CONSTRUCTING CONVEX PROJECTIVE 3-MANIFOLDS WITH GENERALIZED CUSPS

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Abstract. We prove that non-compact finite volume hyperbolic 3-manifolds that satisfy a mild cohomological condition (infinitesimal rigidity) admit a family of properly convex deformations of their complete hyperbolic structure where the ends become generalized cusps of type 1 or type 2. We also discuss methods for controlling which types of cusp occur. Using these methods we produce the first known example of a 1-cusped hyperbolic 3-manifold that admits a convex projective structure with a type 2 cusp. We also use these techniques to produce new 1-cusped manifolds that admit a convex projective structure with a type 1 cusp.

Unless stated otherwise, all manifolds in this paper are orientable. A subset Ω of the projective sphere, $\mathbb{S}^n$, is *properly convex* if it is a bounded convex subset of some affine subspace of $\mathbb{S}^n$. A *properly convex manifold* is a quotient Ω/Γ, where Ω is properly convex and Γ is a discrete, torsion-free subgroup of $\mathrm{SL}(n+1,\mathbb{R})$ that preserves Ω. An important example of a properly convex set is the Klein model of n-dimensional hyperbolic space. As a result, complete hyperbolic manifolds provide a broad and important class of properly convex manifolds.

Suppose M is an n-manifold, a *(marked) convex projective structure on* M is a pair (f, N), where N is a properly convex manifold and $f : M \to N$ is a diffeomorphism. There is a natural equivalence relation on convex projective structures and the *deformation space* of convex projective structures on M, denoted $\mathfrak{B}(M)$, is the set of equivalence classes of convex projective structures. When M is a finite volume hyperbolic manifold and $n \geq 3$ Mostow rigidity implies that there is a distinguished base point in $\mathfrak{B}(M)$ coming from the equivalence of the complete hyperbolic structure on M. A primary focus of this work is to understand the possible geometry of points in a neighborhood of this basepoint.

We now restrict our discussion to dimension 3. Unlike the hyperbolic setting which is extremely rigid, it is sometimes possible to produce a variety of interesting deformations in the properly convex setting. However, there are some (loose) similarities to the hyperbolic setting. In practice, convex projective structures on *closed* manifold tend to be quite rigid. In [12] Cooper–Long–Thistlethwaite analyzed several thousand 3-manifolds with two-generator fundamental group and found that a vast majority ($> 90\%$) do not admit any properly convex deformations of their hyperbolic structure (i.e. the hyperbolic structure is an isolated point of $\mathfrak{B}(M)$). However, they also found a small number of examples that admit positive dimensional families of deformations of their complete hyperbolic structure (see [14]). There are also other isolated examples of closed 3-manifolds whose complete hyperbolic structure can be deformed (see [17, 8, 7]).

There are also similarities between the deformation theory of hyperbolic and convex projective structures when M is non-compact, but has finite volume. In both settings it is possible to find deformations that are "supported near the boundary." In the hyperbolic setting, it is well known (see [25]) that a k-cusped hyperbolic manifold admits a (real) $2k$-dimensional family of deformations of its complete hyperbolic structure. However, these deformations only give rise to *incomplete* hyperbolic structures. Loosely speaking, this is a consequence of there not being any way to deform the cusp of M without losing completeness.

However in the context of properly convex geometry *generalized cusps* provided many interesting ways to deform the cusp of a hyperbolic 3-manifold while preserving completeness (with respect to an appropriate metric). Generalized cusps (see Section 1 for precise definitions) are best thought of as properly convex generalizations of cusps of finite volume hyperbolic manifolds. They were first introduced by Cooper–Long–Tillmann [15] and were recently classified by the author, D. Cooper, and A. Leitner in [3]. In dimension 3

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generalized cusps come in 4 different flavors (type 0, type 1, type 2, and type 3), where the types interpolate between the holonomy of their fundamental group being unipotent (type 0) and diagonalizable (type 3). The main result of this paper is that when M is infinitesimally rigid rel. ∂M (see Section 2 for definition) not only is it possible to deform the hyperbolic structure in $\mathfrak{B}(M)$, but it is also possible to have some control over the geometry near the boundary.

Theorem 0.1. *Let M be a finite volume, non-compact hyperbolic 3-manifold with $k \geq 1$ cusps and let $\mathfrak{B}(M)$ be deformation space of convex projective structures on M. Suppose that M is infinitesimally rigid rel. ∂M then there is a k-dimensional family in $U \subset \mathfrak{B}(M)$ containing the complete hyperbolic structure on M and consisting of convex projective structures on M whose ends are generalized cusps of type 0, type 1 or type 2.*

By carefully analyzing the deformations produced by Theorem 0.1 we also prove:

Theorem 0.2. *Let M be a finite volume, non-compact hyperbolic 3-manifold. Suppose that M is infinitesimally rigid rel. ∂M then there is a convex projective structures on M where each end is a generalized cusp of type 1 or type 2.*

In general, the hypothesis that M is infinitesimally rigidity rel. ∂M is not very restrictive. For instance, in [17], Heusener–Porti prove that infinitely many 1-cusped manifolds arising as surgery on the Whitehead link are infinitesimally rigid rel. ∂M. These examples include infinitely many twist knots and infinitely many once-punctured torus bundles with tunnel number 1. Furthermore, numerical computations performed by the author, J. Danciger, and G.-S. Lee suggest that a majority of manifolds in the SnapPy cusped census [16] are infinitesimally rigid rel. ∂M.

The proof of Theorem 0.1 uses a transversality argument in the space $\mathrm{Hom}(\pi_1 M, \mathrm{SL}(4,\mathbb{R}))$. The idea is to construct a submanifold $\mathcal{S}$ of representations in $\mathrm{Hom}(\pi_1 \partial M, \mathrm{SL}(4,\mathbb{R}))$ whose elements are the holonomy representations of generalized cusps of type 0, type 1, and type 2 (see Section 3 for details). We then show that $\mathcal{S}$ has transverse intersection with the image of a certain "restriction map" in order to construct representations in $\mathrm{Hom}(\pi_1 M, \mathrm{SL}(4,\mathbb{R}))$. We then use a version of the Ehresmann–Thurston principle for properly convex structures due to Cooper–Long–Tillmann [15] in order to show that these representations are holonomies of convex projective structures on M with ends that are generalized cusps.

One application of this theorem is to complete the picture of which generalized cusp types can occur as ends of a convex projective structure on a 1-cusped hyperbolic manifold. Type 0 cusps occur as the ends of finite volume hyperbolic 3-manifolds, and so there are many examples coming from the classical theory of hyperbolic geometry. At the other end of the spectrum the author, along with J. Danciger and G.-S. Lee (see [6]) prove a complementary result which shows that under the same hypothesis as Theorem 0.1, it is possible to find infinite families of convex projective structures on M with type 3 cusps. In particular, it is possible to produce 1 cusped 3-manifolds that admit convex projective structures with type 3 cusps

However, up to this point there have only been isolated examples of manifolds with type 1 or type 2 cusps. One such example is given by the author in [5], where it is shown that the complement in S^3 of the figure-eight knot admits a convex projective structure with a type 1 cusp. Until very recently, there were no known examples of a hyperbolic 3-manifold with type 2 cusps. However, the author was recently made aware of work of M. Bobb [9] in which he produced the first examples of a hyperbolic 3 manifold with a cusp of type 2. His methods are quite different than those of this paper and involve simultaneously bending along multiple embedded totally geodesic hypersurfaces. However, he uses arithmetic methods to produce examples with many totally geodesic hypersurfaces and as a result, the manifolds he constructs have many cusps.

In Section 5 we analyze the geometry of the ends produced by Theorem 0.1. Using these result we are able to show that the complement in S^3 of the 5_2 knot admits a convex projective structure with a type 2 cusp (see Theorem 6.3). To the best of the author's knowledge, this is the first known 1-cusped manifold that admits a convex projective structure with type 2 cusp. Moreover, in Theorem 5.1 we show that a "generic" deformation constructed by Theorem 0.1 will have only type 2 cusps, so in practice Theorem 0.1 should produce infinitely many new examples of 1-cusped manifolds that admit a convex projective structure with a type 2 cusp.

Despite the genericity of type 2 cusps, it is still possible to use Theorem 0.1 to produce examples of properly convex manifolds with type 1 cusps. Specifically, we show in Section 5 that if M satisfies the hypotheses of Theorem 0.1 and admits a certain type of orientation reversing symmetry then Theorem 0.1 produces convex projective structures on M whose cusps are all of type 1. We then apply this result to show that the complement in S^3 of the 6_3 knot admits a convex projective structure with a type 1 cusp (see Theorem 6.5).

Organization of the paper. Section 1 gives some background and definitions related properly convex geometry, generalized cusps, and deformations of convex projective structures. Section 2 discusses infinitesimal deformations and their relationship to twisted cohomology. It also provides some relevant cohomological results in dimension 3. Section 3 defines the slice that will be used in the main transversality argument and outlines some of its important properties. Section 4 is the technical heart of the paper. In this section we provide the main transversality argument and prove Theorem 0.1. Section 5 provides the necessary tools to analyze the geometry of the cusps for the deformations produced by Theorem 0.1. In particular it provides the ingredients to prove Theorem 0.2. Finally, Section 6 outlines the computations necessary to prove the results concerning the 5_2 knot and 6_3 knot.

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1. Properly convex geometry

The *projective n-sphere*, denoted $\mathbb{S}^n$, is the space of rays through the origin in $\mathbb{R}^{n+1}$. More concretely, $\mathbb{S}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$ where $x\sim y$ if an only if there is $\lambda>0$ such that $x=\lambda y$. The group $\mathrm{GL}(n+1,\mathbb{R})$ acts on $\mathbb{S}^n$, however, this action is not faithful. The kernel of the action is $\mathbb{R}^+\,\mathrm{I}$. For each class in $\mathrm{GL}(n+1,\mathbb{R})/\mathbb{R}^+\,\mathrm{I}$ there is a unique representative with determinant ±1. Therefore, if we let

$$G=\mathrm{SL}_{\pm}(n+1,\mathbb{R}) := \{A\in \mathrm{GL}(n+1,\mathbb{R}) \mid \det(A)=\pm 1\}$$

then there is a natural identification of $\mathrm{GL}(n+1,\mathbb{R})/\mathbb{R}^+\,\mathrm{I}\cong G$, and so G is the full group of projective automorphisms of $\mathbb{S}^n$.

The projective n-sphere is related to the more familiar *real projective n-space*, denoted $\mathbb{RP}^n$, which consist of lines through the origin in $\mathbb{R}^{n+1}$ via the 2–to–1 covering given by mapping a ray to the line that contains it. It is possible to work entirely with $\mathbb{RP}^n$ instead of $\mathbb{S}^n$, however the benefit of working with $\mathbb{S}^n$ is that it is orientable for all n and its group of projective automorphisms consists of matrices instead of equivalences classes of matrices. This allows one to use tools from linear algebra, such as eigenvalues, traces, etc., without having to worry about picking representative from equivalence classes.

A *projective hyperplane*, or *hyperplane* for short, is the image of an n-dimensional subspace of $\mathbb{R}^{n+1}$ in $\mathbb{S}^n$. In other words, a projective hyperplane is a great $(n-1)$-sphere. If H is a projective hyperplane then either hemisphere of $\mathbb{S}^n\backslash H$ is naturally identified with $\mathbb{A}^n$ and is thus called an *affine patch* (see Figure 1). The group G acts transitively on the set of affine patches, and so there is model for an affine patch given by

$$\mathbb{A}^n=\{[x_1:\ldots,x_n:1]\mid x_i\in\mathbb{R}\},$$

where $[x_1:\ldots:x_{n+1}]$ is the *homogeneous coordinate* for the ray containing the point $(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1}$. The stabilizer in $\mathrm{GL}(n+1,\mathbb{R})$ of this affine patch is *affine group*, denoted G_a, and consists of matrices that can be written in block form as

$$\begin{pmatrix} A & b\\ 0 & 1\end{pmatrix},$$

where $A\in\mathrm{GL}(n,\mathbb{R})$, $b\in\mathbb{R}^n$. The group G_a acts faithfully on $\mathbb{S}^n$.

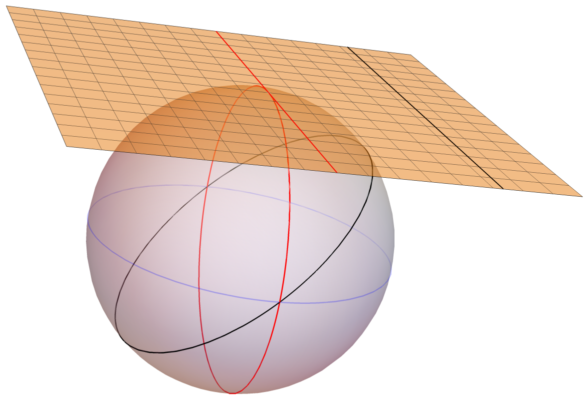

FIGURE 1. An affine in $\mathbb{S}^n$ is identified with $\mathbb{R}^n$ via radial projection

Let $\Omega \subset \mathbb{S}^n$ with non-empty interior, then Ω is *properly convex* if the topological closure, $\overline{\Omega}$, of Ω is a convex subset of some affine patch. Every properly convex set Ω comes with a group $\mathrm{SL}(\Omega)$ consisting of elements of G that preserve Ω.

If Ω is properly convex and $\Gamma \leq \mathrm{SL}(\Omega)$ is discrete and torsion-free then Ω/Γ is a *properly convex manifold.* An important example to keep in mind is the following: let $\mathcal{C}$ be a component of the interior in $\mathbb{R}^{n+1}$ of the light cone of a quadratic form of signature $(n, 1)$ and let $\Omega = \mathcal{C} \cap \mathbb{S}^n$. This is the well known *Klein model* of hyperbolic n-space. In this setting, $\mathrm{SL}(\Omega) = O^+(n, 1) \cong \mathrm{Isom}(\mathbb{H}^n)$ and we see that complete hyperbolic manifolds are examples of properly convex manifolds.

1.1. **Generalized cusps in projective manifolds.** Let M be a finite volume hyperbolic n-manifold. The thick-thin decomposition allows one to decompose write $M = M_K \sqcup \partial$, where M_K is compact manifold (possibly with boundary) homotopy equivalent to M and $\partial = \sqcup_{i=1}^k \partial_i$ is a union of finitely many *cusps*, where each ∂_i is diffeomorphic to $E_i \times (0, \infty)$ for some closed Euclidean $(n-1)$-manifold E_i. As a result, $\Delta_i := \pi_1(\partial_i) = \pi_1(E_i)$ is virtually abelian. It is also possible to describe the geometry of hyperbolic cusps: For each $t \in (0, \infty)$, $E_i \times \{t\}$ is a strictly convex hypersurface in ∂_i. Specifically, the universal cover of $E_i \times \{t\}$ can be identified with a horosphere in $\mathbb{H}^n$. Motivated by the previous discussion of cusps in hyperbolic manifolds we make the following definition:

Definition 1. *A properly convex n-manifold, $C = \Omega/\Gamma$ is a* generalized cusp *if*

- Γ *is virtually abelian*
- $C \cong E \times (0, \infty)$*, where E is a closed Euclidean $(n-1)$-manifold*
- *For each $t \in (0, \infty)$, the universal cover of $E \times \{t\}$ in Ω is strictly convex*

The previous discussion shows that cusps of finite volume hyperbolic n-manifolds are generalized cusps. Generalized cusps were originally introduced in [15] (using a slightly different definition) where they are instrumental in understanding properly convex deformations of non-compact manifolds. The current definition of generalized cusps is the one given by Cooper, Leitner, and the author in [3]. In this work it is shown that the two definitions of generalized cusps are, in fact, equivalent.

The main result from [3] is a classification result for generalized cusps in each dimension. Before providing some specific examples we roughly explain the classification result. In dimension n there are $n+1$ *types* of cusps which are denoted type 0 through type n. Each type determines an n-dimensional Lie subgroup of $\mathrm{GL}(n+1, \mathbb{R})$, T_k (where k is the type), called the *enlarged translation group* which is isomorphic to $\mathbb{R}^n$. Roughly speaking, the larger the type, the closer the enlarged translation group is to being diagonalizable. If $C = \Omega/\Gamma$ is a generalized cusp of type k then Γ contains a finite index subgroup Γ' that is a lattice in a certain codimension 1 Lie subgroup (depending on Γ) of T_k.

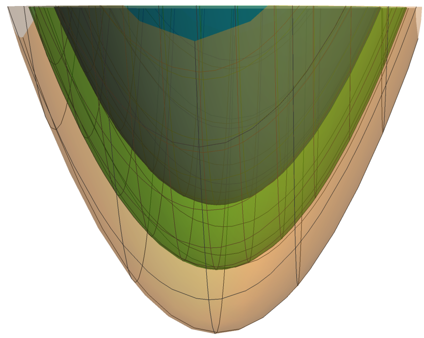

FIGURE 2. $\mathbb{H}^3$ along with some leaves of the horosphere foliation viewed in an affine patch

We now explain the classification in detail in the case where $n = 3$. Since the torus is the only closed Euclidean surface it follows that each 3-dimensional generalized cusp is diffeomorphic to $T^2 \times (0\infty)$. In this case there are 4 types of generalized cusp, and we will primarily concern ourselves with type 0, type 1, and type 2 cusps. For many purposes, it is simpler to work with the Lie algebra $\mathfrak{t}_k$ of the generalized translation group T_k. Nothing is lost working with $\mathfrak{t}_k$ since $\mathfrak{t}_k$ and T_k are isomorphic via the exponential map.

1.1.1. *Type 0 cusps.* Let $x, y, z \in \mathbb{R}$, then the Lie algebra $\mathfrak{t}_0$ consists of elements of the form

$$m_0(x,y,z) = \begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.1}$$

and T_0 consists of elements of the form

$$M_0(x,y,z) = \exp(m_0(x,y,z)) \begin{pmatrix} 1 & x & y & z + \frac{x^2+y^2}{2} \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider the codimension 1 subgroup $T(0)$ of T_0 consisting of elements of the form $M_0(x, y, 0)$. When regarded as elements of G_a, $T(0)$ preserves the properly convex set

$$\Omega_0 = \left\{ [a : b : c : 1] \in \mathbb{S}^3 \mid a > \frac{b^2 + c^2}{2} \right\}.$$

For $s > 0$ let

$$\mathcal{H}^0_s = \left\{ [a : b : c : 1] \in \mathbb{S}^3 \mid a = \frac{b^2 + c^2}{2} + s \right\}.$$

Each $\mathcal{H}_s$ is $T(0)$-invariant and the $\mathcal{H}^0_s$ give a codimension 1 foliation of Ω_0 by strictly convex hypersurfaces. A *type 0 generalized cusp* is a properly convex manifold that is projectively equivalent to Ω_0/Γ where Γ is a lattice in $T(0)$. Such manifolds are easily seen to be generalized cusps since $\mathcal{H}^0_s/\Gamma$ provides a foliation of Ω/Γ by strictly convex tori.

This is a familiar construction in the context of hyperbolic geometry: Ω_0 is the paraboloid model of $\mathbb{H}^3$ (see [13, §3]) and the foliation $\mathcal{H}^0_s$ is a foliation of $\mathbb{H}^3$ by concentric horospheres. The group $T(0)$ consists of parabolic isometries of $\mathbb{H}^3$ with a common fixed point on $\partial\mathbb{H}^3$, and Ω_0/Γ is a hyperbolic torus cusp.

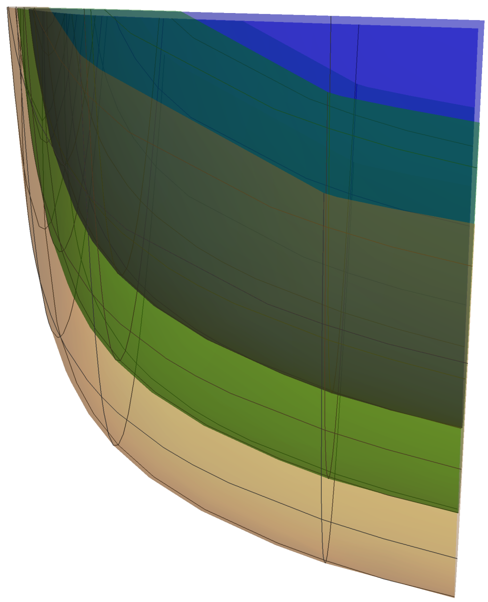

FIGURE 3. The domain Ω_1 and a few leaves of the foliation $\mathcal{H}^1_s$ in an affine patch $d = 1$.

1.1.2. *Type 1 cusps.* Again, let $x, y, z \in \mathbb{R}$, then the Lie algebra $\mathfrak{t}_1$ consists of elements of the form

$$m_1(x,y,z) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & y & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.2}$$

and let T_1 consist of element of the form

$$M_1(x,y,z) = \exp(m_1(x,y,z)) \begin{pmatrix} e^x & 0 & 0 & 0 \\ 0 & 1 & y & z + \frac{y^2}{2} \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\lambda \neq 0$ and let $T(\lambda)$ be the codimension 1 subgroup of T_1 consisting of elements of the form $M_1(\lambda x, y, -\lambda^{-1}x)$. For any $\lambda \neq 0$, the group $T(\lambda)$ preserves both the properly convex set

$$\Omega_1 := \left\{ [a : b : c : 1] \in \mathbb{S}^3 \mid a > 0,\ b > \frac{c^2}{2} - \lambda^{-2}\log(a) \right\}$$

and the strictly convex codimension 1 foliation of Ω_1 by

$$\mathcal{H}^1_s := \left\{ [a : b : c : 1] \in \mathbb{S}^3 \mid a > 0,\ b = \frac{c^2}{2} - \lambda^{-2}\log(a) + s \right\},\ s > 0$$

A *type 1 generalized cusp* is a properly convex manifold that is projective equivalent to Ω_1/Γ where Γ is a lattice in $T(\lambda)$ for some $\lambda \neq 0$. Again, such manifolds are easily seen to be generalized cusps since $\mathcal{H}^1_s/\Gamma$ provides a foliation of Ω_0/Γ by strictly convex tori.

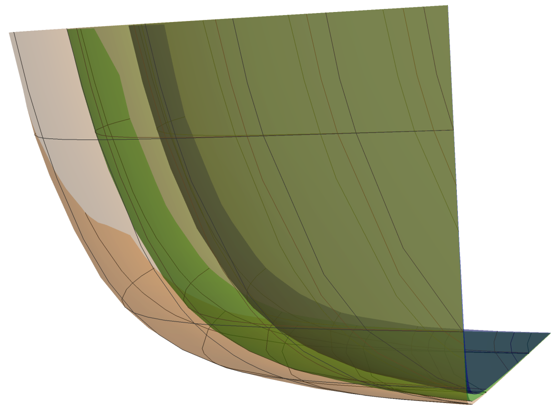

FIGURE 4. The domain Ω_2 and a few leaves of the foliation $\mathcal{H}_s^2$ in an affine patch $d = 1$.

1.1.3. *Type 2 cusps.* Once again, let $x, y, z \in \mathbb{R}$, then the Lie algebra $\mathfrak{t}_2$ consists of elements of the form

$$m_2(x, y, z) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.3}$$

and let T_2 consist of elements of the form

$$M_2(x, y, z) = \exp(m_2(x, y, z)) = \begin{pmatrix} e^x & 0 & 0 & 0 \\ 0 & e^y & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1\lambda_2 > 0$ and let $T(\lambda_1, \lambda_2)$ be the codimension 1 subgroup of T_2 consisting of elements of the form $M_2(\lambda_1 x, \lambda_2 y, -\lambda_1^{-1}x - \lambda_2^{-1}y)$. Each $T(\lambda_1, \lambda_2)$ preserves both the properly convex set

$$\Omega_2 = \left\{[a : b : c : 1] \in \mathbb{S}^3 \mid a, b > 0,\ c > -\lambda_1^{-2}\log(a) - \lambda_2^{-2}\log(b)\right\}$$

and the strictly convex codimension 1 foliation

$$\mathcal{H}_s^2 = \left\{[a : b : c : 1] \in \mathbb{S}^3 \mid a, b > 0,\ c = -\lambda_1^{-2}\log(a) - \lambda_2^{-2}\log(b) + s\right\},\ s > 0$$

A *type 2 generalized cusp* is a properly convex manifold that is projectively equivalent to Ω_2/Γ where Γ is a lattice in $T(\lambda_1, \lambda_2)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1\lambda_2 > 0$. As before, these manifolds are easily seen to be generalized cusps.

Remark 1.1. *If $\lambda_1\lambda_2 < 0$ then it is still possible to define $\mathcal{H}_s^2$, however, in this case the horospheres are not strictly convex and Ω_2 is not properly convex.*

1.2. **Deformation space of convex projective structures.** Let N be the interior of a compact manifold (for instance a finite volume hyperbolic n-manifold) and let $\Gamma = \pi_1 N$. A *(marked) convex projective structure* on N is a pair (f, M) where $M = \Omega/\Gamma$ is a properly convex manifold and $f : N \to M$ is a diffeomorphism called a *marking.* Lifting the marking to the universal cover we get a diffeomorphism $\mathsf{dev} : \tilde{N} \to \Omega$, called a *developing map.* The marking also induces a representation $\rho : \Gamma \to \mathrm{SL}(\Omega) \subset G$ given by $\rho = f_*$ called a *holonomy representation.*

We now define an equivalence relation on marked convex projective structures. Given two marked convex projective structures (f, M) and (f', M') on N with developing maps dev and dev', we say that $(f, M) \sim (f', M')$ if there is a submanifold $N_0 \subset N$ obtained by removing a collar of ∂N and an element $g \in G$ such that the following diagram computes, up to isotopy.

$$\begin{tikzcd} & \mathsf{dev}(N_0) \arrow[dd, "g"] \\ N_0 \arrow[ur, "\mathsf{dev}"] \arrow[dr, "\mathsf{dev}'"'] & \\ & \mathsf{dev}'(N_0) \end{tikzcd}$$

In other words, there is a projective bijection from the complement of a collar of the boundary of M to the complement of a collar of the boundary of M'. If ρ and ρ' are the holonomy representations of (f, M) and (f', M') then $\rho' = g\rho g^{-1}$, and so we see that equivalent marked convex projective structures have conjugate holonomy representations. The *deformation space of convex projective structures on* N, denoted $\mathfrak{B}(N)$, is the set of marked convex projective structures on N, modulo the above equivalence

Let $\mathrm{Rep}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)/G$, where the action of G is by conjugation. For most purposes, it suffices to regard $\mathrm{Rep}(\Gamma, G)$ as given by the naive topological quotient. However, it will sometimes be necessary to endow $\mathrm{Rep}(\Gamma, G)$ with the structure of an affine variety (at least locally). In order to endow $\mathrm{Rep}(\Gamma, G)$ with this type of structure it is necessary to use the Mumford GIT quotient. In general these quotients are not the same, however near representations we will need to consider these two quotients are locally homeomorphic as topological spaces.

By the above discussion, there is a map $\mathsf{hol} : \mathfrak{B}(N) \to \mathrm{Rep}(\Gamma, G)$, called the *holonomy map*, that associates to an equivalence class of convex projective structures the conjugacy class of its holonomy. Using the compact C^∞ topology on developing maps allows us to endow $\mathfrak{B}(N)$ with a topology. The following theorem is a consequence of the Ehresmann-Thurston or holonomy principle (see [11, I.1.7.1] for a statement and proof)

Theorem 1.2. *The map* $\mathsf{hol} : \mathfrak{B}(N) \to \mathrm{Rep}(\Gamma, G)$ *is a continuous local injection.*

We now restrict our attention to the case where N is a finite volume hyperbolic n-manifold, and we let $\rho_{hyp} : \Gamma \to G$ be the holonomy representation of a marked hyperbolic structure on N. If N is closed then work of Koszul [20] shows that hol is a local homeomorphism near $[\rho_{hyp}]$. In other words, if $\rho : \Gamma \to G$ is a representation and $[\rho]$ is sufficiently close to $[\rho_{hyp}]$ in $\mathrm{Rep}(\pi_1 N, G)$ then ρ is also the holonomy of a marked convex projective structure on N. This idea is useful since it reduces the geometric problem of deforming marked convex projective structures on N to the simpler algebraic problem of deforming $[\rho_{hyp}]$ in $\mathrm{Rep}(\Gamma, G)$.

When N is non-compact, Koszul's result breaks down. For example, if N is a cusped hyperbolic 3-manifold then there are representations arbitrarily close to $[\rho_{hyp}]$ in $\mathrm{Rep}(\Gamma, G)$ that correspond to incomplete hyperbolic structures on N. It is easily seen that these are not holonomies of marked convex projective structures on N (for instance they are either indiscrete or non-faithful). However, in recent work of Cooper–Long–Tillmann [15] it is shown that small deformations at the level of representations that preserve certain boundary conditions are guaranteed to be the holonomy of a convex projective structure on N. In order to state their precise result we need to introduce some terminology.

Let $\mathrm{Hom}_{ce}(\Gamma, G)$ be the representations of Γ into G that are holonomies of convex projective projective structures on N such that each end of N is a generalized cusp. A group $\Delta \subset G$ is a *virtual flag group* if it contains a finite index subgroup that is conjugate in G to an upper-triangular group. For instance, the image of the holonomy of a generalized cusp is a virtual flag group. The following is a paraphrasing of part of Theorem 0.2 from [15].

Theorem 1.3. *Suppose* W *is a compact, connected* n*-manifold, let* $N = W \backslash \partial W$*, and let* $\{V_1, \ldots, V_k\}$ *be the set of connected components of* ∂W*. Let* $B_i \cong V_i \times [0, 1)$ *be the end of* N *corresponding to* V_i*. Suppose that* $\rho_0 \in \mathrm{Hom}_{ce}(\Gamma, G)$ *and for* $t \in (-1, 1)$*,* $\rho_t : \Gamma \to G$ *is a continuous path of representation with the property that* $\rho_t(\pi_1 B_i)$ *is a virtual flag group for each* i*. Then there is* $\varepsilon > 0$ *such that for* $t \in (-\varepsilon, \varepsilon)$*,* $\rho_t \in \mathrm{Hom}_{ce}(\Gamma, G)$*.*

Informally, this theorem says that if one performs a small deformation of the holonomy of a properly convex projective structure on N with generalized cusps ends, subject to the constraint that the image of the peripheral subgroups remain virtual flag groups, then the resulting representation is also the holonomy of a properly convex projective structure on N with generalized cusp ends.

2. Infinitesimal deformations and twisted cohomology

Let Γ be a finitely generated group and let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathrm{Hom}(\Gamma, G)$ be the set of homomorphisms from Γ to G. This set is called the *representation variety of* Γ *into* G, or just representation variety if Γ and G are clear from context. If Γ is generated by elements $\gamma_1, \ldots, \gamma_k$ then $\mathrm{Hom}(\Gamma, G)$ can be regarded as a subset of G^k. The relations in Γ give rise to polynomials in the entries of the elements of G and thus $\mathrm{Hom}(\Gamma, G)$ is a algebraic subset of G^k. Let ρ_t be a smooth path of representation in $\mathrm{Hom}(\Gamma, G)$ then for small t we can write

$$\rho_t(\gamma) = \exp(I + u(\gamma)t + O(t^2))\rho_0(\gamma),$$

where $u : \Gamma \to \mathfrak{g}$ is given by $\gamma \mapsto \frac{d}{dt}\big|_{t=0} \rho_t(\gamma)\rho_0(\gamma)^{-1}$. Let $Z^1_{\rho_0}(\Gamma, \mathfrak{g})$ be the set of 1-cocycles with coefficients in $\mathfrak{g}$ twisted by the adjoint of ρ_0. That is the set of functions $v : \Gamma \to \mathfrak{g}$ with the property that

$$v(\gamma_1\gamma_2) = v(\gamma_1) + \gamma_1 \cdot v(\gamma_2),$$

where the action is given by the composition of ρ and the adjoint action (i.e. $\gamma \cdot a = \rho(\gamma)a\rho(\gamma)^{-1}$). Using this formula, it follows that a cocycle is determined by its values of a generating set. The homomorphism condition on ρ_t implies that $u \in Z^1_{\rho_0}(\Gamma, \mathfrak{g})$. For this reason, we will refer to elements of $Z^1_{\rho_0}(\Gamma, G)$ as *infinitesimal deformations of* ρ_0. The space $\mathrm{Hom}(\Gamma, G)$ is an algebraic variety and the above construction gives an identification of $Z^1_{\rho_0}(\Gamma, \mathfrak{g})$ and $T_{\rho_0}\mathrm{Hom}(\Gamma, G)$, where the latter is the *Zariski tangent space* to $\mathrm{Hom}(\Gamma, G)$ at ρ_0 (see [21] for details).

A special class of infinitesimal deformation is given by infinitesimal conjugacies. Let $w \in \mathfrak{g}$ and define $c_t = \exp(tw)$, then $\rho_t = c_t \cdot \rho_0$ is a path of representations, and the resulting infinitesimal deformation is $\gamma \mapsto w - \gamma \cdot w$. The set of deformations of this type consists of 1-coboundaries with coefficients in $\mathfrak{g}$ twisted by the adjoint of ρ_0, which we denote $B^1_{\rho_0}(\Gamma, \mathfrak{g})$. The resulting cohomology group $H^1_{\rho_0}(\Gamma, \mathfrak{g}) = Z^1_{\rho_0}(\Gamma, \mathfrak{g})/B^1_{\rho_0}(\Gamma, \mathfrak{g})$ consists of infinitesimal deformations of ρ_0 modulo the infinitesimal conjugacies.

If ρ_0 is an irreducible representation (for example, if $G = \mathrm{SL}(n+1, \mathbb{R})$, $\Gamma = \pi_1 M$ for a finite volume hyperbolic n-manifold and ρ_0 is the holonomy of a complete hyperbolic structure on M) then near $[\rho_0]$, the topological quotient $\mathrm{Rep}(\Gamma, G)$ agrees with the Mumford GIT quotient $\mathrm{Hom}(\Gamma, G)//G$, endowing $\mathrm{Rep}(\Gamma, G)$ with the structure of an algebraic variety. In this setting $B^1_{\rho_0}(\Gamma, \mathfrak{g})$ can be identified with the Zariski tangent space to the G-orbit in $\mathrm{Hom}(\Gamma, G)$ of ρ_0 and the Zariski tangent space of $\mathrm{Rep}(\Gamma, G)$ at $[\rho_0]$ can be identified with $H^1_{\rho_0}(\Gamma, \mathfrak{g})$.

2.1. **Cohomology of 3-manfiolds.** In this section we discuss the cohomology (with twisted coefficients) of hyperbolic 3-manifolds. Throughout this section let M be a finite volume hyperbolic 3-manifold (typically non-compact), let $\Gamma = \pi_1 M$, $G = \mathrm{SL}_\pm(4, \mathbb{R})$, and let $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$ be its Lie algebra. Let

$$J = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

and let $SO(3,1) = \{A \in \mathrm{SL}(4, \mathbb{R}) \mid A^tJA = J\}$. In this setting, there is a representation $\rho_{hyp} : \Gamma \to \mathrm{SO}(3,1) \subset G$ given by the holonomy of the complete hyperbolic structure on M. By Mostow rigidity, this representation is unique up to conjugacy in G.

There is also a useful splitting of $\mathfrak{g}$ (as a $SO(3,1)$-module). The group $SO(3,1)$ acts on $\mathfrak{g}$ via the adjoint action (i.e. if $g \in \mathrm{SO}(3,1)$ and $a \in \mathfrak{g}$ then $g \cdot a = gag^{-1}$). The map $a \mapsto -Ja^tJ$ is an $SO(3,1)$-module isomorphism. This map is an involution and whose 1-eigenspace is

$$\mathfrak{so}(3,1) = \{a \in \mathfrak{g} \mid a^tJ + Ja = 0\}$$

and the we denote the -1-eigenspace by $\mathfrak{v}$. This gives a splitting

$$\mathfrak{g} = \mathfrak{so}(3,1) \oplus \mathfrak{v}. \tag{2.2}$$

Observe that this is only a splitting of $\mathrm{SO}(3,1)$-modules and not of Lie algebras since $\mathfrak{v}$ is not closed under Lie brackets. The above construction can be repeated using other symmetric matrices, J' of signature $(3,1)$. Using J' will result in a new splitting of $\mathfrak{sl}(4,\mathbb{R})$ that differs from the original splitting by a conjugacy in G. For instance, when executing some of the computation in Section 6 it is convenient to use a slightly different form.

The splitting (2.2) induces a splitting at the level of cohomology:

$$H^1_{\rho_{hyp}}(\Gamma,\mathfrak{g}) \cong H^1_{\rho_{hyp}}(\Gamma,\mathfrak{so}(3,1)) \oplus H^1_{\rho_{hyp}}(\Gamma,\mathfrak{v}) \tag{2.3}$$

and maps $\pi_{\mathfrak{so}(3,1)} : H^1_{\rho_{hyp}}(\Gamma,\mathfrak{g}) \to H^1_{\rho_{hyp}}(\Gamma,\mathfrak{so}(3,1))$ and $\pi_{\mathfrak{v}} : H^1_{\rho_{hyp}}(\Gamma,\mathfrak{g}) \to H^1_{\rho_{hyp}}(\Gamma,\mathfrak{v})$.

A useful way to understand to understand the cohomology groups of our 3-manifold is by restricting to the boundary. Suppose that M has k cusps and let $\partial := \sqcup_{i=1}^k \partial_i$, where ∂_i is the ith cusp of M. If $\Delta_i = \pi_1\partial_i$, then for each i, there is a restriction map, $\mathsf{res}_i : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Delta_i, G)$ given by regarding Δ_i as a subgroup of Γ and restricting representations. By abuse of notation we will denote $\mathsf{res}_i\rho_{hyp}$ by ρ_{hyp}. Each of the above maps descends to $(\mathsf{res}_i)_* : H^1_{\rho_{hyp}}(\Gamma,\mathfrak{g}) \to H^1_{\rho_{hyp}}(\Delta_i,\mathfrak{g})$. Define $Z^1_{\rho_{hyp}}(\Delta,\mathfrak{g}) := \bigoplus_{i=1}^k Z^1_{\rho_{hyp}}(\Delta_i,\mathfrak{g})$ and define $B^1_{\rho_{hyp}}(\Delta,\mathfrak{g})$ and $H^1_{\rho_{hyp}}(\Delta,\mathfrak{g})$ in a similar fashion. Taking the direct sum of the above maps gives

$$\oplus_{i=1}^k (\mathsf{res}_i)_* =: \mathsf{res}_* : Z^1_{\rho_{hyp}}(\Gamma,\mathfrak{g}) \to Z^1_{\rho_{hyp}}(\Delta,\mathfrak{g}).$$

This map sends $B^1_{\rho_{hyp}}(\Gamma,\mathfrak{g})$ into (and in fact onto) $B^1_{\rho_{hyp}}(\Delta,\mathfrak{g})$, and thus descends to a map which, by abuse, we denote $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma,\mathfrak{g}) \to H^1_{\rho_{hyp}}(\Delta,\mathfrak{g})$. The map res_* respects the splitting (2.2) and we get corresponding maps which by further abuse of notation we call $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma,\mathfrak{so}(3,1)) \to H^1_{\rho_{hyp}}(\Delta,\mathfrak{so}(3,1))$ and $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma,\mathfrak{v}) \to H^1_{\rho_{hyp}}(\Delta,\mathfrak{v})$

We begin by discussing cohomology with coefficients in $\mathfrak{so}(3,1)$. These cohomology groups are classically studied and well understood. The following Lemma summarizes some well known properties of $H^1_{\rho_{hyp}}(\Gamma,\mathfrak{so}(3,1))$ that will be important for our purposes.

Lemma 2.1. *Suppose that M has k cusps, then*

- $H^1_{\rho_{hyp}}(\Gamma,\mathfrak{so}(3,1))$ *is $2k$-dimensional*
- $H^1_{\rho_{hyp}}(\Delta,\mathfrak{so}(3,1))$ *is $4k$-dimensional*
- $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma,\mathfrak{so}(3,1)) \to H^1_{\rho_{hyp}}(\Delta,\mathfrak{so}(3,1))$ *is injective*

The first point is follows from Thurston's theory of hyperbolic Dehn surgery [25] and the latter point is often referred to as Calabi–Weil rigidity [10, 26, 27]. As a result we see that the image of $H^1_{\rho_{hyp}}(\Gamma,\mathfrak{so}(3,1))$ is a half-dimensional subspace. This is not coincidental, as there turns out to be a symplectic form on $H^1_{\rho_{hyp}}(\Delta,\mathfrak{so}(3,1))$ induced by the cup produce, for which the image of $H^1_{\rho_{hyp}}(\Gamma,\mathfrak{so}(3,1))$ is a Lagrangian subspace [17, §5].

Cohomology with coefficients in $\mathfrak{v}$ is less well understood, but in this setting we have the following weaker analogue of Lemma 2.1, which can be found in [17]

Lemma 2.2 (Cor. 5.2 & Lem. 5.3 of [17])**.** *Suppose that M has k cusps, then*

- $H^1_{\rho_{hyp}}(\Delta,\mathfrak{v})$ *is $2k$-dimensional*
- $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma,\mathfrak{v})) \subset H^1_{\rho_{hyp}}(\Delta,\mathfrak{v})$ *is k-dimensional.*

We now have the requisite definitions to state our cohomological condition. A manifold M is *infinitesimally rigid rel. ∂M* if the map $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma,\mathfrak{g}) \to H^1_{\rho_{hyp}}(\Delta,\mathfrak{g})$ is injective. To avoid cumbersome phrasing, we will often abbreviate this terminology and say that M is infinitesimally rigid. In other words, there are no infinitesimal deformations of M that are infinitesimal conjugacies when restricted to each cusp. This condition was first introduced in [17]. Some comments regarding this condition are in order. First, by Lemma 2.1, infinitesimal rigidity of M is equivalent to the injectivity of $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma,\mathfrak{v}) \to H^1_{\rho_{hyp}}(\Delta,\mathfrak{v})$. Second, by Lemma 2.2, the dimension of $H^1_{\rho_{hyp}}(\Gamma, v)$ is at least k and so M is infinitesimally rigid if the dimension of $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is *exactly* k.

There are infinitely many infinitesimally rigid cusped hyperbolic 3-manifolds. Specifically, Heusener and Porti [17] show that infinitely many surgeries on the Whitehead link result in manifolds that are infinitesimally rigid. Examples of such families include infinitely many twist knots and infinitely many punctured torus bundles with tunnel number one. Furthermore, based on numerical computation by J. Danciger, G.-S. Lee, and the author it appears that infinitesimal rigidity is a fairly common property amongst 3-manifolds in the SnapPy [16] cusped census.

On the other hand, there are infinitely many cusped 3-manifolds that are not infinitesmially rigid. For example, if M contains a closed, embedded, totally-geodesic hypersurface, then it is possible to perform a type of deformation called bending (see [18] or [4] for details). These deformations are trivial when restricted to any cusp, and so if M contains such a hypersurface then M is not infinitesimally rigid rel. ∂M.

We close this section by describing an important consequence of infinitesimal rigidity. As we have seen, the set $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{g})$ can be interpreted as non-trivial infinitesimal deformations of ρ_{hyp}. Given a cohomology class, $[w] \in H^1_{\rho_{hyp}}(\Gamma, \mathfrak{g})$ one would like to know if there is a family $\rho_t : \Gamma \to G$ of representations that is tangent to w. In the language of algebraic geometry, w is a tangent vector in the Zariski tangent space of the algebraic variety $\mathrm{Hom}(\Gamma, G)$ and this question is equivalent to the question of whether or not ρ_{hyp} is a smooth point. There are numerous examples where ρ_{hyp} fails to be a smooth point (see [12] for explicit examples). There is also a related result of Kapovich–Millson [19] that, roughly speaking, says for 3-manifolds and representations into $\mathrm{SL}(2, \mathbb{C})$ that arbitrary singularities are possible. However the following result from [6] shows that for infinitesimally rigid 3-manifolds, ρ_{hyp} is a smooth point of $\mathrm{Hom}(\Gamma, G)$ and $\mathrm{Rep}(\Gamma, G)$.

Theorem 2.3 (see Thm 3.2 in [6])**.** *Suppose M is a cusped finite volume hyperbolic 3-manifold with $\rho_{hyp} : \Gamma \to SO(3,1)$ the holonomy of the complete hyperbolic structure on M. If M is infinitesimally rigid rel. ∂M then ρ_{hyp} is a smooth point of* $\mathrm{Hom}(\Gamma, G)$ *and* $[\rho_{hyp}]$ *is a smooth point of* $\mathrm{Rep}(\Gamma, G)$.

3. The Slice

Let $G = \mathrm{SL}_{\pm}(4, \mathbb{R})$ and let G_a the of affine transformations of $\mathbb{R}^3$, both of which can be thought of as a subgroups of $\mathrm{GL}(4, \mathbb{R})$ and let $\mathfrak{g}$ and $\mathfrak{g}_a$ be the corresponding Lie algebras. There is a natural injective map from $\varpi : G_a \to G$ given by $M \mapsto |\det(M)|^{-1/4}$. The corresponding map $\varpi : \mathfrak{g}_a \to \mathfrak{g}$ at the level of Lie algebras is given by $v \mapsto v - \frac{\mathrm{tr}(v)}{4}\mathrm{I}$. If Γ is a finitely generated group then the above injection induces an injection from $\mathrm{Hom}(\Gamma, G_a)$ into $\mathrm{Hom}(\Gamma, G)$.

Let

$$S = \{(a, b, x_1, y_1, x_2, y_2) \in \mathbb{R}^6 \mid y_1 x_2 - x_1 y_2 = \pm 1\}$$

It is a simple exercise in differential topology to see that S is a smooth 5-dimensional manifold.

Next, let $C = \{(a, b, x_1, y_1, x_2, y_2) \in S \mid a = b = 0\}$. C is a smooth 3-dimensional submanifold of S and if $s \in S_c$, there is a function $\mathsf{CS} : S_c \to \mathbb{C}$ called the *cusp shape function*, given by $(0, 0, x_1, y_1, x_2, y_2) \mapsto \frac{x_1 + iy_1}{x_2 + iy_2}$.

Here is some information about the calculus of CS.

Lemma 3.1. CS *is a surjective submersion of C onto $\mathbb{C}\backslash\mathbb{R}$. Consequently, $\{CS^{-1}(z) \mid z \in \mathbb{C}\backslash\mathbb{R}\}$ gives a foliation of C by smooth 1-manifolds.*

Proof. First, Let $f > 0$, then $\mathsf{CS}(e/\sqrt{f}, \pm\sqrt{f}, 1/\sqrt{f}, 0) = e \pm if$, and so $\mathbb{C}\backslash\mathbb{R}$ is contained in the image of CS. Furthermore, since $y_1 x_2 - x_1 y_2 = \pm 1$ it follows that $x_1 + iy_1$ and $x_2 + iy_2$ are linearly independent over $\mathbb{R}$ and hence $\mathsf{CS}(x_1, y_1, x_2, y_2) = \frac{x_1 + iy_1}{x_2 + iy_2} \in \mathbb{C}\backslash\mathbb{R}$.

Next, identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way and identify S_c with a subset of $\mathbb{R}^4$. If $h : \mathbb{R}^4 \to \mathbb{R}$ is given by $(x_1, y_1, x_2, y_2) \mapsto y_1 x_2 - x_1 y_2$ and $v \in S_c$ then $T_v S_c = \ker \nabla h(v)$.

Let $f, g : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $f(x_1, y_1, x_2, y_2) = x_2^2 + y_2^2$ and $g(x_1, y_1, x_2, y_2) = x_1 x_2 + y_1 y_2$. Then we can write $\mathsf{CS} = F_1 \circ F_2$ where $F_2 : S_c \to \mathbb{R}^2$ is given by $v \mapsto (f(v), g(v))$ and $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$(f,g) \mapsto \left(\frac{g}{f}, \frac{1}{f}\right)$. Let $v \in S_c$ and $w \in T_v S_c$ be a tangent vector, then using the chain rule we find that

$$\mathsf{CS}_*(w) = \left(\frac{(f(v)\nabla g(v) - g(v)\nabla f(v))(w)}{f(v)^2}, \frac{-\nabla f(v)(w)}{f(v)^2}\right),$$

thus the kernel of CS_* is equal to $\ker \nabla f(v) \cap \ker \nabla g(v) \cap \ker \nabla h(v)$. It is easy to check that $\nabla f(v)$, $\nabla g(v)$, and $\nabla h(v)$ are linearly independent for all $v \in S_c$ and so the kernel of CS_* is 1-dimensional. It follows that CS is a submersion at v and hence a submersion since v was arbitrary. □

We now use S to parameterize families of representations of $\mathbb{Z}^2$ into G_a and G. Fix once and for all a generating set $\{\gamma_1, \gamma_2\}$ for $\mathbb{Z}^2$. For each $s = (a, b, x_1, y_1, x_2, y_2) \in S$ we can define a representation $\rho_s : \mathbb{Z}^2 \to G_a$ via

$$\rho_s(\gamma_1) = \exp\begin{pmatrix} 0 & x_1 & y_1 & 0 \\ 0 & ax_1 & 0 & x_1 \\ 0 & 0 & by_1 & y_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \rho_s(\gamma_1) = \exp\begin{pmatrix} 0 & x_2 & y_2 & 0 \\ 0 & ax_2 & 0 & x_2 \\ 0 & 0 & by_2 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.1}$$

By examining the entries of (3.1) it is easy to see that $F : S \to \mathrm{Hom}(\mathbb{Z}^1, G_a)$ given by $s \mapsto \rho_s$ is an injective immersion of S into $\mathrm{Hom}(\mathbb{Z}^2, G)$ and whose image, which we denote $\mathcal{S}_a$, is an embedded submanifold. Let $\mathcal{C}_a$ be the submanifold of $\mathcal{S}_a$ corresponding to C.

Remark 3.2. *The formula defining ρ_s in* (3.1) *is well defined for all points in $\mathbb{R}^6$*

There is another map $\tilde{F} : S \to \mathrm{Hom}(\mathbb{Z}^2, G)$ given by $s \mapsto \tilde{\rho}_s$, where $\tilde{\rho}_s = \varpi \circ \rho_s$ and we denote the images of S and C under $\tilde{F}$ by $\mathcal{S}$ and $\mathcal{C}$, respectively. It is easy to see that $\mathcal{S}_a$ and $\mathcal{S}$ (resp. $\mathcal{C}_a$ and $\mathcal{C}$) are diffeomorphic via $\rho_s \mapsto \tilde{\rho}_s$. If $\rho_s \in \mathcal{S}$ (or $\mathcal{S}_a$) then we call $s \in S$ the *coordinates of* ρ_s. The reason for using $\mathcal{S}$ is that the transversality argument in Section 4 takes place in $\mathrm{SL}(4, \mathbb{R})$ and not $\mathrm{GL}(4, \mathbb{R})$.

In terms of hyperbolic geometry, CS gives the cusp shape of the representation ρ_s (with respect to the generating set $\{\gamma_1, \gamma_2\}$. It is well known that if $s, s' \in C$ then ρ_s (resp. $\tilde{\rho}_s$) is conjugate to $\rho_{s'}$ (resp. $\tilde{\rho}'_s$) in G_a (resp. G) if and only if $\mathsf{CS}(s) = \mathsf{CS}(s')$. As a result, S **does not** give a parameterization of the image of $\mathcal{S}$ in $\mathrm{Hom}(\mathbb{Z}^2, G)/G$ since there is redundancy coming from representations with the same cusp shape. However, as we will see shortly, no other redundancy arises when projecting $\mathcal{S}$ to $\mathrm{Hom}(\mathbb{Z}^2, G)/G$ near $\mathcal{C}$.

There is another way of viewing the above construction that is also useful: let

$$x_{a,b} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ y_{a,b} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can view $x_{a,b}$ and $y_{a,b}$ as infinitesimal generators of an abelian Lie group $A_{a,b}$ isomorphic to $\mathbb{R}^2$. If $s = (a, b, x_1, y_1, x_2, y_2)$ then the representation ρ_s has image in $A_{a,b}$. Furthermore, if we let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$, then the defining condition for S is a determinant condition on these vectors and thus ensures that v_1 and v_2 are linearly independent vectors in $\mathbb{R}^2$, and so we see that the image of ρ_s is always a lattice in $A_{a,b}$. The group $A_{0,0}$ is equal to $T(0)$, and so we immediately see that many representations in $\mathcal{C}$ are holonomies of type 0 generalized cusps. The following Theorem shows that the remaining representations in $\mathcal{S}$ are also holonomies of generalized cusps.

Theorem 3.3. *Let $\rho \in \mathcal{S}$, then ρ is the holonomy of a generalized cusp of type 0, type 1, or type 2.*

Proof. If $\rho \in \mathcal{C}$ then the image of ρ is a lattice in $T(0$ and so ρ is the holonomy of a type 0 generalized cusp. On the other hand, suppose that $s = (a, b, x_1, y_1, x_2, y_2) \in S$ is such that $(a, b) \neq (0, 0)$. There are two cases: either a or b (but not both) is zero or both a and b are non-zero. We begin with the first case. By performing a conjugacy that permutes the second and third coordinates if necessary, we can assume without loss of generality that $b = 0$. Let

$$N_{a,b}(x,y) = \exp\begin{pmatrix} 0 & x & y & 0 \\ 0 & ax & 0 & x \\ 0 & 0 & by & y \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We will think of $N_{a,b}(x,y)$ as a generic element of the Lie group $A_{a,b}$. Next, let $P_{(12)}$ be the 4×4 matrix that permutes the first two coordinates, let

$$C_a = \begin{pmatrix} 1 & -1/a & 0 & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and and let $\tilde{C}_a = P_{(12)} C_a$. Observe that

$$\tilde{C}_a N_{a,0}(x,y) \tilde{C}_a^{-1} = M_1(ax, y, -a^{-1}x).$$

As a result we see that $A_{a,0}$ is conjugate to $T(a)$ and that the image of $\tilde{C}_a \cdot \rho$ is a lattice in $T(a)$. It follows that ρ is the holonomy of a type 1 generalized cusp.

In the case where both a and b are non-zero we let

$$D_{a,b} = \begin{pmatrix} 1 & -1/a & -1/b & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

let $P_{(123)}$ be the 4×4 matrix that cyclically permutes the first 3 coordinates, and let $\tilde{D}_{a,b} = P_{(123)} D_{a,b}$. In this case we find that

$$\tilde{D}_{a,b} N_{a,b}(x,y) \tilde{D}_{a,b}^{-1} = M_2(ax, by, -a^{-1}x - b^{-1}y),$$

and thus $A_{a,b}$ is conjugate to $T(a,b)$. Arguing as before this implies that $\tilde{D}_{a,b} \cdot \rho$ is the holonomy of a type 2 generalized cusp.

□

We now describe the tangent spaces for $\mathcal{S}$. Since $\mathcal{S}$ is a subvariety of $\mathrm{Hom}(\mathbb{Z}^2, G)$, its tangent space is naturally a subspace of the Zariski tangent space of $\mathrm{Hom}(\mathbb{Z}^2, G)$. For simplicity of notation we will denote $T_{\rho_s}\mathcal{S}$ by $T_s\mathcal{S}$. The tangent bundle TS is pointwise spanned by 5 vector fields each of which can be written as a linear combination of the vector fields $\frac{\partial}{\partial a}$, $\frac{\partial}{\partial b}$, $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial y_1}$, $\frac{\partial}{\partial x_2}$, and $\frac{\partial}{\partial y_2}$. Using $\tilde{F}$ we can push these vector fields on $T\mathcal{S}$, which by abuse of notation we give the same names. Again, these vector fields pointwise span $T\mathcal{S}$.

Recall from Section 2 that $T_\rho \mathrm{Hom}(\mathbb{Z}^2, G)$ can be identified with the space $Z^1_\rho(\mathbb{Z}^2, \mathfrak{g})$ of 1-cocycles with coefficients in $\mathfrak{g}$ twisted by $\mathsf{Ad}(\rho)$. As a result of Remark 3.2, it is possible to think of the elements of $\left\{\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\right\}$ as 1-cocycles in $Z^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{g})$. We now describe this process explicitly when $s \in C$.

Before proceeding, we need the following Lemma

Lemma 3.4. *Let $s \in \mathcal{C}$ and let $u_1, u_2 \in \mathbb{R}$. Then there is a cocycle $z \in Z^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{v})$ with the property that*

$$z(\gamma_i) = \begin{pmatrix} 0 & 0 & 0 & u_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, this cocycle is a coboundary.

Proof. It is easy to check that for any $v_1, v_2 \in \mathbb{R}$ that

$$v = \begin{pmatrix} 0 & v_1 & v_2 & 0 \\ 0 & 0 & 0 & -v_1 \\ 0 & 0 & 0 & -v_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is an element of $\mathfrak{v}$. Since $s \in \mathcal{C}$ we can write $s = (0, 0, x_1, y_1, x_2, y_2)$, and so there is a coboundary in $B^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{v})$ that maps γ_i to

$$v - \rho_s(\gamma_i) \cdot v = \begin{pmatrix} 0 & 0 & 0 & 2(v_1 x_i + v_2 y_i) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $s \in \mathcal{S}$ it is possible to find v_1 and v_2 that satisfy the equations

$$\begin{aligned} 2v_1x_1 + 2v_2y_1 &= u_1 \\ 2v_1x_2 + 2v_2y_2 &= u_2 \end{aligned}$$

Thus there is a cocycle with the required properties and this cocycle is a coboundary. $\square$

We begin with $\frac{\partial}{\partial x_1}$. Writing ρ_s power series formula (with respect to x_1) for the exponential of a matrix we find that

$$\rho_s(\gamma_1) = \begin{pmatrix} 1 - \frac{ax_1}{4} & x_1 & 0 & 0 \\ 0 & 1 + \frac{3ax_1}{4} & 0 & x_1 \\ 0 & 0 & 1 - \frac{ax_1}{4} & 0 \\ 0 & 0 & 0 & 1 - \frac{ax_1}{4} \end{pmatrix} + O(x_1^2)$$
$$\rho_s(\gamma_2) = O'(x_1^2)$$

where $O(x_1^2)$ and $O'(x_1^2)$ are matrices whose derivatives with respect to x_1 at $s = (0, 0, x_1, y_1, x_2, y_2)$ is zero. Thus the infinitesimal deformation $\frac{\partial}{\partial x_1}$ is given by $\frac{\partial}{\partial x_1}(\gamma_1) = \xi_1$ and $\frac{\partial}{\partial x_1}(\gamma_2) = 0$, where

$$\xi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that the image $\left[\frac{\partial}{\partial x_1}\right]$ of $\frac{\partial}{\partial x_1}$ in $H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{g})$ is contained in $H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{so}(3,1))$. A similar computation shows that $\left[\frac{\partial}{\partial y_1}\right]$, $\left[\frac{\partial}{\partial x_2}\right]$, and $\left[\frac{\partial}{\partial y_2}\right]$ are also contained in $H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{so}(3,1))$.

Writing ρ_s power series formula (with respect to a) for the exponential of a matrix we find that

$$\rho_s(\gamma_i) = \begin{pmatrix} 1 - \frac{ax_i}{4} & \frac{x_i^2}{4} & \frac{-x_iy_i}{4} & \frac{x_i(x_i^2 - 3y_i^2)}{24} \\ 0 & 1 + \frac{3ax_i}{4} & 0 & 0 \\ 0 & 0 & 1 - \frac{ax_i}{4} & 0 \\ 0 & 0 & 0 & 1 - \frac{ax_i}{4} \end{pmatrix} + O(a^2),$$

where $O(a^2)$ is a matrix whose derivative with respect to a at $s = (0, 0, x_1, y_1, x_2, y_2)$ is 0.

It follows that the infinitesimal deformation $\frac{\partial}{\partial a}$ is given by $\frac{\partial}{\partial a}(\gamma_1) = \alpha_1$ and $\frac{\partial}{\partial a}(\gamma_2) = \alpha_2$, where

$$\alpha_i = \begin{pmatrix} 0 & \frac{x_i^2}{4} & -\frac{x_iy_i}{4} & 0 \\ 0 & 0 & 0 & \frac{x_i^2}{4} \\ 0 & 0 & 0 & -\frac{x_iy_i}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{x_i}{4} & 0 & 0 & 0 \\ 0 & \frac{3x_i}{4} & 0 & 0 \\ 0 & 0 & -\frac{x_i}{4} & 0 \\ 0 & 0 & 0 & -\frac{x_i}{4} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \frac{x_i(x_i^2-3y_i^2)}{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2}$$

Analogously, we see that $\frac{\partial}{\partial b}$ is given by $\frac{\partial}{\partial b}(\gamma_1) = \beta_1$ and $\frac{\partial}{\partial b}(\gamma_2) = \beta_2$, where

$$\beta_i = \begin{pmatrix} 0 & -\frac{x_i y_i}{4} & \frac{y_i^2}{4} & 0 \\ 0 & 0 & 0 & -\frac{x_i y_i}{4} \\ 0 & 0 & 0 & \frac{y_i^2}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{y_i}{4} & 0 & 0 & 0 \\ 0 & -\frac{y_i}{4} & 0 & 0 \\ 0 & 0 & \frac{3y_i}{4} & 0 \\ 0 & 0 & 0 & -\frac{y_i}{4} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \frac{y_i\left(y_i^2-3x_i^2\right)}{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.3}$$

Again, one can observe that the first term is contained in $\mathfrak{so}(3,1)$ and the second and third terms are contained in $\mathfrak{v}$. Furthermore, by Lemma 3.4, the last term again gives a coboundary. Next, let $[D_a] = \pi_{\mathfrak{v}}\left(\left[\frac{\partial}{\partial a}\right]\right)$ and let $[D_b] = \pi_{\mathfrak{v}}\left(\left[\frac{\partial}{\partial b}\right]\right)$. More precisely, using the middle terms of (3.2) and (3.3) define:

$$\alpha_i' = \begin{pmatrix} -\frac{x_i}{4} & 0 & 0 & 0 \\ 0 & \frac{3x_i}{4} & 0 & 0 \\ 0 & 0 & -\frac{x_i}{4} & 0 \\ 0 & 0 & 0 & -\frac{x_i}{4} \end{pmatrix}, \quad \beta_i' = \begin{pmatrix} -\frac{x_i}{4} & 0 & 0 & 0 \\ 0 & -\frac{x_i}{4} & 0 & 0 \\ 0 & 0 & \frac{3x_i}{4} & 0 \\ 0 & 0 & 0 & -\frac{x_i}{4} \end{pmatrix} \tag{3.4}$$

then $D_a(\gamma_i) = \alpha_i'$ and $D_b(\gamma_i) = \beta_i'$.

Lemma 3.5. *Let $s \in \mathcal{C}$, then $\{[D_a], [D_b]\}$ is a basis for $H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{v})$.*

Proof. Let $s = (0, 0, x_1, y_1, x_2, y_2)$ and suppose that there are c_a, c_b (not both zero) and $u \in \mathfrak{v}$ so that for any $\gamma \in \mathbb{Z}^2$,

$$c_a D_a(\gamma) + c_b D_b(\gamma) = u - \rho_s(\gamma) \cdot u \tag{3.5}$$

A generic element of $u \in \mathfrak{v}$ is of the form

$$u = \begin{pmatrix} -\frac{u_5+u_8}{2} & u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 & -u_1 \\ u_7 & u_6 & u_8 & -u_2 \\ u_9 & -u_4 & -u_7 & -\frac{u_5+u_8}{2} \end{pmatrix}$$

and so $u - \rho_s(\gamma_i) \cdot u$ is

$$\begin{pmatrix} -u_5x_i - u_8y_i - 1/2u_9(x_i^2+y_i^2) & * & * & * \\ -u_9x_i & x_i(2u_5+u_9x_i) & u_8x_i + (u_5+u_9x_i)y_i & * \\ -u_9x_i & u_8x_i + (u_5+u_9x_i)y_i & y_i(2u_8+u_9y_i) & * \\ * & * & * & -u_5x_i - u_8y_i - 1/2u_9(x_i^2+y_i^2) \end{pmatrix}$$

The image of both D_a and D_b consist entirely of diagonal elements of $\mathfrak{v}$. It it follows that the only way that (3.5) can be satisfied is if $u_5 = u_8 = u_9 = 0$. It follows that $c_a = c_b = 0$ $\square$

Using the above description allows us to prove some useful intersection properties of $T_s\mathcal{S}$ when $s \in \mathcal{C}$.

Proposition 3.6. *Let $s \in \mathcal{C}$ and let V and W be the images of $T_s\mathcal{S}$ and $T_s\mathcal{C}$ in $H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{g})$, respectively. Then*

$$H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{so}(3,1)) \cap V = W$$

Proof. The tangent space $T_s\mathcal{C}$ is a subspace of $\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right\rangle$. From the previous paragraph, we know that $\left[\frac{\partial}{\partial x_1}\right], \left[\frac{\partial}{\partial y_1}\right], \left[\frac{\partial}{\partial x_2}\right], \left[\frac{\partial}{\partial y_2}\right] \in H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{so}(3,1))$. It follows that

$$W \subset H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{so}(3,1)).$$

Next, let $v \in V$ and write

$$v = \dot{a}\left[\frac{\partial}{\partial a}\right] + \dot{b}\left[\frac{\partial}{\partial b}\right] + w,$$

where $w \in W$, or alternatively as

$$\dot{a}\,[D_a] + \dot{b}\,[D_b] + \tilde{w},$$

where $\tilde{w} \in H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{so}(3,1))$. Since $\mathfrak{g} = \mathfrak{so}(3,1) \oplus \mathfrak{v}$ (see (2.3)) it follows from Lemma 3.5 that $w \in H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{so}(3,1))$ if an only if $\dot{a} = \dot{b} = 0$, or in other words if $v \in W$.

□

The following proposition shows that at the level of tangent spaces the only redundancy in $\mathcal{S}$ up to conjugacy comes from representations having the same cusp shape.

Proposition 3.7. *Let $s \in \mathcal{C}$, let $z = \mathsf{CS}(s)$, and let $\mathcal{C}_z = \mathsf{CS}^{-1}(z)$, then*

$$B^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{g}) \cap T_s\mathcal{S} = T_s\mathcal{C}_z.$$

Proof. Let $w \in B^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{g}) \cap T_s\mathcal{S}$ and write

$$w = \dot{a}\frac{\partial}{\partial a} + \dot{b}\frac{\partial}{\partial b} + \dot{x_1}\frac{\partial}{\partial x_1} + \dot{y_1}\frac{\partial}{\partial y_1} + \dot{x_2}\frac{\partial}{\partial x_2} + \dot{y_2}\frac{\partial}{\partial y_2}$$

Suppose for the sake of contradiction that $\dot{a} \neq 0$. Looking at the (2,2) entries of $\tilde{\rho}_s(\gamma_1)$ and $\tilde{\rho}_s(\gamma_2)$ we see that $e^{3ax_1/4}$ and $e^{3ax_2/4}$ are eigenvalues of the respective matrices. Since w is tangent to a conjugacy path we see that $e^{3ax_1/4}$ and $e^{3ax_2/4}$ must remain constant up to first order. Since $s \in \mathcal{C}$ this implies that $\dot{a}x_1 = \dot{a}x_2 = 0$. Since $\dot{a} \neq 0$ this implies that $x_1 = x_2 = 0$. However this contradicts the fact that $s \in \mathcal{C} \subset \mathcal{S}$, and so $\dot{a} = 0$. A similar argument shows that $\dot{b} = 0$.

Since $\dot{a} = \dot{b} = 0$ it follows that $w \in T_s\mathcal{C}$. As previously mentioned, CS is a conjugacy invariant and it follows that CS is constant in the direction of w, and so $w \in T_s\mathcal{C}_z$.

On the other hand, suppose that $w \in T_s\mathcal{C}_z$. Clearly, $w \in T_s\mathcal{S}$, and so we must show that w is tangent to a path of conjugations. By conjugating $\tilde{\rho}_s$ by a rotation, we can assume without loss of generality that $s = (0, 0, x_1, 1/x_2, x_2, 0)$. From the proof of Lemma 3.1 we see that $w \in \ker \nabla f(s) \cap \ker \nabla g(s) \cap \ker \nabla h(s)$, and computing the relevant derivatives gives $w = c(0, 0, -1/x_2, x_1, 0, x_2)$, for some $c \in \mathbb{R}$. Next, let

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Conjugating by $R_{c\theta}$ and taking the derivative with respect to θ at 0 gives

$$\left.\frac{d}{d\theta}\right|_{\theta=0} R_{c\theta}\rho_s(\gamma_1)R_{c\theta}^{-1} = \begin{pmatrix} 0 & -c/x_2 & cx_1 & 0 \\ 0 & 0 & 0 & -c/x_2 \\ 0 & 0 & 0 & cx_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \left.\frac{d}{d\theta}\right|_{\theta=0} R_{c\theta}\rho_s(\gamma_2)R_{c\theta}^{-1} = \begin{pmatrix} 0 & 0 & cx_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & cx_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As a result we see that the tangent vector to this conjugacy path is

$$(\dot{a}, \dot{b}, \dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2) = (0, 0, -c/x_2, cx_1, 0, cx_2) = w.$$

Thus we see that w is the tangent vector to a path of conjugations and so $w \in B^1(\mathbb{Z}^2, \mathfrak{g}) \cap T_s\mathcal{S}$ □

Proposition 3.7 has the following immediate corollary.

Corollary 3.8. *If $s \in \mathcal{C}$ the image of $T_s\mathcal{S}$ in $H^1_{\rho_s}(\mathbb{Z}^2, \mathfrak{g})$ is 4-dimensional.*

4. The transversality argument

Recall that M is a finite volume hyperbolic 3-manifold with $k \geq 1$ cusps, Γ is its fundamental group, $\{\Delta_1, \ldots, \Delta_k\}$ is a collection of peripheral subgroups, one for each cusp, and $\rho_{hyp} : \Gamma \to \mathrm{SO}(3,1) \subset G$ is the holonomy of its complete hyperbolic structure.

The main goal of this section is to prove Theorem 0.1. The proof of Theorem 0.1 has two parts. First, we use a transversality argument involving the slice from Section 3 to produce a k-dimensional family of deformations of the holonomy ρ_{hyp} in $\mathrm{Hom}(\Gamma, G)$ whose image in $\mathrm{Rep}(\Gamma, G)$ is also k dimensional. Specifically, we prove:

Theorem 4.1. *Let M be a finite volume, non-compact hyperbolic 3-manifold with $k \geq 1$ cusps. Suppose that M is infinitesimally rigid rel. ∂M. There is a k-dimensional subspace V, a neighborhood, U of 0 in V, and a smooth family of representations $\mathcal{F} = \{\rho_u \mid u \in U\}$ in $\mathrm{Hom}(\Gamma, G)$ such that*

- $\rho_0 = \rho_{hyp}$
- *For each $u \in U$, $\rho_u|_{\Delta_i}$ is the holonomy of a type 0, type 1, or type 2 generalized cusp.*

Furthermore, if $[\mathcal{F}]$ is the image of $\mathcal{F}$ in $\mathrm{Rep}(\Gamma, G)$ then the Zariski tangent space to $[\mathcal{F}]$ at $[\rho_{hyp}]$ is $V \subset H^1_{\rho_{hyp}}(\Gamma, \mathfrak{g})$.

Next, we apply Theorem 1.3 which guarantees that the representations produced in Theorem 4.1 are holonomies of properly convex projective structures on M. We can now prove Theorem 0.1 modulo Theorem 4.1.

Proof of Theorem 0.1 modulo Theorem 4.1. By Theorem 4.1, the restriction $\rho_u \in \mathcal{F}$ to each peripheral subgroup is the holonomy of a generalized cusp of type 0, type 1, or type 2. In particular the peripheral subgroups are virtual flag groups. By Theorem 1.3 we see that after possibly shrinking U we can assume $\mathcal{F} \subset \mathrm{Hom}_{ce}(\Gamma, G)$. Furthermore, since the Zariski tangent space to $[\mathcal{F}]$ in $\mathrm{Rep}(\Gamma, G)$ at $[\rho_{hyp}]$ is a k-dimensional subspace in $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{g})$, we see that $[\mathcal{F}]$ is k-dimensional. Again after possibly shrinking U, we can apply Theorem 1.2 to conclude that $[\mathcal{F}]$ is the image of a k-dimensional family of convex projective structures in $\mathfrak{B}(M)$.

□

4.1. **Proof of Theorem 4.1.** The remainder of this section is dedicated to the proof of Theorem 4.1. We now briefly describe a strategy to construct such a family of representations in Theorem 4.1. For the sake of simplicity we will briefly assume that there is a single cusp and that ρ_{hyp} has been conjugated so that $\mathsf{res}(\rho_{hyp}) \in \mathcal{S}$. First, we show that near ρ_{hyp}, res is an immersion from $\mathrm{Hom}(\Gamma, G)$ to $\mathrm{Hom}(\Delta, G)$ whose image has codimension 3. Next, we show that res is transverse to $\mathcal{S}$. As mentioned before $\mathcal{S}$ has dimension 5 and hence codimension 13 in $\mathrm{Hom}(\mathbb{Z}^2, G)$. Thus the intersection of $\mathcal{S}$ and the image of res is a 2-dimensional submanifold. However, by Proposition 3.7, only 1 of these dimensions is accounted for by conjugacy, and so there must be a path $\rho_t : \Gamma \to G$ of pairwise non-conjugate representations.

We now describe the details of the above construction. The overall strategy is similar to that found in the construction of convex projective structures found in [6]. For this reason we will quote various results from this work.

When addressing the case of multiple cusps (i.e. $k > 1$) one quickly encounters the the following problem: While the restriction map $\mathsf{res} : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Delta, G)$ is an immersion, its codimension is too large (it is $18k - 15$ rather than $3k$). Roughly speaking, this extra codimension is coming from the fact that we are not able to conjugate the restrictions of a representation to each peripheral subgroup independently. To cope with this problem, we construct an *augmented restriction map* that allows us to perform these independent conjugacies. Let M be a finite volume hyperbolic 3-manifold with fundamental group Γ and k cusps. Define $\widetilde{\mathrm{Hom}}(\Gamma, G) := \mathrm{Hom}(\Gamma, G) \times G^{k-1}$ and let

$$\widetilde{\mathsf{res}} : \widetilde{\mathrm{Hom}}(\Gamma, G) \to \mathrm{Hom}(\Delta, G),$$

by $(\rho, g_2, \ldots, g_k) \mapsto (\mathsf{res}_1(\rho), g_2 \cdot \mathsf{res}_2(\rho), \ldots, g_k \cdot \mathsf{res}_k(\rho))$, where the action of G on $\mathrm{Hom}(\Delta, G)$ is the adjoint action. Observe, that when $k = 1$ that $\widetilde{\mathsf{res}} = \mathsf{res}$. The main result concerning the augmented restriction map is that locally it is a submersion with the desired codimension.

Theorem 4.2 (Thm 3.8 in [6])**.** *Let M be a finite volume hyperbolic manifold with $k \geq 1$ cusps and fundamental group Γ, and let ρ_{hyp} be the holonomy of the complete hyperbolic structure. Suppose further that M is infinitesimally rigid rel. ∂M. Then for any $(g_2, \ldots, g_k) \in G^{k-1}$, $\widetilde{\mathsf{res}}$ is a local submersion onto a submanifold of codimension $3k$ near $(\rho_{hyp}, g_2, \ldots, g_k)$*

Picking generators γ_1^i and γ_2^i for Δ_i, we let $\mathcal{S}_i$ be the copy of $\mathcal{S}$ in $\mathrm{Hom}(\Delta_i, G)$, let $\mathcal{C}_i$ be the copy of $\mathcal{C}$ in $\mathcal{S}_i$, let $\Sigma = \mathcal{S}_1 \times \ldots \times \mathcal{S}_k$, and let $\Sigma_c = \mathcal{C}_1 \times \ldots \times \mathcal{C}_k$. Choose $g_i \in G$ so that $g_i \cdot \mathsf{res}_i(\rho_{hyp}) \in \mathcal{S}_i$. Furthermore, by choosing $s_i \in \mathcal{C}_i$ we can arrange that $\rho_{s_i} = \mathsf{res}_i(\rho_{hyp})$. For $s = (s_1, \ldots, s_k)$, let V_Σ be the image of $T_s\Sigma$ in $H^1_{\rho_{hyp}}(\Delta, \mathfrak{g})$.

In this context we can prove the following transversality result involving $\widetilde{\mathsf{res}}$ and Σ.

Proposition 4.3. *The map $\widetilde{\mathsf{res}}$ is transverse to Σ at $(\rho_{hyp}, g_2, \ldots, g_k)$, with $2k$-dimensional local intersection.*

Remark 4.4. *This result is analogous to Lemma 4.6 of* [6]

In order to prove Proposition 4.3 we need the following Lemma:

Lemma 4.5.
$$H^1_{\rho_{hyp}}(\Delta, \mathfrak{g}) = V_\Sigma \oplus \mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{so}(3,1))).$$
Moreover, if $L = \mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{g})) \cap V_\Sigma$ then $\pi_{\mathfrak{v}}|_L$ is an isomorphism between L and $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}))$. the restriction of $\pi_{\mathfrak{v}} : H^1_{\rho_{hyp}}(\Delta, \mathfrak{g}) \to H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$ to L has image $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}))$.

Proof. Lemmas 2.1 and 2.2 imply that $H^1_{\rho_{hyp}}(\Delta, \mathfrak{g})$ is $6k$-dimensional and that $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{so}(3,1))$ is $2k$-dimensional. Corollary 3.8 implies that V_Σ is $4k$-dimensional, and thus the result will follow if we can show that $V_\Sigma \cap \mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{so}(3,1))) = \{0\}$.

By Proposition 3.6, $V_\Sigma \cap H^1_{\rho_{hyp}}(\Delta, \mathfrak{so}(3,1))$ is the image of $T_s\mathcal{C}$ in $H^1_{\rho_{hyp}}(\Delta, \mathfrak{g})$. For each i, choose a nontrivial element $m_i \in \Delta_i$, and let μ_i be the subgroup generated by m_i. The inclusion of μ_i into Δ_i induces a map $(\widehat{\mathsf{res}}_i)_* : H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{so}(3,1)) \to H^1_{\rho_{hyp}}(\mu_i, \mathfrak{so}(3,1))$. Let $H^1_{\rho_{hyp}}(\mu, \mathfrak{so}(3,1)) := \bigoplus_{i=1}^k H^1_{\rho_{hyp}}(\mu_i, \mathfrak{so}(3,1))$, then the sum of these maps from $1 \leq i \leq k$ gives $\widehat{\mathsf{res}}_* : H^1_{\rho_{hyp}}(\Delta, \mathfrak{so}(3,1)) \to H^1_{\rho_{hyp}}(\mu, \mathfrak{so}(3,1))$.

Let $(0,0) \neq (x,y) \in \mathbb{R}^2$, define
$$m_1 = \begin{pmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
and define a cocycle by $z_1(\mu) = m_1$ in $Z^1_{\rho_{hyp}}(\mu, \mathfrak{so}(3,1))$. It is straightforward to check that this cocycle is a coboundary. As a result we see that $V_\Sigma \cap H^1_{\rho_{hyp}}(\Delta, \mathfrak{so}(3,1)) \subset \ker \widehat{\mathsf{res}}_*$.

On the other hand, by Calabi–Weil rigidity (See [10, 26, 27])
$$\widehat{\mathsf{res}}_* \circ \mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{so}(3,1)) \to H^1_{\rho_{hyp}}(\mu, \mathfrak{so}(3,1))$$
is injective, and thus $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{so}(3,1))) \cap V_\Sigma = \{0\}$.

For the last point, the image of $\pi_{\mathfrak{v}}$ restricted to L is contained in $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}))$. The kernel of $\pi_{\mathfrak{v}}|_L$ is easily seen to be $\mathsf{res}_*(H^1(\Gamma, \mathfrak{so}(3,1))) \cap V_\Sigma$, and so by the previous argument, $\pi_{\mathfrak{v}}|_L$ is an injection. By the previous transversality, L is k-dimensional and by Lemma 3.5, $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}))$ is also k-dimensional, and so for dimensional reasons $\pi_{\mathfrak{v}}|_L$ is an isomorphism. $\square$

We can now prove Proposition 4.3.

Proof of Proposition 4.3. Let $\vec{\rho}_{hyp} = (\mathsf{res}_1(\rho_{hyp}), \ldots, \mathsf{res}_k(\rho_{hyp}))$. Near $\vec{\rho}_{hyp}$, the space $\mathrm{Hom}(\Delta, \mathfrak{g})$ is $18k$-dimensional. By construction, Σ has codimension $13k$ and contains $\vec{\rho}_{hyp}$. Let I be the image of $\widetilde{\mathsf{res}}$, then by Theorem 4.2, I has codimension $3k$ near $\vec{\rho}_{hyp}$. Thus if the intersection of Σ and I is transverse at $\vec{\rho}_{hyp}$ then the intersection will have codimension $16k$, or equivalently dimension $2k$.

The tangent space to $\mathrm{Hom}(\Delta, G)$ at $\vec{\rho}_{hyp}$ is $Z^1_{\rho_{hyp}}(\Delta, \mathfrak{g})$ and we can write
$$Z^1_{\rho_{hyp}}(\Delta, \mathfrak{g}) \cong H^1_{\rho_{hyp}}(\Delta, \mathfrak{g}) \oplus B^1_{\rho_{hyp}}(\Delta, \mathfrak{g}).$$
From the construction of $\widetilde{\mathsf{res}}$, it can be seen that at $p = (\rho, g_2, \ldots, g_k) \in \widetilde{\mathrm{Hom}}(\Gamma, G)$,
$$T_p\left(\widetilde{\mathrm{Hom}}(\Gamma, G)\right) = Z^1_\rho(\Gamma, \mathfrak{g}) \oplus \left(\oplus_{i=2}^k B^1_{g_i \cdot \rho}(\Gamma, \mathfrak{g})\right),$$
and that the map $\widetilde{\mathsf{res}}_* : T_p\left(\widetilde{\mathrm{Hom}}(\Gamma, G)\right) \to Z^1_\rho(\Delta, \mathfrak{g})$ is just the componentwise application of res_*. Since ρ_{hyp} is an irreducible representation, it follows that $B^1_{\rho_{hyp}}(\Delta, \mathfrak{g}) \subset T_{\vec{\rho}_{hyp}} I$. On the other hand, from Lemma

4.5 we know that V_Σ and $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{so}(3,1))$ span $H^1_{\rho_{hyp}}(\Delta, \mathfrak{g})$. As a result, $T_{\vec{\rho}_{hyp}} I$ and $T_{\vec{\rho}_{hyp}}\Sigma$ span $Z^1_{\rho_{hyp}}(\Delta, \mathfrak{g})$, and are thus transverse. $\square$

We can now prove Theorem 4.1. The proof is similar to Theorem 4.1 of [6].

Proof of Theorem 4.1. Recall that there are $g_i \in G$ so that $g_i \cdot \mathsf{res}_i(\rho_{hyp}) \in \mathcal{S}_i$, let $p = (\rho_{hyp}, g_2, \dots, g_k) \in \widetilde{\mathrm{Hom}}(\Gamma, G)$, and let $p' = \widetilde{\mathsf{res}}(p)$. By Lemma 4.5 $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{g}))$ intersects V_Σ transversely in a k-dimensional subspace $\tilde{V}$. Let V be the k-dimensional subspace of $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{g})$ such that $\mathsf{res}_*(V) = \tilde{V}$.

As a result, we can find a lift $R : V \to Z^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$ of res_* such that

$$R(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})) \subset W := \widetilde{\mathsf{res}}_*(T_p(\widetilde{\mathrm{Hom}}(\Gamma, G))) \cap T_{p'}(\Sigma).$$

In other words, there is a commutative the diagram

$$\begin{tikzcd} & W \subset Z^1_{\rho_{hyp}}(\Delta, \mathfrak{g}) \arrow[d] \\ V \arrow[ur, "R"] \arrow[r, "\mathsf{res}_*"] & H^1_{\rho_{hyp}}(\Delta, \mathfrak{g}) \end{tikzcd}$$

commutes. The space W is the tangent space to the intersection of the image of $\widetilde{\mathsf{res}}$ and Σ at p'. Thus by Proposition 4.3 we can find a small neighborhood, U, of 0 in V and

- A smooth family $\mathcal{F} = \{\rho_u \mid u \in U\}$ of representation in $\mathrm{Hom}(\Gamma, G)$ such that $\rho_0 = \rho_{hyp}$. The tangent space of $\mathsf{res}(\mathcal{F})$ at $\mathsf{res}(\rho_{hyp})$ is $R(V)$.
- A smooth families $\{g_i^u \mid u \in U\}$ for $2 \le i \le k$, such that $g_i^0 = g_i$ and such that $g_i^u \cdot \mathsf{res}_i(\rho_u) \in \mathcal{S}_i$.

By construction, the image of the space of infinitesimal deformations of $\mathcal{F}$ at ρ_{hyp} in $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{g})$ is V, and so $[\mathcal{F}]$ is k-dimensional. Furthermore, $\mathsf{res}_i(\rho_u)$ is conjugate into $\mathcal{S}_i$. By Theorem 3.3 this implies that the restriction of ρ_u to each peripheral subgroup is the holonomy of a generalized cusp of type 0, type 1, or type 2.

$\square$

5. Controlling the cusps

In this section we describe some theoretical results that make it possible to control the types of the cusps that are produced by Theorem 0.1. This will allow us to prove Theorem 0.2. The first main results of this section is Theorem 5.1 which describes a sufficient condition for ensuring that Theorem 0.2 produces properly convex manifolds with type 2 cusps. The condition in Theorem 5.1 involves the value of certain cohomological quantities. In Section 6 we calculate these invariants for some examples in order to find explicit manifolds that admit properly convex structures with type 2 cusps.

The other main result of this section, Theorem 5.7, shows that in the presence of orientation reversing symmetries it is sometimes possible to guarantee that the deformations produced by Theorem 0.2 have (some) type 1 cusps.

5.1. **Slice bases for $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$.** Recall that M is a finite volume hyperbolic 3-manifold with fundamental group Γ. The manifold M has $k \ge 1$ cusps $\{\partial_1, \dots, \partial_k\}$ and assume that we have chosen a peripherals $\{\Delta_1, \dots, \Delta_k\}$ one for each cusp. For each $\Delta_i \cong \mathbb{Z}^2$ pick a set $\{\gamma_1^i, \gamma_2^i\}$ of generators. Recall that $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v}) := \bigoplus_{i=1}^k H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$. The spaces $H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$ and $H^1_{\rho_{hyp}}(\mathbb{Z}^2, \mathfrak{v})$ are isomorphic vector spaces and we would like to identify a convenient isomorphism between these two spaces.

By Lemma 3.5, $\{[D_a], [D_b]\}$ is a basis of $H^1_{\rho_{hyp}}(\mathbb{Z}^2, \mathfrak{v})$. Using the generating set $\{\gamma_1^i, \gamma_2^i\}$, we can identify Δ_i with $\mathbb{Z}^2$ and $H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$ with $H^1_{\rho_{hyp}}(\mathbb{Z}^2, \mathfrak{v})$. Next, assume that ρ_{hyp} has been conjugated so that $\rho_{hyp}|_{\Delta_i} \in \mathcal{S} \subset \mathrm{Hom}(\mathbb{Z}^2, G)$. Using (3.2) and (3.3), define cocycles $z_{\gamma_1^i}$ and $z_{\gamma_2^i}$ in $Z^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$ by the property that $z_{\gamma_1^i}(\gamma_1^i) = \alpha_1$, $z_{\gamma_1^i}(\gamma_2^i) = \alpha_2$, $z_{\gamma_2^i}(\gamma_1^i) = \beta_1$, $z_{\gamma_2^i}(\gamma_2^i) = \beta_2$. It is easy to see that $\{[z_{\gamma_1^i}], [z_{\gamma_2^i}]\}$ is a basis for $H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$. A basis constructed in this way is called a *slice basis* for $H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$ (with respect to $\{\gamma_1^i, \gamma_2^i\}$). Under the above identification $[z_{\gamma_1^i}] \mapsto [D_a]$ and $[z_{\gamma_2^i}] \mapsto [D_b]$. If we regard elements of a slice basis as elements of $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$ then $\left\{[z_{\gamma_1^1}], [z_{\gamma_2^1}], \ldots, [z_{\gamma_1^k}], [z_{\gamma_2^k}]\right\}$ is a basis for $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$, which we also call a *slice basis*.

Suppose now that M is infinitesimally rigid rel. ∂M and recall that $V = \mathsf{res}_*^{-1}(\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{g}) \cap V_\Sigma)$, which is a k-dimensional subspace. If $[z] \in V$ then $\pi_{\mathfrak{v}} \circ \mathsf{res}_*([z])$ is a non-trivial element of $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$ and so we can write

$$\pi_{\mathfrak{v}} \circ \mathsf{res}_*([z]) = c_{\gamma_1^1}[z_{\gamma_1^1}] + c_{\gamma_2^1}[z_{\gamma_2^1}] + \ldots + c_{\gamma_1^k}[z_{\gamma_1^k}] + c_{\gamma_2^k}[z_{\gamma_2^k}] \tag{5.1}$$

The coordinates of $\pi_{\mathfrak{v}} \circ \mathsf{res}_*([z])$ with respect to the slice basis coming from (5.1) are called *slice coordinates* for $[z]$. The next Theorem describes the relationship between the slice coordinates and the cusp types of the properly convex manifolds produced by Theorem 0.2.

Theorem 5.1. *Suppose that M is infinitesimally rigid rel. ∂M, and suppose $[z] \in V$ has slice coordinates $\left(c_{\gamma_1^1}, c_{\gamma_2^1}, \ldots, c_{\gamma_1^k}, c_{\gamma_2^k}\right)$. Let $I_{[z]} = \{i \mid c_{\gamma_1^i} \neq 0 \text{ and } c_{\gamma_2^i} \neq 0\}$ and $II_{[z]} = \{i \mid c_{\gamma_1^i} \neq 0 \text{ or } c_{\gamma_2^i} \neq 0\}$*

1. *M admits a convex projective structure where if $i \in I_{[z]}$ then the ith cusp is a type 2 generalized cusp and*
2. *M admits a convex projective structure where if $i \in II_{[z]}$ then the ith cusp is a type 1 or a type 2 generalized cusp for each $i \in II_{[z]}$*

Proof. To minimize notation we address the case when M has a single cusp. The multiple cusp case can be treated similarly. From Theorem 4.1, for each $[z] \in V$ there is a family $\rho_t : \pi_1 M \to G$ of representations such that $\rho_0 = \rho_{hyp}$ and whose Zariski tangent vector is z. Furthermore $\rho_t|_\Delta$ is a path in $\mathcal{S}$ with Zariski tangent vector $w = \mathsf{res}_*(z)$. As such we can write

$$w = \dot{a}\frac{\partial}{\partial a} + \dot{b}\frac{\partial}{\partial b} = \tilde{w},$$

where $\tilde{w} \in Z^1_{\rho_{hyp}}(\Delta, \mathfrak{so}(3,1)$, and observe that this implies that $\dot{a}$ and $\dot{b}$ are the slice coordinates of $[z]$.

If either $\dot{a}$ or $\dot{b}$ is non-zero then by examining (3.1) it follows that as t moves away from 0 at least 1 eigenvalue of either $\rho_t(\gamma_1^1)$ or $\rho_t(\gamma_2^1)$ is changing to first order in t. This implies that for $t \neq 0$ that ρ_t is the holonomy of either a type 1 or type 2 cusp, which proves the second claim. Similarly, if both $\dot{a}$ and $\dot{b}$ are non-zero, it follows that as t moves away from 0 that two eigenvalues of both $\rho_t(\gamma_1^1)$ and $\rho_t(\gamma_2^1)$ are changing to first order in t. This implies that for $t \neq 0$ that ρ_t is the holonomy of a type 2 cusp, which proves the first claim. ☐

Remark 5.2. *It is easy to see that if ρ_t is a path in $\mathcal{S}$ that is type 0 when $t = 0$ and type 1 otherwise that the Zariski tangent vector to this path at $t = 0$ has either $\dot{a}$ or $\dot{b} = 0$, but not both. It is tempting to say that if $i \in II \backslash I$ then the ith cusp is type 1. However, this turns out to be the case. The problem is that the slice coordinates are only encoding first order behavior. For instance, the representations*

$$\rho_t(\gamma_1^1) = \exp\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & t & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \rho_t(\gamma_2^1) = \exp\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & t^2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are holonomies of type 2 cusps when $t \neq 0$, however, up to first order the second generator remains constant.

Before proceeding with the proof of Theorem 0.2 we need to recall the following result from [6] that will ensure that we can find a cohomology class in $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ whose restriction to each cusp is non-trivial. We will use this result to ensure that the representations we construct in Theorem 4.1 will be holonomies of type 1 or 2 cusps rather than type 0 (standard hyperbolic cusps).

Lemma 5.3 (See Lem 4.4 of [6])**.** *There exists a cohomology class $[z] \in H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ with the property that $(\mathsf{res}_i)_*[z] \in H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$ is non-trivial for each $1 \leq i \leq k$.*

We can now prove Theorem 0.2.

Proof of Theorem 0.2. By Lemma 5.3 we can find a cohomology class $[w] \in H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ with the property that $(\mathsf{res}_i)_*[w] \in H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$ is non-trivial for each $1 \leq i \leq k$. Furthermore, by Lemma 4.5 there is $[z] \in V$ such that $\pi_{\mathfrak{v}} \circ \mathsf{res}_*([z]) = [w]$. It follows that for each $1 \leq i \leq k$ that either $c_{\gamma_1^i}$ or $c_{\gamma_2^i}$ is non-zero and thus $II = \{1, \dots, k\}$. Applying Theorem 5.1(2) gives the desired conclusion. □

5.2. **Symmetry and type 1 cusps.** One consequence of Theorem 5.1 is that if the slice coordinates of a cohomology class $[z] \in H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ are all non-zero then the resulting convex projective structures corresponding to $[z]$ have all type 2 cusps. A priori, a vector having all non-zero coordinates seems like a generic condition and so it is natural to wonder if Theorem 0.1 ever produces examples with type 1 cusps. In this section we show that in certain circumstances it is possible to produce examples with type 1 cusps. More specifically, we prove a general result (Theorem 5.7) that states that for manifolds admitting certain types of symmetry, Theorem 0.2 produces convex projective manifolds where some of the cusps become type 1 generalized cusps. In Section 6 we use Theorem 5.7 to show that if K is the 6_3 knot then $M = S^3 \backslash K$ admits a properly convex projective structure where the cusp is type 1.

Before proceeding with the proof we discuss how orientation reversing symmetries of M act on ∂M and on $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$. Let $\phi : M \to M$ be an orientation reversing symmetry and define

$$S_\phi = \{i \in \{1, \dots, k\} \mid \phi(\partial_i) = \partial_i\}$$

to be the set of cusps invariant under ϕ.

We first need to address some technicalities regarding how ϕ induces an action on the peripheral subgroups. The map ϕ gives rise to an outer automorphism $[\phi_*] \in \mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$. We now describe how $[\phi_*]$ induces an action on Δ_i for each $i \in S_\phi$. Let $\phi_1, \phi_2 \in [\phi_*]$, and so there is $g \in \Gamma$ such that $\phi_2(\gamma) = g\phi_1(\gamma)g^{-1}$ for any $\gamma \in \Gamma$. Since $\phi(\partial_i) = \partial_i$ there are $g_1, g_2 \in \Gamma$ such that $g_j\phi_j(\Delta_i)g_j^{-1} = \Delta_i$ for $j \in \{1, 2\}$. For $\jmath \in \{1, 2\}$, composing ϕ_j with conjugation by g_j gives an automorphism of Δ_i and we claim that this map is independent of the choice of ϕ_j and g_j. To see this, observe that $g_1(g_2g)^{-1}$ normalizes Δ_i. Since Γ is the fundamental group of a finite volume hyperbolic 3-manifold the normalizer of Δ_i in Γ is equal to the centralizer of Δ_i in Γ. This implies that $g_1(g_2g)^{-1}$ centralizes Δ_i, and thus conjugation by g_1 and by g_2g give rise to the same map from $\phi_1(\Delta_i)$ to Δ_i. As a result,

$$g_2\phi_2(\gamma)g_2^{-1} = g_2g\phi_1(\gamma)g_{-1}g_2^{-1} = g_1\phi_1(\gamma)g_1^{-1},$$

which proves the claim. By abuse of notation we will call this map $\phi_* : \Delta_i \to \Delta_i$.

Lemma 5.4. *If $\phi : M \to M$ is an orientation reversing symmetry. Let $i \in S_\phi$ then $\phi : \partial_i \to \partial_i$ is isotopic an involution. Furthermore, there exists a generating set $\{\gamma_+^i, \gamma_-^i\}$ for $\pi_1(\partial_i)$ such that $\phi_*(\gamma_\pm) = (\gamma_\pm^i)^{\pm 1}$.*

Proof. Since ϕ_* is an automorphism of Δ_i and $\Delta_i \cong \mathbb{Z}^2$, ϕ_* corresponds to an element $M_\phi \in \mathrm{GL}(2, \mathbb{Z})$. Since ϕ is orientation reversing, it follows that $\det(M_\phi) = -1$. As such the characteristic polynomial of M_ϕ is $p_\phi(x) = x^2 - \mathrm{tr}(M_\phi) - 1$.

By Mostow rigidity, the mapping class group of M is finite, and so M_ϕ is a finite order element of $\mathrm{GL}(2, \mathbb{Z})$. It follows that the roots, λ_1, λ_2, of $p_\phi(x)$ are roots of unity. Suppose that the root of p_ϕ are complex, then $\lambda_2 = \overline{\lambda_1}$. However, since $\det(M_\phi) = -1$, we see that $-1 = \lambda\overline{\lambda_1} = |\lambda_1|^2$, which is a contradiction. Thus the roots of p_ϕ are real. Since λ_1 and λ_2 are real and $\det(M_\phi) = -1$, we find that $\{\lambda_1, \lambda_2\} = \{-1, 1\}$. It follows $p_\phi(x) = x^2 - 1$, and is hence ϕ_* is an involution. Since $\mathrm{GL}(2, \mathbb{Z})$ is the mapping class group of ∂_i it follows that *phi* is isotopic to an involution when restricted to ∂_i.

It is clear that there are non-trivial ± 1 eigenspace for the action of ϕ_* on $H^1(\partial_i, \mathbb{R})$, and the proof will be complete if it can be shown that there are eigenvectors in $H^1(\partial_i, \mathbb{Z}) \cong \Delta_i$. Let $\tilde{\gamma}_+^i$ be a non-trivial element of Δ_i that is not a -1-eigenvector of ϕ_*. By the Cayley-Hamilton theorem, M_ϕ is a root of its characteristic polynomial, and so $\gamma_+^i = (M_\phi + \mathrm{I})\tilde{\gamma}_+^i$ is a non-trivial 1-eigenvector of M_ϕ. Using a similar procedure we can construct a non-trivial -1-eigenvector γ_-^i. The set $\{\gamma_+^i, \gamma_-^i\}$ is the desired generating set. Using a similar construction we can produce the appropriate generating sets for the remaining ϕ-invariant cusps, thus completing the proof. □

The generators $\{\gamma_+^i, \gamma_-^i\}$ constructed in Lemma 5.4 are called the *p-curve and m-curve of the ith cusp with respect to* ϕ (or simply the *p-curve and m-curve* is ϕ and i are clear from context).

If $i \in S_\phi$ then ϕ_* is an involution when restricted to Δ_i. It is natural to wonder if the ϕ induces an involution on $H^1(\Delta_i, \mathfrak{v})$. Strictly speaking, ϕ does not induce an action, but instead induces a map

$$\phi^* : H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v}) \to H^1_{\rho_{hyp}\circ\phi_*}(\Delta_i, \mathfrak{v})$$

However, by Mostow rigidity, there is a unique $A_\phi \in O(3,1)$ such that $\rho_{hyp}\circ\phi_* = A_\phi \cdot \rho_{hyp}$. Conjugation by A_ϕ^{-1} provides a map

$$\mathsf{Ad}A_\phi^{-1} : H^1_{\rho_{hyp}\circ\phi_*}(\Delta_1, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\Delta_1, \mathfrak{v}).$$

Composing these two maps gives an automorphism of $H^1_{\rho_{hyp}}(\Delta_1, \mathfrak{v})$ which by abuse of notation we refer to as ϕ^*. In the same way, we can view ϕ^* as an automorphism of $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$.

It turns out that ϕ^* does act as an involution and that using γ_+^i and γ_-^i we can construct a nice eigenbasis for $H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$. We will need the following Lemma, which shows that if M admits an orientation reversing symmetry then the cusp shape of an invariant cusp with respect to $\{\gamma_-^i, \gamma_+^i\}$ is purely imaginary. This is originally due to an observation of Riley [23].

Lemma 5.5. *Let $\phi : M \to M$ be an orientation reversing symmetry and let $i \in S_\phi$. Then the cusp shape of ∂_i with respect to $\{\gamma_-^i, \gamma_+^i\}$ is $z = ic$, where $c > 0$. Consequently, it is possible to conjugate ρ_{hyp} in G so that*

$$\rho_{hyp}(\gamma_-^i) = \begin{pmatrix} 1 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } \rho_{hyp}(\gamma_+^i) = \begin{pmatrix} 1 & 0 & c & c^2/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Proof. We can regard ρ_{hyp} as a representation from Γ to $PSL(2,\mathbb{C})$ in such a way that

$$\rho_{hyp}(\gamma_-^i) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho_{hyp}(\gamma_+^i) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

where z is the cusp shape with respect to $\{\gamma_-^i, \gamma_+^i\}$, which by construction has positive imaginary part. By Mostow rigidity, there is an element $B_\phi \in PSL(2,\mathbb{C})$ such that for each $\gamma \in \Gamma$, $\rho_{hyp}(\phi_*(\gamma)) = \overline{B_\phi \rho_{hyp}(\gamma) B_\phi^{-1}}$, where $\overline{g}$ means entrywise complex conjugation. Since $\phi_*(\gamma_\pm^i) = (\gamma_\pm^i)^{\pm}$, it follows that $-z = \overline{z}$. In other words, z is purely imaginary and thus $z = ic$ for $c > 0$. $\square$

Next, define two cocycles z_+^i, $z_-^i \in Z^1_{\rho_{hyp}}(\Delta_1, \mathfrak{v})$, by $z_\pm^i(\gamma_\mp) = 0$ and $z_\pm^i(\gamma_\pm^i) = a_\pm$, where

$$a_+ = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 3 & \\ & & & -1 \end{pmatrix}, \text{ and } a_- = \begin{pmatrix} -1 & & & \\ & 3 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

A simple computation shows that $[z_\pm^i]$ are nothing more than the slice basis for $H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$ with respect to $\{\gamma_+^i, \gamma_-^i\}$ and the following Lemma shows that $[z_\pm^i]$ are the desired eigenvectors.

Lemma 5.6. *Suppose that $\phi_* : \Delta_i \to \Delta_i$ is induced by an orientation reversing symmetry of M as above. Then $[z_\pm^i]$ is a ± 1-eigenvector for the action of ϕ^* on $H^1_{\rho_{hyp}}(\Delta_1, \mathfrak{v})$. Furthermore, $\{[z_+^i], [z_-^i]\}$ is an eigenbasis for $H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$.*

Proof. To simplify notation we drop the the i scripts on the $z_\pm$, $\gamma_\pm$, and Δ. First, since ϕ_* leaves Δ invariant and $\phi_*(\gamma_\pm) = \gamma_\pm^{\pm 1}$ it follows that there are $x, y \in \mathbb{R}$ so that

$$A_\phi = \begin{pmatrix} 1 & -x & y & \frac{x^2+y^2}{2} \\ 0 & -1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We first show that $[\phi^*(z_+)] = [z_+]$. By the discussion above we see that at the level of cocyles that $(\phi^*(z_+))(\gamma_-) = 0$. Furthermore,

$$\phi^*(z_+)(\gamma_+) = A_\phi^{-1} \cdot z_+(\phi(\gamma_+)) = A_\phi^{-1} \cdot z_+(\gamma_+) = A_\varphi^{-1} \cdot a_+.$$

Therefore,

$$(\phi^*(z_+) - z_+)(\gamma_+) = A_\phi^{-1} \cdot a_+ - a_+ = \begin{pmatrix} 0 & 0 & -4y & -4y^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4y \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now show that $(\phi^*(z_+) - z_+)$ is a coboundary. Let

$$v_+ = \begin{pmatrix} -\frac{y}{c} & 0 & -\frac{2y(c+y)}{c} & 0 \\ 0 & -\frac{y}{c} & 0 & 0 \\ 0 & 0 & \frac{3y}{c} & \frac{2y(c+y)}{c} \\ 0 & 0 & 0 & -\frac{y}{c} \end{pmatrix}$$

Consider the coboundary $w_+(\gamma) = v_+ - \rho_{hyp}(\gamma) \cdot v_+$. Computing, one sees that $\rho_{hyp}(\gamma_-)$ commutes with v_+ and so $w_+(\gamma_-) = 0$, and also that

$$w_+(\gamma_+) = v_+ - \rho_{hyp}(\gamma_+) \cdot v_+ = \begin{pmatrix} 0 & 0 & -4y & -4y^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4y \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so $w_+ = \phi^*(z_+) - z_+$. Thus $[z_+]$ is a 1-eigenvector of φ^*.

The other case is similar. Computing shows that $(\phi^*(z_-) + z_-)(\gamma_+) = 0$. Using the cocycle condition gives $z_-(\gamma_-^{-1}) = -\rho_{hyp}(\gamma_-)^{-1} \cdot z_-(\gamma_-)$ we find that

$$(\phi^*(z_-)+z_-)(\gamma_-) = A_\phi^{-1} \cdot z_-(\gamma_-^{-1}) + z_-(\gamma_-) = -A_\phi^{-1}\rho_{hyp}(\gamma_-)^{-1} \cdot a_- + a_- = \begin{pmatrix} 0 & -4(1+x) & 0 & 4(1+x)^2 \\ 0 & 0 & 0 & 4(1+x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$v_- = \begin{pmatrix} -1-x & 2x(1+x) & 0 & 0 \\ 0 & 3+3x & 0 & -2x(1+x) \\ 0 & 0 & -1-x & 0 \\ 0 & 0 & 0 & -1-x \end{pmatrix}$$

and let $w_-(\gamma) = v_- - \rho_{hyp}(\gamma) \cdot v_-$, then as before we see that $w_- = \phi^*(z_-) + z_-$, and so $[z_-]$ is a -1-eigenvector of ϕ^*. Finally, $H^1_{\rho_{hyp}}(\Delta_1, \mathfrak{v})$ is a 2-dimensional vector space and $[z_\pm]$ are non-trivial eigenvectors with different eigenvalue and so they must be linearly independent, and hence a basis. ☐

We can now state the main theorem of this section that describes when a manifold admitting an orientation reversing symmetry admits a convex projective structure with type 1 cusps.

Theorem 5.7. *Let M be an infinitesimally rigid rel. ∂M and let $\phi : M \to M$ be an orientation reversing symmetry that leaves each cusp invariant. If $\phi^* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is the identity map then there is a properly convex projective structure on M where each cusp is type 1.*

Theorem 5.7 has the following Corollary.

Corollary 5.8. *Suppose that M has a single cusp and that $\phi : M \to M$ is an orientation reversing symmetry and that γ_+ is a p-curve for ϕ. If M is infinitesimally rigid rel. ∂M and the map $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\gamma_+, \mathfrak{v})$ is nontrivial then M admits nearby convex projective structures where the cusp is a type 1 generalized cusp.*

Proof. Since M has a single cusp, $\Delta = \Delta_1$, it follows trivially that each cusp is preserved by ϕ. Since M is infinitesimally rigid rel. ∂M it follows that $H^1_{\rho_{hyp}}(\Gamma, v)$ is 1-dimensional. The image of $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ in $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$ is ϕ^* invariant, and is thus spanned by either $[z_+]$ or $[z_-]$. By hypothesis, $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\gamma_+, \mathfrak{v})$ is non-trivial, but the image of $[z_-]$ is trivial in $H^1_{\rho_{hyp}}(\langle \gamma_+ \rangle, \mathfrak{v})$, and thus $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is spanned by $[z_+]$. It follows that $\phi^* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is the identity. The result follows by applying Theorem 5.7.

□

Before proving Theorem 5.7 we need a couple of auxiliary lemmas. The first Lemma allows us to identify $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}))$ inside $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$.

Lemma 5.9. *Suppose that M is infinitesimally rigid rel. ∂M and let $\phi : M \to M$ be an orientation reversing symmetry that preserves each cusp. Then $\phi^* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is an involution. Moreover, there is an eigenbasis $\mathcal{B} = \{v_1, \ldots, v_k\}$ such that $\mathsf{res}_*(v_i) = [z^i_{e(i)}]$, where $e : \{1, \ldots, k\} \to \{+, -\}$ is the function that returns $+1$ if v_i is a $+1$-eigenvector and $-$ if v_i is a -1-eigenvectors.*

Proof. Recall that $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v}) = \bigoplus_{i=1}^k H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$. We have the commutative diagram

$$\begin{array}{ccc} H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) & \xrightarrow{\mathsf{res}_*} & H^1_{\rho_{hyp}}(\Delta, \mathfrak{v}) \\ \phi^* \big\downarrow & & \big\downarrow \phi^* \\ H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) & \xrightarrow{\mathsf{res}_*} & H^1_{\rho_{hyp}}(\Delta, \mathfrak{v}) \end{array} \tag{5.2}$$

It follows that the image of $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ in $H^1_{\rho_{hyp}}(\Delta_1, \mathfrak{v})$ is ϕ^*-invariant. By hypothesis, $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is k-dimensional and res_* is injective and so there is a basis $\mathcal{B}' = \{v'_1, \ldots, v'_k\}$ for $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}))$ consisting of vectors from the set $\{[z^1_+], [z^1_-], \ldots, [z^k_+], [z^k_-]\}$ of ϕ^*-eigenvectors of $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$. Let $v_i = \mathsf{res}_*^{-1}(v'_i)$. From (5.2), it follows that $\mathcal{B} = \{v_1, \ldots, v_k\}$ is an eigenbasis consisting of ± 1 eigenvectors and $\phi^* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is thus an involution.

Next, suppose that for some $1 \leq i \leq k$ that $[z^i_+], [z^i_-] \in \mathcal{B}'$, then by the pigeonhole principal there must be $j \neq i$ such that

$$(\mathsf{res}_j)_* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\Delta_j, \mathfrak{v})$$

is trivial, however, this contradicts Lemma 5.3. Thus by renumbering the elements of $\mathcal{B}$ we can ensure that $\mathsf{res}_*(v_i) = [z^i_{e(i)}]$

□

The next Lemma gives a sufficient condition for a representation in $\mathcal{S}$ to be the holonomy of a type 0 or type 1 cusp. This criteria will be used in the proof of Theorem 5.7

Lemma 5.10. *Let $A \in \mathrm{GL}(2, \mathbb{Z})$ be such that $A(\gamma_1) = \gamma_1^{-1}$ and $A(\gamma_2) = \gamma_2$ and let $s_0 = (0, 0, 1/c, 0, 0, c)$. There is a neighborhood V of s_0 in S with the property that if $v \in V$ and $\rho_v \in \mathcal{S}_a$ is such that $\rho_v \circ A$ is conjugate to ρ_v then ρ_v is the holonomy of a type 0 or type 1 generalized cusp.*

Proof. Let $s = (a, b, x_1, y_1, x_2, y_2) \in S$ and suppose that $\rho_s \circ A$ is conjugate to ρ_s. Let $\{ax_1, by_1\}$ and $\{ax_2, by_2\}$ be the set of two eigenvalues of largest modulus of $\rho_s(\gamma_1)$ and $\rho_s(\gamma_2)$, respectively. Since ρ_s is conjugate to $\rho_s \circ A$ we find that either

$$ax_1 = -ax_1 \text{ and } by_1 = -by_1 \text{ or} \tag{5.3}$$

$$-ax_1 = by_1 \text{ and } ax_2 = by_2 \tag{5.4}$$

Since $s \in S$, equations (5.3) imply that either a or b is zero. We can choose V such that $x_1 \neq 0$ for all $s \in V$. In this case equations (5.4) imply that $b(x_1 y_2 + y_1 x_2) = 0$. By further shrinking V we can assume that $x_1 y_2 + y_1 x_2 \neq 0$, and thus (5.4) implies that $b = 0$. Thus in either case we see that for $s \in V$ that ρ_s is the holonomy of either a type 0 or type 1 generalized cusp.

□

We can now prove Theorem 5.7.

Proof of Theorem 5.7. Before proceeding with the details, we describe the idea behind the proof. Let $\mathcal{F}$ be the k-dimensional family of representations produced in Theorem 4.1, and recall that $[\mathcal{F}]$ is the image of $\mathcal{F}$ in $\mathrm{Rep}(\Gamma, G)$. We begin by showing that near $[\rho_{hyp}]$, elements $[\rho_u] \in [\mathcal{F}]$ are determined by the eigenvalues of $\rho_u(\gamma_+^i)$. By construction, $\rho_u \circ \phi_*(\gamma_+^i)$ is conjugate to $\rho_u(\gamma_+^i)$ for $1 \leq i \leq k$ and therefore $\rho_u \circ \phi_*$ is conjugate to ρ_u. Restricting ρ_u to each cusp and applying Lemma 5.10 gives the desired result. We now provide the details of this argument.

The symmetry ϕ acts on $\widetilde{\mathrm{Hom}}(\Gamma, G)$ by $\phi \cdot (\rho, g_2, \ldots, g_k) = (\rho \circ \phi_*, g_2, \ldots, g_k)$ and on $\mathrm{Hom}(\Delta, G)$ by $\phi \cdot (\rho_1, \ldots, \rho_k) = (\rho_1 \circ \phi_*, \ldots, \rho_k \circ \phi_*)$. A simple computation shows that $\widetilde{\mathsf{res}}$ is equivariant with respect to these actions. Another simple computation shows that Σ is ϕ-invariant. Combining these facts we find that $\mathcal{F} \subset \mathrm{Hom}(\Gamma, G)$ is also ϕ-invariant. Moreover, the action of ϕ descends to $\mathrm{Rep}(\Gamma, G)$ and the above computation shows that $[\mathcal{F}]$ is also ϕ-invariant. Furthermore, by Mostow rigidity $\phi \cdot [\rho_{hyp}] = [\rho_{hyp}]$.

By hypothesis, $\phi^* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is the identity. Combining this with Lemma 5.9 shows that $\mathsf{res}_*(H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}))$ is spanned by $\{[z_+^1], \ldots, [z_+^k]\}$. Let $U \ni 0$ be the neighborhood in V used to define $\mathcal{F}$ and let $(t_1, \ldots t_k)$ be coordinates on U such that if $u = (t_1, \ldots, t_k)$ then up to conjugacy,

$$\rho_u(\gamma_-^i) = \exp \begin{pmatrix} O(|u|^2) & * & * & * \\ 0 & O(|u|^2) & 0 & * \\ 0 & 0 & O(|u|^2) & * \\ 0 & 0 & 0 & O(|u|^2) \end{pmatrix}, \tag{5.5}$$

$$\rho_u(\gamma_+^i) = \exp \begin{pmatrix} -t_i + O(|u|^2) & * & * & * \\ 0 & -t_i + O(|u|^2) & 0 & * \\ 0 & 0 & 3t_i + O(|u|^2) & * \\ 0 & 0 & 0 & -t_i + O(|u|^2) \end{pmatrix} \tag{5.6}$$

In other words the partial derivative of ρ_u with respect to t_i at ρ_{hyp}, when projected to $H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{v})$, is $[z_+^i]$. The matrix $\rho_u(\gamma_+^i)$ has a unique simple eigenvalue when $t_i \neq 0$ which we denote v_i. From (5.6) it follows that $v_i = \exp(3t_i + O(|u|^2)) = 1 + 3t_i + O(|u|^2)$. Thus by the inverse function theorem the map $(v_1, \ldots, v_k) \mapsto (t_1, \ldots, t_k)$ gives a diffeomorphism from a neighborhood of $(1, \ldots, 1) \in \mathbb{R}^k$ to a neighborhood $\tilde{U}$ of $[\rho_{hyp}]$ in $[\mathcal{F}]$.

Let $[\rho_u] \in \tilde{U} \subset [\mathcal{F}]$, and let $\rho_u \in \mathcal{F}$ be a representative of this conjugacy class. By construction, $\phi_*(\gamma_+^i) = \gamma_+^i$, for $1 \leq i \leq k$. It follows that $\rho_u \circ \phi_*(\gamma_+^i)$ is conjugate to $\rho_u(\gamma_i)$. Since $\tilde{U}$ is parameterized by v_i for $1 \leq i \leq k$ it follows that $\rho_u \circ \phi_*$ is conjugate to ρ_u. By applying Lemmas 5.5 and 5.10 we see that (after possibly shrinking $\tilde{U}$) $\mathsf{res}_i(\rho_u)$ is the holonomy of a type 0 or type 1 cusp for each $[\rho_u] \in \tilde{U}$.

Finally, by Lemma 5.3 there is $[w] \in H^1(\Gamma, \mathfrak{v})$ whose restriction to each cusp is non-trivial and by Lemma 4.5 there is $[z] \in V$ such that $\pi_{\mathfrak{v}} \circ \mathsf{res}_*([z]) = [w]$. If we let ρ_t be a path through ρ_{hyp} in $\mathcal{F}$ tangent to z then Theorem 5.1 implies that these representations will be the holonomies of properly convex structures on M with type 1 cusps.

□

5.3. **Calculating slice coordinates.** In order to apply Theorem 5.1 it is necessary to be able to calculate the slice coordinates, or at least decide when they are non-zero. We close this section with a discussion about calculating slice coordinates using more easily accessible data. Recall that the Lie algebra $\mathfrak{g}$ admits a *Killing form* $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ be the given by $a \otimes b \mapsto 4\mathrm{tr}(ab)$. The Killing form is easily seen to be invariant under the adjoint action of G on $\mathfrak{g}$. If Π is the fundamental group of a closed n-manifold and $\rho : \Pi \to G$ is a representation, the Killing form gives rise to the *Poincaré duality pairing*

$$H^p_\rho(\Pi, \mathfrak{g}) \otimes H^{n-p}_\rho(\Pi, \mathfrak{g}) \xrightarrow{\cup} H^n_\rho(\Pi, \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{B} H^n(\Pi, \mathbb{R}) \cong \mathbb{R} \tag{5.7}$$

It is easy to check that the pairing in (5.7) respects the splitting $\mathfrak{g} \cong \mathfrak{so}(3,1) \oplus \mathfrak{v}$ and so we get

$$H^p_\rho(\Pi, \mathfrak{v}) \otimes H^{n-p}_\rho(\Pi, \mathfrak{v}) \xrightarrow{\cup} H^n_\rho(\Pi, \mathfrak{v} \otimes \mathfrak{v}) \xrightarrow{B} H^n(\Pi, \mathbb{R}) \cong \mathbb{R} \tag{5.8}$$

In both cases, the Poincaré duality pairing is non-degenerate. We will only have occasion to use this pairing in the simple setting where $n = 1$, in which case the construction can be made quite explicit. Specifically, for us $\Pi \cong \mathbb{Z}$ will be generated by the homotopy class γ of a closed loop in ∂M. In this case $H^0_\rho(\Pi, \mathfrak{v})$ can be identified with the $\rho(\gamma)$-invariant elements of $\mathfrak{v}$, which we henceforth denote $\mathfrak{v}^{\rho(\gamma)}$. If $[w] \in H^1_\rho(\Pi, \mathfrak{v})$ and $a \in \mathfrak{v}^{\rho(\gamma)}$ then $\langle [w], a\rangle = 4\mathrm{tr}(w(\gamma)a)$. We will now use these pairings to calculate c_a and c_b.

Once again, for simplicity, assume that M has a single cusp and that $\{\gamma^1_1, \gamma^1_2\}$ is a generating set for Δ. By conjugating we can assume that there are $(u_1, v_1) \in \mathbb{R}^2$ with $v > 0$ such that

$$\rho_{hyp}(\gamma^1_1) = \begin{pmatrix} 1 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \rho_{hyp}(\gamma^1_2) = \begin{pmatrix} 1 & u_1 & v_1 & \frac{u_1^2+v_1^2}{2} \\ 0 & 1 & 0 & u_1 \\ 0 & 0 & 1 & v_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In this setting the complex number $u_1 + iv_1$ is the cusp shape of the ∂_1 with respect to the generating set $\{\gamma^1_1, \gamma^1_2\}$. Let $[z] \in H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ and assume without loss of generality that $z = c_{\gamma^1_1}[z_{\gamma^1_1}] + c_{\gamma^1_2}[z_{\gamma^1_2}]$. From (3.2) and (3.3) it follows that

$$z(\gamma^1_1) = \begin{pmatrix} -c_a & & & \\ & 3c_a & & \\ & & -c_a & \\ & & & -c_a \end{pmatrix},\ z(\gamma^1_2) = \begin{pmatrix} -c_a u_1 - c_b v_1 & & & \\ & 3c_a u_1 - c_b v_1 & & \\ & & 3c_b v_1 - c_a u_1 & \\ & & & -c_a u_1 - c_b v_1 \end{pmatrix}$$

Next, let

$$\delta_{u,v} = \begin{pmatrix} -1 & & & \\ & -\frac{u^2-3v^2}{u^2+v^2} & -\frac{4uv}{u^2+v^2} & \\ & -\frac{4uv}{u^2+v^2} & \frac{3u^2-v^2}{u^2+v^2} & \\ & & & -1 \end{pmatrix}$$

It is easily checked that $\delta_{1,0} \in \mathfrak{v}^{\rho_{hyp}(\gamma^1_1)}$ and $\delta_{u_1,v_1} \in \mathfrak{v}^{\rho_{hyp}(\gamma^1_2)}$. Restricting to the subgroup generated by γ^1_1 (resp. γ^1_2) allows us to regard $[z]$ as an element of $H^1_{\rho_{hyp}}(\langle \gamma^1_1\rangle, \mathfrak{v})$ (resp. $H^1_{\rho_{hyp}}(\langle \gamma^1_2\rangle, \mathfrak{v})$), and computing pairings we find that

$$d^1_1 := \langle [z], \delta_{1,0}\rangle = -16 c_{\gamma^1_1},\ d^1_2 := \langle [z], \delta_{u_1,v_1}\rangle = -16\frac{c_{\gamma^1_1} u_1(u_1^2 - 3v_1^2) + c_{\gamma^1_2} v_1(v_1^2 - 3u_1^2)}{u_1^2 + v_1^2} \tag{5.9}$$

In other words, there is a linear relationship between the slice coordinates and the parings d^1_1 and d^1_2 and this linear relationship is encoded by the matrix

$$M(u_1, v_1) = -16 \begin{pmatrix} 1 & 0 \\ \frac{u_1(u_1^2-3v_1^2)}{u_1^2+v_1^2} & \frac{v_1(v_1^2-3u_1^2)}{u_1^2+v_1^2} \end{pmatrix},$$

By extending this discussion to the multiple cusp setting we can define the pairings d^j_1 and d^j_2 and the matrix $M(u_j, v_j)$, for $1 \leq j \leq k$ and to arrive at the following Proposition

Proposition 5.11. *Suppose that M has k cusps and is infinitesimally rigid rel. ∂M. Let $w_j = u_j + iv_j$ be the cusp shape of the jth cusp of M with respect to $\{\mu_j, \lambda_j\}$, and let*

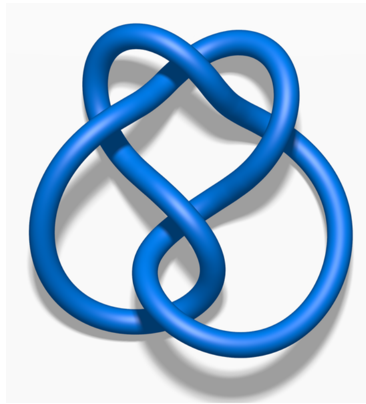

FIGURE 5. The 5_2 knot

and let

$$M = \begin{pmatrix} M(u_1, v_1) & & \\ & \ddots & \\ & & M(u_k, v_k) \end{pmatrix} \tag{5.10}$$

Let $\vec{c} = (c_{\gamma_1^1}, c_{\gamma_2^1}, \dots, c_{\mu_k}, c_{\lambda_k})$ *and let* $\vec{d} = (d_1^1, d_2^1, \dots, d_1^k, d_2^k)$. *If* $\mathsf{Arg}(w_j) \notin \frac{\pi}{3}\mathbb{Z}$ *for each* $1 \leq j \leq k$ *then* M *is invertible and*

$$M^{-1}\vec{d} = \vec{c}$$

Proof. The matrix M is invertible iff $M(u_j, v_j)$ is invertible for each $1 \leq j \leq k$. By examining determinants, it follows that $M(u_j, v_j)$ is singular if and only if $v_j^2 - 3u_j^2 = 0$. The equation $v_j^2 - 3u_j^2 = 0$ is satisfied iff $v_j = \pm\sqrt{3}u_j$ iff $\frac{v_j}{u_j} = \pm\sqrt{3}$. Since $\tan(\mathsf{Arg}(u_j + iv_j)) = \frac{v_j}{u_j}$ it follows that M is singular if and only if $\mathsf{Arg}(u_j + iv_j) \in \frac{\pi}{3}\mathbb{Z}$, thus by hypothesis, M is invertible.

By the discussion of the previous paragraph, $M\vec{c} = \vec{d}$, and the result follows. ☐

Remark 5.12. *By changing generating set for* Δ_i, *it is always possible to ensure that no cusps shape has argument that is an integral multiple of* $\pi/3$.

6. EXAMPLES

This section is dedicated to producing explicit examples of 1-cusped manifolds where Theorem 0.2 produces both type 1 and type 2 cusps. Specifically in Section 6.1, we show that if $M = S^3 \backslash K_{5_2}$ where K_{5_2} is the 5_2 knot (see Figure 5), then M admits a family of convex projective structures where the cusp is a type 2 generalized cusp. Then, in Section 6.2 we show that and that if $M = S^3 \backslash K_{6_3}$, where K_{6_3} is the 6_3 knot (see Figure 6) then M admits a convex projective structure where the cusp is a type 1 generalized cusp.

6.1. **The 5_2 knot complement.**

Before proceeding we mention that there are other recent examples of manifolds admitting type 2 cusps due to Martin Bobb [9], however his examples involve a version of bending for arithmetic manifolds. By work of Reid [22], the figure-eight knot is the only arithmetic knot complement, and so we see that the examples covered in this section are non-arithmetic and hence not covered by Bobb's work.

Let $M = S^3 \backslash K$ where K is the 5_2 knot, let $\Gamma = \pi_1 M$, let $\rho_{hyp} : \Gamma \to G$ be the holonomy of the complete hyperbolic structure on M, and let $\Delta \cong \mathbb{Z}^2$ be a peripheral subgroup of $\pi_1(M)$. In order to apply Theorem 0.2 we first need to check that M is infinitesimally rigid rel. ∂M.

Proposition 6.1. $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ *is 1-dimensional. In particular,* M *is infinitesimally rigid rel.* ∂M.

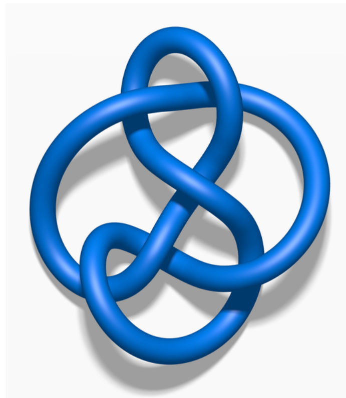

FIGURE 6. The 6_3 knot

Proof. The proof is computational and consists of computing the rank of a certain matrix with entries in a number field. This computation has been implemented in Sage [24] and can be found along with a detailed explanation in the following Sage notebook [1]. We now outline some of the relevant details.

Let $\Gamma = \pi_1 M$, then

$$\Gamma = \langle x, y \mid xwy^{-1}w^{-1} = 1\rangle,$$

where $w = yxy^{-1}x^{-1}yx$

Let $r = xwy^{-1}w^{-1}$, then there is an $\mathbb{R}$-linear map $\mathfrak{v} \times \mathfrak{v} \to \mathfrak{v}$ given by $(a, b) \mapsto \frac{\partial r}{\partial x} \cdot a + \frac{\partial r}{\partial y} \cdot b$, where $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$ are Fox derivatives and the action of $\mathbb{Z}[\Gamma]$ on $\mathfrak{v}$ is given by composing ρ_{hyp} with the adjoint action of $\mathrm{SO}(3, 1)$ on $\mathfrak{v}$. The kernel of this map is naturally isomorphic to the space $Z^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ of 1-cocycles. In [1] the rank of this map is computed to be 8. Since $\dim(\mathfrak{v}) = 9$ this implies that $Z^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is 10-dimensional.

The representation ρ_{hyp} is well known to be irreducible, which implies that $B^1_\rho(\Gamma, \mathfrak{v}) \cong \mathfrak{v}$. This space of coboundaries thus has dimension 9, and hence $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is 1-dimensional. Furthermore, by Lemma 2.2, the image of in $H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$ of $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ under the map res_* has dimension 1, and thus $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\Delta, \mathfrak{v})$ is an injection. □

Let γ_1 and γ_2 be the meridian and homologically determined longitude of 5_2, then it is easily checked (in SnapPy, for instance) that if z is the cusp shape of M with respect to this generating set then z is the unique complex root with positive imaginary part of the polynomial $56 - 4t + 2t^2 + t^3$. It is again easily checked that argument of this root is not an integral multiple of $\pi/3$, and it follows that we can use Proposition 5.11 to calculate the slice coordinates of $[z] \in H^1_{\rho_{hyp}}(M, \mathfrak{v})$.

Lemma 6.2. *If $[z]$ is a generator of $H^1_{\rho_{hyp}}(M, \mathfrak{v})$ and $[z] = c_a \frac{\partial}{\partial a} + c_b \frac{\partial}{\partial b}$ then $c_a, c_b \neq 0$.*

Proof. The proof is again a computation that involves calculating the matrix M from (5.10) and the pairings d_1 and d_2 in (5.9). This calculation is also implemented in the sage notebook [1] where it is shown that c_a and c_b are both non-zero. □

Combining these results we are able to prove the following:

Theorem 6.3. *The manifold M admits a properly convex projective structure whose end is a type 2 generalized cusp.*

Proof. By Lemma 6.2, the generator $[z]$ of $H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ can be written as $[z] = c_a \frac{\partial}{\partial a} + c_b \frac{\partial}{\partial b}$, where $c_a, c_b \neq 0$. The result then follows by applying Theorem 5.1. □

6.2. The 6_3 knot complement.

For this section let $M = S^3 \backslash K$ where K is the 6_3 knot, let $\Gamma = \pi_1 M$, let $\rho_{hyp} : \Gamma \to \mathrm{SL}_{\pm}(4, \mathbb{R})$ be the holonomy of the complete hyperbolic structure on M, and let $\Delta \cong \mathbb{Z}^2$ be a peripheral subgroup of $\pi_1(M)$.

Proposition 6.4. *$H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v})$ is 1-dimensional. In particular, M is infinitesimally rigid rel. ∂M.*

Proof. The proof is essentially the same as that of Proposition 6.1. The details of the computation can be found in [2]. □

Using the above result we can prove the following:

Theorem 6.5. *The manifold M admits a properly convex projective structure whose end is a type 1 generalized cusp.*

Proof. By Proposition 6.4 M is infinitesimally rigid rel. ∂M. The knot 6_3 is a two-bridge knot. It is well known that two-bridge knots are parameterized by a rational number p/q, with p odd and that a two-bridge knot is amphicheiral if and only if $p^2 = -1 \pmod q$. The rational number for the 6_3 knot is $5/13$, and it is thus amphicheiral. As a result, M admits a symmetry that preserves the homologically determined longitude, γ_+, and sends the meridian, γ_-, to its inverse. Furthermore, by the computation in [2] the map $\mathsf{res}_* : H^1_{\rho_{hyp}}(\Gamma, \mathfrak{v}) \to H^1_{\rho_{hyp}}(\gamma_+, \mathfrak{v})$ is non-trivial. The result then follows by applying Corollary 5.8. □

Remark 6.6. *In* [5]*, the author shows, using different methods, that if K is the figure-eight knot then $M = S^3 \backslash K$ a properly convex projective structure with type 1 cusps. However, the figure-eight knot satisfies the hypothesis of Corollary 5.8 and so these structures could also be constructed by the methods in this paper.*

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