DIMENSION FREE PROPERTIES OF STRONG MUCKENHOUPT AND REVERSE HÖLDER WEIGHTS FOR RADON MEASURES

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ABSTRACT. In this paper we prove self-improvement properties of strong Muckenhoupt and Reverse Hölder weights with respect to a general Radon measure on $\mathbb{R}^n$. We derive our result via a Bellman function argument. An important feature of our proof is that it uses only the Bellman function for the one-dimensional problem for Lebesgue measure; with this function in hand, we derive dimension free results for general measures and dimensions.

Keywords: Bellman function, Dimension free estimates, Muckenhoupt weights, Reverse Hölder weights.

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1. INTRODUCTION

It is well known that Muckenhoupt weights on a real line with respect to the Lebesgue measure satisfy self-improvement properties in the following sense: for $p > q$ we always have $A_q \subset A_p$; but also for any function $w \in A_p$, there is an $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$ (we refer to Definition 1 for precise definitions). Besides that, there always exists a q such that $w \in RH_q$. These self-improvement properties allow one to prove many important results in harmonic analysis, see, e.g., [4] or a more recent paper [5]. In [7], the authors considered *strong Muckenhoupt classes*; in particular, it was proven that for a Radon measure μ on $\mathbb{R}^n$ which is absolutely continuous with respect to the Lebesgue measure dx, any weight $w \in A_p^*$ satisfies a Reverse Hölder property with an exponent that does not depend on the dimension n.

For $p > 1$, we say that w belongs to the *strong Muckenhoupt class with respect to* μ, $w \in A_p^*$, if there exists a number $Q > 1$ such that for any rectangular box $R \subset \mathbb{R}^n$ with edges parallel to axis, we have

$$\langle w \rangle_R \langle w^{-1/(p-1)} \rangle_R^{p-1} \leqslant Q,$$

where $\langle \varphi \rangle_R$ denotes the average of the function φ over R:

$$\langle \varphi \rangle_R := \frac{1}{\mu(R)} \int_R \varphi(x) d\mu(x).$$

For $p > 1$, we say that w belongs to the *strong Reverse Hölder class with respect to* μ, $w \in RH_p^*$, if there exists a constant $Q > 1$ such that for any rectangular box R with edges parallel to axis, we have

$$\langle w^p \rangle_R^{1/p} \leqslant Q \langle w \rangle_R.$$

We proceed with the following definition.

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Definition 1. Let μ be a Radon measure on $\mathbb{R}^n$ and w be a function which is positive μ-a.e. For $p > 1$, we denote the *A_p^*-characteristic of w* by

$$[w]_p := \sup_R \langle w\rangle_R \langle w^{-1/(p-1)}\rangle_R^{p-1}$$

and the *Reverse Hölder characteristic of w* by

$$[w]_{RH_p} := \sup_R \langle w^p\rangle_R^{1/p} \langle w\rangle_R^{-1},$$

where both suprema are taken over rectangular boxes R with edges parallel to axis. If $[w]_p < \infty$, we have $w \in A_p^*$ and if $[w]_{RH_p} < \infty$, we have $w \in RH_p^*$.

In [7] it was proved that if μ is an absolutely continuous Radon measure on $\mathbb{R}^n$ and $[w]_p < \infty$, then for some $q > 1$ we have $[w]_{RH_q} < C < \infty$ with an explicit dimension free estimates on q and C. It is of a particular importance that we can take

$$q = 1 + \frac{1}{2^{p+2}[w]_p}.$$

To prove this result, the authors used a clever version of the Calderón–Zygmund decomposition from [6]. The aim of this paper is to derive a sharp result from the one-dimensional case for Lebesgue measure (i.e., for the classical A_p and RH_p classes on $\mathbb{R}$). In this case the result from [7] can be obtained, for example, by means of a so-called Bellman function; i.e., a function of two variables that satisfies certain boundary and concavity conditions in its domain. In the one-dimensional case this function is known explicitly, see [10]. It has been understood for some time that, for classes of functions like A_p, RH_p or BMO_p, when we work with their strong multi-dimensional analogs (e.g., A_p^* and RH_p^*), the one-dimensional Bellman function should prove the higher-dimensional results with dimension free constants. For the Lebesgue measure and the inclusion $RH_p^* \subset A_q^*$, this was done in [1]. The trick of using the Bellman function for one-dimensional problems was also used in [3], [2] and [9] (in a slightly different setting, the same trick was also used in [8]). In this paper, we present a simple version of this trick for general measures; we prove the result from [7] as well as all other results of self-improving type for strong Muckenhoupt and Reverse Hölder weights.

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2. Statement of the main result

2.1. **Properties of Muckenhoupt weights A_p^*.** For $p_1 := -1/(p-1)$ and every $t \in [0,1]$ define $u_{p_1}^{\pm}(t)$ to be solutions of the equation

$$(1-u)(1-p_1 u)^{-1/p_1} = t.$$

The function $u_{p_1}^{+}$ is decreasing and maps $[0,1]$ onto $[0,1]$; the function $u_{p_1}^{-}$ is increasing and maps $[0,1]$ onto $[1/p_1, 0]$. For a fixed $Q > 1$, define

$$s_{p_1}^{\pm} = s_{p_1}^{\pm}(Q) := u_{p_1}^{\pm}(1/Q). \tag{1}$$

Our first main result is the following.

Theorem 2.1. *Let μ be a Radon measure on $\mathbb{R}^n$ with $\mu(H)=0$ for every hyperplane H orthogonal to one of the coordinate axis. Fix numbers $p>1$ and $Q>1$ and set $p_1 := -1/(p-1)$. Then for every weight w with $[w]_p = Q$ we have*

$$w \in A_q^*, \quad 1-s_{p_1}^-(Q) < q < \infty,$$

and

$$w \in RH_q^*, \quad 1 \leqslant q < 1/s_{p_1}^+(Q),$$

where $s_{p_1}^{\pm}(Q)$ are defined in (1)*. These ranges for q are sharp for $n=1$ and $\mu = dx$.*

2.2. **Properties of Reverse Hölder weights.** For $p>1$ and every $t \in [0,1]$ we define $v_p^{\pm}(t)$ to be solutions of the equation

$$(1-pv)^{1/p}(1-v)^{-1} = t.$$

In this case, v_p^+ is a decreasing function that maps $[0,1]$ onto $[0,1/p]$ and v_p^- is an increasing function that maps $[0,1]$ onto $[-\infty,0]$. As before for a fixed $Q>1$, we define

$$s_p^{\pm} = s_p^{\pm}(Q) := v_p^{\pm}(1/Q). \tag{2}$$

Our second main result concerning Reverse Hölder weights is the following.

Theorem 2.2. *Let μ be a Radon measure on $\mathbb{R}^n$ with $\mu(H)=0$ for every hyperplane H orthogonal to one of the coordinate axis. Fix numbers $p>1$ and $Q>1$. Then for every weight w with $[w]_{RH_p} = Q$ we have*

$$w \in A_q^*, \quad 1-s_p^-(Q) < q < \infty,$$

and

$$w \in RH_q^*, \quad 1 \leqslant q < 1/s_p^+(Q),$$

where $s_p^{\pm}(Q)$ are defined in (2)*. These ranges for q are sharp for $n=1$ and $\mu = dx$.*

3. Proof of the main results

We begin with the following Theorem from [10]. This theorem ensures the existence of a certain Bellman function for a one-dimensional problem. In what follows, by letters without sub-indices (e.g., x, $x^{\pm}$) we denote points in $\mathbb{R}^2$ and by letters with sub-indices we denote the corresponding coordinates (e.g., x_1^+ denotes the first coordinate of x^+).

Theorem 3.1 (Theorem 1 in [10]). *Fix $p>1$ and set $p_1 := -1/(p-1)$. Also fix an $r \in (1/s_{p_1}^-, p_1] \cup [1, 1/s_{p_1}^+)$ for $s_{p_1}^{\pm}(Q)$ defined in* (1)*. For every $Q>1$ there exists a non-negative function $B_Q(x)$ defined in the domain $\Omega_Q := \{x=(x_1,x_2) \in \mathbb{R}^2 \colon 1 \leqslant x_1 x_2^{-1/p_1} \leqslant Q\}$ with the following property: $B_Q(x)$ is continuous in x and Q, and for any line segment $[x^-,x^+] \subset \Omega_Q$ and $x = \lambda x^- + (1-\lambda)x^+$, $\lambda \in [0,1]$, we have*

$$B_Q(x) \geqslant \lambda B_Q(x^-) + (1-\lambda) B_Q(x^+).$$

Moreover, $B(x_1, x_1^{p_1}) = x_1^r$ and $B_Q(x) \leqslant c(r,Q) x_1^r$ for some positive constant $c(r,Q)$ and every $x \in \Omega_Q$.

To use the concavity property of the function B_Q for our proof, we need the following lemma. Its proof is given in [10, Lemma4] with an interval instead of the rectangle; however, the proof remains the same in our case.

Lemma 3.2. *Let the measure μ be as before. Fix two numbers $Q_1 > Q > 1$ and a rectangular box $R \subset \mathbb{R}^n$ with edges parallel to the axis. For every coordinate vector* $\mathbf{e}$*, there exists a hyperplane H normal to* $\mathbf{e}$ *that splits R into two rectangular boxes R^1 and R^2 with the following properties:*

(i) For $i = 1,2$ we have $\mu(R^i)/\mu(R) \in (c, 1-c)$ for some constant $c \in (0,1)$;
(ii) For every weight w with $[w]_p \leqslant Q$, we have $[x^1, x^2] \subset \Omega_{Q_1}$ and, therefore,

$$B_{Q_1}(x_1,x_2) \geqslant \frac{\mu(R^1)}{\mu(R)} B_{Q_1}(x_1^1,x_2^1) + \frac{\mu(R^2)}{\mu(R)} B_{Q_1}(x_2^i,x_2^2),$$

where

$$x_1 = \langle w \rangle_R, \qquad x_2 = \langle w^{p_1} \rangle_R$$
$$x_1^i = \langle w \rangle_{R^i}, \qquad x_2^i = \langle w^{p_1} \rangle_{R^i}.$$

We are ready to prove our main result.

Proof of Theorem 2.1. Fix a rectangular box R with edges parallel to the axis, and take any $Q_1 > Q$. We first explain how we split R into two rectangular boxes. Take one of the $(n-1)$-dimensional faces of R, call it R_{n-1}, that has the largest $(n-1)$-area. Among all $(n-2)$-dimensional faces of R_{n-1}, take one of those (call it R_{n-2}) that have the largest $(n-2)$-area. We proceed like this to get $R_{n-1}, \ldots, R_1$. Now take a vector $\mathbf{e}$ that is orthogonal to every R_i, $i = 1, \ldots, n-1$.* We now split R according to Lemma 3.2. Notice that all the corresponding i-dimensional faces of R^1 and R^2 have smaller i-areas than the corresponding i-dimensional faces of R. We now take the boxes R^1 and R^2 and repeat the same procedure. If we repeat this M times, we get a family of rectangular boxes $\mathscr{R} = \{R^{i,M}\}_{i=1\ldots 2^M}$. Denote

$$x_1 = \langle w \rangle_R, \qquad x_2 = \langle w^{p_1} \rangle_R$$
$$x_1^{i,M} = \langle w \rangle_{R^{i,M}}, \qquad x_2^{i,M} = \langle w^{p_1} \rangle_{R^{i,M}}.$$

Abusing the notation, we also define step-functions

$$x_1^M(t) := \sum_{i=1}^{2^M} x_1^{i,M} \mathbb{1}_{R^{i,M}}(t), \qquad x_2^M(t) := \sum_{i=1}^{2^M} x_2^{i,M} \mathbb{1}_{R^{i,M}}(t).$$

From the construction of rectangular boxes, we notice that $x_1^M(t) \to w(t)$ and $x_2^M(t) \to w^{p_1}(t)$ as $M \to \infty$ for μ-a.e. $t \in R$. Indeed, our splitting procedure (and the fact that we have $\mu(R^i)/\mu(R) \in (1-c, c)$ at every step) guarantees that

$$\max_{i=1,\ldots,2^M} \operatorname{diam}(R^{i,M}) \to 0, \;\; M \to \infty,$$

and we obtain the convergence of $x_1^M(t)$ and $x_2^M(t)$ from the Lebegue differentiation theorem for Radon measures. Therefore,

$$B_{Q_1}(x_1,x_2) \geqslant \sum_{i=1}^{2^M} \frac{\mu(R^{i,M})}{\mu(R)} B_{Q_1}(x_1^{i,M}, x_2^{i,M}) = \frac{1}{\mu(R)} \int_R B_{Q_1}(x_1^M(t), x_2^M(t)) d\mu(t).$$

*In the case $n = 2$, we just take $\mathbf{e}$ orthogonal to the longest side of R.

By the Fatou lemma,

$$(3)\quad B_{Q_1}(x_1,x_2) \geqslant \frac{1}{\mu(R)}\int_R \lim_{M\to\infty} B_{Q_1}(x_1^M(t),x_2^M(t))d\mu(t) = \frac{1}{\mu(R)}\int_R B_{Q_1}(w(t),w^{p_1}(t))d\mu(t) = \frac{1}{\mu(R)}\int_R w^r(t)dt = \langle w^r\rangle_R.$$

Since $B_{Q_1}(x_1,x_2)$ is continuous in Q_1 and the above estimate holds for any $Q_1 > Q$, we get

$$c(r,Q)\langle w\rangle_R^r = c(r,Q)x_1^r \leqslant \langle w^r\rangle_R.$$

If we use this estimate for $q = r \in [1, 1/s_{p_1}^+)$, we obtain $w \in RH_q^*$. If we use this estimate for $-1/(q-1) = r \in (1/s_{p_1}^-, p_1]$, we obtain $w \in A_q^*$ for $1 - s_{p_1}^-(Q) < q < \infty$. ☐

To prove Theorem 2.2 we need to use a different Bellman function B_Q. Namely, the following result holds.

Theorem 3.3 (Theorem 1 in [10]). *Fix $p > 1$ and $r \in (1/s_p^-, 1] \cup [p, 1/s_p^+)$ for $s_p^\pm$ defined in* (2). *For every $Q > 1$ there exists a non-negative function $B_Q(x)$ defined in the domain $\Omega_Q := \{x = (x_1,x_2) \in \mathbb{R}^2 \colon 1 \leqslant x_1 x_2^{-1/p} \leqslant Q\}$ with the following property: $B_Q(x)$ is continuous in x and Q, and for any line segment $[x^-,x^+] \subset \Omega_Q$ and $x = \lambda x^- + (1-\lambda)x^+$, $\lambda \in [0,1]$, we have*

$$B_Q(x) \geqslant \lambda B_Q(x^-) + (1-\lambda)B_Q(x^+).$$

Moreover, $B(x_1, x_1^p) = x_1^r$ and $B_Q(x) \leqslant c(r,Q)x_1^r$ for some positive constant $c(r,Q)$ and every $x \in \Omega_Q$.

We also notice that the analog of Lemma 3.2 reads the same, and with this in hand, the proof of Theorem 2.1 is analogous to the proof of Theorem 2.2; we leave the details to the reader.

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