# SPARSE BOUNDS FOR A PROTOTYPICAL SINGULAR RADON TRANSFORM

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Abstract. We use a variant of the technique in [Lac17a] to give sparse $L^p(\log(L))^4$ bounds for a class of model singular and maximal Radon transforms.

## 1. Introduction

Suppose $\mu$ and $\sigma$ are finite signed and positive measures respectively, supported on the unit ball $B(1) \subset \mathbb{R}^n$ with $d\mu = \rho \, d\sigma$ for some bounded density $\rho$, $\mu(\mathbb{R}^n) = 0$, and (using $\hat{}$ to denote the Fourier transform)

$$\max(|\hat{\sigma}(\xi)|, |\hat{\mu}(\xi)|) \lesssim |\xi|^{-\alpha} \tag{1}$$

for some $\alpha > 0$ (Our main examples of interest are when $\sigma$ is surface measure on a compact piece of a finite-type submanifold of $\mathbb{R}^n$ and $\rho$ is a smooth function on $\mathbb{R}^n$ with $\sigma$-mean zero). Define $\mu_j$ by

$$\int f \, d\mu_j = \int f(2^j x) \, d\mu(x).$$

Given a collection of coefficients $\{\epsilon_j\}_{j\in\mathbb{Z}}$ with $|\epsilon_j| \leq 1$ we may consider the singular Radon transform

$$T[f] := \sum_j \epsilon_j \mu_j * f$$

and the maximal averaging operator

$$T^*[f](x) := \sup_j \sigma_j * |f|(x).$$

It is well known that condition (1) implies that $T$ and $T^*$ are bounded on $L^p$ for $1 < p < \infty$.

The following "sparse bound" for $T^*$ was recently proven in [Lac17a] (see also related work [CO])

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2010 *Mathematics Subject Classification.* 42B25.

*Key words and phrases.* Sparse Domination Singular Radon Transform.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2017 semester.

**Theorem 1** (Lacey). *Suppose $\sigma$ is surface measure on the unit sphere in $\mathbb{R}^n$ and $1 < p < q < \infty$ are exponents such that convolution with $\sigma$ is a bounded operator from $L^p$ to $L^q$. For $0 < \theta < 1$ let*

$$\frac{1}{p_\theta} := \frac{1-\theta}{p} + \frac{\theta}{2} \quad \textit{and} \quad \frac{1}{q_\theta} := \frac{1-\theta}{q} + \frac{\theta}{2}.$$

*There is a finite $C_\theta$ such that for every pair of compactly supported $f_1, f_2$ there is a sparse collection of cubes $\mathcal{Q}$ such that*

$$|\langle T^*[f_1], f_2\rangle| \leq C_\theta \sum_{Q\in\mathcal{Q}} |Q| \langle |f_1|\rangle_{Q,p_\theta} \langle |f_2|\rangle_{Q,q'_\theta} \tag{2}$$

*where*

$$\langle |f|\rangle_{Q,p} := \left(\frac{1}{|Q|}\int_Q |f|^p \, dx\right)^{1/p}.$$

Above, we use $|Q|$ to denote the Lebesgue measure of $Q$, and the collection $\mathcal{Q}$ is said to be *sparse* if there is a collection of pairwise disjoint sets $\{F_Q\}_{Q\in\mathcal{Q}}$ with $|F_Q| \geq \frac{1}{2}|Q|$ and $F_Q \subset \mathcal{Q}$. Bounds such as (2) (as well as those which give pointwise or norm domination by sparse operators) have been of much recent interest. See for example [Ler10], [LN15], [Ler16], [DDU16], [BBL16], [BFP16], [CKL16], [CDO16], [Lac17b], [KL17], [NPTV17].

Theorem 1 is nontrivial (given that $T^*$ is known to be bounded on $L^p$) since $q'_\theta < p'_\theta$. Furthermore, the range of exponents is sharp up to the small $\theta$-loss in interpolation (Since there is positive distance between the center of the sphere and the support of the measure, a sparse bound as above implies that convolution with $\sigma$ is bounded from $L^{p_\theta}$ to $L^{q_\theta}$). Lacey's argument does not appear to depend on the geometry of the sphere, and likely extends without modification to compactly supported positive measures satisfying (1).

Our purpose here is to explore the relationship between the method of [Lac17a] and more traditional approaches (which use a regularization of the single scale operator) for bounding $T^*$. This will allow us to push a little closer to the natural endpoint exponents $(p, q)$. We have also organized our argument[1] to facilitate bounds for the singular integral $T$.

Given a cube $Q$, define

$$f_Q^0 := f\cdot 1_{\{x\in Q : |f(x)|\leq \langle |f|\rangle_{Q,p}\}} \quad \text{and} \quad f_Q^m := f\cdot 1_{\{x\in Q: 2^{m-1}\langle |f|\rangle_{Q,p} < |f(x)| \leq 2^m \langle |f|\rangle_{Q,p}\}}, \;\; m > 0. \tag{3}$$

Our bounds will be in terms of the following "restricted-type $L^p\log(L)^4$" averages:

$$\langle |f|\rangle_{Q,p^+} := \sum_{m\geq 0} (m+1)^4 \langle |f_Q^m|\rangle_{Q,p}. \tag{4}$$

It is straightforward to check that for each $\tilde{p} > p \geq 1$

$$\langle |f|\rangle_{Q,p} \leq \langle |f|\rangle_{Q,p^+} \leq C_{\tilde{p}} \langle |f|\rangle_{Q,\tilde{p}}.$$

[1]Specifically, we use a Calderón-Zygmund decomposition of *both* functions, as was done in the original version of [Lac17a]. Later versions feature a streamlined argument which relies instead on the orthogonality of the linearizing functions and does not seem to immediately bound $T$.

**Theorem 2.** *Suppose $\mu, \sigma$ are finite signed and positive measures respectively supported on the unit ball with $\mu(\mathbb{R}^n) = 0$. If $\mu$ and $\sigma$ satisfy* (1) *and $1 < p < q < \infty$ are exponents such that convolution with $\mu$ is a bounded operator from $L^p$ to $L^q$ then there is a finite $C$ such that for every pair of compactly supported functions $f_1, f_2$ there is a sparse collection of cubes $\mathcal{Q}$ such that*

$$|\langle T[f_1], f_2\rangle| \leq C \sum_{Q\in\mathcal{Q}} |Q| \, \langle|f_1|\rangle_{Q,p^+} \langle|f_2|\rangle_{3Q,q'^+} . \tag{5}$$

Essentially the same proof (simply replace the coefficients $\epsilon_j$ by linearizing functions $\epsilon_j(x)$) can be used to bound the maximal operator.

**Theorem 3.** *Suppose $\sigma$ is a finite measure supported on the unit ball satisfying* (1)*, and that $1 < p < q < \infty$ are exponents such that convolution with $\sigma$ is a bounded operator from $L^p$ to $L^q$. There is a finite $C$ such that for every pair of compactly supported $f_1, f_2$ there is a sparse collection of cubes $\mathcal{Q}$ such that*

$$|\langle T^*[f_1], f_2\rangle| \leq C \sum_{Q\in\mathcal{Q}} |Q| \, \langle|f_1|\rangle_{Q,p^+} \langle|f_2|\rangle_{3Q,q'^+} . \tag{6}$$

It was pointed out to us by Jim Wright that our argument also controls the maximally truncated singular integral. Defining

$$T^{**}[f](x) = \sup_j |\sum_{j'\geq j} \mu_{j'} * f(x)|$$

Duoandikoetxea and Rubio de Francia [DRdF86] showed that $T^{**}$ is bounded on $L^p$, $1 < p < \infty$. Using their estimate and a linearization as in Theorem 3 gives

**Theorem 4.** *Suppose $\mu, \sigma$ are finite signed and positive measures respectively supported on the unit ball with $\mu(\mathbb{R}^n) = 0$. If $\mu$ and $\sigma$ satisfy* (1) *and $1 < p < q < \infty$ are exponents such that convolution with $\mu$ is a bounded operator from $L^p$ to $L^q$ then there is a finite $C$ such that for every pair of compactly supported functions $f_1, f_2$ there is a sparse collection of cubes $\mathcal{Q}$ such that*

$$|\langle T^{**}[f_1], f_2\rangle| \leq C \sum_{Q\in\mathcal{Q}} |Q| \, \langle|f_1|\rangle_{Q,p^+} \langle|f_2|\rangle_{3Q,q'^+} . \tag{7}$$

The exponent four in the definition of $\langle|f|\rangle_{Q,p^+}$ is not optimal and could be lowered slightly by following the numerology more carefully. We conjecture (based on parallels in the methods of proof) that the sharp bounds for (7) and (6) may match the (currently unknown) sharp estimates at $L^1$ for $T$ and $T^*$. Specifically, that for a given $\sigma$, (6) should hold with $\langle|f_1|\rangle_{Q,p} \langle|f_2|\rangle_{3Q,q}$ in place of $\langle|f_1|\rangle_{Q,p^+} \langle|f_2|\rangle_{3Q,q'^+}$ if and only if $T^*$ satisfies a weak-type $L^1$ estimate (and similarly for bounds with logarithmic losses). This would suggest that, at the very least, Theorems 2 and 3 should hold with $L^p \log(L)$ in place of $L^p \log(L)^4$.

**Acknowledgements.** The author would like to thank Michael Lacey for sparking his interest in sparse bounds and for several helpful conversations subsequently.

## 2. A review of the $L^p$ theory

We quickly recall a now standard method, which seems to originate in [DRdF86], for proving $L^p$ estimates for $T$ (and $T^*$). This section is purely expository and may be skipped by the experts.

The $L^2 \to L^2$ bound for $T$ is immediate from (1). To prove a bound near $L^1$, perform a Calderón-Zygmund decomposition

$$f = g + \sum_Q b_Q$$

where $b_Q$ is supported on the cube $Q$ and has mean-zero. The contribution from the good function $g$ is handled, as usual, using the $L^2$ estimate.

Let $\ell(Q)$ denote the sidelength of a cube $Q$. If $\mu$ had an integrable derivative, we could deduce a weak-type $L^1$ estimate by leveraging the smoothness of the $\mu_j$ at scale $2^j$ against the cancellation of $b_Q$ for $2^j \geq \ell(Q)$, and by using the decay of the $\mu_j$ at scale $2^j$ against the support of $b_Q$ for $2^j \leq \ell(Q)$ (this, of course, is just the classic Calderón-Zygmund method).

In general, one can write

$$\mu = \sum_{k \leq 0} \mu * \eta_k$$

where $\mu * \eta_k$ is smooth at scale $2^k$. Then $(\mu * \eta_k)_j$ is smooth at scale $2^{j+k}$, and so the contribution from $b_Q$ is acceptable, as above, when $2^{j+k} \geq \ell(Q)$. Here, however, $(\mu * \eta_k)_j$ only has decay at scale $2^j$ and so, other than the trivial bound (i.e. the $(\mu * \eta_k)_j$ are uniformly in $L^1$ and so each of them gives a bounded convolution operator on $L^p$), one is not left with an obvious good option for $\ell(Q) < 2^j < 2^{-k}\ell(Q)$. This gives a weak-type estimate

$$\|T^k[f]\|_{L^{1,\infty}} \lesssim (1-k)\|f\|_{L^1} \tag{8}$$

where

$$T^k[f] := \sum_j \epsilon_j (\mu * \eta_k)_j * f.$$

On the other hand, provided $\eta_k$ is chosen with appropriate cancellation (1) implies

$$\|T^k[f]\|_{L^2} \lesssim 2^{k\alpha}\|f\|_{L^2}. \tag{9}$$

Then $T$ is bounded on $L^p$ for $1 < p < \infty$ from the Marcinkiewicz interpolation theorem. It is not difficult, also using real interpolation, to do a little better (the following is only meant for illustration, and we omit its proof):

**Lemma 1.** *Suppose* $\{T_k\}_{k \leq 0}$ *is* any *sequence of operators satisfying* (8) *and* (9)*. Then*[2] *for* $r > 4$

$$T := \sum_{k \leq 0} T^k$$

[2] It is only coincidence that the four here matches the four in the definition of $\langle |f| \rangle_{Q,p^+}$

*satisfies the "weak-type $L(\log(L))^r$ estimate"*

$$|\{|T[f]| > \lambda\}| \lesssim \int \frac{|f(x)|}{\lambda} \left(\log\left(e + \frac{|f(x)|}{\lambda}\right)\right)^r \, dx, \quad \lambda > 0. \tag{10}$$

In fact, by incorporating the interpolation into the proof rather than crudely using it as a black-box, one finds that our operator $T$ satisfies a weak-type $L\log(L)$ bound, and for many measures $\mu$ one can apply more sophisticated techniques to push even closer to $L^1$. See, for example, [STW04], [CK17], and the references therein.

## 3. Proof of Theorem 2

We will use a sparse bound adaptation (inspired by [Lac17a]) of the method outlined in Section 2. The principle use of the $L^p \to L^q$ estimate for convolution with $\mu$ is to replace the "trivial $L^1$ bound" used for scales $\ell(Q) < 2^j < 2^{-k}\ell(Q)$ above.

Through a limiting argument and appropriate choice of dyadic grid, we may assume that there are finite $N_1, N_2$ such that $\epsilon_j = 0$ for $j$ outside of $[N_1, N_2]$ and that $f_1, f_2$ are supported on $Q_0$ and $3Q_0$ respectively, where $Q_0$ is a dyadic cube with $\ell(Q_0) = 2^{N_2}$(the bounds given will be independent of the $N_j$). Our proof will rely on recursion, each instance of which reduces $N_2$ and the support of the functions. After a finite number of steps, we are left with a null operator.

Write

$$\mathcal{A}_p^r[f](x) := \left(\frac{1}{|B(r)|}\int_{B(r)} |f(x+y)|^p \, dy\right)^{\frac{1}{p}}$$

$$\mathcal{M}_p[f](x) := \sup_{r>0} \mathcal{A}_p^r[f](x)$$

$$T^*_{\text{high}}[f] := \sup_{2^j \le \ell(Q_0)} \sigma_j * |f|$$

$$f_1^m := (f_1)^m_{Q_0} \quad f_2^m := (f_2)^m_{3Q_0} \quad \text{(using notation as in (3))}.$$

We then define

$$E_1 = \{\mathcal{M}_p[f_1] > D\,\langle |f_1|\rangle_{Q_0,p}\} \cup \{\mathcal{M}_1[T^*_{\text{high}}[f_1]] > D\,\langle |f_1|\rangle_{Q_0,p}\} \cup \bigcup_{m\ge 0} \{\mathcal{M}_p[f_1^m] > (m+1)D\,\langle |f_1^m|\rangle_{Q_0,p}\}.$$

and similarly for $E_2$ with $f_2$ in place of $f_1$, $3Q_0$ in place of $Q_0$, and $q'$ in place of $p$.

Choosing $D$ very large (depending on the $L^p, L^{q'}$ bounds for $T^*$ and $\mathcal{M}^p$), we can force $|E| := |E_1 \cup E_2| \le \frac{1}{2}|Q_0|$ and, say, $E \subset 6Q_0$. Using a Whitney decomposition, write $E$ as the disjoint union of a collection of dyadic cubes

$$E = \bigcup_{Q \in \mathcal{Q}_1} Q$$

each of which satisfies

$$5\sqrt{n}\ell(Q) \le \text{distance}\left(Q, \left(\bigcup_{Q' \in \mathcal{Q}_1} Q'\right)^c\right) < 11\sqrt{n}\ell(Q). \tag{11}$$

We then have, for example, that for every cube $Q'$ which contains a cube $Q \in \mathcal{Q}_1$

$$\langle |f_1| \rangle_{Q',p} \lesssim \langle |f_1| \rangle_{Q_0,p}.$$

Perform a Calderón-Zygmund decomposition of $f_1$

$$\begin{aligned} f_1 &=: g_1 + \sum_{Q \in \mathcal{Q}_1} 1_Q(f_1 - \langle f_1 \rangle_{Q,1}) =: g_1 + \sum_{Q \in \mathcal{Q}_1} b_{1,Q} \\ &= g_1 + \sum_{\substack{Q \in \mathcal{Q}_1 \\ Q \subset Q_0}} b_{1,Q} \end{aligned}$$

where, for the last identity, we use that, since $Q_0 \not\subset E$, if $Q \cap Q_0 \neq \emptyset$ then $Q \subset Q_0$. The good function is bounded

$$\|g_1\|_{L^\infty} \lesssim \langle |f_1| \rangle_{Q_0,p}.$$

We will also use repeatedly that for any cube $Q'$ and $r \ge 1$

$$\| \sum_{Q \subset Q'} b_{1,Q} \|_{L^r} \lesssim \|f_1\|_{L^r(Q')}.$$

Decompose

$$|\langle T[f_1], f_2 \rangle| \le |\langle T[g_1], f_2 \rangle| + |\sum_{Q \in \mathcal{Q}_1} \langle T[b_{1,Q}], f_2 \rangle|. \tag{12}$$

The $L^q$ boundedness of $T$ implies that the first term on the right above

$$|\langle T[g_1], f_2 \rangle| \lesssim |Q_0| \langle |f_1| \rangle_{Q_0,p} \langle |f_2| \rangle_{3Q_0,q'}.$$

Writing

$$T_Q[f] := \sum_{2^j \le \ell(Q)} \epsilon_j \mu_j * (1_Q f),$$

the second term of (12)

$$|\sum_{Q \in \mathcal{Q}_1} \langle T[b_{1,Q}], f_2 \rangle| = |\sum_{\substack{Q \in \mathcal{Q}_1 \\ Q \subset Q_0}} \langle (T - T_Q)[b_{1,Q}], f_2 \rangle + \langle T_Q[f_1], f_2 \rangle - \langle f_1 \rangle_{Q,1} \langle T_Q[1_Q], f_2 \rangle|. \tag{13}$$

By induction on $N_2 - N_1$, for each $Q \subset Q_0$ above we can find a sparse collection $\mathcal{Q}_Q$ of dyadic subcubes of $Q$ such that

$$|\langle T_Q[f_1], f_2 \rangle| \lesssim \sum_{Q' \in \mathcal{Q}_Q} |Q'| \langle |f_1| \rangle_{Q',p^+} \langle |f_2| \rangle_{3Q',q'^+}.$$

Setting $F_{Q_0} = Q_0 \setminus E$, we have that

$$\mathcal{Q} := \{Q_0\} \cup \bigcup_{\substack{Q \in \mathcal{Q}_1 \\ Q \subset Q_0}} \mathcal{Q}_Q$$

is sparse, and so it now remains to bound the sums of the first and third terms on the right of (13)

$$\lesssim |Q_0| \, \langle |f_1| \rangle_{Q_0,p^+} \, \langle |f_2| \rangle_{3Q_0,q'^+} \, .$$

Using the $L^q$ boundedness of $T_Q$ and the fact that the $3Q$ are finitely overlapping (from (11)), the sum of the third term is

$$\lesssim \sum_{Q \in \mathcal{Q}_1} \langle |f_1| \rangle_{Q_0,p} \, |Q|^{1/q} \|f_2\|_{L^{q'}(3Q)} \lesssim |Q_0| \, \langle |f_1| \rangle_{Q_0,p} \, \langle |f_2| \rangle_{3Q_0,q'} \, .$$

The last, and main, step of the proof will be to show that

$$| \sum_{Q \in \mathcal{Q}_1} \langle (T - T_Q)[b_{1,Q}], f_2 \rangle \, | \lesssim |Q_0| \, \langle |f_1| \rangle_{Q_0,p^+} \, \langle |f_2| \rangle_{3Q_0,q'^+} \, . \tag{14}$$

Perform a Calderón-Zygmund decomposition of $f_2$

$$f_2 =: g_2 + \sum_{Q \in \mathcal{Q}_1} 1_Q (f_2 - \langle f_2 \rangle_{Q,1}) =: g_2 + \sum_{Q \in \mathcal{Q}_1} b_{2,Q}.$$

The second good function is bounded

$$\|g_2\|_{L^\infty} \lesssim \langle |f_2| \rangle_{3Q_0,q'}$$

which, using the $L^p$ boundedness of $T$ and $T_Q$ (separately), gives

$$\begin{aligned} | \sum_{Q \in \mathcal{Q}_1} \langle (T - T_Q)[b_{1,Q}], g_2 \rangle \, | &\lesssim \| \sum_{Q \in \mathcal{Q}_1} b_{1,Q} \|_{L^p} \|g_2\|_{L^{p'}} + \sum_{Q \in \mathcal{Q}_1} \|b_{1,Q}\|_{L^p} \|g_2\|_{L^{p'}(3Q)} \\ &\lesssim |Q_0| \, \langle |f_1| \rangle_{Q_0,p} \, \langle |f_2| \rangle_{3Q_0,q'} \, . \end{aligned}$$

Expanding $T - T_Q$, (14) will be finished once we estimate

$$| \sum_{Q,Q' \in \mathcal{Q}_1} \sum_{2^j \geq \ell(Q)} \epsilon_j \left\langle \mu_j * b_{1,Q}, b_{2,Q'} \right\rangle |. \tag{15}$$

Then (15) is

$$\begin{aligned} \leq | \sum_{Q,Q' \in \mathcal{Q}_1} \sum_{\substack{j \\ 2^j \geq \max(\ell(Q),\ell(Q'))}} \epsilon_j \left\langle \mu_j * b_{1,Q}, b_{2,Q'} \right\rangle | \\ + | \sum_{Q,Q' \in \mathcal{Q}_1} \sum_{\substack{j \\ \ell(Q) \leq 2^j < \ell(Q')}} \epsilon_j \left\langle \mu_j * b_{1,Q}, b_{2,Q'} \right\rangle |. \end{aligned} \tag{16}$$

If a term in the right sum from (16) is nonzero then $Q \cap 2Q' \neq \emptyset$ and so, by (11), $\ell(Q) = \frac{1}{2}\ell(Q') = 2^j$. For each such $Q, Q'$, rescaling the $L^p \to L^q$ bound for $\mu$ gives

$$|\left\langle \mu_j * b_{1,Q}, b_{2,Q'} \right\rangle| \lesssim |Q'| \left\langle |f_1| \right\rangle_{3Q',p} \left\langle |f_2| \right\rangle_{Q',q'}$$

and thus the right sum from (16) is

$$\begin{aligned}
&\lesssim \sum_{Q' \in \mathcal{Q}_1} |Q'| \left\langle |f_1| \right\rangle_{3Q',p} \left\langle |f_2| \right\rangle_{Q',q'} \\
&\leq \sup_{Q \in \mathcal{Q}_1} \left\langle |f_2| \right\rangle_{Q,q'}^{1-\frac{q'}{p'}} \sum_{Q' \in \mathcal{Q}_1} |Q'| \left\langle |f_1| \right\rangle_{3Q',p} \left\langle |f_2| \right\rangle_{Q',q'}^{\frac{q'}{p'}} \\
&\lesssim \sup_{Q \in \mathcal{Q}_1} \left\langle |f_2| \right\rangle_{Q,q'}^{1-\frac{q'}{p'}} |Q_0| \left\langle |f_1| \right\rangle_{Q_0,p} \left\langle |f_2| \right\rangle_{3Q_0,q'}^{\frac{q'}{p'}} \\
&\lesssim |Q_0| \left\langle |f_1| \right\rangle_{Q_0,p} \left\langle |f_2| \right\rangle_{3Q_0,q'}. \qquad (17)
\end{aligned}$$

We bound the left sum from (16) by two terms which are treated in the same manner (it is irrelevant to the argument whether or not the diagonal $\ell(Q) = \ell(Q')$ is included), one of which is

$$|\sum_{\substack{Q,Q',j \\ \ell(Q) \leq \ell(Q') \leq 2^j}} \epsilon_j \left\langle \mu_j * b_{1,Q}, b_{2,Q'} \right\rangle|. \qquad (18)$$

It will be useful to decompose $\mu$. Let $\tilde{\eta}$ be a Schwartz function with $\hat{\tilde{\eta}}$ identically 1 on $B(1)$ and supported on $B(2)$ and $\eta := \tilde{\eta}_{-1} - \tilde{\eta}$ so that $\hat{\eta}$ is supported on $B(4) \setminus B(1)$ and

$$\hat{\tilde{\eta}} + \sum_{k \leq 0} \widehat{\eta_k} = 1.$$

Then (18)

$$\begin{aligned}
&\leq |\sum_{\substack{Q,Q',j \\ \ell(Q) \leq \ell(Q') \leq 2^j}} \epsilon_j \left\langle (\tilde{\eta} * \mu)_j * b_{1,Q}, b_{2,Q'} \right\rangle| \\
&\qquad + \sum_{k \leq 0} |\sum_{\substack{Q,Q',j \\ \ell(Q) \leq \ell(Q') \leq 2^j}} \epsilon_j \left\langle (\eta_k * \mu)_j * b_{1,Q}, b_{2,Q'} \right\rangle| \\
&=: |\tilde{S}| + \sum_{k \leq 0} |S_k|.
\end{aligned}$$

For $\tilde{S}$ we fix $Q$ and $2^j \geq \ell(Q) =: 2^l$. Using the cancellation of $b_{1,Q}$ we have

$$|\tilde{\eta}_j * b_{1,Q}(x)| \lesssim \left\langle |f_1| \right\rangle_{Q_0,p} 2^{2(l-j)} (1 + |\,\mathrm{distance}(x, Q)|/2^j)^{-N}$$

for large $N$, giving (we will abuse notation by identifying $\mu$ with its conjugate reflection)

$$\begin{aligned}|\left\langle (\tilde{\eta} * \mu)_j * b_{1,Q}, \sum_{\substack{Q' \\ \ell(Q)\le\ell(Q')\le 2^j}} b_{2,Q'} \right\rangle| &\lesssim \langle |f_1|\rangle_{Q_0,p}\, 2^{l-j}|Q|\mathcal{M}_1[\mu_j * \sum_{\substack{Q' \\ \ell(Q)\le\ell(Q')\le 2^j}} b_{2,Q'}](x') \\ &\lesssim 2^{l-j}|Q|\, \langle |f_1|\rangle_{Q_0,p} \, \langle |f_2|\rangle_{3Q_0,q'}\end{aligned}$$

where $x' \in E^c$. (To obtain the second inequality above, we write $\sum b_{2,Q'}$ as the difference of $1_{\bigcup Q'} f_2$ and $\sum 1_{Q'} \langle f_2\rangle_{Q',1}$. The contribution from the former term is bounded by positivity of $\mathcal{M}_1 \circ T^*$ and the fact that $x' \in E^c$, the contribution from the latter term instead uses the $L^\infty$ boundedness of $\mathcal{M}_1[\mu_j * \cdot]$.) Summing over $j$ and $Q'$ then gives

$$|\tilde{S}| \lesssim |Q_0| \, \langle |f_1|\rangle_{Q_0,p} \, \langle |f_2|\rangle_{3Q_0,q'} \, .$$

We now fix $k \le 0$ and turn our attention to $S_k$. We bound the low frequency component

$$\sum_{Q} \sum_{\substack{j \\ 2^j > 2^{-2k}\ell(Q)}} | \sum_{\substack{Q' \\ \ell(Q)\le\ell(Q')\le 2^j}} \epsilon_j \left\langle (\eta_k * \mu)_j * b_{1,Q}, b_{2,Q'} \right\rangle | \lesssim 2^k |Q_0| \, \langle |f_1|\rangle_{Q_0,p} \, \langle |f_2|\rangle_{3Q_0,q'}$$

using the same reasoning as for $\tilde{S}$ (and here, in contrast to $\tilde{S}$, it is important that $x' \in E^c$ since $u_j$ is at a coarser scale than $\eta_{k+j}$).

For $i = 1, 2$ write

$$b^m_{i,Q} := 1_Q \left( f^m_i - \langle f^m_i \rangle_{Q,1} \right).$$

Since

$$f_i = \sum_{m\ge 0} f^m_i$$

we have

$$b_{i,Q} = \sum_{m \ge 0} b^m_{i,Q}.$$

Decompose

$$\begin{aligned}(19)\qquad &\sum_{Q} \sum_{\substack{j \\ 2^j \le 2^{-2k}\ell(Q)}} \sum_{\substack{Q' \\ \ell(Q)\le\ell(Q')\le 2^j}} \epsilon_j \left\langle (\eta_k * \mu)_j * b_{1,Q}, b_{2,Q'} \right\rangle \\ &= \sum_{m_1,m_2\ge 0} \sum_{Q} \sum_{\substack{j \\ 2^j \le 2^{-2k}\ell(Q)}} \sum_{\substack{Q' \\ \ell(Q)\le\ell(Q')\le 2^j}} \epsilon_j \left\langle (\eta_k * \mu)_j * b^{m_1}_{1,Q}, b^{m_2}_{2,Q'} \right\rangle\end{aligned}$$

For pairs $(m_1, m_2)$ with $m_1 + m_2 \le \frac{-k\alpha}{2}$ we use the $L^2$ estimate for convolution with $\eta_k * \mu$. Writing

$$Q^m_i := Q \cap \{f^m_i \neq 0\}$$

for each $0 \le h \le \frac{-k\alpha}{2}$ and $0 \le i \le i' \le -2k$ we have

$$
\begin{aligned}
\sum_{m\le h}\sum_{l} &|\langle(\eta_k*\mu)_{l+i'}*\sum_{\ell(Q)=2^l} b^m_{1,Q}, \sum_{\ell(Q)=2^{l+i}} b^{h-m}_{2,Q}\rangle| \\
&\lesssim 2^{k\alpha}\sum_{m\le h}\sum_{l}\|\sum_{\ell(Q)=2^l} b^m_{1,Q}\|_{L^2}\|\sum_{\ell(Q)=2^{l+i}} b^{h-m}_{2,Q}\|_{L^2} \\
&\lesssim 2^{k\alpha}\sum_{m\le h}\sum_{l}\|\sum_{\ell(Q)=2^l} 1_Q f^m_1\|_{L^2}\|\sum_{\ell(Q)=2^{l+i}} 1_Q f^{h-m}_2\|_{L^2} \\
&\lesssim 2^{k\alpha+h}\,\langle|f_1|\rangle_{Q_0,}\,\langle|f_2|\rangle_{3Q_0,q'}\sum_{m\le h}\sum_{l}|\bigcup_{\ell(Q)=2^l} Q^m_1|^{\frac12}|\bigcup_{\ell(Q)=2^{l+i}} Q^{h-m}_2|^{\frac12} \\
&\lesssim 2^{k\alpha+h}|Q_0|\,\langle|f_1|\rangle_{Q_0,p}\,\langle|f_2|\rangle_{3Q_0,q'}\,.
\end{aligned}
$$

Summing over $i,i'$ and then $h$ we have that the magnitude of the restriction of the sum on the right side of (19) to $m_1+m_2\le\frac{-k\alpha}{2}$ is

$$
\lesssim 2^{\frac{k\alpha}{4}}|Q_0|\,\langle|f_1|\rangle_{Q_0,p}\,\langle|f_2|\rangle_{3Q_0,q'}
$$

which sums over $k\le 0$ to an acceptable contribution.

For $m_1+m_2>\frac{-k\alpha}{2}$ we use the $L^p$ improving property of the $\mu$ averages. Fix $m_1,m_2$ and $0\le i\le i'\le 2k$. Then

$$
\begin{aligned}
\sum_{l}&|\langle(\eta_k*\mu)_{l+i'}*(\sum_{\ell(Q)=2^l} b^{m_1}_{1,Q}),(\sum_{\ell(Q)=2^{l+i}} b^{m_2}_{2,Q})\rangle| \\
&\lesssim \|f^{m_2}_2\|_{L^{q'}}(\sum_{l}\|\mu_{l+i'}*(\sum_{\ell(Q)=2^l} b^{m_1}_{1,Q})\|^q_{L^q})^{1/q}.
\end{aligned}
\tag{20}
$$

The second factor on the right of (20) is

$$
\begin{aligned}
&\lesssim (\sum_{l}\sum_{\substack{Q'\\ \ell(Q')=2^{l+i'}}}\|\mu_{l+i'}*(\sum_{\ell(Q)=2^l} b^{m_1}_{1,Q})\|^q_{L^q(Q')})^{1/q} \\
&\lesssim (\sum_{l}\sum_{\substack{Q'\\ \ell(Q')=2^{l+i'}}}|Q'|\langle|\sum_{\ell(Q)=2^l} b^{m_1}_{1,Q}|\rangle^q_{3Q',p})^{1/q} \\
&\lesssim \sup_{\substack{Q'',l\\ \ell(Q'')=2^{l+i'}}}\langle|\sum_{\ell(Q)=2^l} b^{m_1}_{1,Q}|\rangle^{1-\frac{p}{q}}_{3Q'',p}\;(\sum_{l}\sum_{\substack{Q'\\ \ell(Q')=2^{l+i'}}}|Q'|\langle|\sum_{\ell(Q)=2^l} b^{m_1}_{1,Q}|\rangle^p_{3Q',p})^{1/q} \\
&\lesssim (m_1+1)\,\langle|f^{m_1}_1|\rangle^{1-\frac{p}{q}}_{Q_0,p}\,\|f^{m_1}_1\|^{\frac{p}{q}}_{L^p} \\
&\lesssim (m_1+1)|Q_0|^{\frac1q}\,\langle|f^{m_1}_1|\rangle_{Q_0,p}
\end{aligned}
$$

where, above, we sum over all dyadic cubes $Q'$ of sidelength $2^{l+i'}$. This implies that the sum over $(i,i')$ of (20) is

$$\lesssim k^2(m_1+1)|Q_0| \langle |f_1^{m_1}|\rangle_{Q_0,p} \langle |f_2^{m_2}|\rangle_{3Q_0,q'}$$

and so the sum over $k$ of the magnitude of the restriction of the sum on the right side of (19) to $m_1+m_2 > \frac{-k\alpha}{2}$ is

$$\begin{aligned}
&\lesssim |Q_0| \sum_{m_1,m_2} (m_1+m_2+1)^4 \langle |f_1^{m_1}|\rangle_{Q_0,p} \langle |f_2^{m_2}|\rangle_{3Q_0,q'} \\
&\lesssim |Q_0| (\sum_m (m+1)^4 \langle |f_1^{m}|\rangle_{Q_0,p}) (\sum_m (m+1)^4 \langle |f_2^{m}|\rangle_{3Q_0,q'}) \\
&= |Q_0| \langle |f_1|\rangle_{Q_0,p^+} \langle |f_2|\rangle_{3Q_0,q'^+}
\end{aligned}$$

thus finishing the proof.

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