partial_{ZZ} T^{B,3} = v_1^{B,0} \cdot \partial_x T^{O,1} + v_1^{B,1} \cdot \partial_x T^{O,0} + v_2^{O,0} \cdot \partial_Z T^{B,2} \\ \qquad\qquad + (v_2^{B,0} - \lim\limits_{Z \to +\infty} v_2^{B,0}) \cdot \partial_z T^{O,0}, \\ \partial_Z T^{B,3}(t, x, Z), T^{B,3}(t, x, Z) \to 0, \ Z \to +\infty. \end{cases} \tag{4.19}$$

The initial-boundary layer equation:

$$\begin{cases} \frac{\partial T^{C,3}}{\partial \tau} - \partial_{ZZ} T^{C,3} = -v_1^{C,0} \cdot \partial_x T^{O,1} - v_1^{C,1} \cdot \partial_x T^{O,0} - v_2^{O,0} \cdot \partial_Z T^{C,2} \\ \qquad\qquad - v_2^{I,0} \cdot \partial_Z (T^{B,2} + T^{C,2}) - (v_2^{C,0} - \lim\limits_{Z \to +\infty} v_2^{C,0}) \cdot \partial_z T^{O,0}, \\ T^{C,3}(0, x, Z) = -T^{B,3}(0, x, Z), \\ T^{C,3}(\tau, x, 0) = -T^{I,3}(\tau, x, 0), \quad T^{C,3}(t, x, Z) \to 0, \ Z \to +\infty. \end{cases} \tag{4.20}$$

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