arXiv:1710.03132v1 [math.GT] 9 Oct 2017

# GENERALIZED CUSPS IN REAL PROJECTIVE MANIFOLDS: CLASSIFICATION

SAMUEL A. BALLAS, DARYL COOPER, AND ARIELLE LEITNER

Abstract. A generalized cusp $C$ is diffeomorphic to $[0, \infty)$ times a closed Euclidean manifold. Geometrically $C$ is the quotient of a properly convex domain by a lattice, $\Gamma$, in one of a family of affine groups $G(\psi)$, parameterized by a point $\psi$ in the (dual closed) Weyl chamber for $\mathrm{SL}(n+1,\mathbb{R})$, and $\Gamma$ determines the cusp up to equivalence. These affine groups correspond to certain fibered geometries, each of which is a bundle over an open simplex with fiber a horoball in hyperbolic space, and the lattices are classified by certain Bieberbach groups plus some auxiliary data. The cusp has finite Busemann measure if and only if $G(\psi)$ contains unipotent elements. There is a natural underlying Euclidean structure on $C$ unrelated to the Hilbert metric.

A *generalized cusp* is a properly convex projective manifold $C = \Omega/\Gamma$ where $\Omega \subset \mathbb{RP}^n$ is a properly convex set and $\Gamma \subset \mathrm{PGL}(n+1,\mathbb{R})$ is a virtually abelian discrete group that preserves $\Omega$. We also require that $\partial C$ is compact and strictly convex (contains no line segment) and that there is a diffeomorphism $h : [0, \infty) \times \partial C \longrightarrow C$.

An example is a cusp in a hyperbolic manifold that is the quotient of a horoball. Another generalized cusp $C' = \Omega'/\Gamma'$ is *equivalent* to $C$ if there is a generalized cusp $C''$ and projective embeddings, that are also homotopy equivalences, of $C''$ into both $C$ and $C'$, and they are all diffeomorphic. It follows from the classification theorem that generalized cusps are equivalent if and only if $\Gamma$ and $\Gamma'$ are conjugate subgroups of $\mathrm{PGL}(n + 1,\mathbb{R})$.

It follows from [10] that every generalized cusp in a *strictly* convex manifold of finite volume is equivalent to a *standard cusp*, i.e. a cusp in a hyperbolic manifold. A generalized cusp is *homogeneous* if $\mathrm{PGL}(\Omega)$ (the group of projective transformations that preserves $\Omega$) acts transitively on $\partial\Omega$. It was shown in [11] that every generalized cusp is equivalent to a homogeneous one and, that if the holonomy of a generalized cusp contains no hyperbolic elements, then it is equivalent to a standard cusp. Furthermore, by [11] it follows that generalized cusps often occur as ends of properly convex manifolds obtained by deforming finite volume hyperbolic manifolds.

The holonomy of a generalized cusp is conjugate to a lattice in one of a family of Lie subgroups $G(\psi) \subset \mathrm{PGL}(n + 1,\mathbb{R})$, parameterized by $\psi \in \mathrm{Hom}(\mathbb{R}^n,\mathbb{R})$ with $\psi(e_1) \geq \psi(e_2) \geq \cdots \psi(e_n) \geq 0$. Elements of the unipotent subgroup $P(\psi) \subset G(\psi)$ are called *parabolic*. The type $\mathbf{t} = \mathbf{t}_\psi$, is the number of $i$ with $\psi(e_i) \neq 0$, and the *unipotent rank* is $\mathbf{u}(\psi) = \dim P(\psi) = \max(n - \mathbf{t} - 1, 0)$.

The group $G(\psi)$ is called a *cusp Lie group* and preserves a properly-convex domain, $\Omega(\psi)$, together with a convex function $h_\psi : \Omega(\psi) \to \mathbb{R}$. The level sets $\mathcal{H}_t = h_\psi^{-1}(t)$ are $G(\psi)$-orbits, and are convex hypersurfaces called *horospheres*. The horospheres with $t \leq 0$ foliate $\Omega(\psi)$. There is a transverse $G(\psi)$-foliation by a pencil of lines, and a one parameter subgroup of $\mathrm{PGL}(n+1,\mathbb{R})$, called the *radial flow*, that preserves each line, normalizes $G(\psi)$, and permutes the horospheres.

The interior of $\Omega(0)$ is a model of hyperbolic space, $\mathbb{H}^n$, and $\partial\Omega(0) = S^{n-1}_\infty - z$ where $z$ is some point in $S^{n-1}_\infty$. Moreover $G(0) \subset \mathrm{Isom}(\mathbb{H}^n)$ is the group generated by parabolics and elliptics that fix $z$. At the other extreme, when $\mathbf{t} = n$, then $G(\psi)$ contains a finite index subgroup that is diagonalizable. Moreover, $G(\psi) = \mathrm{PGL}(\Omega(\psi))$ if $\psi \neq 0$. When $\mathbf{t} < n$, the geometry $(\Omega(\psi), G(\psi))$ is fibered over a simplex $\Delta^{\mathbf{t}}$ with fiber a horoball in $\mathbb{H}^{\mathbf{u}+1}$, see (1.32).

If $\Gamma \subset G(\psi)$ is a lattice then $C = \Omega(\psi)/\Gamma$ is called a $\psi$*-cusp*. This is a projective manifold if $\Gamma$ is torsion free, and in general is a projective *orbifold*. In this paper we have chosen to discuss only

*Date*: October 10, 2017.

manifolds, though everything works (suitably modified) for orbifolds. The image of a horosphere in $C$ is called a *horomanifold* and these foliate $C$.

**Theorem 0.1** (uniformization)**.** *Every generalized cusp is equivalent to a $\psi$-cusp.*

The next goal is to classify cusps up to equivalence. For this it is useful to introduce *marked cusps* and *marked* lattices (see Section 3 for the definition and more discussion). A rank-2 cusp in a hyperbolic 3-manifold is determined by a *cusp shape*, which is a Euclidean torus defined up to similarity. This shape is usually described by a complex number $x+iy$ with $y>0$, that uniquely determines a *marked* cusp. Unmarked cusps are described by the modular surface $\mathbb{H}^2/\operatorname{PSL}(2,\mathbb{Z})$.

More generally, a maximal-rank cusp in a hyperbolic $n$-manifold is determined by a lattice in $\operatorname{Isom}(\mathbb{E}^{n-1})$ up to conjugacy and rescaling. We extend this result by showing when $\psi\neq 0$ that a generalized cusp of dimension $n$ with holonomy in $G(\psi)$ is determined by a pair $([\Gamma], A\cdot O(\psi))$ consisting of the conjugacy class of a lattice $\Gamma\subset\operatorname{Isom}(\mathbb{E}^{n-1})$, and an *anisotropy parameter* which we now describe.

The second fundamental form on $\partial\Omega$ is a Euclidean metric that is preserved by the action of $G(\psi)$. This identifies $G(\psi)$ with a subgroup of $\operatorname{Isom}(\mathbb{E}^{n-1})$, and $G(\psi)=T(\psi)\rtimes O(\psi)$ is the semi-direct product of the *translation subgroup*, $T(\psi)\cong\mathbb{R}^{n-1}$, and a closed subgroup $O(\psi)\subset O(n-1)$ that fixes some point $p$ in $\partial\Omega$, see (1.22). The Euclidean structure identifies $\Gamma$ with a lattice in $\operatorname{Isom}(\mathbb{E}^{n-1})$. This lattice is unique up to conjugation by an element of $O(\psi)$. The *anisotropy parameter* is a left coset $A\cdot O(\psi)$ in $O(n)$ that determines the $O(\psi)$-conjugacy class. The group $O(\psi)$ is computed in (1.21).

Given a Lie group $G$, the set of $G$-conjugacy classes of marked lattices in $G$ is denoted $\mathcal{T}(G)$. Define $\mathcal{T}(\operatorname{Isom}(\mathbb{E}^{n-1}),\psi)\subset\mathcal{T}(\operatorname{Isom}(\mathbb{E}^{n-1}))$ to be the subset of conjugacy classes of *marked* Euclidean lattices with rotational part of the holonomy (up to conjugacy) in $O(\psi)$. The classification of generalized cusps (up to equivalence) is completed by:

**Theorem 0.2** (classification)**.**

(1) *If $\Gamma$ and $\Gamma'$ are lattices in $G(\psi)$ TFAE*
  (a) $\Omega(\psi)/\Gamma$ *and* $\Omega(\psi)/\Gamma'$ *are equivalent generalized cusps*
  (b) $\Gamma$ *and* $\Gamma'$ *are conjugate in* $\operatorname{PGL}(n+1,\mathbb{R})$
  (c) $\Gamma$ *and* $\Gamma'$ *are conjugate in* $\operatorname{PGL}(\Omega(\psi))$
(2) *A lattice in $G(\psi)$ is conjugate in* $\operatorname{PGL}(n+1,\mathbb{R})$ *into $G(\psi')$ iff $G(\psi)$ is conjugate to $G(\psi')$.*
(3) *$G(\psi)$ is conjugate in* $\operatorname{PGL}(n+1,\mathbb{R})$ *to $G(\psi')$ iff $\psi'=t\cdot\psi$ for some $t>0$.*
(4) $\operatorname{PGL}(\Omega(\psi))=G(\psi)$ *when $\psi\neq 0$*
(5) *When $\psi\neq 0$ the map $\Theta:\mathcal{T}(\operatorname{Isom}(\mathbb{E}^{n-1}),\psi)\times(O(n-1)/O(\psi))\longrightarrow\mathcal{T}(G(\psi))$ defined in (29) is a bijection.*

One might view (2) in the context of super-rigidity : *an embedding of a lattice determines an embedding of the Lie group that contains it.* Throughout this paper we repeatedly stumble over two exceptional cases. A generalized cusp with $\psi=0$ is projectively equivalent to a cusp in a hyperbolic manifold. This is the only case when $\operatorname{PGL}(\Omega(\psi))$ is strictly larger than $G(\psi)$, and is caused by elements of $\operatorname{PGL}(\Omega(0))\subset\operatorname{Isom}(\mathbb{H}^n)$ that permute horospheres. These elements are hyperbolic isometries of $\mathbb{H}^n$ that fix $z$. This accounts for the fact that the equivalence class of a cusp in a hyperbolic manifold is determined by the *similarity class* ($\operatorname{PGL}(\Omega(\psi))$-conjugacy class) of the lattice, rather than the *$G(\psi)$-conjugacy class*, as in every other case. The other exceptional case is the diagonalizable case $\mathbf{t}=n$, and in this case the radial flow is hyperbolic instead of parabolic. Fortunately both these exceptional cases are easy to understand, but tend to require proofs that consider various cases.

Let $\mathcal{C}^n$ denote the set of equivalence classes of generalized cusps of dimension $n$. Let $\mathcal{M}\mathrm{od}^n$ denote the (disjoint) union over all $\psi$ with $\psi(e_1)=1$ of conjugacy classes of (unmarked) lattice in $G(\psi)$, union lattices in $G(0)\cong\operatorname{Isom}(\mathbb{E}^{n-1})$ up to conjugacy and scaling. It follows from the above that every non-standard generalized cusp is equivalent to one given by a lattice in $G(\psi)$ with $\psi(e_1)=1$, that is unique up to conjugacy in $G(\psi)$ giving:

**Corollary 0.3** (cusps classified by lattices)**.** *There is a bijection* $F : \mathcal{M}\text{od}^n \longrightarrow \mathcal{C}^n$ *defined for* $[\Gamma] \in \mathcal{M}\text{od}^n$ *by* $F([\Gamma]) = [\Omega(\psi)/\Gamma]$ *when* $\Gamma$ *is a lattice in* $G(\psi)$.

If $\alpha$ is an isometry of a metric space $(X, d)$ the *displacement* of $\alpha$ is $\delta(\alpha) = \inf\{d(x, \alpha x) | x \in X\}$.

**Corollary 0.4** (standard parabolics)**.** *Suppose* $M = \Omega/\Gamma$ *is a properly convex* $n$*-manifold such that every end of* $M$ *is a generalized cusp. If* $[A] \in \Gamma$, *and if* $\delta([A]) = 0$ *for the Hilbert metric on* $\Omega$, *then* $[A]$ *is the holonomy of an element of* $\pi_1 C$ *for some generalized cusp* $C \subset M$, *and* $[A]$ *is conjugate in* $\mathrm{PGL}(n + 1, \mathbb{R})$ *into* $PO(n, 1)$.

Generalized cusps are modelled on the geometries $(G(\psi), \Omega(\psi))$, and these are all isomorphic to subgeometries of Euclidean geometry, see (1.33). In fact there is a natural Euclidean metric:

**Theorem 0.5** (underlying Euclidean structure)**.** *There is a Euclidean metric* $\beta$ *on* $\Omega = \Omega(\psi)$ *that is preserved by* $G(\psi)$, *and* $(\Omega, \beta)$ *is isometric to* $\mathbb{R}^{n-1} \times [0, \infty)$ *with the usual Euclidean metric. The restriction of* $\beta$ *to* $\partial\Omega \subset \mathbb{R}^n$ *is the second fundamental form.*

This implies a generalized cusp has an *underlying Euclidean structure*, and also an underlying *hyperbolic structure*, see (2.12). It is a well known that, if $C$ is a maximal rank cusp in a hyperbolic manifold $M$, then $C$ has finite hyperbolic volume.

**Theorem 0.6** (parabolic $\Leftrightarrow$ finite vol)**.** *Suppose* $C = \Omega/\Gamma$ *is a generalized cusp in the interior a properly convex manifold* $M$ *and* $\Gamma$ *is conjugate into* $G(\psi)$. *Then* $C$ *has finite volume in* $M$ *(with respect to the Hausdorff measure induced by the Hilbert metric on* $M$*) iff* $\mathbf{u}(\psi) > 0$.

The original definition [11] of *generalized cusp* differs from the one in the introduction by replacing the word *abelian* by *nilpotent*. The reason *nilpotent* was used originally is the connection between cusps and the Margulis lemma. A consequence of the analysis in this paper is that these definitions are equivalent:

**Theorem 0.7** (nilpotent $\Rightarrow$ abelian)**.** *Suppose* $C = \Omega/\Gamma$ *is a properly convex manifold and* $C \cong \partial C \times [0, \infty)$ *and* $\partial C$ *is compact and strictly convex, and* $\pi_1 C$ *is virtually nilpotent. Then* $C$ *is a generalized cusp and* $\pi_1 C$ *is virtually abelian.*

Another aspect of the definition of *generalized cusp* is that $\partial C$ is *compact*. In the theory of Kleinian groups, rank-1 cusps are important. These are diffeomorphic to $A \times [0, \infty)$ where $A$ is a (non-compact) annulus. For hyperbolic manifolds of higher dimensions there are more possibilities, however the fundamental group of such a cusp is always virtually abelian. This is not the case for properly convex manifolds. In [8] there is an example of a strictly convex manifold with unipotent (parabolic) holonomy, and with fundamental group the integer Heisenberg group. There might to be a nice theory of properly convex manifolds $C \cong \partial C \times [0, \infty)$ with $\pi_1 C$ virtually nilpotent and $\partial C$ strictly convex, but without requiring $\partial C$ to be compact.

The definition of the term *generalized cusp* was the end result of a lot of experimentation with definitions, and was modified as more was discovered about their nature. In retrospect it turns out they are all *deformations* of cusps in hyperbolic manifolds. This theme will be developed in a subsequent paper.

There is a discussion of surfaces in Section 6, and of 3-manifolds in Section 7. The latter is new while the former is well known. The general theory is quite involved, and the reader might wish to skim these sections before the rest of the paper. Choi [6] has studied certain kinds of ends of projective manifolds, and there is some overlap with his work.

Work partially supported by U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 *RNMS: GEometric structures And Representation varieties* (the GEAR Network). Ballas was partially supported by NSF grant DMS 1709097. Cooper was partially supported by NSF grants DMS 1065939, 1207068 and 1045292, and thanks Technische Universität Berlin for hospitality during completion of this work. Leitner was partially supported by the ISF-UGC joint research program framework grant No. 1469/14 and No. 577/15.

## 1. The Geometry of $\psi$-Cusps

We recall some definitions, see [21] for more background. A subset $\Omega \subset \mathbb{RP}^n$ is *properly convex* if the intersection with every projective line is connected, and omits at least 2 points. The *boundary* is used in the sense of manifolds: $\partial\Omega = \Omega \setminus \operatorname{int}(\Omega) \subset \Omega$ and is usually distinct from the *frontier* which is $\operatorname{Fr}(\Omega) := \partial(\operatorname{cl}\Omega) = \operatorname{cl}\Omega \setminus \Omega$. A properly convex domain *has strictly convex boundary* if $\partial\Omega$ contains no line segment. An *affine patch* is the complement of a projective hyperplane. If there is a unique supporting hyperplane to $\Omega$ at $p \in \operatorname{Fr}(\Omega)$ then $p$ is a $C^1$ *point.*

A *geometry* is a pair $(X, G)$ where $G$ is a subgroup of the group of homeomorphisms of $X$ onto itself. In this section we describe a family of geometries parameterized by points in the *(closed dual) Weyl chamber*

$$A = \{\psi \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}) : \psi_i := \psi(e_i) \quad \psi_1 \geq \psi_2 \geq \cdots \geq \psi_n \geq 0\} \tag{1}$$

For each $\psi \in A$, there is a closed convex subset $\Omega(\psi) \subset \mathbb{R}^n$ and a Lie subgroup $G(\psi)$ of $\operatorname{Aff}(\mathbb{R}^n)$, described by (1.22), that preserves $\Omega(\psi)$ and acts transitively on $\partial\Omega$. The pair $(\Omega(\psi), G(\psi))$ is called $\psi$*-geometry.* It is isomorphic to a subgeometry of Euclidean geometry (1.33).

Given $\psi \in A$ the *type* $\mathbf{t} = \mathbf{t}_\psi$ is the largest integer $0 \leq i \leq n$ with $\psi(e_i) > 0$, and

$$V = V_\psi = \mathbb{R}_+^{\mathbf{t}} \times \mathbb{R}^{n-\mathbf{t}}$$

The $\psi$*-horofunction* $h_\psi : V_\psi \to \mathbb{R}$ is defined by

$$h_\psi(x_1, \cdots, x_n) = \begin{cases} -x_{\mathbf{t}+1} - \sum_{i=1}^{\mathbf{t}} \psi_i \log x_i + \frac{1}{2}\sum_{i=\mathbf{t}+2}^{n} x_i^2 & if \quad \mathbf{t} < n \\ -\left(\sum_{i=1}^{n} \psi_i\right)^{-1} \sum_{i=1}^{n} \psi_i \log x_i & if \quad \mathbf{t} = n \end{cases} \tag{2}$$

The $\psi$*-domain* $\Omega = \Omega(\psi) = h_\psi^{-1}((-\infty, 0]) \subset \mathbb{R}^n$ has boundary $\partial\Omega = h_\psi^{-1}(0)$.

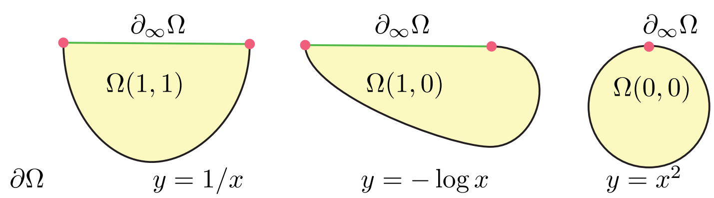


Figure 1. A projective view of some 2 dimensional $\psi$-domains

**Proposition 1.1.** *$\Omega(\psi)$ is a closed subset of $\mathbb{R}^n$ and $\partial\Omega(\psi)$ is a strictly convex hypersurface. Moreover $\Omega(\psi)$ is a properly-convex subset of $\mathbb{RP}^n$.*

*Proof.* $\Omega$ is closed because $h$ is continuous, and it is a smooth manifold with boundary because $h$ is a submersion. Thus $\partial\Omega$ is a smooth hypersurface. The second derivative of $x^2$, and of $-\log(x)$, are both positive on $V_\psi$, so the second derivative $D^2h_\psi$ is positive semi-definite on $V_\psi$. For $\mathbf{t} < n$ it has nullity 1, given by the $x_{\mathbf{t}+1}$ direction. When $\mathbf{t} = n$ it is positive definite.

Suppose $\ell$ is a line segment with endpoints $a, b \in \Omega$. Set $f = h|\ell$ then $f'' \geq 0$ so $f$ attains its maximum at an endpoint. Thus $f \leq \max(f(a), f(b))$ and $f(a), f(b) \leq 0$ since $a, b \in \Omega$. Thus $f \leq 0$ so $\ell \subset \Omega$ and $\Omega$ is convex.

If $\ell$ were contained in $\partial\Omega$ then $h = 0$ everywhere on $\ell$. But $D^2h > 0$ along $\ell$ unless $\mathbf{t} < n$ and $\ell$ is parallel to $e_{\mathbf{t}+1}$, in which case the derivative $Dh \neq 0$ along $\ell$. In every case $h$ is not constant along $\ell$ so $\partial\Omega$ is strictly convex.

Suppose $\ell$ is a complete affine line contained in $\Omega$. Then $\ell$ is contained in $\mathbb{R}_+^{\mathbf{t}} \times \mathbb{R}^{n-\mathbf{t}}$, so $x_i$ is constant along $\ell$ for $i \leq \mathbf{t}$. Thus $\mathbf{t} < n$ and $h_\psi|\ell = C_1 - t + C_2t^2$, where $t$ is an affine coordinate on $\ell$. But $\ell \subset \Omega$ implies this function is nowhere positive, a contradiction. Hence $\Omega$ contains no complete affine line, and is thus properly convex. $\square$

**Remark 1.2.** *Notice all $\psi_i$ must have the same sign, or else $h_\psi$ is not convex. Equivalently, if $|\psi_1| \geq |\psi_2| \geq \cdots \geq |\psi_n| \geq 0$ and if $\Omega(\psi)$ is convex, then either $\psi$ or $-\psi$ is positive.*

For $\mathbf{t} < n$ it is convenient to introduce $\psi^{\mathbf{t}} : \mathbb{R}^{\mathbf{t}} \to \mathbb{R}$ given by

$$\psi^{\mathbf{t}}(x) = \psi(x, 0, \cdots, 0) \tag{3}$$

**Definition 1.3.** *For each $t \in \mathbb{R}$ the hypersurface $\mathcal{H}_t = h_\psi^{-1}(t) \subset V_\psi$ is called a* horosphere.

Since $h_\psi$ is a submersion, these horospheres form a smooth codimension-1 foliation of $V_\psi$. It follows from 1.10 and the discussion in Section 3 of [10] that these are horospheres in the sense of Busemann, and from (8) that they are also *algebraic horospheres* as defined in [10].

**Definition 1.4.** *The $\psi$* cusp Lie group *is the group, $G = G(\psi) \subset \mathrm{Aff}(\mathbb{R}^n)$, of all affine maps that preserves each horosphere.*

This condition is equivalent to $G$ preserving the horofunction. In particular $G$ preserves $\Omega$.

**Definition 1.5.** *A* $\psi$-cusp *is $C = \Omega(\psi)/\Gamma$ where $\Gamma \subset G(\psi)$ is a torsion-free lattice.*

It follows that a $\psi$-cusp is an affine manifold and also a properly-convex manifold. If $\Gamma$ is a lattice in $G(\psi)$ that contains torsion then $C$ is an orbifold.

1.1. **The Radial Flow.** The *parabolic rank* is $\mathbf{u} = \max(n - 1 - \mathbf{t}, 0)$ and the *rank* $\mathbf{r}$ is defined by $\mathbf{r} + \mathbf{u} = n - 1$. Then $\mathbf{r} = \min(\mathbf{t}, n-1)$. A more conceptual interpretation of $\mathbf{r}$ and $\mathbf{u}$ is given by (20). It is convenient to use coordinates on $V_\psi$ given by

$$(x, z, y) \in V_\psi = \begin{cases} \mathbb{R}_+^{\mathbf{r}} \times \mathbb{R} \times \mathbb{R}^{\mathbf{u}} & if \quad \mathbf{t} < n \\ \mathbb{R}_+^{\mathbf{r}} \times \mathbb{R}_+ \times \mathbb{R}^{\mathbf{u}} & if \quad \mathbf{t} = n \end{cases} \tag{4}$$

When $\mathbf{t} = 0$ the $x$-coordinate is empty; and when $\mathbf{t} \geq n-1$ then $\mathbf{u} = 0$ so the $y$-coordinate is empty. The $z$-coordinate is called the *vertical direction.* This terminology is motivated by regarding the horospheres as graphs of functions, see Equation (10).

**Definition 1.6.** *The* basepoint *of $\Omega(\psi)$ is $b = b_\psi = e_1 + \cdots + e_{\mathbf{t}} \in \mathbb{R}^n$.*

Thus for $\mathbf{t} = 0$ the basepoint is $b = 0 \in \mathbb{R}^n$. The basepoint satisfies $h_\psi(b) = 0$ so $b \in \partial\Omega$. When $\mathbf{t} < n$ then $b = (x_0, z_0, y_0)$ where $x_0 = (1, \cdots, 1) \in \mathbb{R}_+^{\mathbf{r}}$ and the remaining coordinates are 0. When $\mathbf{t} = n$ then $b = (1, \cdots, 1)$. In projective coordinates the basepoint is $[b_\psi + e_{n+1}] \in \mathbb{R}\mathrm{P}^n$. Define

$$U = U_\psi = \mathbb{R}_+^{\mathbf{r}} \times \mathbb{R}^{\mathbf{u}}$$

*Radial projection* is the map $\pi = \pi_\psi : V_\psi \to U_\psi$ given by

$$\pi(x, z, y) = \begin{cases} (x, y) & if \quad \mathbf{t} < n \\ (x/z, y/z) & if \quad \mathbf{t} = n \end{cases} \tag{5}$$

**Definition 1.7.** *The* radial flow *$\Phi = \Phi^\psi : V_\psi \times \mathbb{R} \to V_\psi$ is defined by*

$$\Phi_t(x, z, y) := \Phi((x, z, y), t) = \begin{cases} (x, z - t, y) & if \quad \mathbf{t} < n \\ e^{-t}(x, z, y) & if \quad \mathbf{t} = n \end{cases} \tag{6}$$

In the first case the radial flow is called *parabolic* and in the second case it is *hyperbolic.* This terminology agrees with that of [11]. The orbit of a point is called a *flowline.* Each flowline maps to one point under radial projection. When $\mathbf{t} < n$ flowlines are vertical lines, and when $\mathbf{t} = n$ they are open rays that limit on $0 \in \mathbb{R}^n$.

The radial flow is the restriction to $V_\psi$ of a 1-parameter subgroup $\Phi \subset \mathrm{PGL}(n+1, \mathbb{R})$, see (22). The reason for the name *radial flow* is that this group acts on $\mathbb{R}\mathrm{P}^n$ and there is a point $\alpha \in \mathbb{R}\mathrm{P}^n$ called the *center of the radial flow* with the property that, if a point $\beta \in \mathbb{R}\mathrm{P}^n$ is not fixed by the flow, then the orbit of $\beta$ is contained in the projective line containing $\alpha$ and $\beta$. Moreover $\Phi_t(\beta) \to \alpha$ as $t \to \infty$. When $\mathbf{t} = n$ then $\alpha = 0 \in \mathbb{R}^n$ and when $\mathbf{t} < n$ then $\alpha$ is the point at infinity $[e_{\mathbf{t}+1}]$, corresponding to the $z$-axis.

Observe that the radial flow has the following equivariance property:

$$h_\psi(\Phi_t(x)) = h_\psi(x) + t \tag{7}$$

This equation would need to be modified without the first factor in the definition of $h_\psi$ (equation 2) when $\mathbf{t} = n$. It follows that the radial flow permutes the level sets of the horofunction and hence permutes the horospheres and

$$\mathcal{H}_t = \Phi_{-t}(\partial\Omega) \tag{8}$$

**Definition 1.8.** *A* product structure *on a manifold $M$ is a pair of transverse foliations on $M$ determined by a diffeomorphism $P \times Q \to M$. There is a diffeomorphism $f : \partial\Omega(\psi) \times [0, \infty) \to \Omega(\psi)$ given by $f(x, t) = \Phi_{-t}(x)$. This defines a product structure on $\Omega(\psi)$, with a foliation by hypersurfaces called* horomanifolds*, and a transverse foliation by* (half-)flowlines*.*

If $C = \Omega(\psi)/\Gamma$ is a $\psi$-cusp, then $\Gamma$ preserves this product structure, so it covers a product structure on $C$. The set $\Omega$ is *backwards invariant* which means that $\Phi_t(\Omega) \subset \Omega$ for all $t \leq 0$ and $\Omega$ is the backwards orbit of $\partial\Omega$

$$\Omega = \bigcup_{t \leq 0} \Phi_t(\partial\Omega)$$

Define $\ell\text{og} : \mathbb{R} \to \mathbb{R}$ by

$$\ell\text{og}(x) = \begin{cases} 0 & if \quad x \leq 0 \\ \log(x) & if \quad x > 0 \end{cases} \tag{9}$$

and extend this to a map $\ell\text{og} : \mathbb{R}^r \to \mathbb{R}^r$ by applying $\ell\text{og}$ componentwise. Then define $f = f_\psi : U \to \mathbb{R}$ by

$$f_\psi(x, y) \quad = \quad \begin{cases} -\psi^{\mathbf{t}} \circ \ell\text{og}(x) + \|y\|^2/2 & if \quad \mathbf{t} < n \\ \prod_{i=1}^{n-1} x_i^{-\psi_i/\psi_n} & if \quad \mathbf{t} = n \end{cases} \tag{10}$$

The map $F = F_\psi : U \to \partial\Omega$ given by

$$F(x, y) = (x, f(x, y), y)$$

is the inverse of the restriction of *vertical* projection $\pi| : \partial\Omega \to U$, so $\partial\Omega$ is the graph $z = f(x, y)$ of $f$ and $\Omega = \{(x, z, y) : z \geq f(x, y)\}$ is the supergraph of $f$. For $\mathbf{t} < n$ the horofunction is then expressed more compactly as

$$h_\psi(x, z, y) = -z + f_\psi(x, y) \tag{11}$$

When $\mathbf{t} = n$ the vertical coordinate $z$ is *not* the horofunction, so we do not obtain another expression for the horofunction in this case.

1.2. **The Ideal Boundary $\partial_\infty\Omega$.** In what follows $\psi$ is dropped from the notation. We describe the closure $\overline{\Omega}$ in $\mathbb{RP}^n$. Identify affine space $\mathbb{R}^n$ with an affine patch in projective space $\mathbb{RP}^n$ by identifying $(x, z, y)$ in $\mathbb{R}^n$ with $[x : z : y : 1]$ in $\mathbb{RP}^n$. Then

$$\Omega = \{[x : z : y : 1] \mid z \geq f(x, y), \quad x \in \mathbb{R}^{\mathbf{r}}_+\} \subset \mathbb{RP}^n \tag{12}$$

Observe that $\overline{\Omega} \cap \mathbb{R}^n = \Omega$. The *points at infinity* are $\mathbb{RP}^{n-1}_\infty = \mathbb{RP}^n \setminus \mathbb{R}^n$ and

$$\overline{\Omega} = \Omega \sqcup \partial_\infty\Omega \qquad \text{with} \quad \partial_\infty\Omega := \overline{\Omega} \setminus \Omega \subset \mathbb{RP}^{n-1}_\infty \tag{13}$$

The set $\partial_\infty\Omega$ is called the *ideal boundary* or the *boundary at infinity* of $\Omega$. See [13] definition 1.17. The *non-ideal boundary* or just *boundary* of $\Omega$ is $\partial\Omega = \mathbb{R}^n \cap \partial\overline{\Omega}$. Thus

$$\partial\overline{\Omega} = \partial\Omega \sqcup \partial_\infty\Omega.$$

**Lemma 1.9.** *$\partial_\infty\Omega(\psi)$ is the simplex of dimension* $\mathbf{r}$

$$\partial_\infty\Omega(\psi) = \{[x_1 : \cdots : x_{\mathbf{r}+1} : \cdots : 0] \mid x_i \geq 0\} \cong \Delta^{\mathbf{r}}$$

*Proof.* From (12) $\partial_\infty\Omega$ consists of all the points that are the limit of a sequence of points $[x : z : y : 1]$ with $\|(x, z, y)\| \to \infty$ for which $z \geq f(x, y)$.

First assume $\mathbf{t} < n$, and so $\mathbf{t} = \mathbf{r}$. We claim that $y/\|(x, z, y)\| \to 0$ along the sequence. If eventually $\psi\,\ell\text{og}(x) < \|y\|^2/4$ then by (10) it follows that $z > \|y\|^2/4$ and, since $y/\|y\|^2 \to 0$ as $\|y\| \to \infty$, it follows that $y/z \to 0$. Otherwise we may take a subsequence so $\psi\,\ell\text{og}(x) \geq \|y\|^2/4 \to +\infty$. Since $\psi_j > 0$ for all $j \leq \mathbf{r}$, this means for some $i \leq \mathbf{r}$ the coordinate $x_i$ of $x$ is positive and larger than some fixed multiple of $\exp\|y\|$, hence $y/x_i \to 0$. This proves the claim. Hence $\partial_\infty\Omega \subset \Delta^{\mathbf{r}}$.

From (10) we see that $f(e^t x, 0) < 0$ for large $t$. Then by (12)

$$\partial_\infty\Omega \supset \{\lim_{t\to\infty}[e^t x : e^t z : 0 : 1] \mid z \geq 0, \quad x \in \mathbb{R}^r_+\} = \Delta^{\mathbf{r}}$$

which proves the result for $\mathbf{t} < n$.

When $\mathbf{t} = n$ then $\mathbf{r} = n-1$ and $\partial_\infty\Omega \subset \partial_\infty V_\psi = \Delta^{n-1}$. On the other hand if $v \in \operatorname{int}\Delta^{n-1}$ then $v = \lim_{t\to\infty}[tx : tz : 1]$, where $x \in \mathbb{R}^{n-1}_+$ and $z \in \mathbb{R}_+$. From the definition of $f(x)$ (see (10)), it is easy to check, when $t$ is large, that $tz > f(tx)$, hence $(tx, tz) \in \Omega$, and so $\operatorname{int}\Delta^{n-1} \subset \partial_\infty\Omega$. Since $\partial_\infty\Omega$ is closed it follows that $\Delta^{n-1} = \partial_\infty\Omega$. □

**Lemma 1.10.** *There is $q \in \partial_\infty\Omega$ that is a $C^1$-point. Thus $\mathbb{RP}^{n-1}_\infty$ is the unique projective hyperplane in $\mathbb{RP}^n$ that contains $\partial_\infty\Omega$ and is disjoint from $\Omega$. Hence $\mathrm{PGL}(\Omega) \subset \mathrm{Aff}(\mathbb{R}^n)$.*

*Proof.* This is clear for $\mathbf{r} = n-1$ since $\partial_\infty\Omega$ is an $(n-1)$-simplex. Thus we may suppose that $\mathbf{u} \geq 1$ and $\mathbf{r} = \mathbf{t}$. We use coordinates $(x, z, y, w) \in \mathbb{R}^{\mathbf{r}} \oplus \mathbb{R} \oplus \mathbb{R}^{\mathbf{u}} \oplus \mathbb{R} \equiv \mathbb{R}^{n+1}$, so the affine patch used above is $[x : z : y : 1]$, and $\mathbb{RP}^{n-1}_\infty$ is $[x : z : y : 0]$. The idea is to produce a smooth path in $\partial\overline{\Omega}$ through a certain point $q \in \operatorname{int}\partial_\infty\Omega$ and tangent there in an arbitrary direction in the $y$-coordinates.

Define $a = (1, \cdots, 1) \in \mathbb{R}^{\mathbf{r}}$. Given $y \in \mathbb{R}^{\mathbf{u}}$ with $\|y\| = \sqrt{2}$ define a path $\gamma = \gamma_y : [1, \infty) \to \partial\Omega \subset \mathbb{RP}^n$ by

$$\gamma(t) = [t^2 a : f_\psi(t^2 a, ty) : ty : 1]$$

Observe that $\ell\text{og}(t^2 a) = (2\log t)a$ so $f_\psi(t^2 a, ty) = t^2 - \alpha\log(t)$ where $\alpha := 2\psi^{\mathbf{t}}(a) = 2\sum_{i=1}^{\mathbf{t}}\psi_i$, so

$$\gamma(t) = [a : 1 - \alpha t^{-2}\log(t) : t^{-1}y : t^{-2}]$$

Thus

$$q = \lim_{t\to\infty}\gamma(t) = [a : 1 : 0 : 0] \in \operatorname{int}\partial_\infty\Omega$$

Set $s = 1/t$, then $\gamma(1/s)$ extends to a path $\eta = \eta_y : [0, 1) \to \partial\overline{\Omega}$ defined by

$$\eta(0) = q \quad \textit{and} \quad \eta(s) = [a : 1 - \alpha s^2\log(s^{-1}) : sy : s^2] \quad \textit{for } s > 0$$

We now work in the affine patch given by

$$\mathbb{R}^{\mathbf{r}} \oplus \mathbb{R}^{\mathbf{u}} \oplus \mathbb{R} \to \mathbb{RP}^n \qquad (x, y, w) \mapsto [a + x : 1 : y : w]$$

In these affine coordinates observe that $q = \eta(0) = 0$ and for $s > 0$

$$\eta(s) = [1 - \alpha s^2\log(s^{-1})]^{-1}\cdot(0, sy, s^2)$$

is differentiable at $s = 0$ and

$$\eta'(0) = (0, y, 0)$$

Suppose $H$ is an affine hyperplane in this affine patch that contains $\partial_\infty\Omega$ and is disjoint from $\Omega$. Since $q \in \operatorname{int}\partial_\infty\Omega$ it follows that $H$ contains $\mathbb{R}^{\mathbf{r}} \oplus 0 \oplus 0$. We can combine the paths $\eta_y$ and $\eta_{-y}$ to obtain a smooth path through $p$ with tangent vector $\eta'(0)$ at $q$. Thus $H$ contains $\eta'(0)$. Hence $H$ contains the subspace $0 \oplus \mathbb{R}^{\mathbf{u}} \oplus 0 \subset \mathbb{R}^{\mathbf{r}} \oplus \mathbb{R}^{\mathbf{u}} \oplus \mathbb{R}$. So $H$ contains the subspace $\mathbb{R}^{\mathbf{r}} \oplus \mathbb{R}^{\mathbf{u}} \oplus 0$. This has codimension 1 so $H = \mathbb{R}^{\mathbf{r}} \oplus \mathbb{R}^{\mathbf{u}} \oplus 0$. This hyperplane is in $\mathbb{RP}^{n-1}_\infty$, which proves $q$ is a $C^1$ point. Hence $\mathrm{PGL}(\Omega)$ must preserve $\mathbb{RP}^{n-1}_\infty$ and is therefore a subgroup of $\mathrm{Aff}(\mathbb{R}^n)$. □

It is easy to check that $G(\psi)$ acts transitively on the interior of $\partial_\infty\Omega$, so that every point in the interior of $\partial_\infty\Omega$ is also a $C^1$ point.

There is a projectively invariant characterization of the center of the radial flow $\Phi^\psi$ that depends only on $\Omega = \Omega(\psi)$. When $0 < \mathbf{t} < n$ the center of the radial flow, $c = [e_{\mathbf{t}+1}] \in \mathbb{RP}^n$, is a vertex of the simplex $\partial_\infty\Omega = (\operatorname{cl}\Omega \setminus \Omega) \cong \Delta^{\mathbf{r}}$. In the case $n = 2$ and $\mathbf{t} = 1$ there are two vertices, as shown in Figure 1, and $c$ is the one at which $\partial(\operatorname{cl}\Omega)$ is not $C^1$. In general, $\partial(\operatorname{cl}\Omega)$ is not $C^1$ at any vertex of $\partial_\infty\Omega$, however there is one vertex of $\partial_\infty\Omega$ that is distinguished by being *less* $C^1$ than the others in the following sense. Recall (1.6) that $b_\psi$ is the basepoint of $\Omega$. A point $c \in \mathbb{RP}^n$ *has property* $\Delta(c)$ *for* $\Omega$ if:

- $c$ is a vertex of $\Delta^{\mathbf{r}} = \partial_\infty\Omega$
- $\forall\, p \neq c$ if $p$ is a vertex of $\Delta^{\mathbf{r}}$, and $H \cong \mathbb{RP}^2$ contains the points $c, p$ and $b_\psi$, then $c$ is not a $C^1$ point of $H \cap \Omega(\psi)$.

Thus $c$ has property $\Delta(c)$ if it is not a $C^1$ point in certain two dimensional slices of $\Omega$ that look like the middle domain of Figure 1.

**Lemma 1.11.** *Let $c \in \mathbb{RP}^n$ be the center of the radial flow $\Phi^\psi$ and $\Omega = \Omega(\psi)$, and $\mathbf{t} = \mathbf{t}_\psi$. When $0 < \mathbf{t} < n$ then $c$ is the unique point such that $\Delta(c)$ is satisfied. When $\mathbf{t} = 0$ then $\{c\} = \partial_\infty\Omega$. When $\mathbf{t} = n$ there is a unique $n$-simplex $\Delta \subset \mathbb{RP}^n$ that contains $\Omega$, and is not properly contained in any other such simplex; and $c$ is the unique vertex of $\Delta$ that is not in $\partial_\infty\Omega$.*

*Proof.* When $\mathbf{t} < n$ the center of $\Phi^\psi$ is the point $c = [x : z : y] = [0 : 1 : 0] = [e_{\mathbf{t}+1}] \in \mathbb{RP}^{n-1}_\infty$. When $\mathbf{t} = 0$ the result now follows.

Given $s \leq \mathbf{t}$ let $H \cong \mathbb{RP}^2$ be the projective plane that contains the vertices $[e_{\mathbf{t}+1}]$ and $[e_s]$ of $\Delta^{\mathbf{r}}$, and the basepoint $[b_\psi + e_{n+1}]$. The intersection of $H$ with the affine patch $\mathbb{R}^n$ is the affine subspace $V = \langle e_s, e_{\mathbf{t}+1}\rangle + b_\psi$. The restriction of $h = h_\psi$ to $V$ is

$$h(Xe_s + Ye_{\mathbf{t}+1} + b_\psi) = -Y - \psi_s \log(X+1)$$

The affine (hence projective) change of coordinates $(X, Y) = (x - 1, \psi_s y)$ maps the curve $V \cap \partial\Omega$ to $y = -\log x$. It follows that, on the curve $H \cap \partial\overline{\Omega}$, the point $[e_s]$ is $C^1$ and $[e_{r+1}]$ is not $C^1$ (see the middle domain in Figure 1). Since $1 \leq s \leq \mathbf{t}$ was arbitrary it follows that $\Delta([e_{\mathbf{t}+1}])$ is satisfied, and that $\Delta([e_s])$ is not satisfied.

When $\mathbf{t}_\psi = n$ then $c = [e_{n+1}] \in \mathbb{RP}^n$. Let $\Delta \subset \mathbb{RP}^n$ be the closure of $\mathbb{R}^n_+ \subset \mathbb{R}^n$. Then $\Delta = c * \partial_\infty\Omega$. It is easy to see that $\partial\Omega$ is tangent to $\mathbb{R}^n \cap \partial\Delta$ along $\partial(\partial_\infty\Omega)$ thus any simplex that contains $\Omega$ also contains $\Delta$. $\square$

It follows that the product structure defined by 1.8 depends only on the projective equivalence class of $\Omega(\psi)$.

**Corollary 1.12.** *If $A \in \mathrm{PGL}(\Omega(\psi))$ then $A^{-1} \cdot \Phi \cdot A = \Phi$ and $h_\psi \circ A = c \cdot h_\psi$ for some $c > 0$. Hence $A$ preserves the product structure on $\Omega(\psi)$.*

*Proof.* If $A \in \mathrm{PGL}(\Omega(\psi))$ then $A \in \mathrm{Aff}(\mathbb{R}^n)$ by 1.10 so $A$ preserves $\mathbb{RP}^{n-1}_\infty$ which is the stationary hyperplane of $\Phi^\psi$. By 1.11 $A$ also preserves the center of the radial flow. It follows that $A$ normalizes $\Phi$ and, by (7), it follows that $h_\psi \circ A = c \cdot h_\psi$ for some $c = c(A) \neq 0$. Moreover $c > 0$ since $A$ also preserves $\Omega$. Since $A$ preserves $\partial\Omega$, and the foliation by horospheres is the $\Phi$-orbit of $\partial\Omega$, it follows that $A$ preserves this foliation. Similarly $A$ preserves the $\Phi$-orbits of points, which are the flowlines. Thus $A$ preserves the product structure. $\square$

1.3. **The Translation Subgroup.** Recall the standard identification of the affine group $\mathrm{Aff}(\mathbb{R}^n)$ with the subgroup

$$\left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \;:\; A \in \mathrm{GL}(n, \mathbb{R}),\; v \in \mathbb{R}^n \right\} \subset \mathrm{GL}(n+1, \mathbb{R})$$

The affine action on $\mathbb{R}^n$ is realized by the embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $a \mapsto (a, 1)$.

Our next task is to define a subgroup of $G(\psi)$, called the *translation subgroup* $T(\psi) \cong \mathbb{R}^{n-1}$, that acts simply transitively on $\partial\Omega(\psi)$. We first define the *enlarged translation group* $T_{\mathbf{t}} \cong \mathbb{R}^n$ that acts simply transitively on $V_\psi = \mathbb{R}^{\mathbf{t}}_+ \times \mathbb{R}^{n-\mathbf{t}}$. Then $T(\psi) = \ker \psi_*$ for a certain homomorphism $\psi_* : T_{\mathbf{t}} \to \mathbb{R}$ derived from $\psi$. The enlarged translation group is the direct sum of the translation group and the radial flow: $T_{\mathbf{t}} = T(\psi) \oplus \Phi^\psi$.

The *enlarged translation group* $T_{\mathbf{t}}$ has Lie algebra $\mathfrak{t}_{\mathbf{t}}$ that is the image of the map $\Psi_{\mathbf{t}} : \mathbb{R}^n \to \mathfrak{gl}(n+1, \mathbb{R})$ given by

$$\Psi_{\mathbf{t}}(X, Z, Y) := \begin{pmatrix} \operatorname{Diag}(X) & 0 \\ 0 & \begin{pmatrix} 0 & Y^t & Z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \tag{14}$$

Here $X \in \mathbb{R}^{\mathbf{r}}$ and $Z \in \mathbb{R}$ and $Y \in \mathbb{R}^{\mathbf{u}}$, except when $\mathbf{u} = 0$ there is no $Y$, and when $\mathbf{t} = n$ there is no $Z$ and the bottom right block is $(0)$. It is easy to check that all Lie brackets in $\mathfrak{t}_{\mathbf{t}}$ are 0 and so $\mathfrak{t}_{\mathbf{t}}$ is an abelian Lie subalgebra, and $T_{\mathbf{t}} \cong \mathbb{R}^n$ as a Lie group. Define $m_{\mathbf{t}}(X, Z, Y) = \exp \Psi_{\mathbf{t}}(X, Z, Y)$ then $T_{\mathbf{t}}$ consists of all matrices

$$m_{\mathbf{t}}(X, Z, Y) = \begin{pmatrix} \exp \operatorname{Diag}(X) & 0 \\ 0 & \begin{pmatrix} 1 & Y^t & Z + \|Y\|^2/2 \\ 0 & I_{\mathbf{u}} & Y \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \tag{15}$$

**Definition 1.13.** *The* translation group $T := T(\psi)$ *is the kernel of the homomorphism* $\psi_* : T_{\mathbf{t}} \to \mathbb{R}$ *defined for* $\mathbf{t} < n$ *by* $\psi_*(m_{\mathbf{t}}(X, Z, Y)) = \psi^{\mathbf{t}}(X) + Z$, *and for* $\mathbf{t} = n$ *by* $\psi_*(m_n(X)) = (\sum \psi_i)^{-1} \psi(X)$.

For $\mathbf{t} < n$, the translation group $T(\psi)$ consists of the matrices $m_{\mathbf{t}}(X, Y, Z)$ given by (15) for which $Z = -\psi^{\mathbf{t}}(X)$. It will occasionally be convenient write the translation group as the image of a linear map, instead of as the kernel of a linear map. For $\mathbf{t} < n$ the translation group $T(\psi)$ is the image of $m^*_{\mathbf{t}} : \mathbb{R}^{\mathbf{r}} \times \mathbb{R}^{\mathbf{u}} \to \mathrm{GL}(n+1, \mathbb{R})$ given by

$$m^*_{\mathbf{t}}(X, Y) = \begin{pmatrix} \exp \operatorname{Diag}(X) & 0 \\ 0 & \begin{pmatrix} 1 & Y^t & \|Y\|^2/2 - \psi^{\mathbf{t}}(X) \\ 0 & I_{\mathbf{u}} & Y \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \tag{16}$$

and for $\mathbf{t} = n$ the translation group $T(\psi)$ is the image of $m^*_{\mathbf{t}} : \ker \psi \to \mathrm{GL}(n+1, \mathbb{R})$ by

$$m^*_{\mathbf{t}}(X) = \begin{pmatrix} \exp \operatorname{Diag}(X) & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

It is worth pointing out that with this formalism the case $\mathbf{t} = n - 1$ means $\mathbf{u} = 0$ and gives

$$m^*_{\mathbf{t}}(X, Y) = \begin{pmatrix} \exp \operatorname{Diag}(X) & 0 \\ 0 & \begin{pmatrix} 1 & -\psi^{\mathbf{t}}(X) \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{18}$$

**Lemma 1.14.** $T_{\mathbf{t}}$ *acts simply transitively on* $V_\psi = \mathbb{R}^{\mathbf{t}}_+ \times \mathbb{R}^{n-\mathbf{t}}$ *and*

(A) $h_\psi \circ m_{\mathbf{t}} = h_\psi - \psi_* \circ m_{\mathbf{t}}$
(B) $T(\psi)$ *is the the subgroup of* $T_{\mathbf{t}}$ *that preserves* $h_\psi$
(C) $T(\psi)$ *preserves the foliation of* $V_\psi$ *by horospheres*
(D) $T(\psi)$ *preserves the transverse foliation by flowlines.*

*Proof.* It is clear the action is simply transitive, and that (A) implies both (B) and (C), and that (D) holds. We first prove (A) in the case $\mathbf{t} < n$. From (11)

$$-h_\psi(x, z, y) = z + \psi^{\mathbf{t}}(\ell\text{og}\, x) - \|y\|^2/2$$

and

$$\begin{aligned} &m_{\mathbf{t}}(X, Z, Y)(x, z, y)^t \\ &= (\exp(X_1)x_1, \cdots, \exp(X_{\mathbf{r}})x_{\mathbf{r}}, z + Y \cdot y + Z + \|Y\|^2/2, Y_1 + y_1, \cdots, Y_{\mathbf{u}} + y_{\mathbf{u}})^t \end{aligned}$$

so

$$\begin{aligned}
&-h_\psi(m_{\mathbf{t}}(X,Z,Y)(x,z,y)^t)\\
&= z + Y\cdot y + Z + \|Y\|^2/2 + \psi^{\mathbf{t}}\,\ell\mathrm{og}(\exp(X_1)x_1,\cdots,\exp(X_{\mathbf{r}})x_{\mathbf{r}}) - \|Y+y\|^2/2\\
&= z + Z + \psi^{\mathbf{t}}(X+\ell\mathrm{og}\,x) - \|y\|^2/2\\
&= (Z+\psi^{\mathbf{t}}(X)) + \big(z+\psi^{\mathbf{t}}(\ell\mathrm{og}\,x) - \|y\|^2/2\big)\\
&= \psi_*(m_{\mathbf{t}}(X,Z,Y)) - h_\psi(x,z,y)
\end{aligned}$$

A similar but simpler argument applies when $\mathbf{t}=n$, by omitting the $Y$ and $Z$ coordinates. □

**Definition 1.15.** *The* parabolic subgroup *is the subgroup $P(\psi)\subset T(\psi)$ consisting of all unipotent elements, also called* parabolics.

It follows from the above that

$$P(\psi) = \{m^*_{\mathbf{t}}(0,Y) : Y\in\mathbb{R}^{\mathbf{u}}\}$$

Let $T_1,T_2\subset T$ be respectively the subgroup of diagonalizable elements, and the subgroup of elements for which every Jordan block has size at most 2. This description is invariant under conjugacy, and

$$\begin{aligned} T_1 &= \{m^*_{\mathbf{t}}(X,0) : X\in\ker\psi^{\mathbf{r}}\}\\ T_2 &= \{m^*_{\mathbf{t}}(X,0) : X\in\mathbb{R}^{\mathbf{r}}\}\end{aligned} \tag{19}$$

Then $T(\psi)=P(\psi)\oplus T_2$, and $T_1\subset T_2$ and has codimension 1 if $\mathbf{t}>0$. Elements of $T_2$ are called *hyperbolics*. Observe that

$$\mathbf{u}=\dim P(\psi) \qquad \mathbf{r}=\dim T_2 \qquad n-1=\dim T(\psi)=\mathbf{u}+\mathbf{r} \tag{20}$$

If $\mathbf{t}=n$ and $s>0$ then $T(\psi)=T(s\cdot\psi)$, and if $\mathbf{t}<n$ then $T(\psi)$ is conjugate to $T(s^2\cdot\psi)$, by the diagonal matrix

$$\mathrm{Diag}(I_{\mathbf{r}}, s, I_{\mathbf{u}}, s^{-1}) \tag{21}$$

This is the only time such groups are conjugate:

**Proposition 1.16.** *If $\psi,\psi'\in A$ then $T(\psi)$ is conjugate to $T(\psi')$ in $\mathrm{GL}(n+1,\mathbb{R})$ iff $\psi'=s\cdot\psi$ for some $s>0$.*

*Proof.* Suppose $T\subset\mathrm{GL}(n+1,\mathbb{R})$ is conjugate to at least one group in the set $\mathcal{G}=\{T(\psi):\psi\in A\}$. We will construct a codimension-1 subspace $K\subset\mathbb{R}^n$ such that if $T$ is conjugate to $T(\psi)$ then there is a linear map $L\in\mathrm{Isom}(\mathbb{R}^n)$, that permutes the coordinates, and $L(K)=\ker\psi$. Observe that $L$ permutes the coordinates of $\psi$. The result then follows.

Let $\{W_i\subset\mathbb{R}^{n+1} : 0\le i\le\tau\}$ be the set of 1-dimensional subspaces that are preserved by $T$. For each $W_i$ there is a homomorphism (weight) $\lambda_i : T\to\mathbb{R}$ such that $T(g)(w)=\lambda_i(g)w$ for all $g\in T$ and $w\in W_i$. Since $T\in\mathcal{G}$, after reordering the weights, $\lambda_0\equiv 1$. Moreover, if $T$ is conjugate to $T(\psi)$, then $\mathbf{t}(\psi)=\tau$.

Let $T_1\subset T$ to be the subgroup of diagonalizable elements. Define a linear map $\theta : T_1\to\mathbb{R}^\tau$ by $\theta(g)=\ell\mathrm{og}(\lambda_1(g),\cdots,\lambda_\tau(g))$ and set $K=\theta(T_1)\oplus\mathbb{R}^{n-\tau}$. Using (19) it is easy to check $K$ has the required property. □

**Proposition 1.17.** *The surfaces $\partial\Omega(\psi)$ and $\partial\Omega(\psi')$ are projectively equivalent if and only if $T(\psi)$ is conjugate to $T(\psi')$.*

*Proof.* Set $T=T(\psi)$, $T'=T(\psi')$, $\Omega=\Omega(\psi)$ and $\Omega'=\Omega(\psi')$. If $T'$ is conjugate to $T$ then $\psi'=s\cdot\psi$ by 1.16, and by (21) there is a diagonal matrix $g$ with $T'=g\cdot T\cdot g^{-1}$. Moreover $g$ fixes $\mathbb{R}^{\mathbf{t}}$ and so fixes the basepoint $b=b_\psi$. Now $\partial\Omega=T\cdot b$, and $b$ is also the basepoint for $\Omega'$, so $\partial\Omega'=T'\cdot b$, and it follows that $g(\partial\Omega)=\partial\Omega'$.

Conversely, suppose that $\partial\Omega'=g(\partial\Omega)$ for some projective map $g$. Then $g$ conjugates $\mathrm{PGL}(\Omega(\psi))$ to $\mathrm{PGL}(\Omega(\psi'))$. If $\psi\neq 0$ then $\mathrm{PGL}(\Omega(\psi))=G(\psi)$ by (2.7). Since $T(\psi)$ is the subgroup of $\mathrm{PGL}(\psi)$ with positive eigenvalues, if $\psi\neq 0\neq\psi'$ it follows that $g$ conjugates $T(\psi)$ to $T(\psi')$. If $\psi=0$ then $\Omega(\psi)$ is an ellipsoid, hence so is $\Omega(\psi')$, and the result follows. □

From (6) the radial flow $\Phi : \mathbb{R} \to \mathrm{GL}(n+1,\mathbb{R})$ is given by

$$\Phi(s) = \quad \overset{\mathbf{t} < n}{\exp \begin{pmatrix} 0_{\mathbf{t}\times\mathbf{t}} & 0 & 0 \\ 0 & 0 & -s \\ 0 & 0_{n-\mathbf{t}\times n-\mathbf{t}} & 0 \end{pmatrix}} \quad , \overset{\mathbf{t} = n}{\exp \begin{pmatrix} 0_{n-1} & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & 0 \end{pmatrix}} \tag{22}$$

Observe that the one-parameter group $\Phi$ is a subgroup of $T_\mathbf{t}$ and $T_\mathbf{t} = T \oplus \Phi$. In particular $T$ commutes with the radial flow, so $T$ sends radial flows lines to radial flow lines. Thus $T$ induces an action on the space of flowlines in $V_\psi$, and radial projection identifies this space with $U_\psi$. The action of $T$ on $U_\psi$ is affine, and given by omitting row and column $\mathbf{t}+1$ to give

$$\begin{pmatrix} \exp \mathrm{Diag}(X) & 0 & 0 \\ 0 & I_\mathbf{u} & Y \\ 0 & 0 & 1 \end{pmatrix} \quad Y \in \mathbb{R}^\mathbf{u}$$

with both $Y$ and $I_\mathbf{u}$ interpreted as empty for $\mathbf{u} = 0$. This happens when $\mathbf{r} = n-1$. In the case $\mathbf{t} < n$ then $X \in \mathbb{R}^\mathbf{r}$ and when $\mathbf{t} = n$ then $X \in \ker \psi^\mathbf{t}$. From this it follows that:

**Lemma 1.18.** *Under radial projection $\pi : V_\psi \to U_\psi$ the action of $T(\psi)$ on $V_\psi$ is semi-conjugate to a simply transitive affine action of $T$ on $U_\psi$. This action of $T(\psi)$ on $U$ is topologically conjugate to the action of $\mathbb{R}^{n-1}$ on itself by translation.*

*Proof.* The second conclusion follows by conjugating with the map $\mathbb{R}^\mathbf{r}_+ \times \mathbb{R}^\mathbf{u} \to \mathbb{R}^{n-1}$ given by $(x, y) \mapsto (\ell\mathrm{og}\, x, y)$. □

**Lemma 1.19.** *$T(\psi) \subset G(\psi)$ and $T(\psi)$ acts simply transitively on $\partial\Omega(\psi)$.*

*Proof.* By (16) and (15) $T(\psi)$ is the subgroup of $T_\mathbf{t}$ given by $Z = -\psi^\mathbf{t}(X)$. It follows from (1.14)(B) that $T(\psi)$ is the subgroup of $T_\mathbf{t}$ that preserves the horofunction, hence $T(\psi) \subset G(\psi)$. Simple transitivity on $\partial\Omega(\psi)$ also follows from (1.14). □

1.4. **The Orthogonal Subgroup $O(\psi)$.**

**Definition 1.20.** *$O(\psi)$ is the subgroup of $G(\psi)$ that fixes the basepoint $b_\psi$.*

When $\psi = 0$ (the case of a cusp in $\mathbb{H}^n$) then $O(\psi) \cong O(n-1)$ is the subgroup of $O(n) \subset \mathrm{Aff}(\mathbb{R}^n)$ that fixes $e_1$. At the other extreme, when $\mathbf{t} = n$ and all the coordinates of $\psi$ are distinct, then $O(\psi)$ is trivial. The general case is:

**Proposition 1.21.** *Suppose $\psi \in A$ has type $\mathbf{t} = \mathbf{t}(\psi)$. Let $e_1, \cdots, e_{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$ and $S(\psi) \subset \mathrm{GL}(\mathbf{t}, \mathbb{R})$ be the subgroup that permutes $\{e_1, \cdots, e_\mathbf{t}\}$ and preserves the vector $\sum_{i=1}^{\mathbf{t}} \psi_i e_i$. Then $O(\psi)$ is equal to the subgroup $O'(\psi) \subset \mathrm{Aff}(\mathbb{R}^n) \subset \mathrm{GL}(n+1,\mathbb{R})$ given by*

$$O'(\psi) = \quad \overset{\mathbf{t} < n-1}{\begin{pmatrix} S(\psi) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & O(\mathbf{u}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \qquad \overset{\mathbf{t} = n-1}{\begin{pmatrix} S(\psi) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \qquad \overset{\mathbf{t} = n}{\begin{pmatrix} S(\psi) & 0 \\ 0 & 1 \end{pmatrix}}$$

*Proof.* It is easy to check that $O'(\psi)$ fixes the basepoint and preserves the horofunction $h = h_\psi$ so $O'(\psi) \subset O(\psi)$. For the converse, $\mathrm{PGL}(\Omega) \subset \mathrm{Aff}(\mathbb{R}^n)$ so $O(\psi) \subset \mathrm{Aff}(\mathbb{R}^n)$. It is easy to check the result when $\mathbf{t} = n$, so assume $\mathbf{t} < n$. From (2) the horofunction $h : \mathbb{R}^\mathbf{t}_+ \times \mathbb{R}^{n-\mathbf{t}} \to \mathbb{R}$ is

$$h(x, z, y) = -\psi^\mathbf{t}(\ell\mathrm{og}(x)) - z + \|y\|^2/2$$

If $\tau \in O(\psi)$ then $h = h \circ \tau$. Given a unit vector $u = (x, z, y) \in \mathbb{R}^\mathbf{r} \times \mathbb{R} \times \mathbb{R}^\mathbf{u}$ there is an affine line $\ell_u$ in $\mathbb{R}^n$ containing the basepoint that is the image of the map $\gamma_u(t) = b + t \cdot u$. The horofunction is only defined on the subset of this line in $V_\psi$. This gives a function $f = f_u : I_u \to \mathbb{R}$ defined on some maximal interval $I_u \subset \mathbb{R}$ by

$$f_u(t) := h \circ \gamma_u(t) = -tz + t^2\|y\|^2/2 - \sum_{i=1}^{\mathbf{t}} \psi_i \log(1 + tx_i)$$

here $x = (x_1, \cdots, x_{\mathbf{t}})$. We distinguish two classes of line $\ell_u$ according to the behaviour of $f$. The function $f$ is defined on $I_u = \mathbb{R}$ iff $x = 0$, and it is defined on $[0, \infty) \subset I_u$ and grows logarithmically as $t \to \infty$ iff $z = y = 0$ and each coordinate of $x$ is non-negative. Since $\tau$ is affine, it preserves the smallest affine subspace that contains all the lines of a given type. Since $\tau$ fixes the basepoint $b$, and preserves the type of lines, $\tau$ preserves the affine subspaces $P = b + \langle e_1, \cdots, e_{\mathbf{t}}\rangle$ and $Q = b + \langle e_{\mathbf{t}+1}, \cdots, e_n\rangle$. Notice that $P = \langle e_1, \cdots, e_{\mathbf{t}}\rangle$.

The ideal boundary $\partial_\infty \Omega$ is a simplex that is preserved by $\tau$, so $\tau$ permutes the vertices $\{[e_i] : 1 \le i \le \mathbf{t}+1\}$ of this simplex. On $P$ we have $h(x_1 e_1 + \cdots x_{\mathbf{t}} e_{\mathbf{t}}) = -\sum \psi_i \log x_i$. Since $\tau|P$ preserves $h$, it follows that must $\tau$ preserve $\psi|P$. Thus the first $\mathbf{t}$ columns of $\tau$ are as shown in $O'(\psi)$.

The only $u$ for which $f_u$ is linear is when $u = \pm e_{\mathbf{t}+1}$. Since $\tau$ fixes the basepoint and preserves $h$ it follows that $\tau$ maps the line $\ell_{e_{\mathbf{t}+1}}$ to itself by the identity. This gives column $(\mathbf{t}+1)$ in $O'(\psi)$. Finally $f_u$ is a quadratic polynomial with a minimum of 0 at the basepoint exactly when $x = 0$ and $z = 0$ so $u \in \langle e_{\mathbf{t}+2}, \cdots, e_n\rangle$. On this subspace $h(y_1 e_{\mathbf{t}+2} + \cdots + y_{\mathbf{u}} e_n) = \|y\|^2/2$. Since $\tau$ preserves this function, the columns $\mathbf{t}+2$ to $n$ of $\tau$ in $O'(\psi)$ (those that contain $O(\mathbf{u})$) are as shown. Since $\tau$ is affine and fixes the basepoint the last column is as shown in $O(\psi')$. The result now follows. □

**Corollary 1.22.** *Given $\psi \in A$ then $G(\psi) = T(\psi) \rtimes O(\psi)$ is the internal semidirect product of the subgroup $O(\psi)$ defined in (1.21), and the subgroup $T(\psi)$ that is the image of $m^*_{\mathbf{t}}$ defined in (16) and (17).*

**Corollary 1.23.** *Every parabolic in $G(\psi) \subset \mathrm{GL}(n+1, \mathbb{R})$ is conjugate into $O(n, 1)$.*

*Proof.* An element $A \in G(\psi)$ is parabolic if all eigenvalues of $A$ have modulus 1 and $A \notin O(n+1)$. Now $A = B \cdot C$ with $B \in T(\psi)$ and $C \in O(\psi)$. Thus $B \in P(\psi)$ is parabolic. It is then easily checked that $A \in O(n, 1)$. □

1.5. **Domains preserved by $T(\psi)$.** In general $\Omega(\psi)$ is not the only properly convex domain preserved by $T(\psi)$. However, as we will see, when $\mathbf{t} < n$ then any other open $T(\psi)$-invariant properly convex domain is affinely equivalent to the interior of $\Omega(\psi)$. In the diagonalizable case, $\mathbf{t} = n$, there are additional *extended domains*, described in more detail below. Although the same group $T(\psi)$ preserves different properly convex sets, every generalized cusp covered by an extended domain is equivalent to a $\psi$-cusp, see (1.29).

Since the radial flow $\Phi$ centralizes $G(\psi)$ it follows that $G(\psi)$ preserves the properly convex set $\Omega_t(\psi) := \Phi_t(\Omega(\psi))$. As we have seen, these sets are nested, with $\Omega_t(\psi) \subset \Omega_s(\psi)$ if and only if $t \le s$. However there is another way to produce invariant sets.

**Definition 1.24.** *$\mathcal{E}(\mathbf{t}) \subset \mathrm{GL}(n+1, \mathbb{R})$ is the the group of all diagonal matrices $\epsilon$ with $\epsilon_{i,i} = \pm 1$ for $i \le \mathbf{t} = \mathbf{t}(\psi)$ and $\epsilon_{i,i} = 1$ for $i > \mathbf{t}$.*

**Lemma 1.25.** *The subgroup $\mathcal{E}(\mathbf{t}, \psi) \subset \mathcal{E}(\mathbf{t})$ that normalizes $G(\psi)$ consists of all $\epsilon$ such that $\epsilon_{i,i} = \epsilon_{j,j}$ whenever $\psi_i = \psi_j$. Furthermore, $\mathcal{E}(\mathbf{t}, \psi)$ also centralizes $G(\psi)$.*

*Proof.* In this proof, with reference to 1.21, we regard $S(\psi)$ and $O(\mathbf{u})$ as subgroups of $\mathrm{Aff}(\mathbb{R}^n) \subset \mathrm{GL}(n+1, \mathbb{R})$ acting on $\mathbb{R}^n$. It is easy to check that $\mathcal{E}(\mathbf{t})$ centralizes, $T(\psi)$ and $O(\mathbf{u})$. An element $A \in S(\psi)$ permutes the $x_i$ coordinates for $1 \le i \le \mathbf{t}$, and $\epsilon \in \mathcal{E}(\mathbf{t})$ assigns a sign to each of these coordinates so that

$$(\epsilon A \epsilon)_{j,k} = \epsilon_{j,j} \epsilon_{k,k} A_{j,k} \tag{23}$$

is a signed permutation. Thus $\epsilon \in \mathcal{E}(\mathbf{t}, \psi)$ if and only if $\epsilon_{i,i} = \epsilon_{j,j}$ whenever $\psi_i = \psi_j$. □

For $1 \le i \le \mathbf{t}$ let $H_i \subset \mathbb{R}^n$ be the hyperplane $x_i = 0$. For $\epsilon \in \mathcal{E}(\mathbf{t})$ define $V_\psi^\epsilon = \epsilon(V_\psi)$. The $2^{\mathbf{t}}$ sets $V_\psi^\epsilon \subset \mathbb{R}^n$ are pairwise disjoint, and are the components of $X := \mathbb{RP}^n \setminus (\mathbb{RP}^{n-1}_\infty \cup_i H_i)$. It is easy to check that:

**Lemma 1.26.** *$\mathcal{E}(\mathbf{t})$ acts transitively on the components of $X$ and $T_{\mathbf{t}} \oplus \mathcal{E}(\mathbf{t})$ acts transitively on $X$.*

It follows that the only projective hyperplanes that are preserved by $T(\psi)$ are $\mathbb{RP}^{n-1}_\infty$ and the hyperplanes $H_i$ for $1 \le i \le \mathbf{t}$.

Since $\epsilon$ centralizes $T(\psi)$ it follows that $V_\psi^\epsilon$ is preserved by $T(\psi)$ and that $\Omega^\epsilon(\psi) := \epsilon(\Omega(\psi))$ is preserved by $T(\psi)$. Thus $\epsilon$ is an affine map that conjugates the action of $T(\psi)$ on $\Omega(\psi)$ to the action of $T(\psi)$ on $\Omega^\epsilon(\psi)$.

Define $a : \mathbb{R}^n \to \mathbb{R}^n$ by $a(x_1, \cdots, a_n) = (|a_1|, \cdots, |a_n|)$. The extended horofunction is $h_\psi^e := h_\psi \circ a : X \to \mathbb{R}$, and agrees with $h_\psi$ on $V_\psi$, and is preserved by $G(\psi)$. The level sets of $h_\psi^e | V_\psi^\epsilon$ are called *generalized horospheres* and are the images under $\epsilon$ of the horospheres in $V_\psi$.

A properly convex set $U$ that is preserved by the action of $T(\psi)$ is called *reducible* if there is a projective hyperplane $H$ that is preserved by $T(\psi)$ and $H \cap U \neq \emptyset$, otherwise $U$ is *irreducible.* If such $H$ exists then $H$ separates $U$ into two properly convex sets that are preserved by $T(\psi)$. It follows from the above that, if $U$ is irreducible, then $U \subset V_\psi^\epsilon$ for some $\epsilon \in \mathcal{E}(\mathbf{t})$. A *standard $\psi$-domain* is a subset $g(\Omega(\psi)) \subset \mathbb{RP}^n$ where $g \in \mathcal{E}(\mathbf{t}) \oplus \Phi^\psi$. Since $g$ centralizes $T(\psi)$, every standard $\psi$-domain is preserved by $T(\psi)$.

**Lemma 1.27.** *If $U \subset \mathbb{RP}^n$ is an irreducible properly convex set that is preserved by $T(\psi)$, and $U \cong \partial U \times [0, \infty)$, then $U$ is a standard $\psi$-domain. Moreover there is a unique $g \in \mathcal{E}(\mathbf{t}) \oplus \Phi^\psi$ such that $U = g(\Omega(\psi))$.*

*Proof.* There is unique $\epsilon$ such that $U \subset V_\psi^\epsilon$. If $x \in \partial U$ then there is $h \in T_{\mathbf{t}}$ such that $h \circ \epsilon(x) \in \partial\Omega(\psi)$. Since $T(\psi)$ acts simply transitively on $\partial\Omega$, and is the subgroup of $T_{\mathbf{t}} \oplus \Phi^\psi$ that preserves $\partial\Omega$, it follows there is a unique $h \in \Phi^\psi$ with this property, and $g = h \circ \epsilon$. □

Given a subset $Y \subset \mathbb{R}^n$ recall that the ideal boundary of $Y$ is $\partial_\infty Y = \mathrm{cl}(Y) \cap \mathbb{RP}^{n-1}_\infty$ where the closure is taken in $\mathbb{RP}^n$. When $\mathbf{t} = \mathbf{t}(\psi) = n$ the element $-\mathbf{I} \in \mathcal{E}(\mathbf{t})$ is defined to be the affine map of $\mathbb{R}^n$ given by $x \mapsto -x$. In the above notation $-\mathbf{I}_{i,i} = -1$ for all $1 \le i \le n$. In what follows, let $\Omega$ be an irreducible open properly convex set that is preserved by $T(\psi)$, as determined by 1.27. Observe that when $\mathbf{t} = n$ that

$$\Delta^{n-1} \cong \partial_\infty \Omega = \partial_\infty(\Phi_t \Omega) = \partial_\infty(\tau(\Omega))$$

Set $g_t = -\mathbf{I} \circ \Phi_t$ then $g_t\Omega$ is an irreducible $T(\psi)$-invariant domain disjoint from $\Omega$ with $\partial_\infty(g_t\Omega) = \partial_\infty(\Omega)$. As $t \to -\infty$ the domain $g_t\Omega$ converges (in the Hausdorff topology) to an open simplex disjoint from $\Omega$ with one face equal to $\partial_\infty(\Omega)$. This simplex will be referred to as $g_{-\infty}\Omega$. Let

$$U_t := \Omega \sqcup \partial_\infty \Omega \sqcup g_t \Omega$$

then for $t \in \mathbb{R} \cup \{-\infty\}$, $U_t$ is a connected domain in $\mathbb{RP}^n$. It is convex, since it is the union of two convex sets which intersect in $\Delta^{n-1}$. Since $\Phi^\psi \oplus \mathcal{E}(\mathbf{t})$ commutes with $G(\psi)$ it follows that $U_t$ is preserved by $G(\psi)$. The domain, $U_t$, as well as its image under any element of $\Phi^\psi \oplus \mathcal{E}(\mathbf{t})$, is called an *extended domain.*

**Proposition 1.28.** *If $U \subset \mathbb{RP}^n$ is an open properly convex set that is preserved by $T(\psi)$, and $U \cong \partial U \times [0, \infty)$, then either $U$ is a standard $\psi$-domain, or else $\mathbf{t} = n$ and $U$ is an extended domain.*

*Proof.* Since $T(\psi)$ preserves $\mathbb{R}^n$ it preserves each component of $U \cap \mathbb{R}^n$. These components are properly convex so by 1.27 each of these components is $g(\Omega(\psi))$ for some $g \in \mathcal{E}(\mathbf{t}) \oplus \Phi^\psi$. It suffices to show that if there is more than one component, then $\mathbf{t} = n$ and there are exactly two components.

If there is more than one component, then since the union is connected, the closure in $\mathbb{RP}^n$ of two distinct components must intersect. By applying an element of $\mathcal{E}(\mathbf{t}) \oplus \Phi^\psi$ we may assume one component is $\Omega := \Omega(\psi)$ and the other is $g\Omega$ for some $g = \epsilon \circ \Phi_t \in \mathcal{E}(\mathbf{t}) \oplus \Phi^\psi$. This intersection is contained $\partial_\infty\Omega = \Delta^{\mathbf{r}}$. Since the union is open and convex it follows that the intersection has dimension $n-1$ so $\mathbf{r} = n-1$ and $\mathbf{t} = n-1$ or $\mathbf{t} = n$.

We claim that if $\mathbf{t} = n-1$ then the extended domain is not convex. This is because the intersection of $\Omega(\psi)$ with the 2 dimensional affine subspace given by $x_i = 1$ for $i < \mathbf{t}$ and $x_i = 0$ for $i \geq \mathbf{t}+2$ is $x_{\mathbf{t}+1} = -\psi_{\mathbf{t}} \log x_{\mathbf{t}}$ which looks like $y = -\log x$ shown in Figure 1. In this case it is clear that an extended domain is not convex at the right hand fixed point. If $\mathbf{t} = n$ then $g$ must preserve $\partial_\infty\Omega$ which implies $\epsilon = -\mathbf{I}$ completing the proof. □

**Corollary 1.29.** *If $C$ is a generalized cusp with holonomy $\Gamma \subset G(\psi)$ then $C$ is equivalent to a $\psi$-cusp.*

*Proof.* We have $C = U'/\Gamma$ for some $U \cong \partial U \times [0,\infty)$ that is preserved by $\Gamma$. By (4.3) there is a $G(\psi)$-invariant subset $U \subset U'$ and $U/\Gamma$ is equivalent to $C$, so $U \cong \partial U \times [0,\infty)$. By (1.28) either $U$ is a standard $\psi$-domain or else an extended domain. If $U$ is standard, then $C$ is projectively equivalent to a $\psi$-cusp. Otherwise, if $U$ is extended, then $U$ contains a standard domain, $\Omega$, that is $G(\psi)$ invariant, and $C$ is equivalent to the $\psi$-cusp $\Omega/\Gamma$. □

If $C'$ is a generalized cusp that properly contains another generalized cusp $C$, and they have the same boundary, then $\mathbf{t} = n$ and the holonomy is diagonalizable. Equivalent cusps are not always projectively equivalent after removing suitable collars of the boundary. If $\mathbf{t} = n-1$, then $\partial_\infty\Omega(\psi) \cong \Delta^{n-1}$, but there is no larger $G(\psi)$-invariant domain that contains $\partial_\infty\Omega(\psi)$ in its interior.

1.6. **Hex geometry.** In this section $\Delta$ denotes the interior of a simplex. Let $v_0, \cdots, v_{\mathbf{r}} \in \mathbb{R}^{\mathbf{r}+1}$ be a basis, then $[v_i]$ are the vertices of an $r$-simplex $\Delta$. The identity component $D^{\mathbf{r}} \subset \mathrm{PGL}(\Delta)$ is the projectivization of the positive diagonal subgroup, and $\mathrm{PGL}(\Delta) = D^{\mathbf{r}} \rtimes S_{\mathbf{r}+1}$ is an internal semidirect product, where $S_{\mathbf{r}+1}$ is the group of coordinate permutations.

**Definition 1.30.** *The* $\mathbf{r}$-dimensional Hex geometry *is* $\mathbb{H}\mathrm{ex}^{\mathbf{r}} = (\mathrm{PGL}(\Delta), \Delta)$ *where* $\Delta \subset \mathbb{R}\mathrm{P}^{\mathbf{r}}$ *is the interior of an* $\mathbf{r}$*-simplex.*

Let $\{u_i : 0 \leq i \leq \mathbf{r}\} \subset \mathbb{R}^{\mathbf{r}+1}$ be a spanning set of unit vectors with $\sum u_i = 0$. The map $[\sum x_i v_i] \mapsto \sum(\log|x_i|)u_i$ is an isometry taking $(\Delta, d_\Delta)$ to a certain normed vector space $(\mathbb{R}^{\mathbf{r}}, \|\cdot\|)$. The name *Hex geometry* comes from the fact that when $\mathbf{r} = 2$, the unit ball is a regular hexagon. It follows that $(\mathrm{Isom}(\Delta), \Delta)$ is isomorphic to a subgeometry of Euclidean geometry. Moreover $\mathrm{PGL}(\Delta)$ is an index-2 subgroup of $\mathrm{Isom}(\Delta, d_\Delta)$. This is all due to de la Harpe [16].

Recall that $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_{\mathbf{r}} > 0$ and $\psi_i = 0$ for all $\mathbf{r} < i \leq n$. Recall (see Proposition 1.21) that $S(\psi) \subset \mathrm{PGL}(\Delta^{\mathbf{r}})$ is the group of coordinate permutations that preserve the basepoint $b_\psi$. It is clear that $S(\psi)$ is isomorphic to a product of symmetric groups $\prod S_{k_j}$. There is one factor isomorphic to the symmetric group $S_k$ for each maximal consecutive sequence $\psi_i = \psi_{i+i} = \cdots = \psi_{i+k-1}$ of non-zero coordinates in $\psi$.

A *morphisms* between two geometries $(G,X)$ and $(H,Y)$ is a homomorphism $\rho : G \to H$, and an immersion $f : X \to Y$, such that

$$\forall g \in G, x \in X \quad f(g\cdot x) = \rho(g)\cdot(fx)$$

If $f$ and $\rho$ are both inclusions we say $(G,X)$ is a *subgeometry* of $(H,Y)$.

**Definition 1.31.** *The subgeometry* $(D^{\mathbf{r}} \rtimes S(\psi), \Delta^{\mathbf{r}})$ *of* $\mathbb{H}\mathrm{ex}^{\mathbf{r}}$ *is called* $\mathbb{H}\mathrm{ex}^{\mathbf{r}}(\psi)$

When $X$ is a metric space we use the term $X$ *geometry* for $(Isom(X), X)$*-geometry.* For example $\mathbb{H}^n$ is hyperbolic geometry in dimension $n$. The *product geometry* of $(G,X)$ and $(H,Y)$ is $(G \times H, X \times Y)$ with the product action. Given a space $X$ the *trivial geometry on* $X$ is $\mathbb{1}(X) = (G,X)$ with $|G| = 1$. *Horoball geometry* is the subgeometry $\mathrm{Horo}^{\mathbf{u}+1} = (\mathcal{B}, G)$ of $\mathbb{H}^{\mathbf{u}+1}$ where $\mathcal{B} \subset \mathbb{H}^{p+1}$ is a horoball, and $G \subset \mathrm{Isom}(\mathbb{H}^{\mathbf{u}+1})$ is the subgroup that preserves $\mathcal{B}$. In the following theorem interpret both $\mathbb{H}\mathrm{ex}^0(\psi)$, $\mathbb{E}^0$ as the trivial geometry on one point, and $\mathrm{Horo}^0$ as the trivial geometry on $[0,\infty)$.

**Theorem 1.32.** *$(G(\psi), \Omega(\psi))$ is isomorphic to the product geometry $\mathbb{H}\mathrm{ex}^{\mathbf{r}}(\psi) \times \mathrm{Horo}^{\mathbf{u}+1}$ and also to $\mathbb{H}\mathrm{ex}^{\mathbf{r}}(\psi) \times \mathbb{E}^{\mathbf{u}} \times \mathbb{1}([0, \infty))$.*

*Proof.* In what follows most functions and sets should be decorated with $\psi$. This is often omitted for clarity. First assume $\mathbf{t} < n$. Define $\theta : V_\psi \to V_\psi$ by

$$\theta(x, z, y) = (x, z + \psi^{\mathbf{t}}(\ell\mathrm{og}\, x), y)$$

Since $\partial\Omega$ is the graph $z = f(x, y)$ it follows that $\theta(\partial\Omega)$ is the graph of $z = f(x, y) + \psi^{\mathbf{t}}(\ell\mathrm{og}\, x)$. Using (10) this simplifies to $z = \|y\|^2/2$ when $\mathbf{u} > 0$ and to $z = 0$ when $\mathbf{u} = 0$. In each case $\theta(\Omega) = \Delta \times B$ where

$$\Delta := \mathbb{R}^{\mathbf{r}}_+, \qquad B := \{(z, y) \in \mathbb{R} \times \mathbb{R}^{\mathbf{u}} : z \geq \|y\|^2/2\ \}$$

and $G^\theta := \theta \circ G \circ \theta^{-1}$ acts on this set. This gives an isomorphism of geometries $(G, \Omega) \to (G^\theta, \Delta \times B)$.

The subgroup $T^\theta := \theta \circ T(\psi) \circ \theta^{-1}$ of $G^\theta$ acts on $\theta(\Omega)$ by the affine transformations of $\mathbb{R}^n$

$$T^\theta = \begin{pmatrix} \exp \mathrm{Diag}(X) & 0 \\ 0 & \begin{pmatrix} 1 & Y^t & \|Y\|^2/2 \\ 0 & I_{\mathbf{u}} & Y \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \qquad X \in \mathbb{R}^{\mathbf{r}},\ Y \in \mathbb{R}^{\mathbf{u}}$$

The subgroup $(O(\psi))^\theta \subset G^\theta$ acts affinely on $\mathbb{R}^{n+1}$ as before in 1.21. By (2.8) $G(\psi) = T(\psi) \rtimes O(\psi)$ and it follows that the action of $G^\theta = G_\Delta \times G_B$ is affine and splits into the direct sum of actions on $\mathbb{R}^{\mathbf{r}} \oplus \mathbb{R}^{\mathbf{u}+1}$ given by

$$G_\Delta = D^{\mathbf{r}} \rtimes S(\psi), \qquad G_B := \begin{pmatrix} 1 & Y^t & \|Y\|^2/2 \\ 0 & O(\mathbf{u}) & Y \\ 0 & 0 & 1 \end{pmatrix}$$

Then $(B, G_B) \cong \mathrm{Horo}^{\mathbf{u}+1}$, which is obviously isomorphic to $\mathbb{E}^{\mathbf{u}} \times \mathbb{1}([0, \infty))$.

For $\mathbf{t} = n$ the set $\Omega(\psi)$ has a product structure coming from the horospheres, and the radial flow. The group $G(\psi)$ acts trivially on the radial flow factor, and projection along the radial flow gives a $G(\psi)$-equivariant diffeomorphism from each horosphere to $\partial_\infty\Omega(\psi) \cong \Delta^{n-1}$. $\square$

**Corollary 1.33.** *$(G(\psi), \Omega(\psi))$ is isomorphic to a subgeometry of Euclidean geometry.*

*Proof.* Each of the factors in (1.32) is isomorphic to a subgeometry of Euclidean geometry. $\square$

The next section gives a particular isomorphism. It follows from this lemma that $T(\psi)$ can be characterized more abstractly as the subset of all elements in $G(\psi)$ such that all the eigenvalues are positive. As a result $T(\psi)$ is a characteristic subgroup of $G(\psi)$. In the terminology of [10], $T(\psi)$ is an *e-group*, see (4.7).

## 2. Euclidean Structure

This section is devoted to showing a generalized cusp has an underlying Euclidean structure with flat (totally geodesic) boundary. In fact there is a 1-parameter family of such metrics. This is used to provide a natural map from a generalized cusp to a standard cusp, modelled on $\mathbb{H}^n$. A metric is first defined on $V_\psi \subset \mathbb{R}^n$ in terms of a horofunction, and may be viewed as a kind of modified *Hessian metric* [22].

**Theorem 2.1.** *Suppose $\psi \in A$ as in (1), let $h = h_\psi$ be the horofunction on $V = V_\psi$. Given $q \in V$ let $\mathcal{H}$ be the horosphere containing $q$ and $\pi : V \to \mathcal{H}$ be projection along the radial flow. Then there is a quadratic form $\beta = (\mathrm{D}^2 h \circ \mathrm{D}\, \pi) + (\mathrm{D}\, h)^2$ on $T_qV$ that defines a Riemannian metric on $V$ and:*

- (a) *There is an isometry $F : (V, \beta) \to (\mathbb{R}^n, \|\cdot\|^2)$ where $\|\cdot\|^2$ is the standard Euclidean metric.*
- (b) *$F(\Omega(\psi)) = \mathbb{R}^{n-1} \times (-\infty, 0]$.*
- (c) *The horofunction is the $n$'th coordinate of $F$ i.e. $h(p) = F_n(p)$.*
- (d) *The action of $G(\psi)$ on $V$ is by isometries of this metric.*
- (e) *The radial flow $\Phi_t$ on $V$ is conjugated by $F$ to $x \mapsto x + t \cdot e_n$*
- (f) *The radial flow acts on $V$ by isometries.*

(g) *Radial flow lines are orthogonal to horospheres.*
(h) *The action of $T(\psi)$ on $\partial\Omega(\psi)$ is conjugated by $F$ to the group of translations of $x_n = 0$.*

*Proof.* Cleary $\beta$ is symmetric and we first verify that it is also positive definite. Given $q \in V$ let $\mathcal{H} \subset V$ be the horosphere containing $q$. The radial flow line through $q$ is $f : \mathbb{R} \to V$, given by $f(t) = \Phi_t(q)$, and is transverse to $\mathcal{H}$. Thus $T_qV = T_q\mathcal{H} \oplus \mathbb{R}.v$ where $v = f'(0)$ is tangent to the radial flow at $q$. If $w \in T_qV$ then $w = a + t \cdot v$ for some $a \in T_q\mathcal{H}$ and $\mathrm{D}\,\pi(w) = a$.

Observe that $T_q\mathcal{H} = \ker \mathrm{D}_q\, h$. From (7) $\mathrm{D}\, h_q(v) = 1$ so $(\mathrm{D}\, h)^2(a + t \cdot v) = t^2$. Thus

$$\beta(w) = (\mathrm{D}^2\, h)(a) + t^2$$

and it suffices to check that $\mathrm{D}^2\, h$ is positive definite on $\ker \mathrm{D}\, h$.

When $\mathbf{r} = n$

$$\mathrm{D}^2\, h = \left(\sum_{i=1}^{n} \psi_i\right)^{-1} \sum_{i=1}^{n} \psi_i x_i^{-2} dx_i^2 \tag{24}$$

and since all $\psi_i > 0$, and $x_i > 0$ on $V$, it follows that $\mathrm{D}^2\, h$ is positive definite on $T_qV$, and so is positive definite on $\ker \mathrm{D}\, h$.

When $\mathbf{r} < n$

$$\mathrm{D}\, h = -dx_{\mathbf{r}+1} - \sum_{i=1}^{\mathbf{r}} \psi_i x_i^{-1} dx_i + \sum_{i=\mathbf{r}+2}^{n} x_i dx_i \tag{25}$$

$$\mathrm{D}^2\, h = \sum_{i=1}^{\mathbf{r}} \psi_i x_i^{-2} dx_i^2 + \sum_{i=\mathbf{r}+2}^{n} dx_i^2 \tag{26}$$

In this case, by (6), the radial flow is vertical translation and $v = -\partial/\partial x_{\mathbf{r}+1}$. Thus $\mathrm{D}^2\, h$ is positive semi-definite and vanishes only in the $v$-direction, hence it is positive definite on $\ker \mathrm{D}\, h$.

Thus $\beta$ is a Riemannian metric on $V$. Since $G(\psi)$ preserves $h_\psi$, it acts by isometries of $\beta$ proving (d). The radial flow preserves $h$ up to adding a constant, and so preserves $Dh$ and $D^2h$, and is therefore also an isometry of $\beta$ proving (f). Hence the extended translation group $T_{\mathbf{r}}$ acts by isometries of $\beta$. Since this action is simply transitive we may identify $T_{\mathbf{r}}$ with $V$. Since $T_{\mathbf{r}} \cong \mathbb{R}^n$ as a Lie group, it follows it follows that this metric is flat, so there is an isometry $F : (V, \beta) \to (\mathbb{R}^n, \|\cdot\|^2)$, proving (a). We use $(u_1, \cdots, u_n)$ as the coordinates of a point in the codomain $\mathbb{R}^n$.

Each horosphere in $V$ is the orbit of a point under the subgroup $\mathbb{R}^{n-1} \cong T(\psi) \subset T_{\mathbf{r}}$ therefore the horospheres are identified with parallel hyperplanes in $\mathbb{R}^n$. We can choose the isometry $F$ so that the horosphere $\partial\Omega$ is sent to the subspace $u_n = 0$, and so that $\Omega$ is identified with the half space $u_n \leq 0$.

Observe that $T_qV = T_q\mathcal{H} \oplus \mathbb{R} \cdot v$, and $\beta$ is the sum of two quadratic forms, each of which vanishes on one summand and is positive definite on the other. It follows the two summands are orthogonal with respect to $\beta$, which proves (g).

From (g) it follows that flow lines are lines parallel to the $u_n$ direction. Along a flow line $\beta$ is $(\mathrm{D}\, h)^2$ so the distance between $x$ and $\Phi_t(x)$ is $|h(\Phi_t(x)) - h(x)| = |t|$ by (7). Moroever since $\Omega$ is $u_n \leq 0$ the radial flow $\Phi_t$ is conjugated by $F$ to $u \mapsto u + t \cdot e_n$. This proves (b), (c), (e) and (h). ☐

The metric defined on $\Omega(\psi)$ in 2.1 depends on the horofunction $h$. There is a projective transformation $\Omega(\psi) \to \Omega(c \cdot \psi)$ for $c > 0$ (see (21) and the preceding discussion). This map is not an isometry between the corresponding metrics. Thus the metric on $\Omega(\psi)$ is not an invariant of the projective class of $\Omega(\psi)$. The horofunction on $\Omega(c \cdot \psi)$ pullsback to a new horofunction $c \cdot h_\psi$ on $\Omega(\psi)$. This results in replacing $\beta = (\mathrm{D}^2\, h|\ker \mathrm{D}\, h) + (\mathrm{D}\, h)^2$ by

$$\beta^c = (\mathrm{D}^2(c \cdot h)|\ker \mathrm{D}\, c \cdot h) + (\mathrm{D}\, c \cdot h)^2 = c(\mathrm{D}^2\, h|\ker \mathrm{D}\, h) + c^2(\mathrm{D}\, h)^2 \qquad c > 0 \tag{27}$$

**Definition 2.2.** *$\beta^c$ is called a* horofunction metric *on $V_\psi$. The restriction of $\beta^c$ to $\Omega(\psi)$ is called a* horofunction metric *on $\Omega(\psi)$*

**Definition 2.3.** *Suppose $(A, ds_A)$ is a Euclidean manifold and $(\mathbb{R}, dt)$ is a complete Riemannian metric on $\mathbb{R}$. The metric $ds^2 = ds_A^2 + dt^2$ on $A \times \mathbb{R}$ is called a* product Euclidean structure. *Given $c > 0$ there is another product Euclidean metric on $A \times \mathbb{R}$ given by $c.ds_A^2 + c^2 dt^2$ which we call a* horoscaling *of the original metric.*

The point of the next result is that Euclidean metric on $\Omega(\psi)$ is unique up to horoscaling.

**Lemma 2.4.** *Given $U \subset \mathbb{R}\mathrm{P}^n$, let $\mathcal{S}$ be the set of Riemannian metrics $ds^2$ on $U$ such that there is a projective transformation that is an isometry $(U, ds^2) \to (\Omega(\psi), \beta)$ where $\beta$ is a horofunction metric. If $\mathcal{S} \neq \emptyset$ then $\mathcal{S}$ consists of all horoscalings of some (any) element of $\mathcal{S}$.*

*Proof.* Suppose $ds_1^2, ds_2^2 \in \mathcal{S}$. Then there are $P_1, P_2 \in \mathrm{PGL}(n+1, \mathbb{R})$, and $\psi^1, \psi^2 \in A$, and $P_i : U \to \Omega_i$ such that $ds_i^2$ is the pullback, using $P_i$, of $\beta_i$ on $\Omega_i := \Omega(\psi^i)$. Then $P = P_2 \circ P_1^{-1} : \Omega_1 \to \Omega_2$. Hence $\psi^2 = s \cdot \psi^1$ for some $s > 0$. It then follows the pullback of $\beta_1$ to $\Omega_2$ is a horoscaling of $\beta_1$. $\square$

**Remark 2.5.** *A horofunction metric is* never *the same as a Hilbert metric.*

If $C = \Omega(\psi)/\Gamma$ is a generalized cusp then a horofunction metric on $\Omega(\psi)$ covers a Riemannian metric on $C$ which is called a *horofunction metric on $C$*.

**Corollary 2.6.** *A generalized cusp $C$ with a horofunction metric is isometric to the Euclidian manifold $\partial C \times [0, \infty)$ with the product metric, and $\partial C$ is a compact Euclidean manifold.*

**Proposition 2.7.** $\mathrm{PGL}(\Omega(\psi))$ *normalizes $T(\psi)$. Suppose $\beta$ is a horofunction metric on $\Omega(\psi)$ and $\mathbf{t} = \mathbf{t}(\psi)$. If $\mathbf{t} > 0$ then* $\mathrm{PGL}(\Omega(\psi))$ *acts by isometries of $\beta$, and* $\mathrm{PGL}(\Omega(\psi)) = G(\psi)$. *If $\mathbf{t} = 0$ then* $\mathrm{PGL}(\Omega(\psi))$ *acts by horoscalings of $\beta$.*

*Proof.* By 2.1(h) the group $T(\psi)$ acts on $\partial\Omega(\psi)$ as the group of all translations of the Euclidean metric $\beta|$. Suppose $A \in \mathrm{PGL}(\Omega(\psi))$. By 1.12 $A$ normalizes the radial flow. It follows from the above that $A$ is a horoscaling of $\beta$. Thus $A.T(\psi).A^{-1}$ also acts by translations on $\partial\Omega(\psi)$. If $B \in \mathrm{PGL}(\Omega(\psi))$ acts trivially on $\partial\Omega(\psi)$ then $B = id$ because it is a horoscaling. Hence $T(\psi) = A.T(\psi).A^{-1}$.

If $A$ is not an isometry of $\beta$, after replacing $A$ by $A^{-1}$ if needed, we may assume $A$ is a contraction. Hence there is a point $x \in \partial\Omega(\psi)$ fixed by $A$. This gives an identification $T(\psi) \equiv \partial\Omega(\psi)$ via $t \mapsto t(x)$. Under this identification $x$ is identified with the $id \in T(\psi)$.

Let $U \subset \partial\Omega(\psi)$ be the ball of $\beta$ radius 1 center $x$. Then $A(U) \subset U$ is a ball of some radius $r < 1$. Under the identification, $U$ gives a neighborhood $V \subset T$ of the identity in $T$, and $A.V.A^{-1} \subset V$ is a strictly smaller neighborhood. This implies $T$ is unipotent, so $r = 0$. Thus if $r \neq 0$ then $A$ preserves $\beta$. A horosphere in $\Omega(\psi)$ is the characterized as the set of points some fixed distance from $\partial\Omega(\psi)$. Therefore $A$ preserves each horosphere. Hence $A \in G(\psi)$. $\square$

**Corollary 2.8.** *$G(\psi) = T(\psi) \rtimes O(\psi)$ is the internal semi-direct product of $T(\psi)$ and $O(\psi)$.*

*Proof.* By 2.1 the geometry $(G(\psi), \partial\Omega(\psi))$ is isomorphic to a subgeometry of $(\mathrm{Isom}(\mathbb{E}^{n-1}), \mathbb{E}^{n-1})$ and $\mathrm{Isom}(\mathbb{E}^{n-1}) = \mathbb{R}^{n-1} \rtimes O(n-1)$, and moreover that under this isomorphism $G(\psi)$ contains $\mathbb{R}^{n-1}$. $\square$

2.1. **Normalizing the metric.** Given that $\Omega(\psi)$ comes equipped with a family of Euclidean (flat) metrics, it is natural to ask if there is any intrinsic way of distinguishing different metrics. When $\psi = 0$ then the interior of $\Omega(0)$ can be identified with $\mathbb{H}^n$ and for each $c > 0$ there is a (hyperbolic) element $\gamma \in \mathrm{PGL}(\Omega) \subset \mathrm{Isom}(\mathbb{H}^n)$ that rescales the horofunction: $h_0 \circ \gamma = c \cdot h_0$. As a result, there is no projectively invariant way to assign a distinguished metric to $\Omega(0)$. This corresponds to the familiar fact that the complement of a point in the sphere at infinity for $\mathbb{H}^n$ only has an invariant Euclidean *similarity structure* rather than a Euclidean metric. But when $\psi \neq 0$ the story is different.

If $(X, d)$ is a metric space and $f : X \to X$ is an isometry the *displacement distance* of $f$ is $\delta(f) = \inf\{d(x, fx) : x \in X\}$. When $X$ is Euclidean space this infimum is also called the *translation length* of $f$.

If $\psi \neq 0$ define the subset $J(\psi) \subset T_2(\psi)$ to consists of all $A \in T_2(\psi)$ such that the largest eigenvalue of $A$ is $e$. This set is non-empty and compact. A horofunction metric, $\beta^N$, on $\Omega(\psi)$ is *normalized* if $\sup\{\delta(A) : A \in J(\psi)\} = 1$. This metric is Euclidean by 2.1. If $C = \Omega(\psi)/\Gamma$ is a generalized cusp the *normalized horofunction metric* on $C$ is the metric covered by $\beta^N$.

**Corollary 2.9.** *If $\psi \neq 0$ then there is a unique normalized horofunction metric on $\Omega(\psi)$. If $P \in \mathrm{PGL}(n{+}1, \mathbb{R})$ and $P(\Omega(\psi)) = \Omega(\psi')$ then $P$ is an isometry between the normalized horofunction metrics.*

*There is a unique normalized horofunction metric on a $\psi$-cusp $C = \Omega(\psi)/\Gamma$. If $C$ and $C'$ are generalized cusps, and $P : C \to C'$ is a projective diffeomorphism, then $P$ is an isometry between these metrics.*

## 2.2. **The Second Fundamental Form.**

**Definition 2.10.** *Suppose $S \subset \mathbb{R}^n$ is a transversally oriented, smooth hypersurface. The* second fundamental form II *on $S$ is the quadratic form defined on each tangent space $\mathrm{II}_q : T_qS \to \mathbb{R}$ by*

$$\mathrm{II}_q(\gamma'(0)) = \langle \gamma''(0), n_q \rangle$$

*where $\gamma : (-\epsilon, \epsilon) \to S$ is a smooth curve in $S$ with $\gamma(0) = q$ and $n_q$ is a unit normal vector to $S$ at $q$ in the direction given by the transverse orientation.*

It is routine to verify this is well defined. The sign of II depends on a choice of normal orientation. If $S$ is a convex hypersurface and $n_q$ points to the convex side then $\mathrm{II}_q$ is positive definite and defines a Riemannian metric on $S$ see [24]. Suppose $v = \gamma'(0) \in T_q\mathcal{H}$. Then $(\mathrm{D}\,\pi)v = v$ and $v \in \ker \mathrm{D}\,h$ so $\beta(v) = \mathrm{D}^2\,h(v) = \mathrm{II}_q(v)$, thus

$$\beta | T_q\mathcal{H} = \mathrm{II}_q \tag{28}$$

Thus the restriction of the horometric to $T_*\mathcal{H}$ depends only on the hypersurface $\mathcal{H} \subset \mathbb{RP}^n$, and not on the horofunction.

Given $q \in S$ let $n_q$ be a unit normal to $S$ at $q$ oriented towards the convex direction of $S$, then there is a cotangent vector $\eta_q \in T_q^*\mathbb{R}^n$ defined by $\eta_q(v) = \langle v, n_p \rangle$ and

$$\mathrm{II}_q(\gamma'(0)) = \eta_q(\gamma''(0))$$

Observe that $\ker \eta_q = T_qS \subset T_q\mathbb{R}^n$. We refer to $\eta_q$ as the *inward unit cotangent vector* for $S$ at $q$. The following elementary fact does not seem to be well known. It gives another proof of the invariance of the similarity structure on horospheres

**Proposition 2.11.** *Suppose $S \subset \mathbb{R}^n$ is a smooth strictly convex hypersurface and $\tau : \mathbb{R}^n \to \mathbb{R}^n$ is an affine isomorphism and $\tau(S) = S'$. Then $\tau : (S, \mathrm{II}) \to (S', \mathrm{II}')$ is a conformal map.*

*Suppose $\eta_p$ and $\eta'_q$ are the inward unit cotangent vectors to $S$ at $p$, and to $S'$ at $q = \tau(p)$ respectively. Then $\tau^*\eta'_q = \alpha \cdot \eta_p$ for some $\alpha = \alpha(p) > 0$ and $\tau_p^*\,\mathrm{II}'_q = \alpha \cdot \mathrm{II}_p$.*

*Proof.* Given $p \in S$ set $q = \tau(p)$. We must show that $\tau^*(\mathrm{II}'_q) = \alpha(p)\,\mathrm{II}_p$ for some $\alpha(p) > 0$. Let $H \subset \mathbb{R}^n$ be the hyperplane tangent to $S$ at $p$. Translate $H$ infinitesimally so that it intersects $S$ in an infinitesimal ellipsoid centered on $p$. This gives a foliation of an infinitesimal neighborhood of $p$ in $S$ by ellipsoids which we may identify with the levels sets of $\mathrm{II}_p$ in $T_p$. Since affine maps send parallel hyperplanes to parallel hyperplanes, the foliation near $p$ is sent to the foliation near $q$. If two quadratic forms have the same level sets then one is a scalar multiple of the other.

More formally, suppose $\gamma : (-\epsilon, \epsilon) \to S$ is smooth with $\gamma(0) = p$. Then

$$(\tau^*(\mathrm{II}_q))(\gamma'(0)) = \mathrm{II}_q((\tau \circ \gamma)'(0)) = \eta_q((\tau \circ \gamma)''(0))$$

Since $\tau$ is an affine map

$$(\tau \circ \gamma)''(0) = (d\tau)(\gamma''(0))$$

Since $d\tau(T_pS) = T_qS$ it follows that $\eta_q \circ d\tau = \alpha \cdot \eta_p$ for some $\alpha = \alpha(p)$. Thus

$$\eta_q(d\tau(\gamma''(0))) = \alpha \cdot \eta_p(\gamma''(0)) = \alpha \cdot \mathrm{II}_p(\gamma'(0))$$

□

*Proof of 0.5.* The metric $\beta$ with the required properties is given by Theorem (2.1), and the restriction to the boundary is the second fundamental form, see (28). $\square$

**Theorem 2.12** (underlying hyperbolic structure)**.** *Every generalized cusp $C$, with boundary a horomanifold, has a hyperbolic metric $\kappa(C)$ such that $\partial C$ is the quotient of a horosphere in $\mathbb{H}^n$. If $C'$ is another such cusp, and if $P : C \to C'$ is a projective diffeomorphism, then $P$ is an isometry from $\kappa(C)$ to $\kappa(C')$.*

*Proof.* Suppose $C$ is a generalized cusp of dimension $n$ bounded by a horomanifold. Then $C = \Omega(\psi)/\Gamma$ is a $\psi$-cusp. There is a unique horofunction metric, $\beta_C$, on $C$ such that the Euclidean volume of $\partial C$ is 1. This metric is $\kappa(C)$. If $C = \Omega/\Gamma$ and $C' = \Omega'/\Gamma'$ are generalized cusps, and $P : C \to C'$ is a projective diffeomorphism, then $P$ is covered by a projective isomorphism $\Omega \to \Omega'$, which is an isometry between horofunction metrics. Thus $P$ is an isometry.

There is a unique hyperbolic cusp $H$ bounded by a horomanifold, with $\partial H$ isometric to $(\partial C, \kappa(C))$. The restriction of the hyperbolic metric to $\partial H$ equals the restriction of $\kappa(H)$ to $\partial H$. Thus there is an isometry $(C, \kappa(C)) \to (H, \kappa(H))$, that identifies $C$ with a hyperbolic cusp. $\square$

This raises several questions. For example, using this, one can assign a cusp shape, $z \in \mathbb{C}$, to a generalized cusp in a 3-manifold. If a hyperbolic 3-manifold with a cusp can be deformed to have generalized cusps, can this shape change?

## 3. Classification of $\psi$ cusps

This section is devoted to the proof of 0.2 and 0.3. But first we need:

**Lemma 3.1.** *If $C = \Omega/\Gamma$ and $C' = \Omega'/\Gamma'$ are equivalent generalized cusps of dimension $n$, then $\Gamma$ and $\Gamma'$ are conjugate subgroups of* $\mathrm{PGL}(n+1, \mathbb{R})$.

*Proof.* The definition of equivalent cusps given in the introduction is not transitive, though it will follow from the classification that it is transitive. In this proof we use the equivalence relation *generated* by the relation on pairs of cusps: *Given $C$ and $C'$ there is a cusp $C''$, diffeomorphic to both of them, and projective embeddings, that are also homotopy equivalences, into both $C$ and $C'$*. Thus it suffices to prove the lemma when there is a projective embedding of $C$ into $C'$ that is also a homotopy equivalence. We may assume $C \subset C'$ and $\Omega \subset \Omega'$ (this amounts to performing a conjugacy). Since the embeddings are homotopy equivalences it follows that $\Gamma = \Gamma'$. $\square$

**Theorem 0.2** (classification)**.**

(1) *If $\Gamma$ and $\Gamma'$ are lattices in $G(\psi)$ TFAE*
  (a) *$\Omega(\psi)/\Gamma$ and $\Omega(\psi)/\Gamma'$ are equivalent generalized cusps*
  (b) *$\Gamma$ and $\Gamma'$ are conjugate in* $\mathrm{PGL}(n+1, \mathbb{R})$
  (c) *$\Gamma$ and $\Gamma'$ are conjugate in* $\mathrm{PGL}(\Omega(\psi))$
(2) *A lattice in $G(\psi)$ is conjugate in* $\mathrm{PGL}(n+1, \mathbb{R})$ *into $G(\psi')$ iff $G(\psi)$ is conjugate to $G(\psi')$.*
(3) *$G(\psi)$ is conjugate in* $\mathrm{PGL}(n+1, \mathbb{R})$ *to $G(\psi')$ iff $\psi' = t \cdot \psi$ for some $t > 0$.*
(4) $\mathrm{PGL}(\Omega(\psi)) = G(\psi)$ *when $\psi \neq 0$*
(5) *When $\psi \neq 0$ the map $\Theta : \mathcal{T}(\mathrm{Isom}(\mathbb{E}^{n-1}), \psi) \times (O(n-1)/O(\psi)) \longrightarrow \mathcal{T}(G(\psi))$ defined in (29) is a bijection.*

*Proof.* (1) It is clear that $c \Rightarrow a$. Also $a \Rightarrow b$ follows from 3.1.

For (1) $b \Rightarrow c$ and (2). Suppose $\Gamma \subset G(\psi)$ and $\Gamma' \subset G(\psi')$ are lattices and $P \in \mathrm{PGL}(n+1, \mathbb{R})$ with $P \cdot \Gamma \cdot P^{-1} = \Gamma'$. By (4.7) $T(\psi)$ is the unique virtual e-hull of $\Gamma(\psi)$, thus $P \cdot T(\psi) \cdot P^{-1} = T(\psi')$.

Hence $U = P^{-1}(\Omega(\psi'))$ is a properly convex set that is preserved by $T(\psi)$. Moreover $U$ is irreducible, since this property is preserved by projective maps. By 1.27 there is $g \in \mathcal{E}(\psi) \oplus \Phi^{\psi}$ such that $g(U) = \Omega(\psi)$. Since $g$ centralizes $T(\psi)$ we may replace $P$ by $g \circ P$ and assume that $P(\Omega(\psi')) = \Omega(\psi)$. It follows that $P \cdot G(\psi) \cdot P^{-1} = G(\psi')$ proving one direction of (2). The converse of (2) is obvious. If $G(\psi) = G(\psi')$ then $P$ preserves $\Omega(\psi)$ which proves (1) $b \Rightarrow c$.

(3) By 2.8 $T(\psi)$ is a characteristic subgroup of $G(\psi)$: it is the subgroup of elements all of whose eigenvalues are positive. Thus if $P$ conjugates $G(\psi)$ to $G(\psi')$ then it conjugates $T(\psi)$ to $T(\psi')$. By 1.16 this happens if and only if $\psi = t\cdot\psi$ for some $t > 0$.

(4) follows from 2.7. (5) is done below. □

**Corollary 0.3** (cusps classified by lattices)**.** *There is a bijection $F : \mathcal{M}\mathrm{od}^n \longrightarrow \mathcal{C}^n$ defined for $[\Gamma] \in \mathcal{M}\mathrm{od}^n$ by $F([\Gamma]) = [\Omega(\psi)/\Gamma]$ when $\Gamma$ is a lattice in $G(\psi)$.*

*Proof of 0.3.* To show $F$ is surjective, suppose $C$ is a generalized cusp of dimension $n$. By theorem 0.1 there is an equivalent cusp $\Omega(\psi)/\Gamma \in [C]$ for some lattice $\Gamma \subset G(\psi)$. By Theorem 0.2 (1)(c) we may assume $\psi(e_1) = 1$. Then $F([\Gamma]) = [C]$ therefore $F : \mathcal{M}\mathrm{od}^n \to \mathcal{C}^n$ is surjective.

To show $F$ is injective, suppose $F([\Gamma_1]) = F([\Gamma_2])$ for lattices $\Gamma_i \subset G(\psi_i)$. By (3.1) $\Gamma_1$ and $\Gamma_2$ are conjugate subgroups of $\mathrm{PGL}(n+1,\mathbb{R})$. Then by 0.2(2) $G(\psi_1)$ and $G(\psi_2)$ are conjugate in $\mathrm{PGL}(n+1,\mathbb{R})$, and by 0.2(3) this implies $\psi_1 = t\cdot\psi_2$ for some $t > 0$. Since $\psi_1(e_1) = \psi_2(e_1)$ then $\psi_1 = \psi_2$. By theorem 0.2 (1)(c) it follows that $\Gamma_1$ and $\Gamma_2$ are conjugate subgroups of $G(\psi_1)$ so $[\Gamma_1] = [\Gamma_2]$ and $F$ is injective. □

Corollary 0.3 reduces the classification of equivalence classes of generalized cusps to the classification of conjugacy classes of lattices in each of the groups $G(\psi)$. This classification corresponds to *moduli space*. There is a finer classification using the notion of *marking* that results in an analog of *Teichmuller space*. We will show that a *marked* generalized cusp is parameterized by a marked Euclidean cusp, together with a left coset $A\cdot O(\psi) \in O(n-1)/O(\psi)$ called the *anisotropy parameter*. The classification of unmarked cusps is more complicated to state.

One complication is that in general there are finitely many distinct isomorphism types of lattice in $G(\psi)$. To make these subtleties clear requires several definitions.

A discrete subgroup $H$ of a Lie group $G$ is a *lattice* if $G/H$ is compact. The *set of lattices* in $G$ is denoted $\mathrm{Lat}(G)$. The quotient of this set by the action of $G$ by conjugacy gives the set of *conjugacy classes of lattices* in $G$ denoted $\mathcal{M}\mathrm{od}(G) = \mathrm{Lat}(G)/G$. Given a lattice $H$ in $G$, an *$H$-lattice* is a lattice $H'$ in $G$ with $H \cong H'$; and the *set of $H$-lattices* is the subset $\mathrm{Lat}(G,H) \subset \mathrm{Lat}(G)$. The set of conjugacy classes of $H$-lattice is $\mathcal{M}\mathrm{od}(G,H) = \mathrm{Lat}(G,H)/G$ and is a subset of $\mathcal{M}\mathrm{od}(G)$.

A *marking* of an $H$-lattice $H'$ in $G$ is an isomorphism $\theta : H \to H'$, and $\theta$ is also called a *marked $H$-lattice*. The set of all marked $H$-lattices in $G$ is denoted by $\mathrm{Lat}_m(G,H)$. Thus a lattice is a group, but a marked lattice is a homomorphism, and $\mathrm{Lat}_m(G,H)$ is the subset of the representation variety $\mathrm{Hom}(H,\mathrm{PGL}(n+1,\mathbb{R}))$ consisting of those injective homomorphisms with image a lattice. Let $\mathcal{H}$ be a set of lattices in $G$ that contains one lattice in each isomorphism class. The set of marked lattices in $G$ is $\mathrm{Lat}_m(G) = \cup\,\mathrm{Lat}_m(G,H)$ where the union is over $H \in \mathcal{H}$.

Two marked $H$-lattices $\theta_1,\theta_2 : H \to G$ are *conjugate* if there is $g \in G$ with $\theta_2 = g^{-1}\cdot\theta_1\cdot g$, and the set of conjugacy classes of marked $H$-lattices is $\mathcal{T}(G,H) = \mathrm{Lat}_m(G,H)/G$. The set of conjugacy classes of marked lattices in $G$ is $\mathcal{T}(G) = \mathrm{Lat}_m(G)/G$.

As an example, a lattice in $G = \mathrm{Isom}(\mathbb{E}^2)$ is a 2-dimensional Bieberbach group (wallpaper group), and there are 17 isomorphism types for $H$. These are also the isomorphism classes of compact Euclidean 2-orbifolds. There is a natural bijection between $\mathcal{T}(\mathrm{Isom}(\mathbb{E}^2),\mathbb{Z}^2)$ and marked Euclidean structures on a torus $T^2$. It is well known that a marked Euclidean torus of area 1 is parameterized by a point in the upper half plane $\mathbb{H}^2$. Moreover

$$\begin{aligned} \mathcal{T}(\mathrm{Isom}(\mathbb{E}^2),\mathbb{Z}^2) &\cong \mathbb{R}^+ \times \{x+iy \in \mathbb{C} : y > 0\} \equiv \mathbb{R}^+ \times \mathbb{H}^2 \\ \mathcal{M}\mathrm{od}(\mathrm{Isom}(\mathbb{E}^2),\mathbb{Z}^2) &\cong \mathbb{R}^+ \times \mathbb{H}^2/\,\mathrm{PSL}(2,\mathbb{Z}) \end{aligned}$$

the $\mathbb{R}^+$ factor records the area of the torus that is the quotient of $\mathbb{E}^2$ by the action of the lattice.

Before proceeding to the proof of 0.2(5) we give an example for 3-manifolds. For a generic diagonalizable generalized cusp Lie group, such as $\psi = (3,2,1)$, then $G(\psi) \cong \mathbb{R}^2$ and $O(\psi)$ is trivial. A $\mathbb{Z}^2$-lattice in $G(\psi)$ is a subgroup $H = \mathbb{Z}u \oplus \mathbb{Z}v \subset \mathbb{R}^2$ given by a pair of linearly independent vectors $u, v \in \mathbb{R}^2$. Using the $\mathbb{Z}^2$-marking given by $(1,0) \mapsto u$ and $(0,1) \mapsto v$ shows that the $2\times 2$

matrix $M = (u^t, v^t)$ determines a unique marked lattice, so

$$\mathcal{T}(G(\psi), \mathbb{Z}^2) \cong \mathrm{GL}(2, \mathbb{R})$$

There is a natural map $\mathcal{T}(G(\psi), \mathbb{Z}^2) \to \mathcal{T}(\mathrm{Isom}(\mathbb{E}^2), \mathbb{Z}^2)$ and two lattices $M, M' \in GL(2, \mathbb{R})$ have the same image if and only if there is $A \in O(2)$ with $AM = M'$. It follows that

$$\mathcal{T}(G(\psi), \mathbb{Z}^2) \cong O(2) \times \mathcal{T}(\mathrm{Isom}(\mathbb{E}^2), \mathbb{Z}^2)$$

This illustrates 0.2(5): a marked lattice in $G(\psi)$ is parameterized by a marked Euclidean lattice and a left coset of $O(\psi)$. In this case $O(\psi)$ is trivial, so the left coset is just an element of $O(2)$.

Now consider unmarked lattices. A change of marking is a change of basis in $\mathbb{Z}^2$, and this changes the lattice $M$ to $A.M$ where $A \in \mathrm{GL}(2, \mathbb{Z})$. Thus

$$\mathcal{M}\mathrm{od}(G(\psi), \mathbb{Z}^2) \cong \mathrm{GL}(2, \mathbb{Z}) \backslash \mathrm{GL}(2, \mathbb{R})$$

The left action of $\mathrm{GL}(2, \mathbb{Z})$ on $\mathrm{GL}(2, \mathbb{R})$ is free. However the action of $\mathrm{GL}(2, \mathbb{Z})$ on $\mathcal{T}(\mathrm{Isom}(\mathbb{E}^2), \mathbb{Z}^2)$ is not free: a $\pi/2$ rotation fixes an unmarked square torus. Thus

$$\mathcal{M}\mathrm{od}(G(\psi), \mathbb{Z}^2) \ncong O(2) \times \mathcal{M}\mathrm{od}(\mathrm{Isom}(\mathbb{E}^2), \mathbb{Z}^2)$$

which means unmarked lattices in $G(\psi)$ are **not** parametrized by an unmarked lattices in $\mathrm{Isom}(\mathbb{E}^2)$ together with an anisotropy parameter.

*Proof of 0.2(5).* For this proof we will identify $G(\psi)$ with the subgroup $\mathbb{R}^{n-1} \rtimes O(\psi)$ of $\mathrm{Isom}(\mathbb{E}^{n-1})$. Since $\mathrm{Isom}(\mathbb{E}^{n-1})/G(\psi) \cong O(n-1)/O(\psi)$ is compact, every lattice in $G(\psi)$ is also a lattice in $\mathrm{Isom}(\mathbb{E}^{n-1})$. Let $\mathcal{T}(\mathrm{Isom}(\mathbb{E}^{n-1}), \psi) \subset \mathcal{T}(\mathrm{Isom}(\mathbb{E}^{n-1}))$ be the subset of conjugacy classes of lattice with rotational part in $O(\psi)$. The map $\pi : \mathrm{Lat}_m(G(\psi)) \to \mathcal{T}(\mathrm{Isom}(\mathbb{E}^{n-1}), \psi)$ is surjective. Choose a left inverse

$$\sigma : \mathcal{T}(\mathrm{Isom}(\mathbb{E}^{n-1}), \psi) \to \mathrm{Lat}_m(G(\psi))$$

so $\pi \circ \sigma = id$, and define $\Theta : \mathcal{T}(\mathrm{Isom}(\mathbb{E}^{n-1}), \psi) \times (O(n-1)/O(\psi)) \to \mathcal{T}(G(\psi))$ by

$$\Theta([\theta], g.O(\psi)) = [g^{-1} \cdot \sigma([\theta]) \cdot g]. \tag{29}$$

Then 0.2(5) is the assertion that $\Theta$ is a bijection. Set $\mathcal{L} = \mathrm{Im}(\sigma)$ then $\mathcal{L}$ is a set of marked lattices in $G(\psi)$ that contains one representative of each $\mathrm{Isom}(\mathbb{E}^{n-1})$-conjugacy class. There is a map

$$\widetilde{\Theta} : \mathcal{L} \times \mathrm{Isom}(\mathbb{E}^{n-1}) \to \mathcal{T}(G(\psi))$$

given by $\widetilde{\Theta}(\theta, g) = [g^{-1} \circ \theta \circ g]$ which is obviously surjective. Observe that $\widetilde{\Theta}(\theta_1, g_1) = \widetilde{\Theta}(\theta_2, g_2)$ if and only if

$$g_1^{-1} \circ \theta_1 \circ g_1 = k^{-1} \circ (g_2^{-1} \circ \theta_2 \circ g_2) \circ k$$

for some $k \in G(\psi)$. This is equivalent to

$$\theta_1 = g \circ \theta_2 \circ g^{-1} \quad \text{with} \quad g = g_2 \circ k \circ g_1^{-1}$$

Thus $\theta_1, \theta_2$ are conjugate. This implies the domain of $\theta_1$ and of $\theta_2$ is the same lattice $H \in \mathcal{H}$. Since $\theta_1, \theta_2 \in \mathcal{L}$ it follows that $\theta_1 = \theta_2 = \theta$ and

$$\theta = g \circ \theta \circ g^{-1} \tag{30}$$

Therefore $g$ centralizes the lattice $\Gamma = \theta(H)$. It follows that $\widetilde{\Theta}(\theta_1, g_1) = \widetilde{\Theta}(\theta_1, g_2)$ if and only if there is $\theta \in \mathcal{L}$ and $k \in G(\psi)$ such that $\theta_1 = \theta_2 = \theta$ and $g = g_2 \circ k \circ g_1^{-1}$ centralizes $\Gamma$. Observe that if marked lattices are replaced by (unmarked) lattices we can only conclude at this point that $g$ *normalizes* $\Gamma$.

We can express $g \in \mathrm{Isom}(\mathbb{E}^{n-1})$ uniquely as a pair $g = (A, v) \in O(n-1) \times \mathbb{R}^{n-1}$ where $g(x) = Ax + v$, and $A$ is called the *rotational part* of $g$. Indeed, if $g_1(x) = A_1 x + v_1$ and $g_2(x) = A_2 x + v_2$ and $k(x) = Bx + v$ with $B \in O(\psi)$ then

$$g(x) = g_2 \circ k \circ g_1^{-1}(x) = A_2 B A_1^{-1} x + (v_2 - A_2 B A_1^{-1} v_1 + A_2 v) \tag{31}$$

By Bieberbach's first theorem, [5], the subset of the lattice $\Gamma$ consisting of pure translations is a finite index subgroup, $\Gamma_t \subset \Gamma$ that is also a lattice in $\mathbb{R}^{n-1}$. Thus $\Gamma_t$ is centralized by $g$. This means the rotational part of $g$ preserves an ordered basis of $\mathbb{R}^{n-1}$. An element of $O(n-1)$ that preserves an ordered basis of $\mathbb{R}^{n-1}$ is trivial, hence the rotational part of $g$ is trivial, so $A_2BA_1^{-1} = I$, and

$$g(x) = x + (v_2 - v_1 + A_2 v) \tag{32}$$

It follows that $\widetilde{\Theta}(\theta, g_1) = \widetilde{\Theta}(\theta, g_2)$ if and only if $g_1 = (A_1, v_1)$ and $g_2 = (A_1B^{-1}, v_2)$ and there is $v \in \mathbb{R}^n$ such that $g = (I, v_2 - v_1 + A_2v)$ centralizes $\theta$. If we choose $v = A_2^{-1}(v_1 - v_2)$ then $g = (I, 0)$ centralizes $\theta$. It follows that $\widetilde{\Theta}(\theta_1, (A_1, v_1)) = \widetilde{\Theta}(\theta_2, (A_2, v_2))$ if and only if $\theta_1 = \theta_2$ and $A_2 \in A_1O(\psi)$. In other words, $\widetilde{\Theta}(\theta_1, g_1) = \widetilde{\Theta}(\theta_2, g_2)$ if and only if $\theta_1 = \theta_2$ and $g_1G(\psi) = g_2G(\psi)$. As a result $\widetilde{\Theta}$ induces a bijection

$$\Theta' : \mathcal{L} \times \mathrm{Isom}(\mathbb{E}^{n-1})/G(\psi) \to \mathcal{T}(G(\psi))$$

Observe that $\mathrm{Isom}(\mathbb{E}^{n-1})/G(\psi) \cong O(n-1)/O(\psi)$. By definition of $\mathcal{L}$, there is a bijection

$$(\pi|\mathcal{L}) : \mathcal{L} \to \mathcal{T}(\mathrm{Isom}(\mathbb{E}^{n-1}), \psi)$$

given by $\theta \mapsto [\theta]$. Thus $\Theta'$ factors through the bijection $\Theta$ in (29) completing the proof. $\square$

## 4. Generalized Cusps are $\psi$-cusps

As mentioned in the introduction, the idea of a *cusp* in a projective manifold has evolved in a series of papers. Recall that if $\Omega$ is properly convex then $[A] \in \mathrm{PGL}(\Omega)$ is *parabolic* if all the eigenvalues of $A$ have the same modulus and there is no fixed point in $\mathrm{int}(\Omega)$. A definition of the term *cusp* in a properly convex manifold was first given in 5.2 of [10]. There, the holonomy of a cusp $C$ consists of parabolics. The definition used there was dictated by the requirement to establish a thick-thin decomposition for strictly convex manifolds, of possibly infinite volume. In that paper the *rank* of $C$ is defined, and *maximal rank* is equivalent to $\partial C$ being compact. In this paper we only consider cusps of maximal rank, so we have omitted the term *maximal rank* from statements.

A definition of the term *generalized cusp* was first given in [11] definition (6.1). It differs from the definition in the introduction, by using the term *nilpotent* in place of *abelian.* Theorem 0.7 at the end of this section shows that these definitions are equivalent.

**Definition 4.1.** *A* g-cusp *(called a generalized cusp in* [11]*) is a properly convex manifold $C = \Omega/\Gamma$ homeomorphic to $\partial C \times [0, \infty)$ with $\partial C$ a connected closed manifold and $\pi_1 C$ virtually* nilpotent *such that $\partial\Omega$ contains no line segment. The group $\Gamma$ is called a* g-cusp group. *In addition:*

- *If* $\mathrm{PGL}(\Omega)$ *acts transitively on $\partial\Omega$, then $C$ is* homogeneous.
- *If $\pi_1 C$ is virtually abelian, then $C$ is a* generalized cusp.
- *A* cusp *is a generalized cusp with parabolic holonomy.*
- *A* standard cusp *is a cusp that is projectively equivalent to a cusp in a hyperbolic manifold.*

**Theorem 4.2** (0.5 in [10])**.** *Every maximal rank cusp in a properly convex real projective manifold is standard.*

Observe that a finite cover of a *g*-cusp is also a *g*-cusp. In this section we use the following results from of section 6 of [11].

**Theorem 4.3** (6.6 in [11])**.** *Every g-cusp is equivalent to a homogeneous g-cusp.*

**Definition 4.4.** $\mathrm{UT}(n) \subset \mathrm{GL}(n, \mathbb{R})$ *is the subgroup of upper-triangular matrices with positive diagonal entries.*

**Definition 4.5.** *An* e-group *is a subgroup $G \subset \mathrm{GL}(n, \mathbb{R})$ such that every eigenvalue of every element of $G$ is positive. If $\Gamma \subset \mathrm{GL}(n, \mathbb{R})$ is discrete, a* virtual e-hull *for $\Gamma$ is a connected e-group $G \subset \mathrm{GL}(n, \mathbb{R})$ such that $|\Gamma : G \cap \Gamma| < \infty$ and $(G \cap \Gamma)\backslash G$ is compact.*

Observe that $UT(n)$ is an e-group. 6.1, 6.10 and 6.12 in [11] imply

**Proposition 4.6.** *Suppose $P = \Omega/\Gamma$ is a g-cusp of dimension $n$. Then $\Gamma$ contains a finite index subgroup, $\Gamma_1$, that is a lattice in the connected nilpotent group $T(\Gamma) = \exp\langle\log(\Gamma_1)\rangle$. Moreover $T(\Gamma)$ is conjugate in* $\mathrm{GL}(n+1, \mathbb{R})$ *into* $\mathrm{UT}(n+1)$.

In [11] $\Gamma_1 = \text{core}(\Gamma, n)$. The *Zariski closure* of $\Gamma$ generally has larger dimension than $T(\Gamma)$.

**Theorem 4.7** (6.18 [11])**.** *If $\Omega/\Gamma$ is a generalized cusp then $T(\Gamma)$ is the unique virtual e-hull of $\Gamma$.*

**Definition 4.8** (cf. 6.17 in [11])**.** *A* translation group *is a connected nilpotent subgroup $T \subset \mathrm{GL}(n+1,\mathbb{R})$ that is the virtual e-hull of a g-cusp.*

**Definition 4.9** (page 189 [10])**.** *Given a 1-dimensional subspace $U \subset V$ set $p = \mathbb{P}(U)$ and define $\mathcal{D}_p : \mathbb{P}(V) \setminus \{p\} \to \mathbb{P}(V/U)$ by $\mathcal{D}_p([x]) = [x+U]$. The* space of directions *of the subset $\Omega \subset \mathbb{P}(V)$ at $p$ is $\mathcal{D}_p(\Omega \setminus p)$.*

**Theorem 4.10.** *If $G \subset \mathrm{GL}(n+1,\mathbb{R})$ is a translation group then $\exists\ \psi \in A$ such that $G$ is conjugate in $\mathrm{GL}(n+1,\mathbb{R})$ to $T(\psi)$.*

*Proof.* By the above, we may assume $G$ is upper-triangular. By (6.23) and (6.24) in [11] $G$ preserves a properly convex domain $\Omega \subset \mathbb{R}^n$ with $S = \partial\Omega$ strictly convex. Moreover $G$ acts simply transitively on $S$. By scaling we may assume $G \subset \mathrm{Aff}(\mathbb{R}^n)$, so always has 1 is the bottom right entry.

Let $\{e_i \mid 1 \le i \le n+1\}$ be the standard basis of $\mathbb{R}^{n+1}$. Since $G$ is nilpotent, we may further assume there is a decomposition $V := \mathbb{R}^{n+1} = V_1 \oplus \cdots \oplus V_{r+1}$ into $G$-invariant subspaces such that $V_i$ has ordered basis $\mathcal{B}_i = \{e_k \mid m_{i-1} < k \le m_i\}$ where $n_i := \dim V_i = m_i - m_{i-1}$. By reordering the standard basis we may assume $\dim V_{r+1}$ is the maximum of $\dim V_i$ for $i \le r+1$.

Let $UT_1(V_i)$ be the group of unipotent, upper-triangular matrices of size $n_i$. Then there are distinct weights $\lambda_i : G \to \mathbb{R}_+$ and homomorphisms $\rho_i : G \to UT_1(V_i)$ so that $G$ is the image of the inclusion map $\rho : G \to \mathrm{GL}(V)$ given by

$$\rho = \begin{pmatrix} \lambda_1\rho_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2\rho_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & & \lambda_{r+1}\rho_{r+1} \end{pmatrix} \tag{33}$$

Since $G$ is affine, it follows that $\lambda_{r+1} \equiv 1$. Let $w_i = e_{m_i}$ be the last vector in $\mathcal{B}_i$, and $U_i = \langle \mathcal{B}_i \setminus \{w_i\} \rangle$. Then $V_i = U_i \oplus \mathbb{R} \cdot w_i$. The subspace $U = \bigoplus U_i$ is preserved by $G$, and there is a linear projection $\pi : V \to V/U$. Define a subspace $W = \langle w_1, \cdots, w_{r+1} \rangle \subset \mathbb{R}^{n+1}$, so $\mathcal{W} = \{w_1, \cdots, w_{r+1}\}$ is an ordered basis of $W$. There is projection $\pi : V \to V/U$ and an isomorphism $V/U \to W$ defined by $w_i + U \mapsto w_i$. We use the same symbol to denote the induced projection $\pi : \mathbb{P}(V) \setminus \mathbb{P}(U) \longrightarrow \mathbb{P}(V/U)$.

Since $G$ preserves $U$, it acts on $V/U$, and thus on $W$. We denote this action by $\rho_W : G \to \mathrm{GL}(W)$. Using the basis $\mathcal{W}$, this action is diagonal and, recalling that $\lambda_{r+1} \equiv 1$, the action is

$$\rho_W = \begin{pmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & \cdots & & \lambda_r & 0 \\ 0 & 0 & \cdots & & 1 \end{pmatrix} \tag{34}$$

There are $r+1$ projective hyperplanes $P_i$ in $\mathbb{P}(W) \cong \mathbb{RP}^r$ each of which contains all but one of the points $[w_i]$. The complement of these hyperplanes consists of $2^r$ open simplices.

Since $G$ acts transitively on $S$ it also acts (via $\rho_W$) transitively on $\pi(S) \subset \mathbb{P}(W)$. Choose $q = [x] \in S$, then $\pi q$ is in one of these open simplices: $\Delta$. Otherwise, since $S$ is preserved by $G$ it follows that $\pi S$ is contained in some hyperplane $P_i \subset \mathbb{P}(W)$. But this implies $S \subset \pi^{-1}(P_i)$ which is a hyperplane in $\mathbb{P}(V)$. This contradicts that $S$ is a strictly convex hypersurface in $V$.

**Claim 1** Either $G$ is diagonal, or else $H := \rho_W(G)$ acts transitively on $\Delta$.

The fiber $\pi^{-1}(\pi q) \subset \mathbb{P}(V)$ that contains $q$ is the affine subspace $U_q := [x+U]$. If $S$ is transverse to $U_q$ then $\pi S$ contains an open subset of $\Delta$, so $\dim H = \dim \Delta$. But $\pi S$ is the $H$-orbit of a point, and $H$ is the projectivization of a diagonal subgroup, so $H$ acts transitively on $\Delta$.

Thus we may assume $S$ is not transverse to $U_q$. If a strictly convex hypersurface is not transverse to a hyperplane, then it is to tangent to it at one point, so $U_q \cap S = q$. Since $G$ acts transitively on $S$ this condition holds at every $q \in S$. This implies $\pi|S$ is injective so $\dim \Delta \ge \dim S$ thus $r \ge n-1$.

If $r = n$ then $G$ is diagonal as claimed. Otherwise $r = n - 1$. Since $\pi|S$ is injective, and $\dim S = n - 1 = \dim \Delta$, it follows that $\pi(S)$ contains an open subset of $\Delta$. As before this implies $H$ acts transitively, which proves claim 1.

In the case $G$ is diagonal since $\dim G = \dim S = n - 1$ it follows that $G$ is the kernel of some homomorphism $\psi : D \to \mathbb{R}_+$ where $D$ is the diagonal subgroup of $UT(n+1) \cap \mathrm{Aff}(\mathbb{R}^n)$. It follows from 1.2 that $\psi$ or $-\psi$ is positive. This proves the theorem when $r = n$.

Henceforth we assume $r < n$ so $H$ acts simply transitively on $\Delta$. Thus $\dim H = r$ and from (34) it follows that $H \subset \mathrm{GL}(r + 1, \mathbb{R})$ consists of all positive diagonal matrices with 1 in the bottom right corner.

The projection, $\pi$, restricts to a $G$-equivariant surjection $\pi_\Omega : \Omega \longrightarrow \Delta$ and $K = \ker(\rho_W) \subset G$ acts trivially on $\Delta$, and is unipotent. Each fiber $\Omega_q := U_q \cap \Omega = \pi_\Omega^{-1}(q)$ is a properly convex set which is preserved by $K$. Since $G$ acts simply transitively on $\partial\Omega$, it follows that $K$ acts simply transitively on $\partial\Omega_q = U_q \cap \partial\Omega$ for every $q$. Simple transitivity implies that the action of $K$ on $U_q$ is faithful.

The action of $K$ on $\Delta$ is trivial, so $[k(x) + U] = [x + U]$ for all $k \in K$. Since $K$ is unipotent $k(x) \in x + U$ for all $k \in K$. The subspace $U^+ = U \oplus \mathbb{R} \cdot x$ is preserved by $K$ and $U_q = [U + x] \subset \mathbb{P}(U^+) \subset \mathbb{P}(V)$. The action of $K$ on $U^+$ is the restriction of the action on $V$, and is therefore unipotent. Moreover $U^+ = \bigoplus U_i \oplus \mathbb{R} \cdot x$ so the action $K$ on $U^+$ is given by $K' = \rho'(K)$ where

$$\rho' := \rho|U^+ = \begin{pmatrix} \rho_1|U_1 & 0 & \cdots & & & * \\ 0 & \rho_2|U_2 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & 0 & * \\ 0 & 0 & \cdots & & \rho_{r+1}|U_{r+1} & * \\ 0 & 0 & \cdots & & & 1 \end{pmatrix} \tag{35}$$

The properly convex set $\Omega_q = \Omega \cap U_q \subset \mathbb{P}(U^+)$ is preserved by $K'$. Moreover $K'$ is unipotent, nilpotent, upper-triangular, and acts simply transitively on $\partial\Omega_q$. The hyperplane $\mathbb{P}(U) \subset \mathbb{P}(U^+)$ is preserved by $K'$, and the point $s = [e_1] \in \partial_\infty\Omega_q = \mathrm{cl}(\Omega_q) \cap \mathbb{P}(U)$ is fixed by $K'$. Also $\mathcal{D}_s\Omega_q = \mathcal{D}_s(\partial\Omega_q)$, hence $(\mathcal{D}_s\Omega_q)/K' = (\partial\Omega_q)/K'$ is a single point, and thus compact. It now follows from Theorem 5.7 in [10] that $s$ is a round point of $\Omega_q$. Hence $\mathrm{cl}(\Omega_q) = \Omega_q \sqcup \{s\}$.

It follows from Theorem 9.1 in [10] that $\Omega_q$ is an ellipsoid, and $K'$ is conjugate to the parabolic subgroup

$$P = \exp \begin{pmatrix} 0 & y_1 & \cdots & y_u & 0 \\ 0 & \cdots & & & y_1 \\ 0 & \cdots & & & \vdots \\ 0 & \cdots & & & y_u \\ 0 & \cdots & & & 0 \end{pmatrix} \subset GL(u+2, \mathbb{R}) \tag{36}$$

Here, $u + 2 = \dim V_{r+1}$. If $u = 0$ this is the identity matrix, and $K$ is the trivial group. When $u > 0$ the group $P$ does not preserve any non-trivial direct sum decomposition. If follows that $U^+$ is contained in some $V_j$, and for $i \neq j$ that $\dim V_i = 1$. By assumption $V_{r+1}$ has maximum dimension, so $U^+ = V_{r+1}$, and $\dim V_i = 1$ for all $i \leq r$, and

$$K = \begin{pmatrix} I_r & 0 \\ 0 & P \end{pmatrix} \subset \mathrm{GL}(n + 1, \mathbb{R}) \tag{37}$$

This formula also holds when $u = 0$ since $K$ is then trivial. We thus have a short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, "incl"] & G \arrow[r, "\rho_W"] & H \arrow[l, bend left, "\sigma"] \arrow[r] & 1 \end{tikzcd} \tag{38}$$

Since $H \cong \mathbb{R}^r$ there is a splitting $\sigma : H \to G$. From the form of $\rho$ in (33) it follows that

$$\sigma = \begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & & \lambda_r \end{pmatrix} & 0 \\ 0 & \psi \end{pmatrix} \tag{39}$$

where $\psi : H \to UT_1(V_{r+1})$. Since $K$ is a normal subgroup of $G$ it follows that $\psi(H)$ is a subgroup of the normalizer, $N$, of $P$ in $\mathrm{UT}_1(V_{r+1})$. Let $\Phi = \exp \mathbb{R} \cdot a \subset UT_1(V_{r+1})$ be the one-parameter group where $a$ is the elementary matrix with 1 in the top right corner. Then $\Phi$ centralizes $P$.

**Claim 2**: $N = \langle P, \Phi \rangle$, so $\mathfrak{n} = \mathfrak{p} \oplus \mathbb{R} \cdot a$.

The closures of the orbits of $P$ in $\mathbb{P}(V_{r+1})$ consists of a fixed point $p$, a hyperplane $H$ containing $p$, and a one-parameter family of horospheres, each tangent to $H$ at $p$. Since $N$ normalizes $P$ it permutes $P$-orbits. Thus $N$ preserves the fixed set and center of the radial flow, and so normalizes the radial flow. Since $N$ is unipotent, $N$ centralizes the radial flow. The radial flow acts transitively on horospheres, so if $n \in N$ there is $t \in \mathbb{R}$ such that $\phi(t) \circ n$ preserves one horosphere. But, since $N$ centralizes $\Phi$, this implies that $n$ preserves every horosphere. Thus $n$ is an isometry of $\mathbb{H}^k \subset \mathbb{P}(V_{r+1})$ where $k = \dim V_{r+1}$. Since $n$ is unipotent it follows that $n$ is parabolic, thus $n \in P$. So $n \in \Phi_t \cdot P$, which proves the claim 2.

Taking derivatives $\mathrm{D}\,\sigma : \mathfrak{h} \to \mathfrak{g}$. If $f : \mathfrak{h} \to \mathfrak{p}$ is a homomorphism then exponentiating $f + \mathrm{D}\,\sigma$, gives a new section of (38), and so without loss of generality assume that $\mathrm{D}\,\psi$ has image in $\mathbb{R} \cdot a$, thus $\psi(H) \subset \Phi$. It follows that $\mathfrak{k}$ is the Lie algebra of $P(\psi)$ and $\sigma(\mathfrak{h})$ is the Lie algebra of $T_2(\psi)$. As a result $\mathfrak{g} = \mathfrak{k} \oplus \sigma(\mathfrak{h})$ is the Lie algebra of $T(\psi)$.

The strictly convex hypersurface $\partial\Omega$ is a $G$-orbit. It follows from Remark 1.2 that $\psi$ or $-\psi$ is positive. Without loss we may assume $\psi$ is positive, so $\psi \in A$ and $G$ is conjugate to $T(\psi)$. □

*Proof of 0.1 and 0.7.* Suppose $C'$ is a $g$-cusp. By 4.3 $C'$ is equivalent to a homogeneous $g$-cusp $C = \Omega/\Gamma$. Then by 4.6 $\Gamma$ contains a finite index subgroup $\Gamma_1$ that is conjugate to a subgroup of a $T$-group $T$. By 4.10, after a conjugacy, $T = T(\psi)$ for some $\psi$. The interior of $\Omega$ is a domain that is preserved by $T(\psi)$. We may assume it is irreducible, then by 1.27 $\Omega = g(\Omega(\psi))$ for some $g \in \mathcal{E}(\psi) \oplus \Phi^{\psi}$. Thus a conjugate of $\Gamma$ preserves $\Omega(\psi)$, so after conjugacy we may assume $\Gamma \subset \mathrm{PGL}(\Omega(\psi))$. If $\psi \neq 0$ then $\mathrm{PGL}(\Omega(\psi)) = G(\psi)$ and by 2.7, therefore $C$ is $\psi$-cusp. If $\psi = 0$ then $\Gamma_1 \subset T(0)$ so $\Gamma_1 \subset G(0)$. Since $\Gamma/\Gamma_1$ is finite and $\mathrm{PGL}(\Omega(0))/G(0) \cong \mathbb{R}$ it follows that $\Gamma_1 \subset G(0)$ and again $C$ is a $\psi$-cusp. This proves 0.1. It follows $\Gamma$ is virtually abelian, which proves 0.7. □

*Proof of 0.4.* We identify $\pi_1 M \equiv \Gamma$. Since $\delta([A]) = 0$, for every $\epsilon > 0$ there is a loop $\gamma$ in $M$ that has length less than $\epsilon$ and $[\gamma]$ is conjugate in $\pi_1 M$ to $[A]$. It follows that if $X \subset M$ is compact then $[A]$ is represented by a loop in $M \setminus X$. Thus $[A]$ is represented by a loop in an end of $M$, and therefore in a generalized cusp $C \subset M$ with $C = \Omega(\psi)/\Gamma_C$. Since $\delta([A]) = 0$, then we can conjugate so $\det A = \pm 1$, and then $A \in G(\psi)$. The result now follows (1.23). □

## 5. Hilbert Metric in a generalized cusp

In this section we describe how the Hilbert metric of a horomanifold changes as one moves out into the cusp using the radial flow. The horomanifolds shrink, although not uniformly in all directions. Parabolic directions (which only exist when $\mathbf{u} > 0$) shrink exponentially with distance out into the cusp, but hyperbolic directions shrink towards a limiting positive value. Hence the volume of the cusp cross-section (horomanifold) goes to zero exponentially fast when $\mathbf{u} > 0$, and the cusp has finite volume. When $\mathbf{u} = 0$ the cusp cross-section converges geometrically to compact $(n-1)$-manifold, and in this case the cusp has infinite volume. In this discussion volume means *Hausdorff measure.*

If $\Omega$ is an open properly-convex set in $\mathbb{R}\mathrm{P}^n$ the *Hilbert metric* on $\Omega$ is defined as follows. Suppose $p, q \in \Omega$ lie on the line $\gamma : [a,b] \to \mathbb{R}\mathrm{P}^n$ given by $\gamma(t) = [(t-a)\vec{u} + (b-t)\vec{v}]$ with endpoints

$[\vec{u}], [\vec{v}] \in \partial\overline{\Omega}$ and interior in $\Omega$. If $p = [\gamma(x)]$ and $q = [\gamma(y)]$ then

$$d_\Omega(p,q) = \frac{1}{2}\log\left(\frac{|b-x|\,|y-a|}{|b-y|\,|x-a|}\right)$$

Since cross ratios are preserved by projective transformations this is independent of the choice of $\gamma$. This is a Finsler metric. For vectors tangent to this line, the Finsler norm is the pullback of the Riemannian metric on $(a,b)$ given by

$$\frac{1}{2}\left(\frac{1}{x-a} + \frac{1}{b-x}\right)|dx| \tag{40}$$

When $\Omega$ is projectively equivalent to an ellipsoid then $(\Omega, d_\Omega)$ is isometric to hyperbolic space $\mathbb{H}^n$. Otherwise the metric is not Riemannian.

**Lemma 5.1.** *There is a decreasing function $\kappa : \mathbb{R}_+ \to (1,\infty)$ such that $\lim_{x\to\infty}\kappa(x) = 1$ with the following property. Suppose $\Omega' \subset \Omega \subset \mathbb{RP}^n$ are both open and properly-convex. Let $\|\cdot\|'$ and $\|\cdot\|$ be the Hilbert norms on $\Omega'$ and $\Omega$. Suppose $p \in \Omega'$ and $d_\Omega(p, \Omega\setminus\Omega^-) > x$ then $\|\cdot\|_p \le \|\cdot\|'_p \le \kappa(x)\|\cdot\|_p$.*

*Proof.* Since the definition of the Hilbert metric only involves a line segment, it suffices to prove the result in dimension $n = 1$ with $\Omega = (-1,1)$ and $\Omega' = (-u,u)$ and $0 < u < 1$. It is easy to do this. $\square$

Given a metric space $(M,d)$ the $k$-dimensional Hausdorff measure is defined as follows. If $B(x;r,M)$ is the ball of radius $r$ in $M$ center $x$ then $\nu(B(x,r)) = c_k r^k$ where $c_k$ is the volume of the ball of radius 1 in $\mathbb{R}^k$. If $\mathcal{S}$ is a set of balls in $M$ then $\nu(\mathcal{S}) = \sum_{B\in\mathcal{S}}\nu(B)$. Given a subset $X \subset M$ and $\epsilon > 0$ define $\nu_\epsilon(X) = \text{infimum}\,\nu(\mathcal{S})$ where the infimum is over all sets, $\mathcal{S}$, of balls with radius at most $\epsilon$ that cover $X$. Then define an outer measure by $\nu(X) = \lim_{\epsilon\to 0}\nu_\epsilon(X)$. This gives a measure on $M$ in the usual way, called $k$-dimensional Hausdorff measure, denoted $\text{vol}_k$.

If $\alpha$ is an arc in $M$ then $\text{vol}_1(\alpha)$ is the length of the arc. We will use $\text{vol}_{n-1}$ to measure the size of a horomanifold in a generalized cusp.

If $M$ is an $n$-dimensional manifold with a Finsler metric then the measure $\text{vol}_n$ is given by a integrating a certain $n$-form called the *volume form.* Suppose $p \in M$ and $B \subset T_pM$ is the unit ball in the given norm. The volume form on $T_pM$ is normalized so that the volume of $B$ is the Euclidean volume, $c_n$, of the unit $n$-dimensional Euclidean unit ball. Thus if $\omega \ne 0$ is an $n$-form on $T_pM$ then the volume form dvol on $T_pM$ is

$$\text{dvol} = c_n\left(\int_B \omega\right)^{-1}\omega$$

This defines a Borel measure $\text{vol}_M$ on $M$ given by

$$\text{vol}_M(X) = \int_X \text{dvol}$$

For a Riemannian metric this is the usual volume form. For $X \subset M$ we refer to $\text{vol}_n(X)$ as its *volume* written $\text{vol}_n(X;M) = \text{vol}(X)$. For a properly convex projective $n$-dimensional manifold $\text{vol}_n$ is also called *Busemann measure.*

**Lemma 5.2.** *Suppose $M$ and $M'$ are Finsler manifolds of dimension $n$ and $f : M \to M'$ is a diffeomorphism. If $f$ is 1-Lipschitz then* $\text{vol}(f(X);M') \le \text{vol}(X;M)$

*Proof.* Since $f$ is 1-Lipschitz $f(B(x;r,M)) \subset B(f(x);r,M')$ which easily implies the result. $\square$

**Corollary 5.3.** *Suppose $M$ is a Finsler $n$-manifold and $U \subset \mathbb{R}^n$ is convex, and $f : U \to M$ is smooth. Suppose $f$ is 1-Lipschitz, and along lines in the $e_1$ direction that $f$ is $K$-Lipschitz. Then $\text{vol}(f(U);M)) \le K\cdot\lambda(U)$ where $\lambda$ is Lebesgue measure on $\mathbb{R}^n$.*

*Proof.* Let $W = \{(Kx_1, x_2, \cdots, x_n) : (x_1,\cdots,x_n) \in U\}$ then $g : W \to M$ given by $g(x_1,\cdots,x_n) = f(K^{-1}x_1, x_2\cdots, x_n)$ is 1-Lipschitz. The result now follows from (5.2). $\square$

**Corollary 5.4.** *With the hypotheses of (5.1) then*

$$\text{vol}(A;\Omega) \le \text{vol}(A;\Omega') \le (\kappa(x))^n \, \text{vol}(A;\Omega)$$

5.1. **The Radial Flow and the Hilbert metric.** Suppose $(\Omega, G)$ is a $\psi$-geometry. If $p \neq q$ are two points in $\Omega(\psi)$ then $q - p \in \mathbb{R}^n$ is called a *parabolic direction at* $p$ if there is $A \in P(\psi)$ with $A(p) = q$. The infinitesimal version of this is that a *parabolic tangent vector* is a vector $v \in T_p\Omega$ that is tangent to the orbit of point $p$ under the action of a 1-parameter subgroup of $P(\psi)$. If $\mathbf{u} = 0$ there are no parabolic directions, and if $\mathbf{t} = 0$ then every vector tangent to a horosphere is a parabolic direction. In general the parabolic directions correspond to the $y$-coordinates in $(x, z, y)$ coordinates.

**Lemma 5.5.** *Let $\Psi$ be the radial flow on $\Omega = \Omega(\psi)$ and $\mathcal{H}_t = \Phi_t(\partial\Omega)$ the $t$-horosphere. Suppose $p \neq q \in \mathcal{H}_{-1} \subset \mathbb{R}^n$ and define $q(t) = \Phi_{-t}(q)$ and $p(t) = \Phi_{-t}(p)$ and $f(t) = d_\Omega(p(t), q(t))$. Then*

1. *$f(t)$ is decreasing function of $t$*
2. $\lim_{t\to 0} f(t) = 0$ *iff and only if $q - p$ is a parabolic direction at $p$*
3. *If $q - p$ is parabolic then* $\log f(t) \approx -O(d_\Omega(p, p(t)))$.

*Proof.* (1) follows from [23]. First assume $\mathbf{t} < n$ so the radial flow is $\Phi_t(x, z, y) = (x, z - t, y)$ and moves points in the $z$-direction called *vertical*. Let $I(t) \subset \mathbb{R}^n$ be the intersection with $V_\psi$ of the line containing $p(t)$ and $q(t)$, then $I(t) = \Phi_{-t}(I(0))$.

Observe that $I = I(0)$ is a complete affine line if and only if $q - p = (0, z, y)$ in $(x, z, y)$ coordinates, which is equivalent to $q - p$ is a parabolic direction. Also $I$ contains the maximal line segment $J$ in $\Omega$ that contains $p$ and $q$. Thus if $I$ is not a complete line, then $d_\Omega(p, q) = d_J(p, q) \ge d_I(p, q) > 0$. The above implies $f(t) \ge d_I(p, q)$. Hence if $q - p$ is not parabolic then $f(t)$ is bounded below.

Now suppose $q - p$ is parabolic. The radial flow commutes with the translation group, so it suffices to prove the lemma for $p = \Phi_{-1}(b_\psi)$. Then $p(t) = b_\psi + (t + 1)e_{r+1}$ and $q(t) = p(t) + q - p$. Let $P \subset \mathbb{R}^n$ be the affine 2-plane containing the two flow lines $p(t)$ and $q(t)$. Since $q - p = (0, z, y)$ it follows that $x_i$ is constant on $P$ for $i \le r$. Then (2) implies $h_\psi|P$ is quadratic, so $P \cap \Omega$ is a convex set bounded by a parabola.

The rays $p(t)$ and $q(t)$ are parallel in $P \cap \Omega$. Let $U = \{(x, y) : y > x^2\}$. Then $d_U((1, t), (2, t)) = O(t^{-2})$ for large $t$, and $d_U((1, 1), (1, t)) = \log t$. These computations differ from those for $p(t)$ and $q(t)$ by bounded amounts. Hence $f(t) = O(t^{-2})$ so $-\log f(t) = O(t)$ and $d_\Omega(p, p(t)) \approx \log t$.

When $\mathbf{t} = n$ there are no parabolic directions. In this case $p(t)$ and $q(t)$ are rays in $\mathbb{R}^n$ contained in lines through 0. The closure of $\mathbb{R}^n_+$ in $\mathbb{RP}^n$ is an $n$-simplex $\Delta$ that contains $\Omega$, so $d_\Omega \ge d_\Delta$ and $f(t) \ge d_\Delta(p(t), q(t))$. The rays $p(t)$ and $q(t)$ limit on distinct points $p_\infty, q_\infty$ in the interior of $\partial_\infty\Delta$, and $d_\Delta(p(t), q(t))$ is bounded below by the Hilbert distance in $\partial_\infty\Delta$ between $p_\infty$ and $q_\infty$. □

The next result describes how the volume of a subset of a generalized cusp shrinks as it flows out into the end of the cusp using the radial flow. The asymptotic behavior depends only on the parabolic rank $\mathbf{u}$ of the cusp. If $\mathbf{u} > 0$ the volume of the region shrinks exponentially with distance as it flows out, but if $\mathbf{u} = 0$ the volume stays bounded away from 0.

**Proposition 5.6.** *Suppose $C = \Omega/\Gamma$ is a generalized cusp of dimension $n$ and unipotent rank $\mathbf{u}$. For $t \le 0$ let $\Phi_t : C \to C$ be the restriction of the radial flow and $X \subset \partial C$ and $X_t = \Phi_t(X)$ then there exists constants $\gamma$ and $\delta$ such that*
*if $\mathbf{u} = 0$ then* $\text{vol}_{n-1}(X_t; C) > \delta \cdot \text{vol}_{n-1}(X; C)$
*if $\mathbf{u} > 0$ then* $\text{vol}_{n-1}(X_t; C) < \gamma \exp(-d_X(X, X_t)) \cdot \text{vol}_{n-1}(X; C)$.

*Proof.* This follows from 5.5. When $\mathbf{u} > 0$ there are parabolic directions, and it follows from Lemma 5.5 that the map $\Phi_t : X \to X_t$ is a 1-Lipschitz map that exponentially contracts distance in parabolic directions, so the estimate in this case follows from Corollary 5.3.

When $\mathbf{u} = 0$ there are no parabolic directions, and part (2) of Lemma 5.5 implies $\Phi_t^{-1} : X_t \to X$ is a $K$-Lipschitz map (with $K$ is independent of $t$). Again, the estimate follows immediately for $\delta = K^{-n}$. □

**Theorem 5.7.** *Suppose $C$ is a generalized cusp that is contained in the interior of a properly-convex manifold $M$. Then $vol_M(C) < \infty$ if and only if the parabolic rank $\mathbf{u}$ of $C$ satisfies $\mathbf{u} > 0$.*

*Proof.* This follows easily from the estimates in Proposition 5.6. □

*Proof of Theorem 0.6.* Let $C = \Omega/\Gamma$ be a generalized cusp in a properly convex manifold $M$ such that $\Gamma$ is conjugate into $G(\psi)$. From the construction and description of $G(\psi)$ it is easy to see that $G(\psi)$ contains a parabolic element if and only $\mathbf{u} > 0$. Furthermore from Theorem 5.7 it follows that $C$ has finite volume in $M$ if and only if $\mathbf{u} > 0$. □

## 6. Dimension 2

In this section we describe 2-dimensional generalized cusps in a way that illuminates the higher dimensional cases, and can be read before the rest of the paper.

A *generalized cusp*, $C$, in a properly-convex surface, $M$, is a convex submanifold $C \cong S^1 \times [0,\infty)$ of $M$ with $M \setminus C$ connected and $\partial C$ is a strictly convex curve in the interior of $M$. Thus $C = \Omega/\Gamma$, where $\Gamma$ is an infinite cyclic group generated by some element $[A] \in \mathrm{PGL}(3,\mathbb{R})$, and $\Omega$ is properly-convex, and homeomorphic to a closed disc with one point deleted from the boundary, and $\partial\Omega := \Omega \setminus \mathrm{int}(\Omega)$ is a strictly convex curve that covers $\partial C$. The proof of the following is routine.

**Theorem 6.1.** *A generalized cusp has holonomy conjugate to a group generated by $[A]$ where either $A$ is diagonal with three distinct positive eigenvalues, or else is one of*

$$\begin{pmatrix} e^a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad a \neq 0$$

We regard $\mathrm{Aff}(\mathbb{R}^2)$ as a subgroup of $\mathrm{PGL}(3,\mathbb{R})$. For each $\psi = (\psi_1, \psi_2) \in \mathbb{R}^2$ with $\psi_1 \geq \psi_2 \geq 0$ there is a one-dimensional subgroup $T(\psi) \subset \mathrm{Aff}(\mathbb{R}^2)$

$$T(\psi) = \begin{array}{ccc} \psi_1 \geq \psi_2 > 0 & \psi_1 > \psi_2 = 0 & \psi_1 = \psi_2 = 0 \\ \begin{pmatrix} e^x & 0 & 0 \\ 0 & e^{-x.\psi_2/\psi_1} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} e^x & 0 & 0 \\ 0 & 1 & -\psi_1 x \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & x & x^2/2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \end{array} \qquad x \in \mathbb{R}$$

The holonomy of a generalized cusp is conjugate in $\mathrm{PGL}(3,\mathbb{R})$ into one of these groups. The orbit of the basepoint (1.6) under each of these Lie groups is a convex curve $\gamma$ in $\mathbb{R}^2$ and the convex hull of $\gamma$ is a properly-convex closed set $\Omega = \Omega(\psi) \subset \mathbb{R}^2$ as shown in Figure (2), that is preserved by the group.

The closure of $\Omega$ in $\mathbb{RP}^2$ is $\overline{\Omega} = \Omega \sqcup \partial_\infty\Omega$ where $\partial_\infty\Omega \subset \mathbb{RP}^\infty_1$, and $\partial_\infty = [e_1]$ for $T(0,0)$, and it is the closed line segment $\{[te_1 + (1-t)e_2] : 0 \leq t \leq 1\}$ with endpoints $[e_1]$ and $[e_2]$ in the remaining cases as shown in Figure (2).

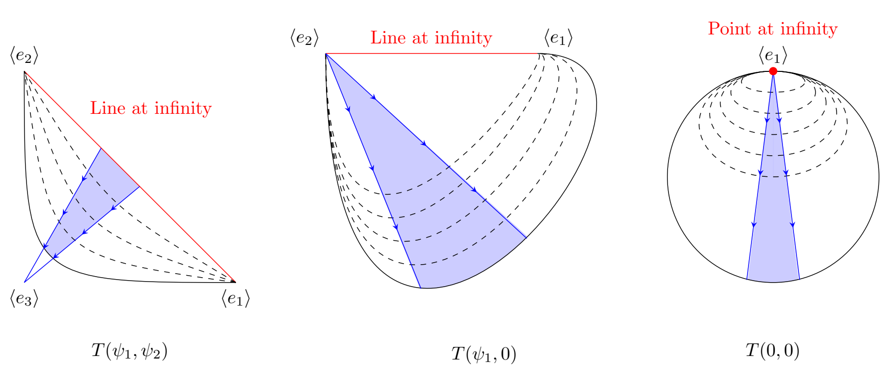


FIGURE 2. Generalized Cusps: Projective View

Goldman classified convex projective structures on closed surfaces [15], and Marquis [20], [19] shows that if $S$ is a finite type surface without boundary, then a properly-convex projective structure on $S$ has finite area if and only if the holonomy of each end of $S$ is unipotent: conjugate into $T(0,0)$.

Each domain $\Omega(\psi)$ has two foliations that are preserved by $T(\psi)$. A *horocycle* is the orbit of a point under $T(\psi)$. The *radial flow* is a one parameter subgroup $\Phi^\psi \subset \mathrm{PGL}(3,\mathbb{R})$ that only depends on $\mathbf{t} = \mathbf{t}(\psi)$, which is the number of non-zero coordinates of $\psi$.

$$\begin{array}{cccc} & \mathbf{t}=2 & \mathbf{t}=1 & \mathbf{t}=0 \\ \Phi^\psi(t) = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \textit{center} = & [e_3] & [e_2] & [e_1] \end{array}$$

This group centralizes $T(\psi)$. The $\Phi$-orbit of a (non-stationary) point is called a *radial flow line* and is contained in a projective line. All these lines meet at a single point called the *center of the radial flow*. The foliation of $\Omega$ by (subarcs of) radial flow lines is transverse to the horocycle foliation. The domain $\Omega$ is backwards invariant under the radial flow: $\Phi_t(\Omega) \subset \Omega$ for $t \leq 0$.

The group $T(\mathbf{t}) := T(\psi) \oplus \Phi^\psi$ is called the *enlarged translation group* (14) is

$$\begin{array}{ccccc} & \mathbf{t}=2 & \mathbf{t}=1 & \mathbf{t}=0 & \\ T(\mathbf{t}) = & \begin{pmatrix} e^x & 0 & 0 \\ 0 & e^y & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} e^x & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} & \qquad x, y \in \mathbb{R} \end{array}$$

and $T(\psi)$ is the kernel of a homomorphism $T(\mathbf{t}) \to \mathbb{R}$ derived from $\psi$.

A fundamental domain for a generalized cusp is obtained by taking an interval $J \subset \partial\Omega$ that is a fundamental domain for the action there, and taking the backward orbit $\cup_{t\leq 0}\Phi_t(J)$ under the radial flow. We now describe these foliations, see Figures 2 and 3.

For $T_0$, the domain $\Omega = \{(x_1, x_2) : x_1 \geq x_2^2/2\}$, and the horocycles are $x_1 = C + x_2^2/2$, and the radial flowlines are $x_2 = C$. There is an identification of $\Omega$ with a horoball $B \subset \mathbb{H}^2$. The action of $T_0$ on $\Omega$ is then conjugated to the action of those parabolic isometries that preserve $B$. Horocycles in $\Omega$ map to horocycles in $B$ and radial flow lines in $\Omega$ map to hyperbolic geodesics that are orthogonal to the horocycles. In $\mathbb{RP}^2$ the horocycles for $\Omega$ are ellipses of unbounded eccentricity, all tangent at $[e_1]$.

The group $T_2(\psi_1, \psi_2)$ preserves the positive quadrant $\Delta = \{(x_1, x_2) : x_1, x_2 > 0\}$. The domain $\Omega$ is the subset of $\Delta$ with $x_1^{\psi_2} x_2^{\psi_1} \geq 1$, and is foliated by the horocycles $x_1^{\psi_2} x_2^{\psi_1} = C$. Each horocycle limits on the points $[e_1], [e_2] \in \mathbb{RP}^1_\infty$ that are the attracting and repelling fixed points of the holonomy. The radial flow lines in $\mathbb{R}^2$ are straight lines through the origin, which is the neutral fixed point of the holonomy.

For $T_1(\psi_1)$ the domain $\Omega = \{(x_1, x_2) : x_2 \geq -\psi_1 \log x_1, \quad x_1 > 0\}$. The horocycles are $x_2 = -\psi_1 \log x_1 + C$. At $[e_2] \in \partial_\infty\Omega$ the horocycles are transverse to $\partial_\infty\Omega$, but at $[e_1]$ they are tangent to $\partial_\infty\Omega$. The radial flow lines are the straight lines $x_1 = C$.

The subgroup $O(\psi) \subset \mathrm{PGL}(\Omega(\psi))$ is the stabilizer of a point. This group is trivial unless $\psi_1 = \psi_2$ in which case $O(\psi) \cong \mathbb{Z}_2$. The action of $O(\psi)$ is easily described in homogeneous coordinates on $\mathbb{RP}^2$. When $\lambda = (0,0)$ it is generated by the reflection $[x_1 : x_2 : x_3] \mapsto [x_1 : -x_2 : x_3]$ and otherwise by $[x_1 : x_2 : x_3] \mapsto [x_2 : x_1 : x_3]$. In each case this preserves $\Omega(\psi)$. If $\psi_1 \neq \psi_2$ then $\mathrm{PGL}(\Omega(\psi)) = T(\psi)$ and acts freely on $\Omega(\psi)$.

In all dimensions, a generalized cusp is determined by a lattice in a generalized cusp Lie group. For a surface, a lattice is infinite cyclic, and is determined by a nontrivial element of some $T(\psi)$ up to replacing the element by its inverse. A *marked lattice* is a lattice with a choice of basis. Thus conjugacy classes of lattices correspond to moduli space and conjugacy classes of marked lattices to Teichmuller space.

There is an equivalence relation on *marked* generalized cusps generated by projectively embedding one in another. Let $\mathcal{T}$ be the (Teichmuller) space of equivalence classes of *marked* generalized cusps

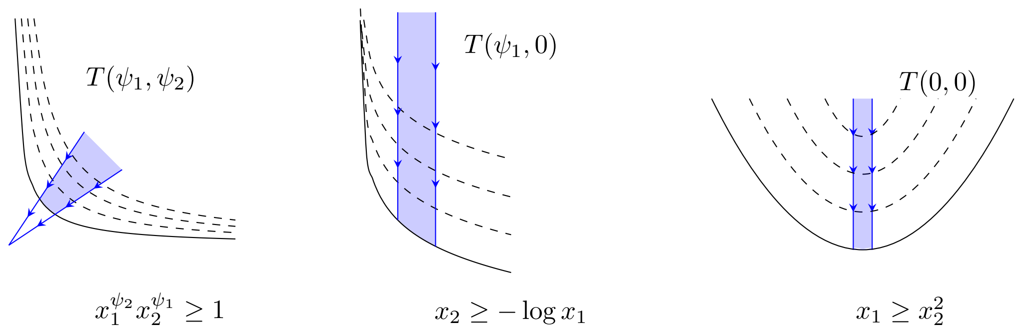


FIGURE 3. Generalized Cusps: Affine View

for surfaces. There is an identification of $\mathcal{T}$ with a subspace of $\mathrm{SL}(3,\mathbb{R})$ modulo conjugacy that sends a marked generalized cusp to the conjugacy class, $[A]$, of the holonomy of the chosen generator. The eigenvalues $\{\exp(x_1), \exp(x_2), \exp(x_3)\}$ of $A$ determine $[A]$ and satisfy $x_1 + x_2 + x_3 = 0$. Thus a generalized cusp is determined by $\{x_1, x_2, x_2\}$ up to permutations.

Let $X$ be the 2-orbifold $\mathbb{R}^2/S_3$ (closed Weyl chamber) where we identify $\mathbb{R}^2$ with the plane $x_1+x_2+x_3 = 0$ in $\mathbb{R}^3$, and the quotient is by the action of the symmetric group $S_3$ on the coordinates. Then $X$ can be identified with the fundamental domain for this action: $X = \{(x_1, x_2, x_3) : x_1+x_2+x_3 = 0, \quad x_1 \geq x_2 \geq x_3\}$; which can be identified with $Y = \{(y_1, y_2) : y_2 \geq y_1 \geq 0\}$ via $y_2 = x_1 - x_3$ and $y_1 = x_2 - x_3$.

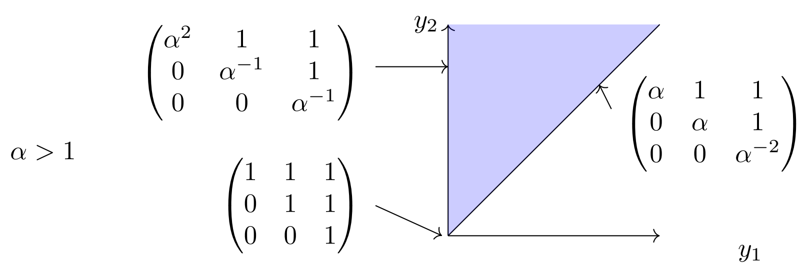


FIGURE 4. Y $\equiv$ Parameter space of 2 dimensional cusps

**Proposition 6.2.** *There is a homeomorphism $f : Y \to \mathcal{T}$ given by*

$$f(y_1, y_2) = \left[ \begin{pmatrix} \exp((2y_2 - y_1)/3) & 1 & 1 \\ 0 & \exp((2y_1 - y_2)/3) & 1 \\ 0 & 0 & \exp((-y_1 - y_2)/3) \end{pmatrix} \right]$$

*Proof.* By Theorem 6.1 the matrix shown determines a generalized cusp. Clearly $f$ is continuous. It is easy to check that $f$ is surjective. Consideration of eigenvalues shows $f$ is injective.

Suppose $A \in \mathrm{SL}(3,\mathbb{R})$ and $[A] \in \mathcal{T}$, then $A$ has real positive eigenvalues. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $A$ in decreasing order and define $g([A]) = (\log \lambda_1, \log \lambda_2, \log \lambda_3)$. Since the eigenvalues of a matrix are continuous functions of the matrix, $g$ is continuous. But $g$ is the inverse of $f$, so $f$ is a homeomorphism. $\square$

The groups $T(\psi)$ and $T(\psi')$ are conjugate in $\mathrm{PGL}(3,\mathbb{R})$ if and only if $\psi = t\psi'$ for some $t > 0$. It follows that the space of conjugacy classes of translation subgroup is the non-Hausdorff space obtained by taking the quotient of $X$ by this equivalence relation. This is the union of a compact Euclidean interval $[0,1]$ and one extra point which only has one neighborhood.

## 7. Dimension 3

Let $C = \Omega/\Gamma$ be an orientable 3-dimensional generalized cusp, then $C$ is diffeomorphic to $T^2 \times [0, \infty)$. Given $\psi = (\psi_1, \psi_2, \psi_3)$ with $\psi_1 \geq \psi_2 \geq \psi_3 \geq 0$ there is a Lie subgroup $G(\psi) = T(\psi) \rtimes O(\psi)$ of $\mathrm{PGL}(4, \mathbb{R})$, where $T(\psi) \cong \mathbb{R}^2$ is called the *translation group*, and $O(\psi)$ is compact. Then $\Gamma$ is conjugate to a lattice in some $T(\psi)$, and $\psi$ is unique up to multiplication by a positive scalar.

The Lie groups $T(\psi)$ fall into 4 families, depending on the *type* $\mathbf{t} = \mathbf{t}_\psi$, which is the number of non-zero components of $\psi$.

$$\begin{array}{cc}
\mathbf{t} = 0 & \mathbf{t} = 1 \\
\begin{pmatrix} 1 & y_1 & y_2 & \frac{1}{2}(y_1^2 + y_2^2) \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} &
\begin{pmatrix} e^{x_1} & 0 & 0 & 0 \\ 0 & 1 & y_1 & \frac{1}{2}y_1^2 - \psi_1 x_1 \\ 0 & 0 & 1 & y_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\mathbf{t} = 2 & \mathbf{t} = 3 \\
\begin{pmatrix} e^{x_1} & 0 & 0 & 0 \\ 0 & e^{x_2} & 0 & 0 \\ 0 & 0 & 1 & -\psi_1 x_1 - \psi_2 x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} &
\begin{pmatrix} e^{x_1} & 0 & 0 & 0 \\ 0 & e^{x_2} & 0 & 0 \\ 0 & 0 & e^{(-\psi_1 x_1 - \psi_2 x_2)/\psi_3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}$$

The group $T(\psi)$ preserves a properly convex domain $\Omega(\psi) \subset \mathbb{R}^3$ that is the convex hull of the $T(\psi)$-orbit of the *basepoint*, see (1.6). It has a foliation by convex surfaces called *horospheres*, that are $T(\psi)$-orbits. Moreover $\Omega(\psi)$ is the epigraph of a convex function, see (10), and is shown in Figure 5.

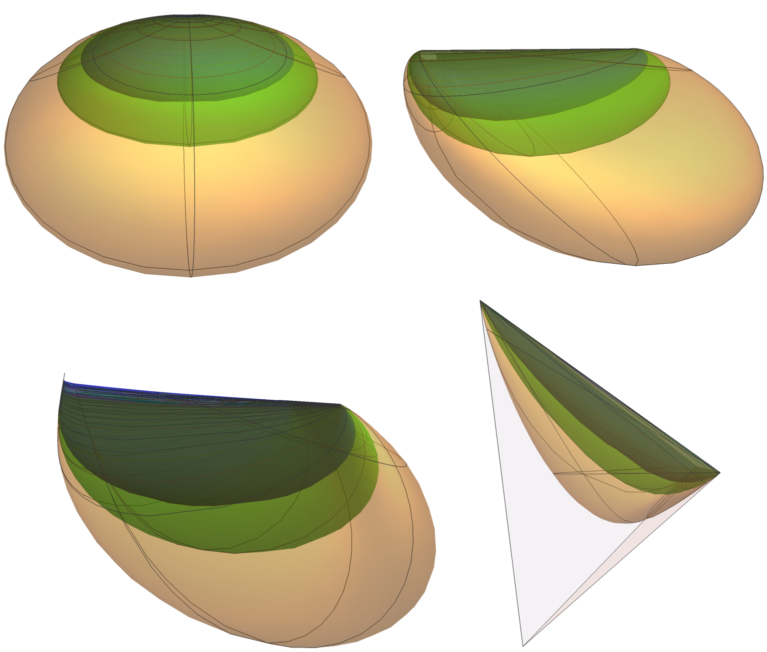

Figure 5. 3-dimensional generalized cusp domains and their foliation by horospheres in projective space. From left to right, top to bottom the domains are $\Omega(0,0,0)$, $\Omega(1,0,0)$, $\Omega(1,1,0)$, finally $\Omega(1,1,1)$ is shown inside a simplex

The radial flow (6) is a one-parameter affine group $\Phi^\psi$ that centralizes $T(\psi)$, and $\Phi^\psi$-orbits give a foliation by a pencil of lines transverse to the horospheres. The *enlarged translation group* (14) is $T_{\mathbf{t}} = T(\psi) \oplus \Phi^\psi \cong \mathbb{R}^3$. It is obtained by replacing the most complicated term in the matrix for $T(\psi)$ by $z$. There are 4 such groups, depending only on $\mathbf{t}$. Then $T_\psi$ is the kernel of a homomorphism $T_{\mathbf{t}} \to \mathbb{R}$ obtained from $\psi$. The group $T(\mathbf{t})$ acts simply transitively on $\mathbb{R}^{\mathbf{r}}_+ \times \mathbb{R}^{3-\mathbf{r}}$, and the latter contains $\Omega(\psi)$.

The group, $O(\psi)$, is the subgroup of $G(\psi)$ that fixes the basepoint (see (1.6)). It is computed in (1.21), and $O(0,0,0) \cong O(2)$, and $O(\psi_1, 0, 0) \cong O(1)$ when $\psi_1 \neq 0$. For the remaining cases, $O(\psi)$ is the group of coordinate permutations of $\mathbb{R}^3$ that preserve $\psi$. In particular $O(\psi)$ is finite unless $\psi = 0$.

There is a 6 parameter family of marked, 3-dimensional generalized cusps. As described in 0.2 they are parameterized by a triple $(\psi, \Gamma, A \cdot O(\psi))$ with $\psi$ as above, and $\Gamma$ is a marked lattice of co-area 1 in $\mathbb{R}^2$, and $A \cdot O(\psi) \in O(2)/O(\psi)$ is a left coset.

In [18], the third author showed that in dimension 3, every translation group, as defined in 4.8, is conjugate into one of these 4 families. This, together with [1], provided the *impetus* for the present paper.

We now describe some geometric properties of these domains and discuss relevant examples from the literature. The interior of $\Omega(0,0,0)$ is projectively equivalent to $\mathbb{H}^3$. If $C \cong \Omega(0,0,0)/\Gamma$ then $\Gamma$ is conjugate into $\mathrm{PO}(3,1)$. Cusps of finite volume hyperbolic 3-manifolds give rise to generalized cusps of this type. The ideal boundary, see (13), of $\Omega(0,0,0)$ consists of a single point which is stabilized by $G(\psi)$, and $C$ admits a compactification by a singular projective manifold obtained by adjoining this ideal boundary point.

For a generalized cusp $C = \Omega/\Gamma_C$ modelled on $\Omega = \Omega(1,0,0)$ the ideal boundary, $\partial_\infty \Omega$ is a projective line segment $J$. The action of $\Gamma_C$ on $I = \mathrm{int}(J)$ is discrete iff $\Gamma$ contains a parabolic. In this case $C$ has a compactification $\overline{C} = (\Omega \cup I)/\Gamma_C$ that is a projective manifold that is singular along the circle $S^1 = I/\Gamma_C$.

In [1], the first author found, for $t \in [0, \infty)$, a continuous family of properly convex manifolds projectively equivalent to $M_t = \Omega_t/\Lambda_t$, and diffeomorphic to the figure-8 knot complement, $X = S^3 \setminus K$, and $M_0$ is the complete hyperbolic structure. Moreover the end of $M_t$ is projectively equivalent to $\Omega(t,0,0)/\Gamma_t$, where $\Gamma_t \subset T(t,0,0)$ is a lattice containing parabolics. As a result, for $t > 0$, there is a compactification $\overline{M}(t) = \Omega_t^+/\Lambda_t$ that is a projective structure on $S^3$ that is singular along $K$, and $M_t = \overline{M}(t) \setminus K$ is a properly convex structure on $X$. Here $\Omega_t^+ \supset \Omega_t$ and also contains the $\Gamma_t$-orbit of an open segment in $\partial_\infty \Omega(t,0,0)$. The cusp of the hyperbolic manifold $M_0$ deforms to a generalized cusp of a different type. As the deformation proceeds, an ideal boundary point of $\mathbb{H}^3$ *opens up* into an ideal boundary segment. This is an example of a *geometric transition*; the hyperbolic cusp $\Omega(0,0,0)/\Gamma_0$ geometrically transitions to the non-hyperbolic cusp $\Omega(t,0,0)/\Gamma_t$ as $t$ moves away from zero, cf. [12] and [9]. Higher dimensional examples of hyperbolic manifolds deforming to properly convex manifolds with type 1 cusps can be found in [3]. Furthermore, in subsequent work, the authors will show that every generalized cusp arises as a deformation of a hyperbolic cusp in this way.

The domains of the form $\Omega(\psi_1, \psi_2, 0)$ have ideal boundary a 2-simplex, $\Delta$. The interior of one of the edges of $\Delta$ consists of $C^1$ points, and the remainder of the 1-skeleton of $\Delta$ consists of non-$C^1$ points. In particular, the fixed point of the radial flow is the intersection of the two edges of non-$C^1$ points of $\Delta$, see (1.11). Any lattice in $T(\psi_1, \psi_2, 0)$ acts properly discontinuously on $\Delta$. Thus $C = \Omega(\psi_1, \psi_2, 0)/\Gamma$ has a manifold compactification by adjoining $\Delta/\Gamma$. There are currently no known examples where the hyperbolic structure on a finite volume 3-manifold deforms to a manifold of this type. However, Gye-Seon Lee [17] has produced some numerical deformations of the complete hyperbolic structure on both the figure-eight knot complement and the figure-eight knot sister that appear to have ends of this type.

Finally, the domains of the form $\Omega(\psi_1, \psi_2, \psi_3)$ also have ideal boundary consisting of a 2-simplex $\Delta$. However, in this case each point of the 1-skeleton of $\Delta$ is a non-$C^1$ point. As in the previous case, if $\Gamma$ is a lattice in $T(\psi_1, \psi_2, \psi_3)$ then $\Gamma$ acts properly discontinuously on $\Delta$ and there is a compactification of $C$ by adjoining $\Delta/\Gamma$. There are examples of properly convex deformations of the complete hyperbolic structure on finite volume hyperbolic 3-manifolds whose topological ends are of the form $\Omega(\psi_1,\psi_2,\psi_3)/\Gamma$, where $\Gamma \leq T(\psi_1,\psi_2,\psi_3)$. The first such examples were constructed for the figure-eight knot complement and the figure-eight sister by Gye-Seon Lee [17]. His examples are constructed by gluing together two projective *ideal* tetrahedra using the combinatorial pattern that produces the figures-eight knot complement (see Chapter 3 of [25] for details). Subsequent work of the first author, J. Danciger and G-S. Lee showed that any finite volume hyperbolic 3-manifold that satisfies a mild cohomological condition (that is known to be satisfied by infinitely many hyperbolic 3-manifolds) also admits deformations all of whose ends are projectively equivalent to $\Omega(1,1,1)/\Gamma$, where $\Gamma \leq T(1,1,1)$, thus producing many additional examples.

Furthermore, as explained in (1.5), the lack of $C^1$ points in the 1-skeleton of the ideal boundary allows properly convex manifolds with ends projectively equivalent to quotients of $\Omega(\psi_1,\psi_2,\psi_3)$ to sometimes be glued together to produce new properly convex manifold. This idea is explored in detail in [2] and using these techniques it is possible to find properly convex projective structures on non-hyperbolic 3-manifolds. This was first done by Benoist [4] using Coxeter orbifolds.

## References


[1] Ballas, S. *Finite Volume properly-convex Deformations of the Figure-Eight Knot* Geom. Dedicata 178 (2015), 49-73

[2] Ballas, S., Danciger, J., and Lee, G. *Convex Projective Structures on Non-Hyperbolic 3-Manifolds* Preprint August 2015. http://arxiv.org/pdf/1508.04794v1.pdf

[3] Ballas, S. and Marquis, L., *Properly Convex Bending of Hyperbolic Manifolds* Preprint September 2016. https://arxiv.org/pdf/1609.03046.pdf

[4] Benoist, Yves, *Convexes divisibles. IV. Structure du bord en dimension 3*, Invent. Math. ,164, (2)249-278, (2006)

[5] Charlap, L. *Bieberbach groups and flat manifolds* Springer-Verlag, (1986)

[6] Choi, S. *A Classification of Radial or Totally Geodesic Ends of Real Projective Orbifolds I: A Survey of Results.* ArXiv e-prints, January 2015. http://arxiv.org/pdf/1507.00809v1.pdf

[7] S. Choi, G. Lee, and L. Marquis. *Deformations of convex real projective structures on manifolds and orbifolds.* Survey for the Handbook of Group Actions edited by Lizhen Ji, Athanase Papadopoulos and Shing-Tung Yau

[8] Cooper, D. *The Heisenberg group acts on a strictly convex domain.* Conformal Geometry and Dynamics 21 (2017) 101-104

[9] Cooper, D., Danciger, J., and Wienhard, A. *Limits of Geometries.* to appear in Trans. Amer. Math. Soc. 2017

[10] Cooper, D., Long, D., and Tillmann, S. *On Convex Projective Manifolds and Cusps.* Advances in Mathematics 277 (2015), 181-251.

[11] Cooper, D., Long, D., and Tillmann, S., *Deforming Convex Projective Manifolds*, 2015. to appear: Geometry and Topology, http://arxiv.org/pdf/1511.06206.pdf,

[12] Danciger, J. *A geometric transition from hyperbolic to anti-de Sitter geometry*, Geom. Topol. 17 (2013) (5) ,3077-3134

[13] Danciger, J., Kassel, F., and Guéritaud, F. *Convex Cocompact Actions in Real Projective Geometry* 2017 https://arxiv.org/pdf/1704.08711.pdf

[14] Goldman, W. *Projective Geometry on Manifolds.* http://www.math.umd.edu/∼wmg/gstom.pdf

[15] Goldman, W. *Convex real projective structures on compact surfaces.* J. Differential Geom 31, (1990) (3), 791-845

[16] P. de la Harpe. On Hilbert's metric for simplices. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 97–119. Cambridge Univ. Press, Cambridge, 1993.

[17] Lee, G.-S., Personal Communication

[18] Leitner, A. *A Classification of Subgroups of $SL(4,\mathbb{R})$ Isomorphic to $\mathbb{R}^3$ and Generalized Cusps in Projective 3-Manifolds.* Topology and its Applications 206, (2016), 241-254

[19] Marquis, L. *Espaces des modules des surfaces projectives proprement convexes de volume fini.* Geometry and topology 14, (2010), p. 2103-2149

[20] Marquis, L. *Surface projective proprement convexe de volume fini.* Annales de l'institut Fourier 62, 1 (2012), p. 325-392.

[21] Marquis, L. Around Groups in Hilbert Geomety. In *A Handbook of Hilbert Geometry*, eds Papadopoulos, A., and Troyanov, M. IRMA Lectures in Mathematics and Theoretical Physics (22) 2014 p.207-261.

[22] Shima, H. *Homogeneous Hessian manifolds* Ann. Inst. Fourier (Grenoble), 30 (1980) 3, 91-128

[23] E. Socié-Méthou. *Behaviour of distance functions in Hilbert-Finsler geometry.* Differential Geom. Appl., 20(1):1–10, 2004.

[24] Schneider, R. *Closed convex hypersurfaces with second fundamental form of positive curvature.* Proc. Amer. Math. Soc., 35, 1972, 230-233.

[25] Thurston, W. *The Geometry and Topology of Three-Manifolds.* Princeton Univ. Math. Dept. Available from http://msri.org/publications/books/gt3m/, 1979.