SHADOWS OF CHARACTERISTIC CYCLES, VERMA MODULES, AND POSITIVITY OF CHERN-SCHWARTZ-MACPHERSON CLASSES OF SCHUBERT CELLS

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Abstract. Chern-Schwartz-MacPherson (CSM) classes generalize to singular and/or non-compact varieties the classical total homology Chern class of the tangent bundle of a smooth compact complex manifold. The theory of CSM classes has been extended to the equivariant setting by Ohmoto. We prove that for an arbitrary complex algebraic manifold X, the homogenized, torus equivariant CSM class of a constructible function φ is the restriction of the characteristic cycle of φ via the zero section of the cotangent bundle of X. This extends to the equivariant setting results of Ginzburg and Sabbah. We specialize X to be a (generalized) flag manifold G/B. In this case CSM classes are determined by a Demazure-Lusztig (DL) operator. We prove a 'Hecke orthogonality' of CSM classes, determined by the DL operator and its Poincaré adjoint. We further use the theory of holonomic $\mathcal{D}_X$-modules to show that the characteristic cycle of the Verma module, restricted to the zero section, gives the CSM class of a Schubert cell. Since the Verma characteristic cycles naturally identify with the Maulik and Okounkov's stable envelopes, we establish an equivalence between CSM classes and stable envelopes; this reproves results of Rimányi and Varchenko. As an application, we obtain a Segre type formula for CSM classes. In the non-equivariant case this formula is manifestly positive, showing that the extension in the Schubert basis of the CSM class of a Schubert cell is effective. This proves a previous conjecture by Aluffi and Mihalcea, and it extends previous positivity results by J. Huh in the Grassmann manifold case. Finally, we generalize all of this to partial flag manifolds G/P.

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1. Introduction

According to a conjecture attributed to Deligne and Grothendieck, there is a unique natural transformation $c_* : \mathcal{F} \to H_*$ from the functor of constructible functions on a complex algebraic variety X to homology, where all morphisms are proper, such that if X is a smooth then $c_*(1\!\!1_X) = c(TX) \cap [X]$. This conjecture was proved by MacPherson [39]; the class $c_*(1\!\!1_X)$ for possibly singular X was shown to coincide with a class defined earlier by M.-H. Schwartz [54, 55, 11]. For any constructible subset $W \subset X$, we call the class $c_{SM}(W) := c_*(1\!\!1_W) \in H_*(X)$ the *Chern-Schwartz-MacPherson* (CSM) class of W in X. The theory of CSM classes was later extended to the equivariant setting by Ohmoto, [42]. We denote by $c^T_{\text{SM}}(W) := c^T_*(1\!\!1_W)$ the torus equivariant CSM class.

The main object of study in this paper are the (torus) equivariant CSM classes of Schubert cells in flag manifolds. These classes were computed in various generality: for Grassmannians, in the non-equivariant specialization, by Aluffi and Mihalcea [4, 41], and Jones [27]; for type A partial flag manifolds by Rimányi and Varchenko [48], using the fact that they coincide with certain weight functions studied in [49, 47, 45], and using interpolation properties obtained by Weber [60]; and for flag manifolds in all Lie types by Aluffi and Mihalcea [5] using Bott-Samelson desingularizations of Schubert varieties.

The CSM classes satisfy an impressive range of properties. On the algebraic side, these properties stem from two sources: relations to Hecke algebras, and to stable envelopes. We briefly explain this next. It was proved in [5] that the equivariant CSM classes arise from a twisted representation of the Weyl group on the equivariant cohomology of the flag manifold, via certain Demazure-Lusztig operators. In type A, this representation was studied by Lascoux, Leclerc and Thibon [34, 35], and in all types these operators have been considered by Ginzburg [22] in relation to the (affine, degenerate) Hecke algebra. Second, Rimányi and Varchenko [48] (see also Su's thesis [59]) showed that the CSM classes are essentially equivalent to the *stable envelopes* of Maulik and Okounkov [40] for the cotangent bundle of flag manifolds. The interplay between these two points of view allows us to establish precise orthogonality results among these CSM classes and associated 'dual' CSM classes.

The natural geometric framework where these points of view converge is that of the characteristic cycles on the cotangent bundle. Classical results of Ginzburg [21] and Sabbah [50] (see also [52]) showed that the (non-equivariant) MacPherson's transformation factors through the group of conic Lagrangian cycles in the cotangent bundle. We revisit this theory, and extend it to the equivariant setting. In fact, we prove that Sabbah-Ginzburg map relating conic Lagrangian cycles to homology of flag manifolds has a simple intersection theoretic meaning: it is the restriction to the zero section of a conic Lagrangian cycle, taking into account the $\mathbb{C}^*$-action on the cotangent fibres. For flag manifolds, a rich source of characteristic cycles is given by the proof of Brylinski-Kashiwara [14] and Beilinson-Bernstein [7] of Kazhdan-Lusztig conjectures. In this context, the Lagrangian cycles corresponding to CSM classes are the characteristic cycles of the Verma modules.

We combine the algebraic and geometric frameworks to prove a Segre type formula for the CSM classes of Schubert cells, in tems of the characteristic cycle of the Verma modules. The formula is manifestly positive in the non-equivariant case, and it proves the non-equivariant version of a positivity conjecture from [5] for the CSM classes of Schubert cells in flag manifolds. This generalizes a similar positivity result proved by J. Huh [25] for the Grassmannian case (in turn proving an earlier conjecture posed in [4]).

1.1. **Statement of results.** Next we give a more precise account of the results we are proving. The first part of the paper consists of a general discussion of Ohmoto's torus equivariant CSM classes from the point of view of Lagrangian cycles in the cotangent bundle of a smooth complex algebraic variety X. We extend the formalism of *shadows* of characteristic classes developed in [2] to build a dictionary between $\mathbb{C}^*$-equivariant classes in a vector bundle E endowed with a $\mathbb{C}^*$-action by fiberwise dilation and *homogenizations* of shadows of corresponding classes in the projective completion of E (Proposition 2.7). Here, the homogenization of a class $c = \sum_{i=0}^n c_i$ with $c_i \in H_i(X)$, with respect to the character χ defining the action, is

$$c^\chi := c_0 + c_1\chi + \cdots + c_n\chi^n \in H_0^{\mathbb{C}^*}(X) \quad .$$

Note that $H_*^{\mathbb{C}^*}(X) \cong H_*(X)[\hbar]$, where $\hbar := c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1)) \in H^2_{\mathbb{C}^*}(pt)$ corresponds to the standard representation of $\mathrm{Lie}(\mathbb{C}^*)$, since the given action is trivial on X.

In §2.2 we extend the formalism of shadows to smooth varieties X endowed with the action of a torus T. This notion allows us to define a morphism from the group of T-equivariant conical Lagrangian cycles in the cotangent bundle of X to $H_*^T(X)$, and we prove (Proposition 3.3) that the equivariant shadow of the characteristic cycle of a constructible function φ equals Ohmoto's equivariant CSM class of φ:

$$\mathrm{Shadow}^T(\mathrm{CC}(\varphi)) = \check{c}_*^T(\varphi) \in H_*^T(X).$$

Here $\check{c}_*^T(\varphi)$ denotes a 'signed' version of Ohmoto's natural transformation (see (7)). In §4 we use these results to prove (Theorem 4.3):

Theorem 1.1. *Let X be a complex algebraic manifold, with a T-action. Consider the $\mathbb{C}^*$-action dilating the cotangent fibres with character $\hbar^{-1}$ on $T^*(X)$. Let $\iota : X \to T^*X$ be the zero section. Then*

$$\iota^*[\mathrm{CC}(\varphi)]_{T\times\mathbb{C}^*} = c_*^{T,\hbar}(\varphi) \in H_0^{T\times\mathbb{C}^*}(X).$$

Here $c_*^{T,\hbar}$ is the homogenization of Ohmoto's equivariant MacPherson transformation. Theorem 1.1 generalizes to the equivariant case results of Ginzburg [21, Appendix] and Sabbah [50]. In fact, even in the non-equivariant case, the theorem gives another formula of Ginzburg's analogous map from *loc. cit.* (avoiding the use of equivariant K-theory and the equivariant Riemann-Roch transformation).

This result informs the rest of the paper, which is focused on the study of (homogenized) equivariant CSM classes of Schubert cycles in generalized flag manifolds $X = G/B$, where G is a complex, simple Lie group, and B a Borel subgroup. Let $T \subset B$ be the maximal torus, and W the associated Weyl group. Let R^+ denote the set of positive roots associated to (G, B). Denote by $X(w)^\circ := BwB/B$ the Schubert cell for the Weyl group element $w \in W$. Further, let $\mathcal{M}_w$ be the Verma $\mathcal{D}_X$-module determined by the Verma module of highest weight $-w(\rho) - \rho$, where ρ is half the sum of positive roots. Denote by $\mathrm{Char}(\mathcal{M}_w)$ the characteristic cycle (sometimes called *singular support*) of the holonomic $\mathcal{D}_X$-module $\mathcal{M}_w$; see §6.1. This is a conic $T \times \mathbb{C}^*$-stable Lagrangian cycle in $T^*(X)$.

The relevance of the Verma module comes from the proof of Kazhdan-Lusztig conjectures by Brylinski-Kashiwara [14] and Beilinson-Bernstein [7]. It was shown therein that $\mathrm{Char}(\mathcal{M}_w)$ is (up to a sign) equal to the the characteristic cycle $CC(1\!\!1_{X(w)^\circ})$. Then Theorem 1.1 applied to the indicator function of the Schubert cell $\varphi = 1\!\!1_{X(w)^\circ}$ implies that

$$c^{T,\hbar}_{\mathrm{SM}}(X(w)^\circ) = (-1)^{\ell(w)} \iota^* \,\mathrm{Char}(\mathfrak{M}_w).$$

The Verma characteristic cycle in the right hand side equals Maulik and Okounkov's *stable envelope* $stab_+(w)$. This is stated without proof in [40, Remark 3.5.3, p. 69], and for completeness we sketch one in Lemma 6.5 below. Combining the two facts one immediately obtains the corollary (cf. Corollary 6.6):

Corollary 1.2. *Let $w \in W$. Then $\iota^*(stab_+(w)) = (-1)^{\dim X} c^{T,\hbar}_{SM}(X(w)^\circ)$, as elements in $H_0^{T\times\mathbb{C}^*}(X)$.*

This equality was proved earlier by Rimányi and Varchenko [48, Thm. 8.1 and Rmk. 8.2] (see also [59]), using interpolation properties of CSM classes stemming from Weber's work [60], and the defining localization properties of the stable envelopes recalled (cf. §6.2 below).

The main result of the paper involves a formula for the CSM classes in terms of a Segre operator, which we define next. Consider the following diagram:

$$\begin{array}{ccc} T^*(X) & \hookrightarrow & \mathbb{P}(T^*(X) \oplus 1\!\!1) \\ \pi \searrow \nwarrow \iota & & \downarrow \overline{q} \\ & & X \end{array}$$

Here $\mathbb{P}(T^*(X) \oplus 1\!\!1)$ is the completion of the cotangent bundle, $\pi, \overline{q}$, the natural projections, and ι the inclusion of the zero section. As before, consider a $\mathbb{C}^*$-action on $T^*(X)$ acting on the cotangent fibres with character $\hbar^{-1}$. Let $C \subset T^*(X)$ be an equivariant cycle in T^*X and $\overline{C} \subset \mathbb{P}(T^*(X) \oplus 1\!\!1)$ be its Zariski closure. Associated to this data consider the Segre operator given by

$$Segre^T_{T^*(X)\oplus\mathbf{1}}(C) := \overline{q}_*\Big(\frac{[\overline{C}]}{c^T(\mathcal{O}(-1))}\Big) = \overline{q}_*(\sum_{j\geq 0} c_1^T(\mathcal{O}(1))^j \cap [\overline{C}]).$$

Here c^T denotes the total torus equivariant Chern class of a T-equivariant vector bundle. In the non-equivariant homology, the Chern classes are nilpotent, and this operator has values in homology. In the equivariant context the operator has values in the localization $H_*^T(X)_{loc}$ of $H_*^T(X)$ at the equivariant ring $H^*_T(pt)$; see §8 below. The main result of this paper is the following (Theorem 8.3):

Theorem 1.3. *Let $w \in W$ be an element in the Weyl group. Then the following equality holds:*

$$c^T_{SM}((X(w)^\circ) = \prod_{\alpha\in R^+}(1+\alpha) Segre^T_{T^*(G/B)\oplus\mathbf{1}}(\mathrm{Char}(\mathcal{M}_w)) \in H_*^T(G/B)_{loc}.$$

In particular, in the non-equivariant homology,

$$c_{SM}(X(w)^\circ) = Segre_{T^*(G/B)\oplus \mathbf{1}}(\mathrm{Char}(\mathcal{M}_w)) \in H_*(G/B).$$

In other words, the equivariant CSM class of a Schubert cell is (up to a factor) equal to the Segre operator applied to the corresponding Verma cycle.

As advertised, in the non-equivariant case the Segre class is manifestly effective and it implies the positivity of CSM classes. The proof of this corollary (but *not* of the theorem 1.3) is independent of other facts in the paper, and will be given next. Let $[X(w)] \in H_*(X)$ be the fundamental class of the Schubert variety $X(w) := \overline{X(w)^\circ}$. The set of fundamental classes $\{[X(w)]\}_{w\in W}$ form a basis of the module $H_*(X)$.

Corollary 1.4 (Positivity of CSM classes)**.** *Let $w \in W$. Then the non-equivariant CSM class $c_{SM}(X(w)^\circ)$ is effective, i.e. in the Schubert expansion*

$$c_{SM}(X(w)^\circ) = \sum_{v\leq w} c(v;w)[X(v)] \in H_*(X),$$

the coefficients $c(v;w)$ are non-negative. Further, let $P \subset G$ be any parabolic subgroup. Then the CSM classes of Schubert cells in $H_*(G/P)$ are effective, i.e., the similar Schubert expansion involves nonnegative coefficients $c(v;w)$.

Proof. Consider first $X = G/B$. The Verma $\mathcal{D}_X$-module $\mathcal{M}_w$ is holonomic, thus its characteristic cycle $\mathrm{Char}(\mathcal{M}_w) \subset T^*(X)$ is effective [24, p. 119] (or by [51, Rem. 6.0.4 on p.389] for the corresponding result in terms of perverse sheaves), and so is its closure in $\mathbb{P}(T^*(X)\oplus 1\!\!1)$. The tautological line bundle $\mathcal{O}_{T^*(X)\oplus\mathbf{1}}(1)$ is globally generated, i.e. it is a quotient of a trivial bundle. Indeed, $\mathcal{O}_{T^*(X)\oplus\mathbf{1}}(1)$, is a quotient of $T(X)\oplus 1\!\!1$, and by homogeneity of X, $T(X)$ is globally generated. Then $c_1(\mathcal{O}_{T^*(G/B)\oplus\mathbf{1}}(1)) \cap [\overline{C}]$ is effective, for any effective cycle $\overline{C}$ [20, Ex. 12.1.7 (a)]. The result follows from Theorem 1.3. Effectivity of CSM classes in G/B implies effectivity in G/P by [5, Proposition 3.5]. □

This generalizes the positivity of CSM classes of Schubert cells for Grassmann manifolds, which was proved by Aluffi and Mihalcea [4, 41] and Jones [27], Stryker [56] in several cases, and for any Schubert cell in any Grassmann manifold by J. Huh [25]. Huh used that in this situation the Schubert varieties can be desingularized by varieties with finitely many Borel orbits. Unfortunately, the known desingularizations of Schubert varieties in arbitrary flag manifolds do not satisfy this property in general. Seung Jin Lee [36, Thm. 1.1] proved that the positivity of CSM classes for type A flag manifolds is *implied* by a positivity property satisfied in a certain subalgebra of Fomin-Kirillov algebra [18] generated by Dunkl elements. Thus CSM positivity can be also regarded as evidence for the Fomin-Kirillov conjecture.

The proof of the theorem 1.3 follows from theorem 1.1 together with the 'Hecke' and 'stable envelopes' orthogonality properties of CSM classes. To explain the Hecke orthogonality, consider two families of (Demazure-Lusztig) homogeneous operators

$$\mathbf{L}_i := \hbar\partial_i - s_i, \mathbf{L}_i^\vee := \hbar\partial_i + s_i : H_*^{T\times\mathbb{C}^*}(X) \to H_*^{T\times\mathbb{C}^*}(X).$$

where $\partial_i : H_*^T(X) \to H_{*+2}^T(X)$ are the Bernsteĭn-Gelfand-Gelfand operators ([8]) and s_i stand for the right actions of simple reflections in the Weyl group W on $H_*^T(X)$. These operators satisfy the braid relations, hence we obtain by composition operators $\mathbf{L}_w$, $\mathbf{L}_w^\vee$ for all $w \in W$. They have been studied in relation to Hecke algebras [34, 35, 22] and they are adjoint to each other with respect to Poincaré pairing; cf. Lemma 5.2 below. The specialization $L_i := (\mathbf{L}_i)_{\hbar\mapsto 1}$ is an operator which was shown in [5] to generate recursively the CSM classes. In the homogenized context, this means that

$$c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ) = \mathbf{L}_{w^{-1}}[X(id)]_{T\times\mathbb{C}^*}$$

(Definition 5.3 and Proposition 5.4). Similarly, applying $\mathbf{L}_{w^{-1}}^\vee$ determine alternative 'signed' classes $c_{\mathrm{SM}}^{T,\hbar,\vee}(X(w)^\circ)$; these classes can be defined similarly to CSM classes, but would normalize to the (equivariant) class of the *cotangent* bundle for a smooth compact variety. We also study the action of these operators on opposite Schubert cells $Y(w)^\circ := B^-wB/B$. The main result is the following (Theorem 5.7):

Theorem 1.5 (Hecke orthogonality). *The homogenized equivariant CSM classes satisfy the following orthogonality property:*

$$\langle c_{SM}^{T,\hbar}(X(u)^\circ), c_{SM}^{T,\hbar,\vee}(Y(v)^\circ)\rangle = \delta_{u,v}\prod_{\alpha\in R^+}(\hbar+\alpha).$$

Here $\langle\cdot,\cdot\rangle$ is the usual Poincaré (intersection) pairing. Note that the Schubert classes satisfy a similar orthogonality property (there the product over the roots is omitted). Thus, equivariant CSM classes consist of 'corrections' of ordinary fundamental classes by lower dimensional terms, in a way preserving the basic orthogonality of Schubert varieties.

The 'stable envelopes' orthogonality follows from the orthogonality of the stable envelopes proved by Maulik and Okounkov [40]. That orthogonality is with respect to the *localization pairing*, derived from Atiyah-Bott localization theory. An expression of this pairing in terms of Poincaré pairing is recalled in §7 below. Then the orthogonality of stable envelopes (Prop. 7.1), together with Corollary 1.2 implies:

$$\left\langle c_{\mathrm{SM}}^T(X(u)^\circ), \frac{c_{\mathrm{SM}}^T(Y(v)^\circ)}{c^T(T(G/B))}\right\rangle = \delta_{u,v}.$$

(See Theorem 9.4, and compare also with Remark 9.6 for another approach to this orthogonality result, not depending on theory of stable envelopes). Similar orthogonality results, in the language of *weight functions* were obtained in [45, 47, 49, 23], for the type A flag manifolds.

Comparing with the orthogonality result from theorem 1.5, we obtain a relation between homogenized equivariant CSM classes of Schubert cells and their signed counterparts (Theorem 7.1). Even in its non-equivariant specialization this result is highly non-trivial:

$$c_{\mathrm{SM}}(X(v)^\circ) = c(TX)\cap c_{\mathrm{SM}}^\vee(X(v)^\circ)$$

(Corollary 7.4). This shows that for flag manifolds G/B one can view the signed classes $c_{\mathrm{SM}}^\vee(X(v)^\circ$ as 'Schwartz-MacPherson Segre classes' of Schubert cells. Using these formulas and some manipulations involving shadows of characteristic cycles ultimately yields Theorem 1.3.

Further applications include: a localization formulas for the CSM classes at *any* fixed point (Cor. 6.7) and a Chevalley formula (§9.3) for the CSM classes (this uses results of Su [58, 57] about similar formulas for stable envelopes); formulas for the characteristic class associated to the intersection cohomology sheaf of a Schubert variety (Remark 6.3); a remarkable statement relating the transition matrix between CSM and Schubert classes and its inverse (Prop. 5.10).

These results are generalized in §9 to the case of *partial* flag manifolds G/P, $P\supseteq B$ a parabolic subgroup. The identification between dual Chern classes and Segre classes is no longer valid, as simple examples show. Nevertheless, stable envelopes behave similarly for G/P, therefore the 'stable envelopes' orthogonality property extends to G/P, as one expects. In this case, the statement replaces the dual classes with the corresponding Segre classes (Theorem 9.4):

$$\left\langle c^T_{\text{SM}}(X(uW_P)^\circ), \frac{c^T_{\text{SM}}(Y(vW_P)^\circ)}{c^T(T(G/P))} \right\rangle_{G/P} = \delta_{u,v} \quad ,$$

where W_P stands for the subgroup of W generated by the simple reflections in P. We give an alternate proof, using the functoriality of CSM classes and a 'Verdier-Riemann-Roch' theorem for (equivariant) Chern classes due to Yokura [61] and Ohmoto [42] (see also [52]).

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2. Shadows and equivariant cohomology

2.1. **Definition and basic properties.** In this paper we work in the complex algebraic context, with A_* the Chow group respectively H_* the Borel-Moore homology. Moreover, in case we speak of codimension we always assume that our spaces are pure dimensional.

Let $\pi : E \to X$ be a vector bundle of rank $e+1$ on X, and consider the projective bundle of lines $\mathbb{P}(E)$. The pull-back via the projection $q : \mathbb{P}(E) \to X$ realizes $A_*(\mathbb{P}(E))$ as a module over $A_*(X)$. In fact,

$$A_*(\mathbb{P}(E)) \cong A_*(X)[\zeta]/(\zeta^{e+1} + \zeta^e q^* c_1(E) + \cdots + \zeta q^* c_e(E) + q^* c_{e+1}(E))$$

where $\operatorname{rk} E = e+1$ and $\zeta = c_1(\mathcal{O}_E(1))$. Every class α of codimension k in $A_*(\mathbb{P}(E))$ may be written as

$$\alpha = \sum_{j=0}^{e} \zeta^j q^*(\underline{\alpha}^{k-j}) \tag{1}$$

for uniquely defined classes $\underline{\alpha}^{k-j}$ of codimension $k-j$ in $A_*(X)$ ([20, Theorem 3.3(b)]). Following [2], we call the (non-homogeneous) class

$$\text{Shadow}_E(\alpha) := \underline{\alpha}^{k-e} + \underline{\alpha}^{k-e+1} + \cdots + \underline{\alpha}^{k} \tag{2}$$

the *shadow* of α in X. By (1), a homogeneous class $\alpha \in A_*(\mathbb{P}(E))$ may be reconstructed from its shadow and its codimension k. We will omit the subscript E from the notation if the ambient projective bundle is understood from the context. The following lemma is useful in relating shadows of classes in different projective bundles, with $c(-)\cap$ the total Chern class operator.

Lemma 2.1. *For a class α in $A_*(\mathbb{P}(E))$ the following hold in $A_*(X)$.*

(i) *The shadow* $\text{Shadow}_E(\alpha)$ *equals*

$$c(E) \cap q_*(c(\mathcal{O}_E(-1))^{-1} \cap \alpha) = c(E) \cap q_* \sum_{j \geq 0} c_1(\mathcal{O}_E(1))^j \cap \alpha \quad ;$$

(ii) *If F is a subbundle of E, and $\alpha \in A_*(\mathbb{P}(F)) \hookrightarrow A_*(\mathbb{P}(E))$, then*

$$\text{Shadow}_E(\alpha) = c(E/F) \cap \text{Shadow}_F(\alpha) \quad ;$$

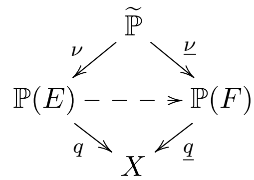
(iii) *If* $\mathrm{Shadow}_E(\alpha) = \sum_{j=0}^{e} \underline{\alpha}^{k-j}$*, and* L *is a line bundle on* X*, then*

$$\mathrm{Shadow}_{E\otimes L}(\alpha) = \sum_{j=0}^{e} c(L)^j \cap \underline{\alpha}^{k-j} \quad ;$$

(iv) *Further, let* $E \to F$ *be a surjection of bundles, with kernel* K*, and let* C_E *be a cycle in* $\mathbb{P}(E)$ *disjoint from* $\mathbb{P}(K)$*. Let* C_F *be the cycle in* $\mathbb{P}(F)$ *obtained by pushing forward* C_E*. Then*

$$\mathrm{Shadow}_E([C_E]) = c(K) \cap \mathrm{Shadow}_F([C_F]) \quad .$$

Proof. Part (i) is [2, Lemma 4.2]. Part (ii) follows immediately from (i). Part (iii) is a straightforward computation, which we leave to the reader. For part (iv), let $\underline{q} : \mathbb{P}(F) \to X$ be the projection. The given surjection $E \to F$ induces a rational map $\mathbb{P}(E) \dashrightarrow \mathbb{P}(F)$, which is resolved by blowing up along $\mathbb{P}(K)$; let $\nu : \widetilde{\mathbb{P}} \to \mathbb{P}(E)$ be this blow-up, and let $\underline{\nu} : \widetilde{\mathbb{P}} \to \mathbb{P}(F)$ be the induced morphism:



Since C_E is disjoint from $\mathbb{P}(F)$ and ν is an isomorphism over $\mathbb{P}(E) \smallsetminus \mathbb{P}(F)$, the cycle C_E determines a cycle $\widetilde{C}_E$ in $\widetilde{\mathbb{P}}$, disjoint from the exceptional divisor, such that $[C_E] = \nu_*[\widetilde{C}_E]$ and $[C_F] = \underline{\nu}_*[\widetilde{C}_E]$. By part (i) and the projection formula,

$$\mathrm{Shadow}_E([C_E]) = c(E) \cap q_*\nu_*(c(\nu^*\mathcal{O}_E(-1))^{-1} \cap [\widetilde{C}_E]) \quad .$$

Now note that $\nu^*\mathcal{O}_E(-1)$ and $\underline{\nu}^*\mathcal{O}_F(-1)$ differ by a term supported on the exceptional divisor in $\widetilde{\mathbb{P}}$, hence they agree on $\widetilde{C}_E$. Therefore

$$\begin{aligned}\mathrm{Shadow}_E([C_E]) &= c(E) \cap \underline{q}_*\underline{\nu}_*(c(\underline{\nu}^*\mathcal{O}_F(-1))^{-1} \cap [\widetilde{C}_E]) \\ &= c(E) \cap \underline{q}_*(c(\mathcal{O}_F(-1))^{-1} \cap [C_F]) \\ &= c(K) \cap \mathrm{Shadow}_F([C_F])\end{aligned}$$

again by the projection formula and part (i). □

The formula in part (i) may be expressed concisely in terms of a 'Segre class operator'; see §3.2.

Shadows are compatible with the operation of taking a cone. More precisely, let $1\!\!1$ denote the trivial rank-1 line bundle on X, and consider the projective completion $\mathbb{P}(E\oplus 1\!\!1)$; E may be identified with the complement of $\mathbb{P}(E \oplus 0)$ in $\mathbb{P}(E \oplus 1\!\!1)$. Consider a $\mathbb{C}^*$-action on E by fibrewise dilation, and the trivial $\mathbb{C}^*$-action on $1\!\!1$. This induces a $\mathbb{C}^*$-action on $\mathbb{P}(E\oplus 1\!\!1)$ such that the inclusion $E \subset \mathbb{P}(E\oplus 1\!\!1)$ is $\mathbb{C}^*$-equivariant, and the trivial action on $\mathbb{P}(E) = \mathbb{P}(E\oplus 0)$. A class α in $A_*(\mathbb{P}(E))$ determines a $\mathbb{C}^*$-invariant class $C(\alpha)$ in $A_*(\mathbb{P}(E \oplus 1\!\!1))$, obtained by taking the cone with vertex the zero-section $X = \mathbb{P}(0 \oplus 1\!\!1)$.

Lemma 2.2. $\mathrm{Shadow}_E(\alpha) = \mathrm{Shadow}_{E\oplus\mathbf{1}}(C(\alpha))$.

Proof. This follows from Lemma 2.1 (i). Indeed $c(E \oplus 1\!\!1) = c(E)$, and $\mathbb{P}(E) \cong \mathbb{P}(E \oplus 0)$ represents $c_1(\mathcal{O}_{E\oplus\mathbf{1}}(1))$, so that

$$\sum_{j\geq 1} c_1(\mathcal{O}_{E\oplus\mathbf{1}}(1))^j \cap C(\alpha) = \sum_{j\geq 0} c_1(\mathcal{O}_E(1))^j \cap \alpha;$$

(the remaining term vanishes in the push-forward for dimensional reasons). □

Remark 2.3. Not all $\mathbb{C}^*$-invariant classes in $A_*(\mathbb{P}(E \oplus \mathbb{1}))$ are obtained from classes in $A_*(\mathbb{P}(E))$ as in Lemma 2.2. For instance, the class of the zero section $X = \mathbb{P}(0 \oplus \mathbb{1})$ is $\mathbb{C}^*$-fixed, and not of this form. For any subvariety $V \subseteq X = \mathbb{P}(0 \oplus \mathbb{1}) \subseteq \mathbb{P}(E \oplus \mathbb{1})$,

$$\mathrm{Shadow}_{E\oplus\mathbf{1}}([V]) = c(E) \cap [V]$$

by Lemma 2.1 (i), since $\mathcal{O}_{E\oplus\mathbf{1}}(-1)$ is trivial along the zero-section. ⌟

Denote by $i : E \to \mathbb{P}(E \oplus \mathbb{1})$ the embedding of E as the complement of $\mathbb{P}(E \oplus 0)$, and by $\overline{q} : \mathbb{P}(E \oplus \mathbb{1}) \to X$ the projection.

Lemma 2.4. *If $\alpha \in A_*(\mathbb{P}(E \oplus \mathbb{1}))$ has codimension k, then $i^*(\alpha) = \pi^*(\underline{\alpha}^k)$ where $\underline{\alpha}^k$ is the component of* $\mathrm{Shadow}_{E\oplus\mathbf{1}}(\alpha)$ *of codimension k.*

Proof. Indeed, $\alpha = \sum_{j=0}^{e+1} \eta^j \overline{q}^*(\underline{\alpha}^{k-j})$, where $\mathrm{Shadow}(\alpha) = \sum_{j=0}^{e+1} \underline{\alpha}^j$ and $\eta = c_1(\mathcal{O}_{E\oplus\mathbf{1}}(1))$. Since $\mathbb{P}(E \oplus 0)$ represents η and is disjoint from E, $i^*(\eta) = 0$. Therefore $i^*(\alpha) = i^*\overline{q}^*(\underline{\alpha}^k) = \pi^*(\underline{\alpha}^k)$ as stated. □

2.2. **Equivariant cohomology.** Let T be a torus and let X be a variety with a T-action. Then the equivariant cohomology $H_T^*(X)$ is the ordinary cohomology of the Borel mixing space $X_T := (ET \times X)/T$, where ET is the universal T-bundle and T acts by $t \cdot (e, x) = (et^{-1}, tx)$.[1] It is an algebra over $H_T^*(pt)$, the polynomial ring $\mathbb{Z}[t_1, \dots, t_s]$, where $t_i, \dots, t_s$ are generators for the weight lattice of T. We may also define T-equivariant *(Borel Moore) homology* and *Chow groups*; see e.g., [17]. Every closed subvariety $Y \subseteq X$ that is invariant under the T action determines equivariant fundamental classes $[Y]_T$ in $H_*^T(X)$ and $A_*^T(X)$. Whenever X is smooth, we can and will identify the equivariant homology $H_*^T(X)$ with the equivariant cohomology $H_T^*(X)$. We address the reader to [6, 33], or [42] for basic facts on equivariant cohomology and homology. Equivariant vector bundles have equivariant Chern classes $c^T(-)$; see [6, §1.3], [17, §2.4]. In Chow, $c_j^T(E)\cap -$ is an operator $A_i^T(X) \to A_{i-j}^T(X)$.

As in §2.1, let $\pi : E \to X$ be a vector bundle of rank $e+1$ on X. We consider the action of $\mathbb{C}^*$ on E by fiberwise dilation with character χ, and denote by E^χ the vector bundle E endowed with this $\mathbb{C}^*$-action. The natural projection $\pi : E^\chi \to X$ is equivariant, where $\mathbb{C}^*$ acts trivially on X. Since the action of $\mathbb{C}^*$ on X is trivial, the Borel mixing space $X_{\mathbb{C}^*}$ is isomorphic to $B\mathbb{C}^* \times X = \mathbb{P}^\infty \times X$. Here and in the following, we will denote by $\mathbb{P}^\infty$ any approximation $\mathbb{P}^N$ with $N \gg 0$ sufficiently large; see e.g., [6], §1.2. We will give the results in the ordinary and equivariant Chow groups; they imply the corresponding results in ordinary/equivariant (Borel-Moore) homology, via the map defined in [17, §2.8]. By definition,

$$A_i^{\mathbb{C}^*}(X) = A_{N+i}(\mathbb{P}^N \times X)$$

for $N \gg 0$. We will call $\dim X - i$ the 'codimension' of a class in this group. Since $X_{\mathbb{C}^*} \cong \mathbb{P}^\infty \times X$,

$$A_*^{\mathbb{C}^*}(X_{\mathbb{C}^*}) \cong A_*(X)[\hbar] \quad ,$$

where $\hbar := c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1))$ corresponds to the identity character. Next, denote by

$$\rho : X_{\mathbb{C}^*} = \mathbb{P}^\infty \times X \longrightarrow X$$

[1] In general, the Borel mixing space is only a separated algebraic space, but if X is a quasi-projective scheme, then X_T is again quasi projective; see [42, §2.2] or [43].

the projection. If χ is the character $z \mapsto z^a$, a standard computation shows that the mixing space $E^\chi_{\mathbb{C}^*}$, along with the natural projection to $X_{\mathbb{C}^*}$, is isomorphic to the vector bundle $\pi^\chi : \rho^* E \otimes \mathcal{O}_{\mathbb{P}^\infty}(-a) \to X_{\mathbb{C}^*}$. We have the diagram

$$\begin{array}{ccc} E^\chi_{\mathbb{C}^*} = \rho^* E \otimes \mathcal{O}_{\mathbb{P}^\infty}(-a) & & E^\chi \\ \downarrow{\scriptstyle \pi^\chi} & & \downarrow{\scriptstyle \pi} \\ X_{\mathbb{C}^*} = \mathbb{P}^\infty \times X & \xrightarrow{\rho} & X \end{array} \tag{3}$$

Lemma 2.5. *The projection π induces by flat pull-back a codimension-preserving isomorphism $\pi^* : A^{\mathbb{C}^*}_*(X) \xrightarrow{\sim} A^{\mathbb{C}^*}_*(E^\chi)$. The embedding $\iota : X \to E$ of the zero section induces a codimension-preserving isomorphism*

$$\iota^* : A^{\mathbb{C}^*}_*(E^\chi) \xrightarrow{\sim} A^{\mathbb{C}^*}_*(X) \cong A_*(X)[\hbar] \quad ,$$

satisfying $\iota^* = (\pi^*)^{-1}$.

Proof. This follows from [20, Theorem 3.3(a)] applied to the projection $\pi^\chi : E^\chi_{\mathbb{C}^*} \to X_{\mathbb{C}^*}$. □

2.3. **Homogenization of shadows and $\mathbb{C}^*$-equivariant cohomology.** Consider a (non-homogeneous) class $\alpha = \sum_{j=0}^{\ell} \alpha^j \in A_*(X)$, where α^j denotes the homogeneous component of α of codimension j in X. The choice of (a codimension) $k \geq \ell$ and of a character χ determine the homogeneous class

$$\alpha^\chi := \sum_{j=0}^{\ell} \chi^{k-j} \alpha^j \in A^{\mathbb{C}^*}_{\dim X - k}(X); \tag{4}$$

here we write $\chi = a\hbar = c_1(\mathcal{O}_{\mathbb{P}^\infty}(-a))$ if χ is the character $z \mapsto z^a$. We will call α^χ the *'(χ-)homogenization'* of degree k of α; the fixed codimension k will be clear from the context.

Example 2.6. A key example is given by the homogenization of the total Chern class of the bundle E^χ. If $x_1, \ldots, x_{e+1}$ are the (non-equivariant) Chern roots of E then the ($\mathbb{C}^*$-equivariant) Chern roots of E^χ are $x_1 + \chi, \ldots, x_{e+1} + \chi$. It follows that for every subvariety $V \subseteq X$,

$$c^{\mathbb{C}^*}_{e+1}(E^\chi) \cap [V]_{\mathbb{C}^*} = (c(E) \cap [V])^\chi \in A^{\mathbb{C}^*}_{\dim V-(e+1)}(X) \tag{5}$$

(note that $[V]$ may be identified with $[V]_{\mathbb{C}^*}$ since the $\mathbb{C}^*$-action on X is trivial). I.e., in this case the homogenization is naturally an equivariant top Chern class. ⌟

Now let C be a $\mathbb{C}^*$-invariant cycle of codimension k in $E = E^\chi$. Viewing E as an open subset of $\mathbb{P}(E \oplus 1\!\!1)$ as above, C determines a codimension-k cycle $\overline{C}$ in $\mathbb{P}(E \oplus 1\!\!1)$. The next result compares the class $[C]_{\mathbb{C}^*}$ of C in the equivariant Chow group $A^{\mathbb{C}^*}_*(E^\chi)$ and the class $[\overline{C}]$ in the ordinary Chow group $A_*(\mathbb{P}(E \oplus 1\!\!1))$.

Proposition 2.7. *Let C be a $\mathbb{C}^*$-invariant cycle of codimension k in E^χ, as above. Then $[C]_{\mathbb{C}^*} \in A^{\mathbb{C}^*}_*(E^\chi) \overset{\iota^*}{\cong} A_*(X)[\hbar]$ is the χ-homogenization of degree k of the shadow of $[\overline{C}]$:*

$$\iota^*([C]_{\mathbb{C}^*}) = (\mathrm{Shadow}([\overline{C}]))^\chi \quad .$$

Remark 2.8. In particular, this shows that equivariant fundamental classes of invariant subvarieties of a vector bundle E^χ of rank $e+1$ are of the form

$$\alpha^k + \chi\, \pi^*(\alpha^{k-1}) + \cdots + \chi^{e+1}\, \pi^*(\alpha^{k-e-1})$$

i.e., combinations of powers χ^j with $0 \le j \le \operatorname{rk} E$. In other words, among all equivariant classes, proposition 2.7 distinguishes the fundamental classes of fixed codimension equivariant subvarieties in E as those classes determined by no more than $\operatorname{rk} E + 1$ homogeneous classes in $A_*(X)$, a quantity independent of $\dim X$. It will in fact follow from the proof (cf. (6)) that if such a subvariety is not supported within the zero section of E, then $\alpha^{k-e-1} = 0$. ⌟

Proof. By linearity we may assume that $C = V$ for a $\mathbb{C}^*$-invariant subvariety V of E. First assume that the subvariety is contained in the zero-section, so that $C = \iota_*([V])$ for a subvariety V of X. By the (equivariant) self-intersection formula,

$$\iota^*([C]_{\mathbb{C}^*}) = \iota^*(\iota_*[V]_{\mathbb{C}^*}) = c_{e+1}^{\mathbb{C}^*}(E^\chi) \cap [V]_{\mathbb{C}^*} = (c(E) \cap [V])^\chi,$$

where the last equality follows from (5) ($[V]_{\mathbb{C}^*}$ may be identified with $[V]$ since the $\mathbb{C}^*$-action on X is trivial). On the other hand, $\overline{C} = C$ is also the push-forward of V to the zero-section $X = \mathbb{P}(0 \oplus 1\!\!1) \subset \mathbb{P}(E \oplus 1\!\!1)$. As in Remark 2.3, we deduce that

$$\operatorname{Shadow}(\overline{C}) = c(E \oplus 1\!\!1) \cap [V] = c(E) \cap [V]$$

from which the claim follows.

Next, assume V is *not* supported on the zero-section of E and has codimension k. Since V is $\mathbb{C}^*$-invariant, V determines and is determined by a subvariety $\mathbb{P}(V)$ of $\mathbb{P}(E)$; in this case, $\overline{V}$ is the cone over $\mathbb{P}(V)$ with center $\mathbb{P}(0 \oplus 1\!\!1)$. Let $\operatorname{Shadow}_E(\mathbb{P}(V)) = \sum_{j=0}^{e} \underline{\alpha}^{k-j}$, with $\underline{\alpha}^{k-j}$ of codimension $(k-j)$. By Lemma 2.2 this is also the shadow of $\overline{V}$, so that

$$(\operatorname{Shadow}_{E \oplus \mathbf{1}}(\overline{V}))^\chi = \sum_{j=0}^{e} \chi^j \underline{\alpha}^{k-j}$$

in $A_{\dim X - k}^{\mathbb{C}^*}(X)$. Denote by $i^\chi : E^\chi \to \mathbb{P}(E^\chi \oplus 1\!\!1)$ the open embedding with complement $\mathbb{P}(E^\chi \oplus 0)$. This is a flat $\mathbb{C}^*$-equivariant map, therefore $[V]_{\mathbb{C}^*} = (i^\chi)^*[\overline{V}]_{\mathbb{C}^*}$. We calculate $(i^\chi)^*[\overline{V}]_{\mathbb{C}^*}$ using mixing spaces. First, observe that the mixing space $\overline{V}_{\mathbb{C}^*}$ equals the cone over $\mathbb{P}^\infty \times \mathbb{P}(V)$ in $\mathbb{P}(E^\chi_{\mathbb{C}^*} \oplus 1\!\!1)$. Then Lemma 2.4 applied to the open embedding $E^\chi_{\mathbb{C}^*} \to \mathbb{P}(E^\chi_{\mathbb{C}^*} \oplus 1\!\!1)$ implies that the equivariant class $(i^\chi)^*[\overline{V}]_{\mathbb{C}^*}$ is the limit of the codimension k component of

$$(q^\chi)^*(\operatorname{Shadow}_{E^\chi_{\mathbb{C}^*} \oplus \mathbf{1}}(\overline{V}_{\mathbb{C}^*})) = (q^\chi)^*(\operatorname{Shadow}_{E^\chi_{\mathbb{C}^*}}(\mathbb{P}^\infty \times \mathbb{P}(V)));$$

where $q^\chi : E^\chi_{\mathbb{C}^*} \to \mathbb{P}^\infty \times X$ denotes the projection. (This equality follows from Lemma 2.2.) If χ is the character $z \mapsto z^a$ then with the notation from diagram (3) above

$$\mathbb{P}(E^\chi_{\mathbb{C}^*}) = \mathbb{P}(\rho^*(E) \otimes \mathcal{O}_{\mathbb{P}^\infty}(-a)) \cong \mathbb{P}(\rho^*(E)) = \mathbb{P}^\infty \times \mathbb{P}(E).$$

We have $\operatorname{Shadow}_{\rho^*(E)}([\mathbb{P}^\infty \times \mathbb{P}(V)]) = \sum_{j=0}^{e} \rho^*(\underline{\alpha}^{k-j})$; therefore

$$\operatorname{Shadow}_{E^\chi_{\mathbb{C}^*}}([\mathbb{P}^\infty \times \mathbb{P}(V)]) = \operatorname{Shadow}_{\rho^*(E) \otimes \mathcal{O}(-a)}([\mathbb{P}^\infty \times \mathbb{P}(V)]) = \sum_{j=0}^{e} c(\mathcal{O}(-a))^j \cap \rho^*(\underline{\alpha}^{k-j})$$

by Lemma 2.1 (iii). It follows that the codimension k component of $\operatorname{Shadow}_{E^\chi_{\mathbb{C}^*}}(\mathbb{P}^\infty \times \mathbb{P}(V))$ equals $\sum_{j=0}^{e} c_1(\mathcal{O}(-a))^j \cap \rho^*(\underline{\alpha}^{k-j})$. Interpreting the result in the equivariant Chow group, we deduce

$$\iota^*([V]_{\mathbb{C}^*}) = \iota^* i^{\chi *}([\overline{V}_{\mathbb{C}^*}]) = \iota^*(q^\chi)^* \sum_{j=0}^{e} c_1(\mathcal{O}(-a))^j \cap \rho^*(\underline{\alpha}^{k-j}) = \sum_{j=0}^{e} \chi^j \underline{\alpha}^{k-j} \tag{6}$$

as needed. □

Remark 2.9. If X is endowed with the action of a torus T, and $E \to X$ is a T-equivariant vector bundle on X, then shadows of equivariant classes in $A_*^T(\mathbb{P}(E))$ may be defined in $A_*^T(X)$. If in addition $E = E^\chi$ is given a $\mathbb{C}^*$-action by fibrewise dilation with character χ, the definition from (4) generalizes to give the homogenization of an *equivariant* (non-homogenous) class: for $\alpha = \sum_{j=0}^{\ell} \alpha^j \in A_*^T(X)$ and the choice of (a codimension) $k \geq l$, the homogenization $\alpha^\chi = \sum_{j=0}^{\ell} \chi^{k-j}\alpha^j$ is a class in $A^{T\times\mathbb{C}^*}_{\dim X - k}(X)$. The results of this section remain valid in this more general context, and this is the situation we will consider in the sections that follow. ⌟

3. Equivariant Chern-Schwartz-MacPherson classes

3.1. **Preliminaries.** Let X be a scheme with a torus T action. The group of constructible functions $\mathcal{F}(X)$ consists of functions $\varphi = \sum_W c_W 1\!\!1_W$, where the sum is over a finite set of constructible subsets $W \subset X$ and $c_W \in \mathbb{Z}$ are integers. A group $\mathcal{F}^T(X)$ of *equivariant* constructible functions (for tori and for more general groups) is defined by Ohmoto in [42, §2]. We recall the main properties that we need:

(1) If $W \subseteq X$ is a constructible set which is invariant under the T-action, its characteristic function $1\!\!1_W$ is an element of $\mathcal{F}^T(X)$. We will denote by $\mathcal{F}^T_{inv}(X)$ the subgroup of $\mathcal{F}^T(X)$ consisting of T-invariant constructible functions on X. (The group $\mathcal{F}^T(X)$ also contains other elements, but this will be immaterial for us.)
(2) Every proper T-equivariant morphism $f: Y \to X$ of algebraic varieties induces a homomorphism $f_*^T: \mathcal{F}^T(X) \to \mathcal{F}^T(Y)$. The restriction of f_*^T to $\mathcal{F}^T_{inv}(X)$ coincides with the ordinary push-forward f_* of constructible functions. See [42, §2.6].

Ohmoto proves [42, Theorem 1.1] that there is an equivariant version of MacPherson transformation $c_*^T : \mathcal{F}^T(X) \to H_*^T(X)$ that satisfies $c_*^T(1\!\!1_X) = c^T(T_X) \cap [X]_T$ if X is a non-singular variety, and that is functorial with respect to proper push-forwards. The last statement means that for all proper T-equivariant morphisms $Y \to X$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^T(Y) & \xrightarrow{c_*^T} & H_*^T(Y) \\ {\scriptstyle f_*^T}\downarrow & & \downarrow{\scriptstyle f_*^T} \\ \mathcal{F}^T(X) & \xrightarrow{c_*^T} & H_*^T(X) \end{array}$$

In [42], Ohmoto defines c_*^T for quasi-projective schemes X; in particular, no compactness is required (this is already observed by MacPherson [39]). The quasi-projective hypothesis is further relaxed in [43] to that of separated algebraic spaces, using the technique of Chow envelopes. In our main application X will be a projective (flag) manifold.

Definition 3.1. Let Z be a T-invariant constructible subset of X. We denote by $c^T_{\mathrm{SM}}(Z) := c_*^T(1\!\!1_Z) \in H_*^T(X)$ the *equivariant Chern-Schwartz-MacPherson (CSM) class,* and for Z a T-invariant irreducible algebraic subvariety of X by $c^T_{\mathrm{Ma}}(Z) := c_*^T(\mathrm{Eu}_Z) \in H_*^T(X)$ the *equivariant Chern-Mather* class of Z. ⌟

Here, Eu_Z is MacPherson's local Euler obstruction ([39, §3]). Both classes *depend* on the chosen ambient space X. However, if $\overline{Z}$ is the closure of Z in X, then the inclusion $\overline{Z} \subset X$ is proper, and one may view these classes as (non-homogenous) elements of $H_*^T(\overline{Z})$; the corresponding classes in any T-invariant subvariety W of X containing Z may be obtained by pushing-forward these classes (by the functoriality of c_*^T). We will often omit the dependence of the ambient space, when this space is clear from the context. Both the CSM and the

Chern-Mather classes equal $[\overline{Z}]_T+$ lower dimensional terms, and both classes agree with $c^T(TZ)\cap[Z]_T$ if Z is projective and non-singular. In [42, §4.3] Ohmoto gives an explicit geometric construction of the equivariant Chern-Mather class.

It will be useful to consider a 'signed' version $\check{c}_*^T$ of Ohmoto's equivariant MacPherson's transformation, by changing the sign of its components of odd (complex) dimension: for all invariant constructible functions φ, define

$$\check{c}_*^T(\varphi)=\sum_{k\ge 0}\check{c}^T(\varphi)_k:=\sum_{k\ge 0}(-1)^k c^T(\varphi)_k \quad . \tag{7}$$

Note that $c^T(\varphi)_k=0$ for $k<0$ by [44, Sect.4.1]. In the non-equivariant setting, this signed Chern class transformation appears implicitly in e.g., work of Sabbah [50] and Schürmann [52] (see also [31, 44, 52, 53]) where MacPherson's transformation is constructed via Lagrangian cycles in the cotangent bundle of X. The equivariant version of this construction is discussed below in §3.2. 'Dual' CSM and Chern-Mather classes are defined by setting

$$c_{\mathrm{SM}}^{T,\vee}(Z):=(-1)^{\dim Z}\check{c}_*^T(1\!\!1_Z) \quad , \quad c_{\mathrm{Ma}}^{T,\vee}(Z):=(-1)^{\dim Z}\check{c}_*^T(\mathrm{Eu}_Z) \quad . \tag{8}$$

If Z is T-invariant, irreducible and nonsingular, then both classes agree with the equivariant Chern class of the cotangent bundle of Z, $c^T(T^*Z)\cap[Z]_T$. For complete flag manifolds, a geometric interpretation of signed CSM classes, in terms of Poincaré duality, will be given in §5 below.

3.2. **CSM classes and shadows of characteristic cycles.** In this section we recall a construction of MacPherson's natural transformation by means of *characteristic cycles,* and extend this construction to the equivariant setting. In the non-equivariant case this construction appears in (among others) [50, 31, 44, 2, 53]. Our main ingredient is Ohmoto's construction of the equivariant MacPherson's transformation, [42].

In this section X will denote a smooth (complex) algebraic variety with an action of a torus T. We will state our results in equivariant Borel-Moore homology; *mutatis mutandis,* they hold in the Chow group. (The two theories coincide for complex flag manifolds.) The construction is illustrated in the following diagram (cf. [53, §3]); the notation is explained next.

$$\begin{array}{ccccc} \mathcal{F}^T_{inv}(X) & \xleftarrow[\sim]{\check{\mathrm{Eu}}} & Z_*^T(X) & \xrightarrow{c_{\mathrm{Ma}}^{T,\vee}} & H_*^T(X) \\ \| & & \downarrow{\scriptstyle \mathrm{cn}\,\wr} & & \| \\ \mathcal{F}^T_{inv}(X) & \xrightarrow[\sim]{\mathrm{CC}} & L_T(X) & \xrightarrow{\mathrm{Shadow}^T} & H_*^T(X) \end{array} \tag{9}$$

Here $Z_*^T(X)$ denotes the group of T-invariant cycles in X, while $L_T(X)$ denotes the additive group of T-invariant conic Lagrangian cycles in the cotangent bundle T^*X of X. (This is a T-equivariant bundle, where the T-action is induced from the left T-action on X.) The elements of $L_T(X)$ are linear combinations of T-invariant subvarieties $V\subseteq T^*X$ of dimension $\dim X$ which are also invariant under the natural $\mathbb{C}^*$ dilation action on the fibers, and such that the natural symplectic form on T^*X restricts to 0 on V.

The top maps are the 'signed' Euler obstruction, defined on irreducible varieties Z by $\check{\mathrm{Eu}}_Z:=(-1)^{dim Z}\,\mathrm{Eu}_Z$, and the signed equivariant Chern-Mather class $\check{c}_{\mathrm{Ma}}^T$, defined as in §3.1 on invariant irreducible projective varieties and extended by linearity to invariant cycles.

The homomorphism $\check{\mathrm{E}}\mathrm{u}$ is an isomorphism, and the composition

$$c_{\mathrm{Ma}}^{T,\vee} \circ \check{\mathrm{E}}\mathrm{u}^{-1} = \check{c}_*^T$$

is the signed equivariant MacPherson transformation. (Cf. [42, Proposition 4.3]. Ohmoto works with non-signed classes, but the signed versions are convenient for us as they come up naturally in the context of characteristic cycles in the cotangent bundle T^*X.) The map $\mathrm{cn} : Z_*^T(X) \to L_T(X)$ takes an irreducible cycle Z to its *conormal space* $T_Z^*X := \overline{T_{Z^{reg}}^*X}$, where Z^{reg} is the smooth part of Z. This map is a group isomorphism; see e.g., [31, Lemma 3] or [24, Thm. E.3.6]. By composition we obtain an induced 'characteristic cycle' map $\mathrm{CC} : \mathcal{F}_{inv}^T(X) \to L_T(X)$ determined on irreducible T-invariant cycles Z by

$$\mathrm{CC}(\mathrm{Eu}_Z) = (-1)^{dim Z}[T_Z^*X]$$

(see [44, eq. (11), page 67]). Since both maps cn, $\check{\mathrm{E}}\mathrm{u}_Z$ are isomorphisms, the characteristic cycle map CC is a group isomorphism as well. For a constructible function φ, the image $\mathrm{CC}(\varphi)$ is a conic Lagrangian cycle in T^*X called the *characteristic cycle* of φ; this cycle is clearly T-invariant if $\varphi \in \mathcal{F}_{inv}^T(X)$.

Remark 3.2. The left commutative square of (9) and the definitions above can be refined, if we are working with an algebraic Whitney stratification $\mathcal{S} := \{S \subset X\}$ of X such that each stratum S is connected and T-invariant (see [52]). In later sections we will take X to be a flag manifold with the stratification given by Schubert cells. Then the characteristic cycle map CC can also be directly defined in terms of stratified Morse theory for constructible functions, inducing an isomorphism between the subgroup of $\mathcal{F}_{inv}^T(X)$ generated by all $1\!\!1_S$ and the subgroup of $L_T(X)$ generated by all $\overline{T_S^*X}$ (with S a stratum).

The map $\mathrm{Shadow}^T : L_T(X) \to H_*^T(X)$ in the diagram is defined in terms of the 'shadow' operation studied in §2. Explicitly, let $(T^*X)_T$ denote the mixing space of the cotangent bundle T^*X; $(T^*X)_T$ is a bundle over X_T. A Lagrangian cycle $C \in L_T(X)$ determines a T-invariant cycle $\overline{C}$ in $\mathbb{P}(T^*X \oplus 1\!\!1)$, and hence a cycle $\overline{C}_T$ in $\mathbb{P}((T^*X)_T \oplus 1\!\!1)$. We let

$$\mathrm{Shadow}^T(C) := \mathrm{Shadow}_{(T^*X)_T \oplus \mathbf{1}}([\overline{C}_T]); \tag{10}$$

this class lives in the (ordinary) homology of the mixing space X_T, and is therefore naturally an element of $H_*^T(X)$.

Proposition 3.3. *Diagram* (9) *commutes, i.e.:*

$$\mathrm{Shadow}^T(\mathrm{CC}(\varphi)) = \check{c}_*^T(\varphi)$$

for every invariant constructible function φ*.*

Proposition 3.3 also shows that the map Shadow^T coincides with a map defined by Ginzburg in [21, §A.3], using $\mathbb{C}^*$-equivariant K-theory on T^*X. (There are some sign differences between our approach and Ginzburg's, due to the fact that Ginzburg often uses the signed version of the CSM or Mather classes without explicitly acknowledging this.) In this respect, Proposition 2.7 above can be regarded as an alternative to Ginzburg's construction; see also Proposition 4.1 below.

The non-equivariant version of Proposition 3.3 is [2, Lemma 4.3]; this is essentially a reformulation of [44, eq. (12), page 67], which in turn is based on a calculation of Sabbah [50]. For another approach to this non-equivariant version of Proposition 3.3 see also [52]. In the present context, the connection between shadows and these formulas is the equivariant version of Lemma 2.1 (i): for an invariant cycle C in T^*X,

$$\mathrm{Shadow}^T(C) = c^T(T^*X) \cap s^T(C) \quad . \tag{11}$$

Here, $s^T(-)$ is an 'equivariant Segre class' operator, defined as follows: as above, the cycle C determines a T-invariant cycle $\overline{C}$ in $\mathbb{P}(T^*X \oplus \mathbb{1})$, and

$$s^T(C) := \overline{q}_*(c^T(\mathcal{O}_{T^*X\oplus\mathbf{1}}(-1))^{-1} \cap [\overline{C}]) \quad , \tag{12}$$

where $\overline{q} : \mathbb{P}(T^*X \oplus \mathbb{1}) \to X$ is the projection. In the non-equivariant case, the Chern classes are nilpotent, therefore the operator is well defined. Equivariantly, this operator has values in an appropriate completion. The Segre operator will be key in §8 below, where $X = G/B$ is a flag manifold. In that case, theorem 8.3 shows that it has values in a localization of the equivariant homology ring. We refer to [20, Chapter 4] and [32] for detailed information on Segre classes and operators in ordinary Chow groups. Formula (11) follows directly by applying its non-equivariant version to approximating spaces.

Proof of Proposition 3.3. Let U denote an approximation space to ET (see [17], [42]), and denote by u the projection $U \times_T X \to U/T$. Note that the relative cotangent bundle T^*u is the U-approximation of the bundle $(T^*X)_T$. Every invariant constructible function φ on X determines a constructible function on $U \times_T X$, agreeing with φ on the fibers of u. We denote this function by φ_U. By Ohmoto's definition, $\check{c}_*^T(\varphi)$ is the limit of classes

$$\check{c}_*^T(\varphi) := \varinjlim_U c(T^*U_T)^{-1} \cap \check{c}_*(\varphi_U)$$

where T^*U_T is the vector bundle $(T^*U) \times_T X \to U \times_T X$ ([42, p. 122 and Definition 3.2]). This bundle is an extension of $u^*(T^*(U/T))$ by a trivial bundle, therefore

$$\check{c}_*^T(\varphi) := \varinjlim_U u^*c(T^*(U/T))^{-1} \cap \check{c}_*(\varphi_U)\,.$$

By [2, Lemma 4.3],

$$\check{c}_*(\varphi_U) = \mathrm{Shadow}_{T^*(U\times_T X)\oplus\mathbf{1}}(\overline{\mathrm{CC}(\varphi_U)})$$

in $H_*(U \times_T X)$. Now we claim that

$$\mathrm{Shadow}_{T^*(U\times_T X)\oplus\mathbf{1}}(\overline{\mathrm{CC}(\varphi_U)}) = u^*c(T^*(U/T)) \cap \mathrm{Shadow}_{T^*u\oplus\mathbf{1}}(\overline{C^U}), \tag{13}$$

where C^U is the cycle in the relative cotangent bundle T^*u determined by the invariant cycle $\mathrm{CC}(\varphi)$ in T^*X. Indeed, by linearity we may assume that $\mathrm{CC}(\varphi)$ equals the conormal bundle $T^*_Z X$ of a subvariety Z of X. If $Z = X$, then the conormal bundle is the zero-section of T^*X, and the cycles $\mathrm{CC}(\varphi_U)$, C^U are the zero-sections of $T^*(U \times_T X)$, T^*u, respectively. In this case, (13) amounts to

$$c(T^*(U \times_T X)) \cap [U \times_T X] = u^*c(T^*(U/T))c(T^*u) \cap [U \times_T X]$$

(Remark 2.3), which follows from the Whitney formula since we have an exact sequence

$$0 \longrightarrow u^*T^*(U/T) \longrightarrow T^*(U \times_T X) \longrightarrow T^*u \longrightarrow 0\,.$$

If $Z \neq X$, then $\mathrm{CC}(\varphi_U) = T^*_{U\times_T Z}(U \times_T X) \subseteq T^*(U \times_T X)$, C^U is the image of $\mathrm{CC}(\varphi_U)$ in T^*u, and $\mathrm{CC}(\varphi_U)$ only meets $u^*T^*(U/T)$ at 0. In this case (13) follows by Lemma 2.1 (iv) and Lemma 2.2. We can conclude that

$$\check{c}_*^T(\varphi) = \varinjlim_U \mathrm{Shadow}_{T^*u\oplus\mathbf{1}}(\overline{C^U}) \quad ,$$

and this is the statement. □

Proposition 3.3 and formula (11) imply:

Corollary 3.4. *Let $\varphi \in \mathcal{F}^T_{inv}(X)$ be an invariant constructible function. There is an identity in $H^T_*(X)$,*

$$c^T(T^*X) \cap s^T(\mathrm{CC}(\varphi)) = \check{c}^T_*(\varphi).$$

4. Homogenized CSM classes are pull backs of characteristic cycles

We can now apply Proposition 2.7 and obtain another construction of the map $L_T(X) \to H^T_*(X)$, using the equivariant pull-back via the zero section $\iota : X \to T^*X$ of the cotangent bundle.

Again assume that X is nonsingular, and let T^*X denote the cotangent bundle of X. View T^*X as a $T \times \mathbb{C}^*$-equivariant bundle, where the T-action is induced from the left T-action on X as in §3.2 and the $\mathbb{C}^*$ factor acts on the fibers of T^*X by dilation with character χ. Let $\pi : T^*X \to X$ be the natural projection and $\iota : X \to T^*X$ the zero section; they are both $T \times \mathbb{C}^*$-equivariant. Observe that every $C \in L_T(X)$ is also $T \times \mathbb{C}^*$ invariant. Since $\mathbb{C}^*$ acts trivially on X, $H^{T\times\mathbb{C}^*}_*(X) \cong H^T_*(X)[\hbar]$.

Proposition 4.1. *Let $\mathbb{C}^*$ act on fiber of T^*X by dilation with character χ. Then for all $C \in L_T(X)$,*

$$\mathrm{Shadow}^T(C)^\chi = \iota^*([C]_{T\times\mathbb{C}^*}) \quad .$$

Proof. The equivariant homology $H^{T\times\mathbb{C}^*}_*(X)$ may be viewed as the $\mathbb{C}^*$-equivariant homology of the mixing space X_T: $H^{T\times\mathbb{C}^*}_*(X) \cong H^{\mathbb{C}^*}(X_T)$. By Proposition 2.7,

$$\iota^*([C]_{T\times\mathbb{C}^*}) = \iota^*([C_T]_{\mathbb{C}^*}) = \mathrm{Shadow}([\overline{C}]_T)^\chi = \mathrm{Shadow}^T(C)^\chi$$

by definition of Shadow^T. □

As a consequence, we obtain the promised re-interpretation of the map $L_T(X) \to H^T_*(X)$ in diagram (9).

Corollary 4.2. *Let $\mathbb{C}^*$ act on fiber of T^*X by dilation with character $\hbar$. Then for all $C \in L_T(X)$,*

$$\mathrm{Shadow}^T(C) = \iota^*([C]_{T\times\mathbb{C}^*})|_{\hbar\mapsto 1} \quad .$$

By the commutativity of diagram (9), the same result implies a direct realization of the homogenized CSM class of a constructible function in terms of the equivariant pull-back:

Theorem 4.3. *Let $\iota : X \to T^*X$ be the zero section, and let $\mathbb{C}^*$ act on the fibers of T^*X by the character $\hbar^{-1}$. Then*

$$\iota^*[\mathrm{CC}(\varphi)]_{T\times\mathbb{C}^*} = c^T_*(\varphi)^\hbar \in H^{T\times\mathbb{C}^*}_0(X).$$

Proof. By Propositions 4.1 and 3.3,

$$\iota^*[\mathrm{CC}(\varphi)]_{T\times\mathbb{C}^*} = \mathrm{Shadow}^T(C)^{-\hbar} = \check{c}^T_*(\varphi)^{-\hbar} \quad .$$

By definition of homogenization (cf. (4)) and of signed Chern class,

$$\check{c}^T_*(\varphi)^{-\hbar} = \sum_{j=0}^{\dim X} (-\hbar)^j (-1)^j c^T_*(\varphi)_j = \sum_{j=0}^{\dim X} \hbar^j c^T_*(\varphi)_j = c^T_*(\varphi)^\hbar \quad .$$

(The class in (4) is indexed by codimension, while here the index denotes dimension). □

Example 4.4. Let $X = \mathbb{P}^1$ and consider the constructible function $1\!\!1_{\mathbb{P}^1}$. For simplicity, we will only work $\mathbb{C}^*$-equivariantly. Then

$$c_*(1\!\!1_{\mathbb{P}^1}) = [\mathbb{P}^1] + 2[pt] = c(T\mathbb{P}^1) \cap [\mathbb{P}^1].$$

By definition of homogenization,

$$c_*(1\!\!1_{\mathbb{P}^1})^\hbar = \hbar[\mathbb{P}^1] + 2[pt].$$

On the other hand, by the self-intersection formula,

$$\iota^*(\iota_*[\mathbb{P}^1]_{\mathbb{C}^*}) = c_1^{\mathbb{C}^*}(T^*\mathbb{P}^1) \cap [\mathbb{P}^1]_{\mathbb{C}^*} = (c_1(T^*\mathbb{P}^1) - \hbar) \cap [\mathbb{P}^1] = -\hbar[\mathbb{P}^1] - 2[pt].$$

Together with the fact that $\mathrm{CC}(1\!\!1_{\mathbb{P}^1}) = -[T^*_{\mathbb{P}^1}\mathbb{P}^1] = -\iota_*[\mathbb{P}^1]_{\mathbb{C}^*}$, this implies that

$$\iota^*[\mathrm{CC}(1\!\!1_{\mathbb{P}^1})]_{\mathbb{C}^*} = c_*(1\!\!1_{\mathbb{P}^1})^\hbar$$

as claimed. ⌟

Specializing Theorem 4.3 to the constructible functions $\varphi = 1\!\!1_Z$ and $\varphi = \mathrm{Eu}_Z$ gives the following.

Corollary 4.5. *Let $Z \subseteq X$ be a T-stable constructible subvariety, and let $\mathbb{C}^*$ act on the fibers of T^*X by the character $\hbar^{-1}$. Then the homogenized CSM class satisfies*

$$c^T_{SM}(Z)^\hbar = \iota^*[\mathrm{CC}(1\!\!1_Z)]_{T\times\mathbb{C}^*} \;\in H_0^{T\times\mathbb{C}^*}(X).$$

If $Z \subseteq X$ is a T-stable irreducible subvariety then the homogenized Chern-Mather class satisfies

$$c^T_{Ma}(Z)^\hbar = (-1)^{\dim Z}\iota^*[T^*_Z X]_{T\times\mathbb{C}^*} \;\in H_0^{T\times\mathbb{C}^*}(X).$$

Remark 4.6. If one further specializes Theorem 4.3 to the characteristic function $\varphi = 1\!\!1_X$ and considers the nonequivariant case, i.e. when $\hbar \mapsto 1$ and elements in $H^*_T(pt)$ are specialized to 0, then one obtains the classical index formula

$$\int_X \iota^*[T^*_X X] = (-1)^{\dim X}\chi(X),$$

where $\chi(X)$ is the Euler characteristic. ⌟

5. Hecke orthogonality of homogenized CSM classes for complete flag manifolds

We will now focus on the case in which $X = G/B$ is a flag manifold, where G is a complex simple Lie group and B is a Borel subgroup. The flag manifold has two transversal algebraic Whitney stratifications given by the B and B^- orbits, where B^- is the opposite Borel subgroup.The goal of this section is to study the torus-equivariant CSM classes of these orbits. Formulas for these classes were computed in [5], in terms of a certain Demazure-Lusztig operator. Here we continue this investigation and we show that the *dual* homogenized CSM classes of Schubert cells are determined respectively by the adjoint of the Demazure-Lusztig operator. The two Demazure-Lusztig operators have been studied in several papers, in relation to degenerate Hecke algebras, and to equivariant K theory of G/B; see [22, 34, 35, 38].

The main application, is an orthogonality property enjoyed by the (homogenized) CSM classes. Rimányi and Varchenko [48, §7] proved that CSM classes of Schubert cells satisfy an orthogonality property, inherited from the orthogonality of Maulik and Okounkov's stable envelopes [40]. However, the orthogonality we will prove in this section is related to orthogonality properties in the (degenerate) Hecke algebras, as discussed e.g. in [34] and it appears to be complementary to that from [48, 40]. The two orthogonality properties are the key ingredients needed later to prove certain identities relating the dual and ordinary CSM classes. In turn, these identities will lead to a proof in §8 of a positivity property of CSM classes conjectured in [5].

In what follows, we will use the notation from [5]. The Weyl group $N_G(T)/T$ is denoted by W, $\ell : W \to \mathbb{N}$ is the length function, and w_0 denotes the longest element. Notice that $B^- = w_0 B w_0$. For $w \in W$, define the Schubert cells $X(w)^\circ := BwB/B \simeq \mathbb{C}^{\ell(w)}$ and the opposite Schubert cells $Y(w)^\circ := B^- wB/B \simeq \mathbb{C}^{\dim X - \ell(w)}$. Their closures are the Schubert varieties $X(w) = \overline{BwB/B}$ and the opposite Schubert varieties $Y(w) = \overline{B^- wB/B}$. These are complex projective algebraic varieties such that $\dim_{\mathbb{C}} X(w) = \operatorname{codim}_{\mathbb{C}} Y(w) = \ell(w)$. The Bruhat order $\leq$ is a partial order on the Weyl group W; it may be defined by declaring that $u \leq v$ if and only if $X(u) \subseteq X(v)$. Also recall that the opposite Schubert classes $[Y(w)]$ are Poincaré dual to the Schubert classes $[X(w)]$, in the sense that

$$\langle [X(u)], [Y(v)] \rangle := \int_{G/B} [X(u)] \cdot [Y(v)] = \delta_{uv} \tag{14}$$

with respect to the usual intersection pairing (see e.g., [13, Proposition 1.3.6]). In fact, (14) also holds for equivariant classes, since the intersections are generically transverse. We also note that $[Y(w)] = [X(w_0 w)]$. This equality for ordinary classes does *not* extend to the equivariant setting, but of course one can translate from B to B^--stable Schubert classes using the left multiplication by w_0; the details will be recalled below.

5.1. **An operator relating CSM classes on flag manifolds.** The main result of [5] calculates the equivariant CSM classes $c^T_{\mathrm{SM}}(X(w)^\circ)$. For each simple reflection $s_i \in W$ one can associate two operators on $H^*_T(X)$. The first is the BGG operator $\partial_i : H^T_*(X) \to H^T_{*+2}(X)$ and it is obtained as $\partial_i = \pi_i^*(\pi_i)_*$ where π_i is the projection $\pi_i : G/B \to G/P_i$, and P_i is the minimal parabolic subgroup associated to i. The BGG operator ∂_i is $H^*_T(pt)$-linear. The second operator is the homogeneous algebra automorphism $s_i : H^T_*(X) \to H^T_*(X)$ obtained by the *right* Weyl group multiplication by s_i on $G/B = K/T_{\mathbb{R}}$. Here $K \subset G$ is the maximal compact subgroup, $T_{\mathbb{R}} := T \cap K$ is the maximal torus and (abusing notation) $H^T_*(X) = H^*_T(X)$ by Poincaré duality. For each simple reflection $s_i \in W$ define the (non-homogeneous) operator $\mathcal{T}_i := \partial_i - s_i$ acting on the homology $H^T_*(X)$. The operators $\mathcal{T}_i$ satisfy the braid relations for W and $\mathcal{T}_i^2 = id$ ([5, Proposition 4.1]). Thus, we may define an operator $\mathcal{T}_w$ for any element w of the Weyl group. The next result was proved in [5, Corollary 4.2].

Theorem 5.1. *Let $w \in W$ be an element of the Weyl group. Then*

$$\mathcal{T}_i(c^T_{SM}(X(w)^\circ)) = c^T_{SM}(X(ws_i)^\circ).$$

Therefore, for every $w \in W$, we have

$$c^T_{SM}(X(w)^\circ) = \mathcal{T}_{w^{-1}}([pt]_T).$$

5.2. **Demazure-Lusztig operators.** As in §4, consider a $\mathbb{C}^*$ acting on X trivially and let $\hbar \in H^2_{\mathbb{C}^*}(pt) \subset H^*_{T\times\mathbb{C}^*}(X) = H^*_T(X)[\hbar]$. We will denote by $\mathbb{T}$ the torus $T \times \mathbb{C}^*$. We define two operators

$$\mathbf{L}_i := \hbar\partial_i - s_i; \quad \mathbf{L}_i^\vee := \hbar\partial_i + s_i,$$

acting on $H^{\mathbb{T}}_*(X)$. These are *homogeneous* operators on $H^{\mathbb{T}}_*(X) \cong H^*_{\mathbb{T}}(X)$. Notice that $\mathbf{L}_i$ is the homogenized version of the operator $\mathcal{T}_i = \partial_i - s_i$ from [5] recalled above. The 'dual' operator $\mathbf{L}_i^\vee$ is precisely the Demazure-Lusztig operator discussed by Ginzburg in [22, Eq. (47)]. It follows from the basic commutation properties of ∂_i and s_i (see e.g., [5, Lemma 2.2]) that both operators $\mathbf{L}_i$ and $\mathbf{L}_i^\vee$ satisfy the braid relations and that $\mathbf{L}_i^2 = (\mathbf{L}_i^\vee)^2 = id$. In particular for each $w \in W$ there are well defined operators $\mathbf{L}_w$ and $\mathbf{L}_w^\vee$. Let $\langle \cdot, \cdot \rangle : H^*_{\mathbb{T}}(X) \otimes H^*_{\mathbb{T}}(X) \to H^*_{\mathbb{T}}(pt)$ be the intersection pairing defined by $\langle a, b \rangle = \int_X a \cdot b$.

Lemma 5.2. *The operators $\mathbf{L}_i$ and $\mathbf{L}_i^\vee$ are adjoint to each other, i.e., for any $a, b \in H^*_{\mathbb{T}}(X)$ there is an identity in $H^*_{\mathbb{T}}(pt)$:*

$$\langle \mathbf{L}_i(a), b \rangle = \langle a, \mathbf{L}_i^\vee(b) \rangle.$$

Therefore, $\langle \mathbf{L}_w(a), b \rangle = \langle a, \mathbf{L}_{w^{-1}}^\vee(b) \rangle$ for all $w \in W$.

Proof. It suffices to show that the BGG operator ∂_i is self-adjoint and that the adjoint of s_i is $-s_i$. We first verify that ∂_i is self-adjoint; while this is well-known, we include a proof for completeness. Let P_i be the minimal parabolic group and $\pi_i : G/B \to G/P_i$ the natural projection. Recall that $\partial_i = \pi_i^*(\pi_i)_*$. Then

$$\langle \partial_i(a), b \rangle = \int_{G/B} \pi_i^*(\pi_i)_*(a) \cdot b = \int_{G/P_i} (\pi_i)_*(a) \cdot (\pi_i)_*(b) = \langle a, \partial_i(b) \rangle,$$

where the last equality follows by symmetry.

In order to verify that s_i and $-s_i$ are adjoint, let $e_w := wB \in G/B$ denote the T-fixed point in X corresponding to w (so $e_{id} = 1.B$ is the B-fixed point). Then $\int_X a \cdot b$ is the coefficient of $[X(id)]_{\mathbb{T}} = [e_{id}]_{\mathbb{T}}$ in the expression for $a \cdot b$ with respect to the Schubert basis. Recall also that

$$s_i[e_{id}]_{\mathbb{T}} = -[e_{s_i}]_{\mathbb{T}} = P(t)[X(s_i)]_{\mathbb{T}} - [e_{id}]_{\mathbb{T}}$$

where $P(t) \in H^2_{\mathbb{T}}(pt)$. (Cf. e.g., [5, (4) and §6.3].) Then

$$\langle s_i(a), b \rangle = \int_X s_i(a) \cdot b = \int_X s_i(a) \cdot s_i s_i(b) = \int_X s_i(a \cdot s_i(b)) = -\int_X a \cdot s_i(b) = -\langle a, s_i(b) \rangle.$$

Here we also used that s_i squares to the identity. □

Next we use formulas for ∂_i and s_i to calculate the actions of $\mathbf{L}_i$ and $\mathbf{L}_i^\vee$ on Schubert classes. Of course, these actions can be obtained by homogenizing the action of $(\mathbf{L}_i)_{\hbar \mapsto 1}$ and $(\mathbf{L}_i^\vee)_{\hbar \mapsto 1}$ to Schubert classes, and the latter can be obtained as in [5, §6.3], based on the results of [33]. Therefore we will be brief, and indicate only the most salient points.

(15)
$$\mathbf{L}_k([X(w)]_{\mathbb{T}}) = \begin{cases} -[X(w)]_{\mathbb{T}} & \text{if } \ell(ws_k) < \ell(w) \\ (\hbar + w(\alpha_k))[X(ws_k)]_{\mathbb{T}} + [X(w)]_{\mathbb{T}} + \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)]_{\mathbb{T}} & \text{if } \ell(ws_k) > \ell(w) \end{cases}$$

where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(w) = \ell(ws_k s_\beta)$, and $w(\alpha_k)$ denotes the natural W action $w \cdot \alpha_k$ on T-weights; note that $w(\alpha_k) > 0$ since $ws_k > w$. The corresponding formula for the dual operator $\mathbf{L}_k^\vee$ is

(16)
$$\mathbf{L}_k^\vee([X(w)]_{\mathbb{T}}) = \begin{cases} [X(w)]_{\mathbb{T}} & \text{if } \ell(ws_k) < \ell(w) \\ (\hbar - w(\alpha_k))[X(ws_k)]_{\mathbb{T}} - [X(w)]_{\mathbb{T}} - \sum \langle \alpha_k, \beta^\vee \rangle [X(ws_k s_\beta)]_{\mathbb{T}} & \text{if } \ell(ws_k) > \ell(w) \end{cases}$$

where the sum is as before. For further use, we calculate the actions of the two operators on the opposite Schubert basis elements $[Y(w)]_{\mathbb{T}}$. For that, we use the well known technique of applying the automorphism induced by left multiplication by w_0, which we recall next.

Let $n_{w_0} \in G$ be a representative of $w_0 \in W = N_G(T)/T$. Left-multiplication by n_{w_0} induces an automorphism $G/B \to G/B$, $gB \mapsto n_{w_0} gB$. The image of any T-stable subvariety under this morphism is independent of the choice of the representative n_{w_0}, and by abuse of language we denote n_{w_0} by w_0 and the morphism simply by φ_{w_0}. This morphism satisfies

$$\varphi_{w_0}(X(w)^\circ) = \varphi_{w_0}(BwB/B) = w_0Bw_0w_0wB/B = B^-w_0wB/B = Y(w_0w)^\circ. \tag{17}$$

Let $\chi_0 : T \to T$ be the automorphism obtained by conjugating by n_{w_0}, i.e., $\chi_0(t) = n_{w_0}tn_{w_0}^{-1}$. This is independent of the choice of n_{w_0}, hence we denote it simply by $\chi_0(t) = w_0tw_0$. (Note: $w_0 = w_0^{-1}$ in W.) The morphism φ_{w_0} is not T-equivariant, and it twists the T characters via χ_0:

$$\varphi_{w_0}(t.gB) = w_0tgB = \chi_0(t)\varphi_{w_0}(gB).$$

Together with identity (17) this implies that φ_{w_0} induces an automorphism $\varphi_{w_0}^* : H_T^*(G/B) \to H_T^*(G/B)$ mapping $[Y(w)]_T$ to $\varphi_{w_0}^*[Y(w)]_T = [X(w_0w)]_T$. As before, this is *not* a homomorphism of $H_T^*(pt)$-algebras, but it 'twists' the coefficients in the base ring $H_T^*(pt)$ according to the automorphism χ_0. The ring automorphism $\varphi_{w_0}^*$ can be extended to one of $H_{\mathbb{T}}^*(X)$ by letting $\varphi_{w_0}^*(\hbar) = \hbar$. Recall also that the operator s_i is given by the right W-action on $H_T^*(G/B)$, i.e., by the right multiplication on $G/B = K/T_{\mathbb{R}}$ and as such it commutes with left-multiplication. From this we deduce that both operators $\mathbf{L}_k$ and $\mathbf{L}_k^\vee$ commute with $\varphi_{w_0}^*$, thus

$$\mathbf{L}_k[Y(w)] = \varphi_{w_0}^*\mathbf{L}_k[X(w_0w)]; \quad \mathbf{L}_k^\vee[Y(w)] = \varphi_{w_0}^*\mathbf{L}_k^\vee[X(w_0w)]_{\mathbb{T}}.$$

Explicitly, we obtain

$$\mathbf{L}_k([Y(w)]_{\mathbb{T}}) = \begin{cases} -[Y(w)]_{\mathbb{T}} & \text{if } \ell(ws_k) > \ell(w) \\ (\hbar + w(\alpha_k))[Y(ws_k)]_{\mathbb{T}} + [Y(w)]_{\mathbb{T}} + \sum\langle\alpha_k, \beta^\vee\rangle[Y(ws_ks_\beta)]_{\mathbb{T}} & \text{if } \ell(ws_k) < \ell(w) \end{cases}$$

where the sum is over all positive roots $\beta \neq \alpha_k$ such that $\ell(w) = \ell(ws_ks_\beta)$, and $w(\alpha_k) := w \cdot \alpha_k$ denotes the natural W action on T-weights. In this case $w(\alpha_k) < 0$, since $ws_k < w$. Similarly

$$\mathbf{L}_k^\vee([Y(w)]_{\mathbb{T}}) = \begin{cases} [Y(w)]_{\mathbb{T}} & \text{if } \ell(ws_k) > \ell(w) \\ (\hbar - w(\alpha_k))[Y(ws_k)]_{\mathbb{T}} - [Y(w)]_{\mathbb{T}} - \sum\langle\alpha_k, \beta^\vee\rangle[Y(ws_ks_\beta)]_{\mathbb{T}} & \text{if } \ell(ws_k) < \ell(w) \end{cases}$$

5.3. **Hecke orthogonality of homogenized CSM classes.** We consider next homogeneous equivariant CSM classes of Schubert cells and of opposite Schubert cells. As we will see, with suitable positions these classes are (Poincaré) duals of each other.

Definition 5.3. For $w \in W$, we let

$$c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ) := \mathbf{L}_{w^{-1}}[X(id)]_{\mathbb{T}}; \quad c_{\mathrm{SM}}^{T,\hbar}(Y(w)^\circ) := \mathbf{L}_{w^{-1}w_0}[Y(w_0)]_{\mathbb{T}};$$

and

$$c_{\mathrm{SM}}^{T,\hbar,\vee}(X(w)^\circ) := \mathbf{L}_{w^{-1}}^\vee[X(id)]_{\mathbb{T}}; \quad c_{\mathrm{SM}}^{T,\hbar,\vee}(Y(w)^\circ) := \mathbf{L}_{w^{-1}w_0}^\vee[Y(w_0)]_{\mathbb{T}}.$$

Because both operators $\mathbf{L}_i$, $\mathbf{L}_i^\vee$ are homogeneous of degree 0, it follows that these classes all belong to $H_0^{\mathbb{T}}(X)$. ⌟

The following proposition follows immediately from the definitions of the operators $\mathbf{L}_i$ and $\mathbf{L}_i^\vee$ and from Theorem 5.1.

Proposition 5.4. *The class $c_{SM}^{T,\hbar}(X(w)^\circ)$ and the class $c_{SM}^{T,\hbar,\vee}(Y(w)^\circ)$ coincide respectively with the $\hbar$-homogenizations of the CSM class $c_{SM}^{T}(X(w)^\circ)$ and of the dual class $c_{SM}^{T,\vee}(Y(w)^\circ)$ from equation* (8). *Explicitly,*

$$c_{SM}^{T,\hbar}(X(w)^\circ) = c_{SM}^{T}(X(w)^\circ)^\hbar = \sum \hbar^k c_{SM}^{T}(X(w)^\circ)_k,$$

and

$$c_{SM}^{T,\hbar,\vee}(Y(w)^\circ)= c_{SM}^{T,\vee}(Y(w)^\circ)^\hbar = (-1)^{\dim X-\ell(w)}\sum \hbar^k(-1)^k c_{SM}^T(Y(w)^\circ)_k,$$

where the subscript k denotes the component of dimension k.

In particular, $c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ)$ is the class expressed as an equivariant pull-back of the characteristic cycle in Theorem 4.3; analogous statements can be made concerning the other classes introduced in Definition 5.3. Also, the classes $c_{\mathrm{SM}}^{T,\hbar}(Y(w)^\circ)$, $c_{\mathrm{SM}}^{T,\hbar,\vee}(X(w)^\circ)$ are related to the others by the automorphism φ_{w_0}. We record the result.

Proposition 5.5. *The following hold:*

$$c_{SM}^{T,\hbar}(Y(w)^\circ) = \varphi_{w_0}^* c_{SM}^{T,\hbar}(X(w_0w)^\circ)$$
$$c_{SM}^{T,\hbar,\vee}(X(w)^\circ) = \varphi_{w_0}^* c_{SM}^{T,\hbar,\vee}(Y(w_0w)^\circ)$$

Another description of the dual CSM classes can be found in Theorem 7.3 below. The explicit expressions (15), (16) yield the following computation of the leading terms of the main classes introduced in Definition 5.3 (also cf. [5, Proposition 6.5]).

Lemma 5.6. *For $w \in W$, let $e(w) := \prod_{w^{-1}(\alpha)<0}(\hbar+\alpha)$ and $\check{e}(w) := \prod_{w^{-1}(\alpha)<0}(\hbar-\alpha)$. Then*

$$c_{SM}^{T,\hbar}(X(w)^\circ) = e(w)[X(w)]_{\mathbb{T}} + \textit{terms involving } [X(v)]_{\mathbb{T}} \textit{ for } v<w.$$
$$c_{SM}^{T,\hbar,\vee}(X(w)^\circ) = \check{e}(w)[X(w)]_{\mathbb{T}} + \textit{terms involving } [X(v)]_{\mathbb{T}} \textit{ for } v<w.$$

Next is the main result of this section. It establishes that the classes $c_{\mathrm{SM}}^{T,\hbar}(X(u)^\circ)$, $c_{\mathrm{SM}}^{T,\hbar,\vee}(Y(v)^\circ)$ are dual to each other, in the sense made precise in the following statement.

Theorem 5.7. *The homogenized CSM classes satisfy the following orthogonality property:*

$$\langle c_{SM}^{T,\hbar}(X(u)^\circ), c_{SM}^{T,\hbar,\vee}(Y(v)^\circ)\rangle = \delta_{u,v}\prod_{\alpha>0}(\hbar+\alpha).$$

An analogous orthogonality property holds for opposite Schubert cells:

$$\langle c_{SM}^{T,\hbar}(Y(u)^\circ), c_{SM}^{T,\hbar,\vee}(X(v)^\circ)\rangle = \delta_{u,v}\prod_{\alpha>0}(\hbar-\alpha).$$

Remark 5.8. Observe that

$$e := \prod_{\alpha\in R_+}(\hbar+\alpha) = (-1)^{\dim G/B} e^{T\times\mathbb{C}^*}(T_{w_0}^*(G/B)), \tag{18}$$

i.e., e can be interpreted as the the signed $T\times\mathbb{C}^*$ equivariant Euler class of $T_{w_0}^*(G/B)$; equivalently, it is the equivariant Euler class of $T_{w_0}(G/B)$. ⌟

Proof of Theorem 5.7. To prove the first equality, we use Lemma 5.2 and Definition 5.3 to obtain the following:

$$\begin{aligned}\langle c_{\mathrm{SM}}^{T,\hbar}(X(u)^\circ), c_{\mathrm{SM}}^{T,\hbar,\vee}(Y(v)^\circ)\rangle &= \langle \mathbf{L}_{u^{-1}}[X(id)]_{\mathbb{T}}, \mathbf{L}^\vee_{v^{-1}w_0}[Y(w_0)]_{\mathbb{T}}\rangle\\ &= \langle \mathbf{L}_{w_0v}\mathbf{L}_{u^{-1}}[X(id)]_{\mathbb{T}}, [Y(w_0)]_{\mathbb{T}}\rangle\\ &= \langle \mathbf{L}_{w_0vu^{-1}}[X(id)]_{\mathbb{T}}, [Y(w_0)]_{\mathbb{T}}\rangle\\ &= \text{coefficient of } [X(w_0)]_{\mathbb{T}} \text{ in } c_{\mathrm{SM}}^{T,\hbar}(X(uv^{-1}w_0)^\circ).\end{aligned}$$

By Lemma 5.6, this coefficient is 0 unless $u = v$, and it equals $\prod_{\alpha>0}(\hbar+\alpha)$ if $u = v$. This verifies the first equality. The second equality follows from the same argument, using now that

$$\langle \mathbf{L}_{u^{-1}w_0}[Y(w_0)]_{\mathbb{T}}, \mathbf{L}^{\vee}_{v^{-1}}[X(id)]_{\mathbb{T}}\rangle = \langle [Y(w_0)]_{\mathbb{T}}, \mathbf{L}^{\vee}_{w_0uv^{-1}}[X(id)]_{\mathbb{T}}\rangle = \delta_{u,v}\prod_{\alpha>0}(\hbar-\alpha),$$

again from Lemma 5.6. □

Corollary 5.9 (CSM Poincaré duality)**.** *Ordinary CSM classes are Poincarè dual to dual CSM classes of opposite cells. That is:*

$$\langle c_{SM}(X(u)^\circ), c^{\vee}_{SM}(Y(v)^\circ)\rangle = \delta_{u,v}. \tag{19}$$

Proof. This follows from the previous theorem by specializing $\hbar \mapsto 1$ and $\alpha \mapsto 0$. □

In ordinary homology, the leading terms of $c_{\mathrm{SM}}(X(u)^\circ)$ and $c^{\vee}_{\mathrm{SM}}(Y(v)^\circ)$ are $[X(u)]$, $[Y(v)]$, respectively: we may view these CSM classes as 'deformations' of the fundamental classes by lower dimensional terms. Corollary 5.9 states that these deformations preserve the intersection pairing: cf. (14) and (19).

5.4. **The transition matrix between Schubert and CSM classes.** We present two consequences of the orthgonality property, in the non-equivariant setting.

The CSM class of each Schubert cell may be written in terms of the Schubert basis:

$$c_{\mathrm{SM}}(X(v)^\circ) = \sum_{u\in W}\mathbf{c}(u;v)[X(u)] \quad . \tag{20}$$

with $\mathbf{c}(u;v) \in \mathbb{Z}$. A natural question is to find the inverse of the matrix $(\mathbf{c}(u;v))_{u,v}$ where u, v vary in the Weyl group W.

Proposition 5.10. *The inverse of the matrix $\big(\mathbf{c}(u;v)\big)_{u,v}$ is the matrix*

$$\big((-1)^{\ell(u)-\ell(v)}\mathbf{c}(w_0v;w_0u)\big)_{u,v} \quad .$$

Proof. Let $(d(u;v))_{u,v}$ be the inverse matrix. In other words,

$$[X(v)] = \sum_{u\in W} d(u;v)c_{\mathrm{SM}}(X(u)^\circ) \quad .$$

By Corollary 5.9, $d(u;v) = \langle [X(v)], c^{\vee}_{\mathrm{SM}}(Y(u)^\circ)\rangle$. This is the coefficient of $[Y(v)]$ in the expansion of $c^{\vee}_{\mathrm{SM}}(Y(u)^\circ)$ in the basis of (opposite) Schubert classes. This coefficient can calculated from the Proposition 5.4 above, using that for any $w \in W$, $[Y(w)] = [X(w_0w)]$. □

Example 5.11. Consider the complete flag manifold Fl(4) in type A, $W = S_4$. We consider the matrix whose (i, j) entry is the coefficient $\mathbf{c}(w_i, w_j)$, where we list the permutations $w_i \in S_4$, $i = 1, \dots, 24$ in lexicographic order (in 'window' format):

$$1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431,$$
$$3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321 \quad .$$

Applying [5, Corollary 4.2], we obtain the matrix shown in Figure 1 at the end of the paper. The inverse of this matrix is the matrix shown in Figure 2. As claimed in Proposition 5.10, this matrix is (up to signs) the anti-transpose of (38).

Consider now the problem of defining a constructible function Θ_V on a given variety V, such that $c_*(\Theta_V) = [V]$. Such a function is of course not uniquely defined, but there are interesting situations in which a particularly natural function satisfies this property. For example, if V is a toric variety compactifying a torus V°, then $\Theta_V = 1\!\!1_{V^\circ}$ is such a function ([3, Théorème 4.2]). Proposition 5.10 implies that Schubert varieties offer another class of examples.

Corollary 5.12. *Let* $\Theta_w = \sum_v (-1)^{\ell(w)-\ell(v)} \mathbf{c}(w_0 v; w_0 w) 1\!\!1_{X(v)^\circ}$. *Then* $c_*(\Theta_w) = [X(w)]$.

Proof. Recall that $Y(w)^\circ$ is the w_0-translation of $X(w_0 w)^\circ$, cf. (17). It follows that

$$c^\vee_{\mathrm{SM}}(Y(v)^\circ) = c^\vee_{\mathrm{SM}}(X(w_0 v)^\circ) = \sum_{u \in W} (-1)^{\ell(v)-\ell(u)} \mathbf{c}(w_0 u; w_0 v)[X(w_0 u)]$$

with notation as in (20). Therefore, (19) amounts to the identity

$$\sum_w \mathbf{c}(w,u)(-1)^{\ell(v)-\ell(w)} \mathbf{c}(w_0 w; w_0 v) = \delta_{uv}.$$

□

To illustrate, let $w = w_0$, the maximum length element in the Weyl group. Then according to Corollary 5.12

$$[G/B] = c_* \left(\sum_v (-1)^{\ell(w_0)-\ell(v)} 1\!\!1_{X(v)^\circ} \right) \quad . \tag{21}$$

This identity is independently proven in [5, Proposition 5.5].

Remark 5.13. Consider the Schubert expansion of the equivariant CSM class:

$$c^T_{\mathrm{SM}}(X(w)^\circ) = \sum \mathbf{c}^T(v;w)[X(v)]_T,$$

where $\mathbf{c}^T(v;w) \in H^*_T(pt) = (Lie\ T)^*$. There is an interpretation of the coefficients $\mathbf{c}^T(u;v)$ in the affine nil-Hecke algebra, found by S.J. Lee in [36]. We briefly recall this interpretation, and refer to *loc. cit.* for all the details. The affine nil-Hecke algebra $\mathcal{H}_{nil}$ is generated by the elements $\bar{\partial}_w$ and $\lambda \in (\mathrm{Lie}\, T)^*$, subject to the following relations:

(1) $\lambda\mu = \mu\lambda$, for any $\lambda, \mu \in (\mathrm{Lie}\, T)^*$;
(2) $\bar{\partial}_w \bar{\partial}_y = \delta_{\ell(wy), \ell(w)+\ell(y)} \bar{\partial}_{wy}$;
(3) $\bar{\partial}_i \lambda = s_i \lambda \cdot \bar{\partial}_i - \langle \lambda, \alpha_i^\vee \rangle$.

For each simple root α_i, define the element $\bar{s}_i := 1 + \alpha_i \bar{\partial}_i \in \mathcal{H}_{nil}$. The elements $\{\bar{\partial}_i\}$ and $\{\bar{s}_i\}$ satisfy the braid relations in the Weyl group W. Let $w = s_{i_1} \cdot \ldots \cdot s_{i_k} \in W$ be a reduced decomposition. Then according to [36, Thm. 6.2] there is an identity

$$(\bar{s}_{i_1} + \bar{\partial}_{i_1}) \cdot \ldots \cdot (\bar{s}_{i_k} + \bar{\partial}_{i_k}) = \sum_v \mathbf{c}^T(v;w) \bar{\partial}_v.$$

Geometrically, $\bar{\partial}_i$ corresponds to the BGG operator ∂_i, $\bar{s}_i$ to $-s_i$, and the weight λ to the Chevalley multiplication by the equivariant Chern class $c_1^T(G \times^B \mathbb{C}_{-\lambda})$ (the class of a G-equivariant line bundle over G/B).

6. CSM classes and characteristic cycles for flag manifolds

In the classical results of Beilinson-Bernstein [7], Brylinski-Kashiwara [14], and Kashiwara-Tanisaki [28], the theory of characteristic cycles associated to holonomic D-modules on flag manifolds becomes a powerful geometric tool to study the representation theory related to Kazhdan-Lusztig polynomials. In turn, characteristic cycles for Verma modules are (up to a sign) equal to the stable basis elements of Maulik and Okounkov [40]. Relations of stable basis to integrable systems were studied in a series of papers by Rimányi, Tarasov and Varchenko [47, 49, 46]. Since characteristic cycles and CSM classes are closely related (cf. Theorem 4.3), it is natural to ask for the precise relations between these classes, Verma modules, and stable basis elements. This is the goal of this section. Hints of this connection were given in [5], where it was observed that the operator $\mathcal{T}_i = \partial_i - s_i$ determining CSM classes for Schubert cells in flag manifolds is closely related to the Demazure-Lusztig operator; this is now better explained in section 5.2 above where we studied the homogenized CSM classes. In the same vein, Rimányi and Varchenko [48] observed that localizations of equivariant CSM classes, calculated earlier by Weber [60], and those of the stable basis elements, coincide.

6.1. **CSM classes and characteristic cycles of Verma modules.** Let X be a smooth complex variety. We start by recalling a commutative diagram considered by Ginzburg in [21, Appendix], which is largely based on results from [7, 14, 28].

$$\begin{array}{ccccc} Perv(X) & \xleftarrow[\sim]{\mathrm{DR}} & Mod_{rh}(\mathcal{D}_X) & & \\ \downarrow{\scriptstyle \chi_{stalk}} & & \downarrow{\scriptstyle \mathrm{Char}} & & \\ \mathcal{F}(X) & \xrightarrow[\sim]{\mathrm{CC}} & L(X) & \xrightarrow{\check{c}_*^{Gi}} & H_*(X). \end{array} \tag{22}$$

Here $Mod_{rh}(\mathcal{D}_X)$ denotes the Abelian category of algebraic holonomic $\mathcal{D}_X$-modules with regular singularities, and $Perv(X)$ is the Abelian category of perverse (algebraically) constructible complexes (of sheaves of $\mathbb{C}$-vector spaces) on X. The functor DR is defined on $M^\cdot \in Mod_{rh}(\mathcal{D}_X)$ by

$$\mathrm{DR}(M^\cdot) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, M^\cdot)[\dim X],$$

that is, it computes the DeRham complex of a holonomic module (up to a shift), viewed as an *analytic* $\mathcal{D}_X$-module. This functor realizes the Riemann-Hilbert correspondence, and is an equivalence. We refer to e.g., [28, 21] for details. The left map χ_{stalk} computes the stalkwise Euler characteristic of a constructible complex, and the right map Char gives the characteristic cycle of a holonomic $\mathcal{D}_X$-module. The map CC is the characteristic cycle map for constructible functions from diagram (9) above. The commutativity of diagram (9) is shown in [21] using some deep $\mathcal{D}$-module techniques; it also follows from [51, Ex.5.3.4 on pp.359-360]. Also note that the upper transformations in (9) factors over the corresponding Grothendieck groups, so they also apply to complexes of such $\mathcal{D}$-modules. The map $\check{c}_*^{Gi}$ is defined in [21, §A.3] (where it is denoted c_*). Ginzburg observes the identity

$$\check{c}_*^{Gi}[T^*_Y X] = \check{c}_{\mathrm{Ma}}(Y) \tag{23}$$

for all closed subvarieties $Y \subseteq X$ ([21, Lemma A3.2], [50, Lemma 1.2.1]). By Corollary 4.5, it follows that in fact $\check{c}_*^{Gi} = \iota^*_{\hbar=-1}$, the specialization at $\hbar = -1$ of the pull-back via the zero-section map $\iota : X \to T^*X$. It is proved in [21, Theorem A5] that the map $\check{c}_*^{Gi}$ commutes with proper push-forwards. This can now also be shown by combining [52, Sec. 4.6] with Proposition 4.1 of this paper. It follows that the composition $\check{c}_*^{Gi} \circ \mathrm{CC} = \check{c}_*$ coincides with (the signed) MacPherson's natural transformation from constructible functions to homology; this is the non-equivariant version of Theorem 4.3.

Now let $X = G/B$ be the generalized flag manifold. We recall some results from [28]. Let $\rho \in Lie(T)^*$ denote half the sum of positive roots. For $w \in W$ let M_w be the Verma module of highest weight $-w\rho - \rho$, a module over the universal enveloping algebra $U(\mathfrak{g})$. Let $\mathfrak{M}_w$ denote the holonomic $\mathcal{D}_X$-module

$$\mathfrak{M}_w = \mathcal{D}_X \otimes_{U(\mathfrak{g})} M_w.$$

Consider the constructible complex $\mathrm{DR}(\mathfrak{M}_w)$. According to [28, Theorem 3] (where it is attributed to Brylinski-Kashiwara [14] and Beilinson-Bernstein [7]) there is an identity

$$\mathrm{DR}(\mathfrak{M}_w) = \mathbb{C}_{X(w)^\circ}[\ell(w)]\,; \tag{24}$$

also cf. [24, Corollary 12.3.3(i)]. (Note that the definition of DR from [28] differs from the one from [21] and [24] by a shift of $\dim X$.) It follows that the constructible function associated to the Verma module M_w is

$$\chi(\mathrm{DR}(\mathfrak{M}_w)) = (-1)^{\ell(w)} 1\!\!1_{X(w)^\circ}.$$

By the commutativity of diagram (22),

$$\mathrm{Char}(\mathfrak{M}_w) = (-1)^{\ell(w)}\,\mathrm{CC}(1\!\!1_{X(w)^\circ});$$

therefore, Corollary 4.5 implies the following.

Corollary 6.1. *Let $w \in W$. There is an identity*

$$c^{T,\hbar}_{SM}(X(w)^\circ) = (-1)^{\ell(w)} \iota^* \,\mathrm{Char}(\mathfrak{M}_w).$$

Corollary 6.1 gives a cohomological formula for the coefficients of the expansion of the (equivariant) CSM class into Schubert classes.

Proposition 6.2. *Consider the expansion*

$$c^{T}_{SM}(X(w)^\circ) = \sum c(u;w)[X(u)]_T.$$

Then

$$c(u;w) = (-1)^{\ell(w)} \langle \iota^* \,\mathrm{Char}(\mathcal{M}_w), [Y(u)]_{T\times\mathbb{C}^*}\rangle_{\hbar=1}$$

where $\langle \cdot,\cdot\rangle$ is the Poincaré pairing in $H^*_{T\times\mathbb{C}^*}(X)$.

Proof. This follows immediately from the fact that $\langle [X(u)]_T, [Y(v)]_T\rangle = \delta_{u,v}$. ☐

Remark 6.3. Let $\mathcal{L}_w$ be the holonomic $\mathcal{D}_X$-module associated to L_w, the quotient by the maximal proper highest weight module of the Verma module M_w. By the proof of the Kazhdan-Lusztig conjectures [7, 14] (see also [24, Chapter 12]) we have

$$\mathrm{Char}(\mathcal{L}_w) = \sum_{u\le w} (-1)^{\ell(w)-\ell(u)} P_{u,w}(1)\,\mathrm{Char}(\mathcal{M}_u)$$

where $P_{u,w}(q)$ is the Kazhdan-Lusztig polynomial. Then Corollary 6.1 implies that

$$\iota^* \,\mathrm{Char}(\mathcal{L}_w) = (-1)^{\ell(w)} \sum_{u\le w} P_{u,w}(1)\, c^{T,\hbar}_{\mathrm{SM}}(X(u)^\circ). \tag{25}$$

Kazhdan and Lusztig ([30]; cf. [24, Theorem 12.2.5]) proved that

$$P_{u,w}(1) = (-1)^{\ell(w)} \sum_j (-1)^j \dim H^j(IC(\mathbb{C}_{X(w)}))_u \quad ,$$

where $H^j(IC(\mathbb{C}_{X(w)}))_u$ denotes the stalk of the j-th cohomology sheaf of the intersection cohomology complex of the Schubert variety $X(w)$ at points of $X(u)^\circ$. That is,

$$\chi_{stalk}(IC(\mathbb{C}_{X(w)})) = (-1)^{\ell(w)} \sum_{u \le w} P_{u,w}(1) 1\!\!1_{X(u)^\circ} \quad , \tag{26}$$

and by (25) this implies that

$$\iota^* \operatorname{Char}(\mathcal{L}_w) = c_*^T(\chi_{stalk}(IC(\mathbb{C}_{X(w)})))^\hbar \tag{27}$$

is the characteristic class associated to the intersection cohomology complex. In fact, the equality

$$\mathrm{DR}(\mathcal{L}_w) = IC(X(w))$$

holds, [28, Theorem 3] (or [24, (12.2.13) and Corollary 12.3.3(ii)]). This directly implies (27), by the commutativity of diagram (22).

Further, there are situations in which $\operatorname{Char}(\mathcal{L}_w)$ is known to be irreducible, and hence equal to the conormal cycle $T^*_{X(w)}X$of $X(w)$; for example, in type A this is the case if w is a Grassmannian permutation, by a result of Bressler, Finkelberg, and Lunts ([12]; see [10] for generalizations). If $\operatorname{Char}(\mathcal{L}_w) = [T^*_{X(w)}X]$, Corollary 4.5 gives

$$\iota^*(\operatorname{Char}(\mathcal{L}_w)) = (-1)^{\ell(w)} {c_{\mathrm{Ma}}}^T(X(w))^\hbar \quad .$$

Alternately, this follows from (25) and the equality

$$P_{u,w}(1) = \mathrm{Eu}_{X(w)}(p)$$

for $p \in X(u)^\circ$, $u \le w$, which holds by the microlocal index formula ([29, Theorem 6.3.1], [16, Théorème 3]) and (26), when $\operatorname{Char}(\mathcal{L}_w)$ is irreducible. For the micro-local index formula in terms of constructible sheaves or functions see also [51, Rem.5.0.4 on pp. 294-295] and [52, Thm. 3.9]. ⌟

6.2. **CSM classes and the stable basis for T^*X.** Stable *envelopes* are conic Lagrangian cycles in T^*X introduced by Maulik and Okounkov [40] in relation to the study of symplectic resolutions. A key observation from *loc. cit.* is that these are (up to a sign) characteristic cycles of Verma modules, hence the results of §6.1 apply. We briefly recall next the definition of stable envelopes for T^*X; see [40, Chapter 3] or [58] for more details. The definition depends on a choice of chamber in the Lie algebra of the maximal torus and we let $+$ denote the positive chamber determined by the Borel subgroup B. As before, we let the torus $\mathbb{C}^*$ act on T^*X by dilation of cotangent fibers by a non-trivial character $\hbar^{-1}$; hence it acts trivially on X.

Theorem 6.4 ([40, 58])**.** *There exist unique $T \times \mathbb{C}^*$-equivariant Lagrangian cycles $\{\mathrm{stab}_+(w) \,|\, w \in W\}$ in T^*X which satisfy the following properties:*

1. *supp* $\mathrm{stab}_+(w) \subset \bigcup_{u \le w} T^*_{X(u)}X$*;*
2. $\mathrm{stab}_+(w)|_w = \prod_{\alpha \in R^+, w\alpha < 0} (w\alpha - \hbar) \prod_{\alpha \in R^+, w\alpha > 0} w\alpha$*;*
3. $\mathrm{stab}_+(w)|_u$ *is divisible by $\hbar$, for any $u < w$ in the Bruhat order.*

Here, $|_u$ denotes localization, i.e., restriction in equivariant cohomology to the fixed point in $X(u)$. It is easy to see that the transition matrix between $\{\mathrm{stab}_+(w) \,|\, w \in W\}$ and the fixed point basis in $H^*_{T\times\mathbb{C}^*}(T^*X)$ is triangular with nontrivial diagonal terms. Hence the stable envelopes $\{\mathrm{stab}_+(w) \,|\, w \in W\}$ form a basis after localization, called the *stable basis* for T^*X. The following lemma was observed by Maulik and Okounkov [40, p. 69, Remark 3.5.3], but for completeness we include a sketch of the proof, using Corollary 6.1 above.

Lemma 6.5. *Let $w \in W$ be a Weyl group element and char$(\mathcal{M}_w)$ and stab$_+(w)$ be the characteristic cycle of the Verma module, respectively the stable envelope associated to w. Then*

$$\mathrm{Char}(\mathcal{M}_w) = (-1)^{\dim X - \ell(w)} \,\mathrm{stab}_+(w).$$

Sketch of the proof. We need to check that conditions (1)–(3) are satisfied. The support condition (1) follows from the definition of the characteristic cycle. To check (2) and (3) we notice first that that $\iota : X \to T^*X$ is $T \times \mathbb{C}^*$-equivariant, and that the fixed loci satisfy $(T^*X)^{T\times\mathbb{C}^*} = X^T$, since $\mathbb{C}^*$ acts trivially on X. We deduce from Corollary 6.1 that for any $u \leq w$ the localization of $\mathrm{Char}(\mathcal{M}_w)$ is given by

$$\mathrm{Char}(\mathcal{M}_w)|_u = (-1)^{\ell(w)} c^{T,\hbar}_{\mathrm{SM}}(X(w)^\circ)|_u.$$

Recall now that the homogenized CSM class can be written as

$$c^{T,\hbar}_{\mathrm{SM}}(X(w)^\circ) = \sum_{u\leq w, 0\leq k} \hbar^k (c^T(u;w)[X(u)])_k \tag{28}$$

where $c^T(u;w)$ are polynomials in $H^*_T(pt)$ of degree $\leq \ell(u)$ (cf. [5, Proposition 6.5(a)]) and $(c^T(u;w)[X(u)])_k$ is the component of $c^T(u;w)[X(u)]$ in $H^T_{2k}(X)$. Since the localization $[X(u)]|_w$ equals 0 for $u < w$, it follows that the localization $\mathrm{Char}(\mathcal{M}_w)|_w$ equals the homogenization

$$\mathrm{Char}(\mathcal{M}_w)|_w = (-1)^{\ell(w)} c^{T,\hbar}_{\mathrm{SM}}(X(w)^\circ)|_w = (-1)^{\ell(w)} (c^T(w,w)[X(w)]|_w)^\hbar.$$

One can calculate the localization $c^{T,\hbar}_{\mathrm{SM}}(X(w)^\circ)|_w$ using Definition 5.3 together with formula (15). Alternatively, the localization $[X(w)]|_w$ is the Euler class of the normal bundle of $X(w)$ at the smooth point w; see e.g., [33, §2] for a combinatorial formula for this. A combinatorial formula for the coefficient $c(w;w)$ is given in [5, Proposition 6.5(c)], and we leave it as an exercise to use these formulas in order to check the correct normalization from condition (2). (Similar results were also obtained by Rimányi and Varchenko [48] using Weber's localization formulas [60].)

Let $c_0 := c^T_{\mathrm{SM}}(X(w)^\circ)_{\deg 0}$ be the degree 0 part of the non-homogenized class $c^T_{\mathrm{SM}}(X(w)^\circ)$. To check condition (3) it suffices to show $(c_0)|_u = 0$ for any $u < w$. By [5, Proposition 6.5(d)] $c_0 = [e_w]$, the equivariant class of the T-fixed point w. Clearly $[e_w]|_u = 0$ for $u \neq w$ and this finishes the proof. □

Corollary 6.6. *Let $w \in W$ be a Weyl group element. Then*

$$\iota^*(\mathrm{stab}_+(w)) = (-1)^{\dim X} c^{T,\hbar}_{SM}(X(w)^\circ). \tag{29}$$

In [58], Su found localization formulas for the stable basis elements to *any* torus fixed point, generalizing formulas of Billey [9] for the localization of Schubert classes. Corollary 6.6 implies similar localization formulas for the homogenized CSM classes. We record this next.

Corollary 6.7. *Fix $u, w \in W$ two elements such that $w \leq u$ in Bruhat ordering, and fix a reduced decomposition $u = s_{i_1} \dots s_{i_\ell}$. Then the restriction $c^{T,\hbar}_{SM}(X(w)^\circ)|_u$ equals*

$$c^{T,\hbar}_{SM}(Y(w)^\circ)_{|u} = (-1)^{\dim G/B - \ell(u)} \prod_{\alpha\in R^+\setminus R(u)} (\alpha - \hbar) \sum \hbar^{\ell-k} \prod_{t=1}^{k} \beta_t, \tag{30}$$

where the sum is over all subwords $\sigma_{j_1}\sigma_{j_2}\dots\sigma_{j_k}$ such that $w = \sigma_{j_1}\sigma_{j_2}\dots\sigma_{j_k}$; if $j_t = i_p$, then β_t is defined by $\beta_t = s_{i_1}\cdots\sigma_{i_{p-1}}\alpha_{i_p}$ with $\beta_1 = \alpha_{i_1}$; and $R(u) = \{\beta_i | 1 \leq i \leq \ell\}$.

Note that the set $R(u)$ coincides to the set of *inversions* of u^{-1}, i.e. the set of those positive roots α such that $u^{-1}(\alpha) < 0$; cf. [26, p. 14]. In particular, the sum in the equation (30) does not depend on the reduced expression for y. A similar formula for the localization of the CSM class $c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ)$ can be obtained by applying the automorphism φ_{w_0} to (30) and using Proposition 5.5(a).

6.3. **Dual CSM classes and opposite stable envelopes.** In this section consider the stable basis elements $stab_-(w)$ determined by the negative chamber in the Lie algebra of T. We use the automorphism $\varphi_{w_0} : G/B \to G/B$ from equation (17) above to relate the $+$ and $-$ versions of the stable basis elements, and also to relate the dual CSM classes to the elements $stab_-(w)$. Start by observing that the left multiplication by w_0 on X extends to one on T^*X and it induces a ring automorphism denoted again by $\varphi_{w_0} : H^*_{T\times\mathbb{C}^*}(T^*X) \to H^*_{T\times\mathbb{C}^*}(T^*X)$ which preserves $\hbar$ and acts on $H^*_T(pt)$ by w_0. The following follows immediately from the definition of the stable basis.

Lemma 6.8. *The automorphism φ_{w_0} satisfies*

$$\varphi_{w_0}(stab_+(w)) = stab_-(w_0w).$$

Proof. One can easily check the (duals of the) three conditions from Theorem 6.4. □

We then obtain a parallel to Corollary 6.6.

Proposition 6.9. *The following equalities hold:*

(i) $\iota^*(stab_-(w))|_{\hbar\mapsto-\hbar} = (-1)^{\ell(w)} c_{SM}^{T,\hbar,\vee}(Y(w)^\circ)$;
(ii) $\iota^*(stab_-(w)) = (-1)^{\dim X} c_{SM}^{T,\hbar}(Y(w)^\circ)$.

Proof. The equality (ii) follows from (i), using the definition of dual CSM classes from Proposition 5.4. To prove (i), first calculate that

$$\begin{aligned}\varphi_{w_0}(c_{\mathrm{SM}}^{T,\hbar}(X(w_0w)^\circ))|_{\hbar\to-\hbar} =&\varphi_{w_0}\left(\sum \hbar^k c_{\mathrm{SM},k}^{T}(X(w_0w)^\circ)\right)|_{\hbar\to-\hbar}\\ =&\sum(-\hbar)^k c_{\mathrm{SM},k}^{T}(Y(w)^\circ)\\ =&(-1)^{\dim X-\ell(w)} c_{\mathrm{SM}}^{T,\hbar,\vee}(Y(w)^\circ).\end{aligned}$$

The second equality follows from Proposition 5.5. From this, Lemma 6.8, and the fact that $\iota^*\varphi_{w_0} = \varphi_{w_0}\iota^*$ we obtain that

$$\begin{aligned}\iota^*(stab_-(w))|_{\hbar\to-\hbar} =&\iota^*(\varphi_{w_0} stab_+(w_0w))|_{\hbar\to-\hbar}\\ =&\varphi_{w_0}(\iota^*(stab_+(w_0w))|_{\hbar\to-\hbar})\\ =&(-1)^{\dim X}\varphi_{w_0}(c_{\mathrm{SM}}^{T,\hbar}(X(w_0w)^\circ))|_{\hbar\to-\hbar}\\ =&(-1)^{\ell(w)} c_{\mathrm{SM}}^{T,\hbar,\vee}(Y(w)^\circ).\end{aligned}$$

by applying Corollary 6.6. □

7. Application: dual and ordinary CSM classes

As we noted in §5, as a particular case of the orthogonality Theorem 5.7 we have identity (21), which we may rewrite as

$$\sum_v (-1)^{\dim X(v)} c_{\mathrm{SM}}(X(v)^\circ) = (-1)^{\dim X}[X]$$

where $X = G/B$. Equivalently,

$$\sum_v c^\vee_{\mathrm{SM}}(X(v)^\circ) = [X] \quad . \tag{31}$$

Capping (31) by the Chern class of the tangent bundle of X, we see that

$$\sum_v c(TX) \cap c^\vee_{\mathrm{SM}}(X(v)^\circ) = \sum_v c_{\mathrm{SM}}(X(v)^\circ) \tag{32}$$

since both sides agree with $c(TX) \cap [X]$. In this section we will combine Theorem 5.7 and the results of §6, together with an orthogonality statement for stable envelopes due to Maulik and Okounkov, to obtain a sharp formula relating dual and ordinary equivariant CSM classes. As a particular (non-equivariant) case, we will prove that the *individual summands* in (32) agree.

First, let X be a smooth, projective, complex variety, and $E \to X$ a vector bundle. Assume that E admits the action of a torus $\mathbb{T}$, such that each component of the fixed locus is compact. For instance, $\mathbb{T}$ could be a $\mathbb{C}^*$ with the action given by fibre dilation; in this case the fixed locus is equal to X. In general, let F_j be the fixed loci with inclusions $\iota_j : F_j \to X$. As explained in e.g. in [37, Definition 16], even though E is not compact, one can still define a degree via Atiyah-Bott localization: if $\gamma \in H_{\mathbb{T}}(E)$, then we can set

$$\int_E \gamma := \sum_j \int_{F_j} \frac{\iota_j^* \gamma}{e^{\mathbb{T}}(N_{F_j} E)},$$

where $e^{\mathbb{T}}(N_{F_j}E)$ is the $\mathbb{T}$-equivariant Euler class of the normal bundle of F_j in E. This degree has values in the fraction field of $H^*_{\mathbb{T}}(pt)$. To simplify the set-up we will also assume that each component of the fixed locus is included in X. (This assumption will hold for our application.) Then instead of restricting over each component of the fixed locus one may simply restrict to X. Therefore, if $\iota : X \to E$ denotes the zero section, one can define a pairing $\langle \cdot, \cdot \rangle : H^*_{\mathbb{T}}(E) \otimes H^*_{\mathbb{T}}(E) \to Frac(H^*_{\mathbb{T}}(pt))$ given by

$$\langle \gamma_1, \gamma_2 \rangle_E := \left\langle \iota^* \gamma_1, \frac{\iota^* \gamma_2}{e^{\mathbb{T}}(E)} \right\rangle_X , \tag{33}$$

where the pairing on the right is the usual Poincaré pairing on X.

We now specialize to the situation when $X := G/B$, $E := T^*X$ and the torus is $\mathbb{T} := T \times \mathbb{C}^*$. As before, the $\mathbb{C}^*$ component acts on T^*X by the character $\hbar^{-1}$, and the action of T is the natural action induced from left multiplication on X. Recall the stable envelopes $stab_+(w), stab_-(w) \in H^*_{T\times\mathbb{C}^*}(T^*X)$ defined in §6.2. The following result was proved by Maulik and Okounkov [40, Theorem 4.4.1]:

Proposition 7.1. *Let $u, v \in W$. Then the following orthogonality relation holds:*

$$\langle stab_+(u), stab_-(v) \rangle_{T^*X} = (-1)^{\dim X} \delta_{u,v}.$$

Remark 7.2. The corresponding orthogonality relation for characteristic cycles

$$\langle CC(1\!\!1_{X(u)^\circ}), CC(1\!\!1_{Y(v)^\circ}) \rangle_{T^*X} = (-1)^{\dim X} \delta_{u,v}$$

for $u, v \in W$ also follows from the T-equivariant version of [52, Cor.1.5 resp. Cor.3.5] for the transversal intersecting algebraic Whitney stratifications of X given by the (opposite) Schubert cells:

$$\langle CC(1\!\!1_{X(u)^\circ}), CC(1\!\!1_{Y(v)^\circ}) \rangle_{T^*X} = (-1)^{\dim X} \chi(X(u)^\circ \cap Y(v)^\circ) ,$$

with $\chi(X(u)^\circ \cap Y(v)^\circ) = \chi((X(u)^\circ \cap Y(v)^\circ)^T) = \delta_{u,v}$ by localization of the Euler characteristic at the T-fixed points [51, Cor.3.2.2 on p 174]. Note that such a fixed point localization of the Euler characteristic also underlies Weber's localization theorem [60][Thm.20] for torus equivariant CSM classes (used in [48]). Moreover, [53, Cor.1.5 resp. Cor.3.5] implies that

$$supp(CC(1\!\!1_{X(u)^\circ})) \cap supp(CC(1\!\!1_{Y(v)^\circ}))$$

is contained in the zero section of T^*X, so that here no localization for the intersection number of these characteristic cycles is needed. Finally, the arguments from [53] directly apply also to the torus equivariant context by using [42, Sec. 4.1], together with the multiplicativity of the (torus equivariant) Chern classes with respect to cross products as well as Theorem 4.3 of this paper.

We use this to prove the main result of this section:

Theorem 7.3. *Let $v \in W$. Then the following equality holds in $H_T^*(G/B)$:*

$$\frac{c_{SM}^{T,\hbar,\vee}(Y(v)^\circ)}{\prod_{\alpha\in R_+}(\hbar+\alpha)} = (-1)^{\dim G/B}\frac{c_{SM}^{T,\hbar}(Y(v)^\circ)}{e^{T\times\mathbb{C}^*}(T^*(G/B))}.$$

Proof. By the definition (33) of the localization pairing, Corollary 6.6, and Proposition 6.9,

$$(-1)^{\dim G/B}\delta_{u,v} = \langle stab_+(u), stab_-(v)\rangle_{T^*X} = \left\langle c_{\mathrm{SM}}^{T,\hbar}(X(u)^\circ), \frac{c_{\mathrm{SM}}^{T,\hbar}(Y(v)^\circ)}{e^{T\times\mathbb{C}^*}(T^*(G/B))}\right\rangle_X. \tag{34}$$

Then the result follows from the orthogonality in Theorem 5.7 together with the fact that the Poincaré pairing is non-degenerate. ☐

Combining Theorem 7.3 with the interpretation of the quantity $e = \prod_{\alpha\in R_+}(\hbar+\alpha)$ from (18), we obtain:

$$c_{\mathrm{SM}}^{T,\hbar,\vee}(Y(v)^\circ) = \frac{e^{T\times\mathbb{C}^*}(T^*_{w_0}(G/B))}{e^{T\times\mathbb{C}^*}(T^*(G/B))}\, c_{\mathrm{SM}}^{T,\hbar}(Y(v)^\circ). \tag{35}$$

We can specialize (35) by letting all equivariant parameters $\alpha \mapsto 0$ and $\hbar \mapsto \pm 1$. This yields the equality of individual summands in (31) announced earlier.

Corollary 7.4. *For all $v \in W$, the following hold:*

(1) $c_{SM}(X(v)^\circ) = c(T(G/B)) \cap c_{SM}^\vee(X(v)^\circ)$;
(2) $c_{SM}^\vee(X(v)^\circ) = c(T^*(G/B)) \cap c_{SM}(X(v)^\circ)$.

Proof. Set $X = G/B$. After taking $\alpha \mapsto 0$ in (35), and using that $X(v)^\circ$ is a G-translate of $Y(w_0v)^\circ$, we obtain

$$c_{\mathrm{SM}}^{\hbar,\vee}(X(v)^\circ) = \frac{(-1)^{\dim X}\hbar^{\dim X}}{e^{\mathbb{C}^*}(T^*X)} c_{\mathrm{SM}}^{\hbar}(X(v)^\circ).$$

Observe that $e^{\mathbb{C}^*}(T^*X) = \prod(-\hbar - x_i)$, where x_i are the Chern roots of TX. Therefore

$$e^{\mathbb{C}^*}(T^*X)|_{\hbar\mapsto 1} = (-1)^{\dim X}c(TX);\quad e^{\mathbb{C}^*}(T^*X)|_{\hbar\mapsto -1} = c(T^*X).$$

Then the corollary follows from the specializations at $\hbar \mapsto 1$ respectively $\hbar \mapsto -1$. ☐

Example 7.5. For $X = \mathrm{Fl}(2) = \mathbb{P}^1$ and $v = w_0$ the longest Weyl group element, $c_{\mathrm{SM}}^\vee(X(v)^\circ) = [\mathbb{P}^1] - [pt]$, $c_{\mathrm{SM}}(X(v)^\circ) = [\mathbb{P}^1] + [pt]$ and $c(T\mathbb{P}^1) = [\mathbb{P}^1] + 2[pt]$. Corollary 7.4 is immediate. More generally, an algorithm calculating CSM classes of Schubert cells $X(v)^\circ$ (and therefore their duals as well) was obtained in [5], and this may be used to verify the identities from Corollary 7.4 explicitly in many concrete cases. For instance, the following are the (non-equivariant) CSM classes of the Schubert cells in Fl(3), the variety parametrizing flags in $\mathbb{C}^3$:

$$\begin{aligned}
c_{\mathrm{SM}}(X(w_0)^\circ) &= [\mathrm{Fl}(3)] + [X(s_1s_2)] + [X(s_2s_1)] + 2[X(s_1)] + 2[X(s_2)] + [pt];\\
c_{\mathrm{SM}}(X(s_1s_2)^\circ) &= [X(s_1s_2)] + [X(s_1)] + 2[X(s_2)] + [pt];\\
c_{\mathrm{SM}}(X(s_2s_1)^\circ) &= [X(s_2s_1)] + 2[X(s_1)] + [X(s_2)] + [pt];\\
c_{\mathrm{SM}}(X(s_1)^\circ) &= [X(s_1)] + [pt];\\
c_{\mathrm{SM}}(X(s_2)^\circ) &= [X(s_2)] + [pt];\\
c_{\mathrm{SM}}(X(id)^\circ) &= [pt].
\end{aligned}$$

The total Chern class of Fl(3) is

$$c(T\,\mathrm{Fl}(3)) = \sum_v c_{\mathrm{SM}}(X(v)^\circ) = [\mathrm{Fl}(3)]+2[X(s_1s_2)]+2[X(s_2s_1)]+6[X(s_1)]+6[X(s_2)]+6[pt]$$

Again one checks Corollary 7.4 using the multiplication table in $H^*(\mathrm{Fl}(3))$. ⌟

Remark 7.6. The proof of Theorem 7.3 depends on both the orthogonality of CSM classes (Theorem 5.7), and that of stable envelopes (Proposition 7.1) resp. that of characteristic cycles (Remark 7.2). It would be interesting to relate directly Theorem 5.7 to (34). ⌟

8. The CSM class as a Segre class: proof of the main theorem

In this section we combine the orthogonality properties from Theorems 5.7 and 7.3 to prove Theorem 1.3 from the introduction. Recall that this is the main ingredient to prove that in the non-equivariant case the CSM classes are effective, thus proving the positivity conjecture stated in [5]. We start by proving the following lemma, which is known among experts, but for which we could not find a reference.

Lemma 8.1. *The following equality holds in $H^*_T(G/B)$:*

$$c^T(T(G/B)) \cdot c^T(T^*(G/B)) = \prod_{\alpha>0}(1-\alpha^2).$$

Proof. The Chern class of G/B is given by $c^T(T(G/B)) = \prod_{\alpha>0}(1+c_1^T(\mathcal{L}_\alpha))$ where $\mathcal{L}_\alpha$ is the G-equivariant line bundle on G/B with fibre of weight $-\alpha$ over $1.B$; see e.g. [5, Remark 6.6]. Then the localization at the T-fixed point e_w is given by $c^T(T(G/B))_{|w} = \prod_{\alpha>0}(1-w(\alpha))$. From this we obtain that

$$(c^T(T(G/B)) \cdot c^T(T^*(G/B)))_{|w} = \prod_{\alpha>0}(1-w(\alpha))(1+w(\alpha)) = \prod_{\alpha>0}(1-\alpha)(1+\alpha),$$

because w permutes the set of roots. □

Remark 8.2. The lemma implies that

$$c(T(G/B))\, c(T^*(G/B)) = 1 \tag{36}$$

in $H^*(G/B)$. In particular, for the cotangent bundle $T^*(G/B)$, the localization pairing from (33) specialized to $\hbar \mapsto 1$ coincides with the twisted Poincaré pairing studied by Lascoux, Leclerc and Thibon [34] in relation to the degenerate Hecke algebra.

Recall the following construction from §3.2. Consider the projection $\overline{q} : \mathbb{P}(T^*(G/B) \oplus \mathbb{1}) \to G/B$ and the tautological subbundle $\mathcal{O}_{T^*(G/B)\oplus\mathbf{1}}(-1) \subset T^*(G/B) \oplus \mathbb{1}$; this is an inclusion of T-equivariant bundles. If C is a cycle in $T^*(G/B)$, then the Segre operator is defined by

$$s^T_{T^*(G/B)\oplus\mathbf{1}}(C) = \overline{q}_*\Big(\frac{[\overline{C}]}{c^T(\mathcal{O}_{T^*(G/B)\oplus\mathbf{1}}(-1))}\Big);$$

cf. equation (12) above. Let $H^*_T(G/B)_{loc}$ be the localization of $H^*_T(G/B)$ at the multiplicative system $H^*_T(pt)$. We will see below that the Segre operator is naturally valued in a localization of the cohomology ring $H^*_T(G/B)$.

Theorem 8.3. *Let $w \in W$. The following equality holds in the localized ring $H^*_T(G/B)_{loc}$:*

$$c^T_{SM}(X(w)^\circ) = \prod_{\alpha>0}(1+\alpha)s^T_{T^*(G/B)\oplus\mathbf{1}}(\mathrm{Char}(\mathcal{M}_w)).$$

Proof. A standard calculation based on Whitney formula and Lemma 8.1 shows that the right hand side can be rewritten as

$$\frac{c^T(T(G/B))\cdot c^T(T^*(G/B))}{\prod_{\alpha>0}(1-\alpha)}s_{T^*(G/B)\oplus\mathbf{1}}(\mathrm{Char}(\mathcal{M}_w)) = \frac{c^T(T(G/B))}{\prod_{\alpha>0}(1-\alpha)}\overline{q}_*\big(c^T(Q)\cap[\overline{\mathrm{Char}(\mathcal{M}_w)}]\big).$$

Here Q denotes the universal quotient bundle. In particular, this quantity belongs in the claimed localized ring. To prove the equality we use Theorem 7.3 specialized at $\hbar = 1$ together with the definition of dual CSM classes to obtain that

$$c^T_{\mathrm{SM}}(Y(w)^\circ) = \frac{c^T(T(G/B))}{\prod_{\alpha>0}(1+\alpha)}c^{T,\vee}_{\mathrm{SM}}(Y(w)^\circ) = \frac{c^T(T(G/B))}{\prod_{\alpha>0}(1+\alpha)}(-1)^{\dim Y(w)}\check{c}^T_*(\mathbb{1}_{Y(w)^\circ}).$$

By Corollary 3.4,

$$\begin{aligned}\check{c}^T_*(\mathbb{1}_{Y(w)^\circ}) &= c^T(T^*(G/B))\cap s^T_{T^*(G/B)\oplus\mathbf{1}}(CC(\mathbb{1}_{Y(w)^\circ}))\\ &= (-1)^{\dim Y(w)}c^T(T^*(G/B))\cap s^T_{T^*(G/B)\oplus\mathbf{1}}(\mathrm{Char}(\widetilde{\mathcal{M}}_w)),\end{aligned}$$

where the sign change is from the definition of the appropriate characteristic cycles. Here $\widetilde{\mathcal{M}}_w = \varphi^*_{w_0}\mathcal{M}_{w_0w}$ is the Verma module associated to the opposite Schubert cell $Y(w)^\circ$ (i.e. $\chi_{stalk}(DR(\widetilde{\mathcal{M}}_w)) = \mathbb{1}_{Y(w)^\circ}$), and φ_{w_0} is the automorphism from (17). Combining the previous two equations, we deduce that

$$c^T_{\mathrm{SM}}(Y(w)^\circ) = \frac{c^T(T(G/B))\cdot c^T(T^*(G/B))}{\prod_{\alpha>0}(1+\alpha)}s^T_{T^*(G/B)\oplus\mathbf{1}}(\mathrm{Char}(\widetilde{\mathcal{M}}_w)).$$

The last quantity equals $\prod_{\alpha>0}(1-\alpha)s_{T^*(G/B)\oplus\mathbf{1}}(\mathrm{Char}(\widetilde{\mathcal{M}}_w))$ by Lemma 8.1. The equality in the theorem follows by applying the cohomology ring automorphism $\varphi^*_{w_0}$ to both sides. □

9. Partial flag manifolds

The goal of this section is to indicate how the results of the previous sections extend to the case of partial flag manifolds, focusing on a generalization of the orthogonality result. Let $P \subset G$ be a parabolic subgroup containing B and let G/P be the corresponding partial flag manifold. Let $f : G/B \to G/P$ be the natural projection. For the following generalities we refer the reader to [13]. Let W_P be the subgroup of W generated by the simple reflections in P and denote by W^P the set of minimal length representatives for the cosets of W_P in W. For each $w \in W^P$ there are Schubert cells $X(wW_P)^\circ := BwP/P$ and $Y(wW_P)^\circ := B^-wP/P$ in G/P, whose closures are corresponding Schubert varieties $X(wW_P)$ and $Y(wW_P)$. If $w \in W^P$ and w_P denotes the longest element in W_P then f restricts to isomorphisms $X(w)^\circ \to X(wW_P)^\circ$, $Y(ww_P)^\circ \to Y(wW_P)^\circ$; in particular,

$\dim_{\mathbb{C}} X(wW_P) = \operatorname{codim}_{\mathbb{C}} Y(wW_P) = \ell(w)$. The Bruhat order on W restricts to a partial ordering on W^P: for $u, v \in W^P$, $u \leq v$ iff $X(u) \subseteq X(v)$. CSM classes of Schubert cells in partial flag manifolds are computed in [5, §3.3].

9.1. **CSM classes and stable envelopes.** There is an analogue of the Existence Theorem 6.4 which yields the set of stable envelopes $\{stab^P_\pm(u) : u \in W^P\}$, with the $\pm$ sign depending on a chamber of $Lie(T)$. For each sign choice, the set of stable envelopes forms a basis for the cohomology ring $H^*_{T\times\mathbb{C}^*}(G/P)$ localized at $H^*_{T\times\mathbb{C}^*}(pt)$. See e.g. [58] for more details. We can consider diagram (22) for G/P; as in §6.1, there are well defined push-forwards for each of the (associated Grothendieck) groups in this diagram, and the given maps commute with (proper) push-forward. The next proposition states that the relation between the stable envelopes and the CSM classes extends to partial flag manifolds.

Proposition 9.1. *Let $w \in W^P$ be a minimal length representative. The following hold:*

(a) *The constructible function associated to the direct image complex $f_*(\mathfrak{M}_w)$ of the holonomic module $\mathfrak{M}_w$ is $(-1)^{\ell(w)} 1\!\!1_{X(wW_P)^\circ}$.*
(b) *There is an identity*

$$\iota^*(\mathrm{stab}^P_+(w)) = (-1)^{\dim(G/P)} c^{T,\hbar}_{SM}(X(wW_P)^\circ).$$

Proof. Using that DR and χ_{stalk} commute with push-forward we obtain

$$f_*((\chi_{stalk} \circ \mathrm{DR})\mathfrak{M}_w) = (-1)^{\ell(w)} f_*(1\!\!1_{X(w)^\circ}).$$

Then (a) follows because for $w \in W^P$ the restriction $f : X(w)^\circ \to X(wW_P)^\circ$ is an isomorphism. Part (b) follows because $\mathrm{stab}^P_+(w) = (-1)^{\dim(G/P)-\ell(w)} \mathrm{Char}(f_*(\mathfrak{M}_w))$ (with a proof similar to that from Lemma 6.5), and from Corollary 4.5. ☐

9.2. **Orthogonality.** In this section we study orthogonality of CSM classes in the G/P case. Recall that a morphism $g : Z_1 \to Z_2$ induces a pull-back on constructible functions $g^* : \mathcal{F}(Z_2) \to \mathcal{F}(Z_1)$ defined by

$$g^*(\varphi)(z_1) = \varphi(g(z_1)),$$

for all $\varphi \in \mathcal{F}(Z_2)$, $z_1 \in Z_1$. We will need to calculate the pull-back of the CSM class of a Schubert cell. In order to do that, we need the following Verdier-Riemann-Roch (VRR) type formula due to Yokura (see also [53]) and generalized to the equivariant case by Ohmoto:

Theorem 9.2 ([61, Theorem 2.3], [42, Theorem 4.1])**.** *Let $g : Z_1 \to Z_2$ be a smooth T-equivariant morphism of algebraic T-varieties. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{F}^T(Z_2) & \xrightarrow{c^T_*} & A^T_*(Z_2) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle c^T(T_g)\cap g^*} \\ \mathcal{F}^T(Z_1) & \xrightarrow{c^T_*} & A^T_*(Z_1) \end{array}$$

where T_g denotes the relative tangent bundle of g.

Our goal is to obtain an equivariant version of formula (19) from Corollary 5.9, generalized to the case of partial flag manifolds. Observe first that (19) does *not* hold in general for partial flag manifolds, even in the non-equivariant formulation.

Example 9.3. View $\mathbb{P}^2$ as a partial flag manifold. The Schubert cells are isomorphic to $\mathbb{A}^i$, $i = 0, 1, 2$, and we have

$$c_{\mathrm{SM}}(\mathbb{A}^2) = c_{\mathrm{SM}}(\mathbb{P}^2) - c_{\mathrm{SM}}(\mathbb{P}^1) = [\mathbb{P}^2] + 2[\mathbb{P}^1] + [\mathbb{P}^0] \quad .$$

The cell $\mathbb{A}^2$ is *not* its own opposite, yet

$$\langle c_{\mathrm{SM}}(\mathbb{A}^2), c^\vee_{\mathrm{SM}}(\mathbb{A}^2)\rangle_{\mathbb{P}^2} = \int ([\mathbb{P}^2] + 2[\mathbb{P}^1] + [\mathbb{P}^0]) \cdot ([\mathbb{P}^2] - 2[\mathbb{P}^1] + [\mathbb{P}^0]) = 1 - 4 + 1 = -2 \neq 0 \quad .$$

On the other hand, note that in the case of complete flag manifolds G/B, by Corollary 7.4(i) we may replace $c^\vee_{\mathrm{SM}}(Y(v)^\circ)$ with $c_{\mathrm{SM}}(Y(v)^\circ)/c(T(G/B))$ in (19). In the parabolic case, Corollary 7.4 does not hold in general, so these two formulations are not equivalent. For instance,

$$\frac{c_{\mathrm{SM}}(\mathbb{A}^2)}{c(T\mathbb{P}^2)} = [\mathbb{P}^2] - [\mathbb{P}^1] + [\mathbb{P}^0] \neq c^\vee_{\mathrm{SM}}(\mathbb{A}^2).$$

Now note that

$$\left\langle c_{\mathrm{SM}}(\mathbb{A}^2), \frac{c_{\mathrm{SM}}(\mathbb{A}^2)}{c(T\mathbb{P}^2)} \right\rangle_{\mathbb{P}^2} = \int ([\mathbb{P}^2] + 2[\mathbb{P}^1] + [\mathbb{P}^0]) \cdot ([\mathbb{P}^2] - [\mathbb{P}^1] + [\mathbb{P}^0]) = 1 - 2 + 1 = 0$$

as we would expect from orthogonality. The main result of this section proves that the alternative formulation of (19) *does* hold for arbitrary partial flag manifolds, in the equivariant setting. ⌟

Theorem 9.4. *Let $u, v \in W^P$. Then*

$$\left\langle c^T_{SM}(X(uW_P)^\circ), \frac{c^T_{SM}(Y(vW_P)^\circ)}{c^T(T(G/P))} \right\rangle_{G/P} = \delta_{u,v}.$$

Remark 9.5. If $\varphi \in \mathcal{F}^T(G/P)$ is a T-invariant constructible function, the class

$$s^T(\varphi) := \frac{c^T_*(\varphi)}{c^T(T(G/P))}$$

can be interpreted as a Segre class–compare with (11) and Corollary 3.4, which deals with the 'signed' version of this notion, and for more general manifolds. These classes, sometimes called *Schwartz-MacPherson Segre* classes (see [42, §5.3]), are related to the study of Thom polynomials. In the non-equivariant case they have been studied in [1]. In this context, Theorem 9.4 states that the equivariant Poincaré duals of the CSM classes are Schwartz-MacPherson Segre classes. These classes coincide with the signed CSM classes for Schubert cells in G/B, but in general they are different, cf. Example 9.3. ⌟

Proof of Theorem 9.4. Let w_P be the longest element in W_P, so that as recalled earlier f restricts to an isomorphism $Y(vw_P)^\circ \to Y(vW_P)^\circ$, and in particular

$$f_*(1\!\!1_{Y(vw_P)^\circ}) = 1\!\!1_{Y(vW_P)^\circ}.$$

An analysis of the T-fixed points yields $f^{-1}(X(uW_P)^\circ) = \coprod_{w\in W_P} X(uw)^\circ$, from which we deduce that

$$f^*(1\!\!1_{X(uW_P)^\circ}) = \sum_{w\in W_P} 1\!\!1_{X(uw)^\circ}.$$

By the VRR formula from Theorem 9.2

$$c^T(T_f) \cap f^*(c^T_{\mathrm{SM}}(X(uW_P)^\circ)) = \sum_{w\in W_P} c^T_{\mathrm{SM}}(X(uw)^\circ). \tag{37}$$

Then we calculate:

$$\begin{aligned}
&\left\langle c^T_{\mathrm{SM}}(X(uW_P)^\circ), \frac{c^T_{\mathrm{SM}}(Y(vW_P)^\circ)}{c^T(T(G/P))} \right\rangle_{G/P} \\
&= \int_{G/P} c^T_{\mathrm{SM}}(X(uW_P)^\circ) \cdot \frac{f_*(c^T_{\mathrm{SM}}(Y(vw_P)^\circ))}{c^T(T(G/P))} \quad \text{(by functoriality of CSM classes)} \\
&= \int_{G/B} f^* c^T_{\mathrm{SM}}(X(uW_P)^\circ) \cdot \frac{c^T_{\mathrm{SM}}(Y(vw_P)^\circ)}{f^* c^T(T(G/P))} \quad \text{(by the projection formula)} \\
&= \sum_{w\in W_P} \int_{G/B} c^T_{\mathrm{SM}}(X(uw)^\circ) \cdot \frac{c^T_{\mathrm{SM}}(Y(vw_P)^\circ)}{c^T(T_f)\cdot f^*(c^T(T(G/P)))} \quad \text{(by the VRR formula (37))} \\
&= \sum_{w\in W_P} \left\langle c^T_{\mathrm{SM}}(X(uw)^\circ), \frac{c^T_{\mathrm{SM}}(Y(vw_P)^\circ)}{c^T(T(G/B))} \right\rangle_{G/B} \quad \text{(by the Whitney formula for } c^T(T_f)) \\
&= \sum_{w\in W_P} \left\langle c^T_{\mathrm{SM}}(X(uw)^\circ), \frac{c^{T,\vee}_{\mathrm{SM}}(Y(vw_P)^\circ)}{\prod_{\alpha>0}(1+\alpha)} \right\rangle_{G/B} \quad \text{(by Theorem 7.3 specialized to } \hbar\mapsto 1) \\
&= \delta_{u,v}
\end{aligned}$$

as needed. The last equality holds by Theorem 5.7, since $uw = vw_P$ implies $u = v$ (and hence $w = w_P$). Indeed, $u, v \in W^P$ and W^P consists of representatives of distinct cosets of W_P in W. □

Remark 9.6. Theorem 9.4 also follows from the T-equivariant version of [53, Thm. 1.2] for the transversal intersecting algebraic Whitney stratifications of G/P given by the (opposite) Schubert cells:

$$\begin{aligned}
\left\langle c^T_{\mathrm{SM}}(X(uW_P)^\circ), \frac{c^T_{\mathrm{SM}}(Y(vW_P)^\circ)}{c^T(T(G/P))} \right\rangle_{G/P} &= \int_{G/P} c^T_{\mathrm{SM}}(X(uW_P)^\circ) \cap Y(vW_P)^\circ) \\
&= \chi(X(uW_P)^\circ \cap Y(vW_P)^\circ) = \delta_{u,v}
\end{aligned}$$

using again the localization of the Euler characteristic at the T-fixed points [51, Cor. 3.2.2, p. 174]. Finally, the arguments form [53] directly apply also to the torus equivariant context by using [42, Sec. 4.1], together with the multiplicativity of the (torus equivariant) Chern classes with respect to cross products as well as Theorem 4.3 of this paper.

Example 9.7. We illustrate Theorem 9.4 by working out its statement for $\mathbb{P}^n$, in the non-equivariant case. Schubert cells are isomorphic to $\mathbb{A}^i$, $i = 0, \dots, n$, and $\mathbb{A}^i$ is opposite to $\mathbb{A}^{n-i}$. In this case, Theorem 9.4 claims that

$$\left\langle c_{\mathrm{SM}}(\mathbb{A}^i), \frac{c_{\mathrm{SM}}(\mathbb{A}^{n-j})}{c(T\mathbb{P}^n)} \right\rangle_{\mathbb{P}^n} = \delta_{ij} \quad .$$

Let $\xi \in A^1\mathbb{P}^n$ be the hyperplane class. Then

$$c_{\mathrm{SM}}(\mathbb{A}^i) = (1+\xi)^i \cdot [\mathbb{P}^i] = \xi^{n-i}(1+\xi)^i \cdot [\mathbb{P}^n] \quad ,$$

and

$$\frac{c_{\mathrm{SM}}(\mathbb{A}^{n-j})}{c(T\mathbb{P}^n)} = \frac{(1+\xi)^{n-j}\cdot[\mathbb{P}^{n-j}]}{(1+\xi)^{n+1}} = \xi^j(1+\xi)^{-(j+1)} \cdot [\mathbb{P}^n] \quad ,$$

so that

$$\left\langle c_{\mathrm{SM}}(\mathbb{A}^i), \frac{c_{\mathrm{SM}}(\mathbb{A}^{n-j})}{c(T\mathbb{P}^n)} \right\rangle_{\mathbb{P}^n} = \int_{\mathbb{P}^n} \xi^{n-i+j}(1+\xi)^{i-j-1} \cdot [\mathbb{P}^n] \quad .$$

If $i = j$, this degree equals $\int \xi^n(1+\xi)^{-1} = 1$. If $i < j$, then $n - i + j > n$ and the degree equals 0. If $i > j$, then $\xi^{n-i+j}(1+\xi)^{i-j-1}$ is a polynomial of degree $n-1 < n$ in ξ, so that the degree is also 0, as prescribed by Theorem 9.4. ⌟

Remark 9.8. The orthogonality from Theorem 9.4 can also be deduced from the parabolic analogue of the orthogonality of stable envelopes (the analogue of Proposition 7.1) together with the analogue of Proposition 6.9(ii). We leave the details to the reader. ⌟

9.3. **A CSM Chevalley formula.** The Chevalley formula gives the Schubert expansion of a product of a divisor Schubert class $[Y(s_\beta)]$ by another class $[Y(w)]$, in an appropriate cohomology ring of G/P; here $s_\beta \in W \setminus W_P$ is a simple reflection and $w \in W^P$. We refer to [19], in the non-equivariant setting, and e.g. to [15, §8] for the formula in the equivariant ring $H_T^*(G/P)$. For stable envelopes, a Chevalley formula was found by Su in [57, Thm. 3.7]. Then proposition 9.1 determines a formula to multiply the CSM class $c_{\mathrm{SM}}^{T,\hbar}(Y(w)^\circ)$ by any divisor class. We record the result next. Let ϖ_β be the fundamental weight corresponding to the simple root β, and let R_P^+ denote the set of positive roots in P.

Theorem 9.9. *Let $w \in W^P$, and β be a simple root not in P. Then the following identity holds in $H^*_{T\times\mathbb{C}^*}(G/P)$:*

$$[Y(s_\beta)] \cup c_{SM}^{T,\hbar}(Y(w)^\circ) = (\varpi_\beta - w(\varpi_\beta))c_{SM}^{T,\hbar}(Y(w)^\circ) + \quad \hbar \sum (\varpi_\beta, \alpha^\vee) c_{SM}^{T,\hbar}(Y(ws_\alpha W_P)^\circ),$$

where the sum is over roots $\alpha \in R^+ \setminus R_P^+$ such that $\ell(ws_\alpha W_P) > \ell(w)$, $\alpha^\vee$ is the coroot of α, and $(\cdot,\cdot)$ is the evaluation pairing.

The classical Chevalley formula Theorem (see e.g. [15, Thm. 8.1]) can be deduced from Theorem 9.9 via a limiting process as follows. Write

$$c_{\mathrm{SM}}^{T,\hbar}(Y(w)^\circ) = \sum_{u \geq w} c(u;w)[Y(u)],$$

where $u \in W^P$ and the coefficients $c(u;w) \in H_{T\times\mathbb{C}^*}^{2(\dim G/P - \ell(u))}(pt)$. If we set all the T-equivariant parameters to 0, and set $\hbar$ to 1, the leading coefficient $c(w;w) = 1$. Therefore,

$$\lim_{\hbar\to\infty} \frac{c_{\mathrm{SM}}^{T,\hbar}(Y(w)^\circ)}{(\hbar)^{\dim G/P - \ell(w)}} = [Y(w)].$$

For any root $\alpha \in R^+ \setminus R_P^+$, such that $\ell(ws_\alpha W_P) > \ell(w)$ we have

$$\lim_{\hbar\to\infty} \frac{\hbar c_{\mathrm{SM}}^{T,\hbar}(Y(ws_\alpha W_P)^\circ)}{(\hbar)^{\dim G/P - \ell(w)}} = [Y(ws_\alpha W_P)]$$

if and only if $\ell(ws_\alpha W_P) = \ell(w) + 1$. Otherwise, the limit is 0. Hence, if we divide both sides of the equation in Theorem 9.9 by $(\hbar)^{\dim G/P - \ell(w)}$, and let $\hbar$ go to ∞, we obtain

$$[Y(s_\beta)] \cup [Y(w)] = (\varpi_\beta - w(\varpi_\beta))[Y(w)] + \sum (\varpi_\beta, \alpha^\vee)[Y(ws_\alpha W_P)],$$

where the sum is over those roots $\alpha \in R^+ \setminus R_P^+$ such that $\ell(ws_\alpha W_P) = \ell(w) + 1$. This is the classical Chevalley formula; see e.g. [15, Thm. 8.1].

Figure 1. CSM matrix, Fl(4)

$$\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&2&1&2&0&1&0&3&1&3&0&2&0&3&2&3&1&2&1&3&2&3\\
0&0&1&1&2&2&0&0&2&2&3&3&1&1&2&2&4&4&2&2&3&3&4&4\\
0&0&0&1&0&1&0&0&0&3&0&3&0&1&0&3&1&3&0&1&0&3&1&3\\
0&0&0&0&1&1&0&0&0&0&2&2&0&0&0&0&4&4&1&1&2&2&4&4\\
0&0&0&0&0&1&0&0&0&0&0&3&0&0&0&0&2&5&0&1&0&3&2&5\\
0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2&3&3&3&3&3&3\\
0&0&0&0&0&0&0&1&0&2&1&2&0&3&0&4&3&4&2&4&2&6&4&6\\
0&0&0&0&0&0&0&0&1&1&2&2&0&0&1&1&4&4&0&0&2&2&4&4\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&1&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&3&3&0&0&1&1&3&3\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&2&0&0&0&1&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&3&3&3&3&3&3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&2&1&2&0&2&0&5&2&5\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&0&0&3&3&5&5\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&2&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&2&1&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix} \tag{38}$$

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FIGURE 2. Inverse CSM matrix, Fl(4)

$$
\begin{pmatrix}
1 & -1 & -1 & 2 & 2 & -1 & -1 & 1 & 2 & -5 & -5 & 3 & 2 & -3 & -1 & 4 & 6 & -3 & -5 & 4 & 3 & -4 & -3 & 1 \\
0 & 1 & 0 & -2 & -1 & 1 & 0 & -1 & 0 & 5 & 2 & -3 & 0 & 3 & 0 & -4 & -4 & 3 & 2 & -4 & -1 & 4 & 2 & -1 \\
0 & 0 & 1 & -1 & -2 & 1 & 0 & 0 & -2 & 3 & 5 & -3 & -1 & 1 & 1 & -2 & -6 & 3 & 3 & -2 & -3 & 3 & 3 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & -1 & 0 & 2 & 2 & -3 & 0 & 2 & 0 & -3 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -2 & 3 & 0 & 0 & 0 & 0 & 4 & -3 & -1 & 1 & 1 & -2 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & -2 & 3 & 0 & -1 & 0 & 2 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 2 & 2 & -1 & -2 & 3 & 1 & -4 & -4 & 2 & 5 & -4 & -3 & 4 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 & -3 & 0 & 4 & 3 & -2 & -2 & 4 & 1 & -4 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 1 & 0 & 0 & -1 & 1 & 4 & -2 & 0 & 0 & 3 & -2 & -3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -3 & 2 & 0 & 0 & -1 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 2 & 2 & -1 & -3 & 2 & 3 & -3 & -3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -1 & 1 & 0 & -2 & 0 & 3 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 1 & 0 & 0 & -3 & 2 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 2 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

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