ADIC TROPICALIZATIONS AND COFINALITY OF GUBLER MODELS

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Abstract. We introduce adic tropicalizations for subschemes of toric varieties as limits of Gubler models associated to polyhedral covers of the ordinary tropicalization. Our main result shows that Huber's adic analytification of a subscheme of a toric variety is naturally isomorphic to the inverse limit of its adic tropicalizations, in the category of locally topologically ringed spaces. The key new technical idea underlying this theorem is cofinality of Gubler models, which we prove for projective schemes and also for more general compact analytic domains in closed subschemes of toric varieties. In addition, we introduce a G-topology and structure sheaf on ordinary tropicalizations, and show that Berkovich analytifications are limits of ordinary tropicalizations in the category of topologically ringed topoi.

1. Introduction

Let X be a separated scheme of finite type over a complete and algebraically closed, nontrivially valued field K. In previous work, we related the underlying topological space of the nonarchimedean analytification X^{an}, in the sense of Berkovich [Ber90], to limits of tropicalizations for inverse systems of toric embeddings [Pay09, FGP14]. Here, we deepen those results by studying analogous limits in the category of locally topologically ringed spaces, establishing connections to adic analytifications in the sense of Huber, and accounting for the structure sheaf on X^{an}.

Our starting point is the well-known fact that tropical geometry provides a systematic method for producing formal models of X from a toric embedding $\iota : X \hookrightarrow Y_\Sigma$ plus combinatorial data. Roughly speaking, a polyhedral complex Δ that covers the ordinary tropicalization $\mathrm{Trop}(X, \iota)$ gives rise to a formal model $\mathfrak{X}_\Delta$, which we refer to as a *Gubler model*. Furthermore, any refinement Δ' of Δ gives rise to a natural map of Gubler models $\mathfrak{X}_{\Delta'} \to \mathfrak{X}_\Delta$. See §3 for details of these constructions. We define the *adic tropicalization* to be the locally ringed space

$$\mathfrak{Trop}(X, \iota) = \varprojlim_\Delta \mathfrak{X}_\Delta,$$

where the limit is taken over all morphisms of Gubler models induced by polyhedral refinements.

We focus on inverse systems of toric embeddings $\mathcal{S}$ of X that satisfy the following condition.

(†) There is an affine open cover $X = U_1 \cup \cdots \cup U_r$ such that, for any finite sets of regular functions $R_1 \subset K[U_1], \ldots, R_r \subset K[U_r]$, there is an embedding $\iota \in \mathcal{S}$ such that U_j is the preimage of a torus invariant affine open set U_{σ_j} and each element of R_j is the pullback of a character that is regular on U_{σ_j}, for each $1 \le j \le r$.

This condition is stronger than the condition $(\star)$ appearing in [FGP14]. While $(\star)$ is sufficient to recover the underlying topological space of X^{an} as a limit of tropicalizations, a stronger condition seems necessary for recovering finer information, such as structure sheaves. We emphasize that the inverse system of all toric embeddings of X does satisfy (†), with respect to any affine open cover, provided that X admits a single toric embedding [FGP14, Theorem 4.2].

Given an inverse system of toric embeddings $\mathcal{S}$ satisfying (†), our first result expresses the adic analytification X^{ad} as a limit of adic tropicalizations in the category of locally topologically ringed spaces. (See §2 for background on Huber's adic spaces, including the construction of X^{ad}.)

Theorem 1.1. *Let $\mathcal{S}$ be an inverse system of toric embeddings that satisfies (†). Then the induced map*

$$(X^{\mathrm{ad}}, \mathscr{O}^{\circ}_{X^{\mathrm{ad}}}) \longrightarrow \varprojlim_{\mathcal{S}} \big(\mathfrak{Trop}(X, \iota), \mathscr{O} \big)$$

is an isomorphism of locally topologically ringed spaces.

This theorem is one way of quantifying the idea that tropical geometry produces many formal models of algebraic varieties. When X is projective, we prove that every formal model is dominated by a Gubler model; see Theorem 4.1.3.

Tropical geometry also gives natural methods for constructing formal models of analytic subdomains in X^{an}. Indeed, if $\mathfrak{X}_{\Delta}$ is a Gubler model, then each subcomplex $\Delta' \subset \Delta$ naturally gives rise to a formal open subscheme $\mathfrak{X}_{\Delta'} \subset \mathfrak{X}_{\Delta}$. Then $\mathfrak{X}_{\Delta'}$ is an admissible formal model of the analytic subdomain $\mathrm{Trop}^{-1}(|\Delta'|) \subset X^{\mathrm{an}}$, which is compact if Δ' is finite.

Theorem 1.2. *Let $\mathcal{S}$ be an inverse system of toric embeddings of X that satisfies (†), and let $V \subset X^{\mathrm{an}}$ be a compact analytic domain. Then, for any formal model $\mathfrak{V}$ of V, there is a Gubler model $\mathfrak{X}_{\Delta}$ associated to some $\iota \in \mathcal{S}$, and a finite subcomplex $\Delta' \subset \Delta$ such that $\mathfrak{X}_{\Delta'}$ is an admissible formal model of V that dominates $\mathfrak{V}$.*

This does not prove existence of Gubler models with desirable properties such as mild singularities; even if X is smooth and projective and $\mathfrak{V}$ is semistable, one does not have control over the singularities of the Gubler models $\mathfrak{X}_{\Delta'}$ that dominate $\mathfrak{V}$.

Arguments similar to those used in the proof of Theorem 1.1 also give a natural way of realizing the Berkovich analytification X^{an} as a limit of tropicalizations in a way that accounts for the structure sheaf. Recall that the structure sheaf on X^{an} is defined with respect to the G-topology (a Grothendieck topology in which the "admissible open subsets" are actual subsets, fiber products are intersections, and admissible covers are a distinguished class of set-theoretic covers by admissible subsets), given by analytic domains and admissible covers.

In §5, we define an analogous G-topology on the ordinary tropicalization $\mathrm{Trop}(X, \iota)$, along with a structure sheaf $\mathscr{O}^{\mathrm{trop}}$ of topological R-algebras in this G-topology, such that the tropicalization map $\mathrm{Trop} : X^{\mathrm{an}} \longrightarrow \big(\mathrm{Trop}(X, \iota), K \otimes_R \mathscr{O}^{\mathrm{trop}} \big)$ is a morphism of topologically G-ringed spaces, as are the projections between tropicalizations associated to morphisms of toric embeddings.

Theorem 1.3. *Let $\mathcal{S}$ be an inverse system of toric embeddings that satisfies (†). Then the natural map*

$$X^{\mathrm{an}} \longrightarrow \varprojlim_{\mathcal{S}} \big(\mathrm{Trop}(X, \iota)_G, K \otimes_R \mathscr{O}^{\mathrm{trop}} \big)$$

induces an equivalence of locally ringed topoi.

Remark 1.4. If the residue field of K is $\mathbb{C}$, and if $\iota \colon X \hookrightarrow Y_\Sigma$ is a closed embedding into a toric variety such that each initial degeneration $\mathrm{in}_v(X)$ is smooth, then the disjoint union of the initial degenerations $\mathrm{in}_v(X)(\mathbb{C})$, for $v \in \mathrm{Trop}(X, \iota)$, is naturally identified with an object in Parker's category of exploded manifolds [Par12]. Thus our construction of the adic tropicalization $\mathfrak{Trop}(X, \iota)$ provides both a Zariski topology and a sheaf of topological rings on any exploded manifold arising in this way. In situations where some of the initial degenerations $\mathrm{in}_v(X)$ are not smooth, or the residue field is not $\mathbb{C}$, our construction provides an algebro-geometric analogue of Parker's exploded manifolds, allowing singularities and non-reduced structures. When $\dim X = 1$, adic tropicalization is closely related to the metrized curve complexes of Amini and Baker [AB15]. For additional

remarks on the relationship between adic tropicalization and exploded manifolds, and a thorough discussion of the relation between adic tropicalization and metrized curve complexes, see [Fos16].

Remark 1.5. See also [FR16b, FR16a] for relations between adic analytifications and limits of tropicalizations over higher rank valued fields.

2. Preliminaries

Throughout, we fix an algebraically closed[1] nonarchimedean field K, which is complete with respect to a nontrivial valuation

$$\mathrm{val}\colon K \longrightarrow \mathbb{R} \sqcup \{\infty\}.$$

Let R be the ring of integers in K, with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Fix a real number $0 < \varepsilon < 1$, and let $|-|\colon K \longrightarrow \mathbb{R}_{\geqslant 0}$ denote the induced norm, given by

$$|a| = \varepsilon^{\mathrm{val}(a)}.$$

In this section, we briefly recall the basic notions that we need from the theory of K-analytic spaces, in the sense of Berkovich [Ber90], admissible formal models, in the sense of Raynaud [Bos14], and adic spaces in the sense of Huber [Hub94]. For an expanded expository presentation, motivated by the results of this paper and including further references, see also [Fos16, §2].

2.1. **Berkovich spectra.** Let $A \cong K\langle T_1, \ldots, T_n\rangle/\mathfrak{a}$ be an affinoid algebra. The Berkovich spectrum $\mathscr{M}(A)$ is the set of continuous seminorms $|-|\colon A \to \mathbb{R}_{\geqslant 0}$ that extend the given norm on K, equipped with the subspace topology for the inclusion $\mathscr{M}(A) \subset A^{\mathbb{R}_{\geqslant 0}}$. The Berkovich spectrum $\mathscr{M}(A)$ also carries a natural G-topology, which refines the ordinary topology, and can be described as follows.

Given elements $f_1, \ldots, f_m, g$ of A with no common zero, let

$$B = A\langle X_1, \ldots, X_m\rangle/(gX_1 - f_1, \ldots, gX_m - f_m)$$

be the corresponding *rational algebra*. The presentation $A\langle X_1, \ldots, X_m\rangle \longrightarrow\!\!\!\!\!\rightarrow B$ induces an inclusion $\mathscr{M}(B) \subset \mathscr{M}(A)$, and the subsets that occur in this way are called *rational subdomains*. A subset $V \subset \mathscr{M}(A)$ is *admissible* in the G-topology if every point $x \in V$ has a neighborhood of the form $V_1 \cup \cdots \cup V_n$, where each V_i is a rational domain, and $x \in V_1 \cap \cdots \cap V_n$. Similarly, the *admissible covers* in the G-topology are covers of admissible subsets by admissible subsets $V = \bigcup_{i \in I} V_i$ such that every point $x \in V$ has a neighborhood which is a finite union of sets in the cover $V_{i_1} \cup \cdots \cup V_{i_n}$, with $x \in V_{i_1} \cap \cdots \cap V_{i_n}$. The structure sheaf on $\mathscr{M}(A)$ is the unique sheaf of topological rings

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[1] The assumption that K is algebraically closed is used in the following ways. First, in the introduction, we mention that the system of all toric embeddings of any closed subscheme of a toric variety satisfies (†), referring to [FGP14, Theorem 4.2]. The statement of that theorem should have included the hypothesis that K be algebraically closed; it is required in the proof in order to apply the embedding algorithm from [Wło93]. Next, K being algebraically closed guarantees that the value group Γ is divisible, which simplifies the discussion of (Γ, Σ)-admissible polyhedra in §3.2. If Γ is neither discrete nor divisible, then these should also be required to have vertices in N_Γ. Finally, in the proofs of our main results, we use Friedenberg's theorem on existence of locally finite completions of (Γ, Σ)-admissible complexes [Fri19]. His construction produces complexes with vertices in the divisible hull of Γ, and we do not know how to overcome the resulting difficulties when Γ is neither discrete nor divisible.

in this G-topology whose value on a rational domain $\mathscr{M}(B)$ is the rational algebra B, and whose restriction maps between these rational algebras are the natural ones. See [Ber90, Ber93] for further details.

2.2. Adic spectra. Given an affinoid algebra $A \cong K\langle T_1, \ldots, T_n \rangle / \mathfrak{a}$, let $A^\circ \subset A$ be the subring of power bounded elements. The *adic spectrum* of the pair (A, A°), denoted $\mathrm{Spa}(A, A^\circ)$, is the set of all equivalence classes of continuous seminorms $\| - \|_x \colon A \longrightarrow \{0\} \sqcup \Gamma$ such that $\|a\|_x \leq 1$ for all $a \in A^\circ$.[2] Here Γ denotes any totally ordered abelian group, written multiplicatively, and $\{0\} \sqcup \Gamma$ is the totally ordered abelian semigroup in which $0 \cdot \gamma = 0$ and $0 < \gamma$ for all $\gamma \in \Gamma$.

The equivalence relation is the smallest such that continuous seminorms

$$\| - \|_x \colon A \longrightarrow \{0\} \sqcup \Gamma \qquad \text{and} \qquad \| - \|_{x'} \colon A \longrightarrow \{0\} \sqcup \Gamma',$$

are equivalent whenever there is an inclusion $\alpha \colon \{0\} \sqcup \Gamma \hookrightarrow \{0\} \sqcup \Gamma'$ of ordered abelian semigroups such that $\alpha \circ \| - \|_x = \| - \|_{x'}$.

If $B = A\langle X_1, \ldots, X_m \rangle / (gX_1 - f_1, \ldots, gX_m - f_m)$ is a rational algebra, with the topology induced by the quotient norm, then the presentation $A\langle X_1, \ldots X_m \rangle \longrightarrow B$ induces an inclusion $\mathrm{Spa}(B, B^\circ) \subset \mathrm{Spa}(A, A^\circ)$. The subsets that occur in this way are called *rational subsets*, and rational subsets generate the topology on $\mathrm{Spa}(A, A^\circ)$.

The structure sheaf $\mathscr{O}$ is the unique sheaf that takes the value B on a rational subset $\mathrm{Spa}(B, B^\circ)$, and whose restriction maps between such rational algebras are the natural ones. The stalk of this structure sheaf $\mathscr{O}$ at any point $x \in \mathrm{Spa}(A, A^\circ)$ is a topological local ring, and the seminorm $\| - \|_x$ induces a continuous seminorm on $\mathscr{O}_x$. We also consider the subsheaf $\mathscr{O}^\circ$ of power bounded functions, whose value on an open subset U is the ring of sections $f \in \mathscr{O}_X(U)$ for which $\|f\|_x \leq 1$ at every point $x \in U$. If $U = \mathrm{Spa}(B, B^\circ)$ is a rational subset, then $\mathscr{O}^\circ(U) = B^\circ$.

2.3. Analytic spaces. A K-*analytic space*, in the sense of Berkovich, is a topologically G-ringed space X with an atlas of admissible subspaces $V_i \subset X$, whose charts are isomorphic to Berkovich spectra of affinoid K-algebras, with the additional condition that each point has a neighborhood which is a finite union of affinoids in the atlas.

Similarly, an *adic space* over K is a locally topologically ringed space $(Z, \mathscr{O}_Z)$ with an atlas $\{V_i \hookrightarrow Z\}$, whose charts are adic spectra of K-affinoid algebras.

There is a natural functor from K-analytic spaces to adic spaces, defined as follows. Suppose V is a K-analytic space, with an atlas of Berkovich spectra $V_i \cong \mathscr{M}(A_i)$, glued along inclusions induced by morphisms of affinoid algebras $f_{ij} \colon A_i \to A_j$ for $V_j \subset V_i$. Then the associated adic space V^{ad} has an atlas of adic spectra $\mathrm{Spa}(A_i, A_i^\circ)$, glued along the inclusions induced by the $\{f_{ij}\}$. If X is a separated scheme of finite type over K, we write X^{ad} for the *adic analytification* of X, i.e., the adic space associated to the Berkovich analytification X^{an}.

2.4. Admissible formal models. Many of the relations between adic spaces and K-analytic spaces, including those arising through analytification and tropicalization of algebraic schemes, are best understood in terms of Raynaud's theory of admissible formal models. For an accessible presentation of this theory, see [Bos14].

A topological R-algebra A is *admissible* if there is an isomorphism of topological rings

$$A \cong R\langle t_1, \ldots, t_n \rangle / \mathfrak{a},$$

where $R\langle t_1, \ldots, t_n \rangle / \mathfrak{a}$ has its $\mathfrak{m}$-adic topology, such that:

(i) the ideal $\mathfrak{a} \subset R\langle t_1, \ldots, t_n \rangle$ is finitely generated;

(ii) the ring $R\langle t_1, \ldots, t_n \rangle / \mathfrak{a}$ is free of $\mathfrak{m}$-torsion.

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[2]The adic spectrum is defined similarly for pairs (A, A^+), where A^+ is a subring of the power bounded elements [Hub96, §1.1], but we will only need the case where $A^+ = A^\circ$.

An *admissible formal R-scheme* is a formal scheme over $\mathrm{Spf}\, R$ that admits an open covering by formal spectra $\mathfrak{U}_i = \mathrm{Spf}\, A_i$, such that each A_i is an admissible R-algebra.

Let $\mathfrak{X}$ be an admissible formal R-scheme, and let $\{\mathfrak{U}_i \hookrightarrow \mathfrak{X}\}$ be an open cover of $\mathfrak{X}$ by formal spectra of admissible R-algebras A_i. Each topological algebra $K \otimes_R A_i$ is K-affinoid [Bos14, §7.4]. Moreover, the Berkovich spectra $\mathscr{M}(K \otimes_R A_i)$ of these K-affinoid algebras glue to produce a K-analytic space over K, in the sense of [Ber90, §3.1]. We denote this K-analytic space $\mathfrak{X}^{\mathrm{an}}$; it is the *Raynaud fiber* of $\mathfrak{X}$.

Definition 2.4.1. (Admissible formal models of an analytic space). *Let V be any K-analytic space over K. An* admissible formal model of V *is an admissible formal R-scheme $\mathfrak{V}$ together with an isomorphism of K-analytic spaces $\mathfrak{V}^{\mathrm{an}} \xrightarrow{\sim} V$.*

A *morphism* of admissible formal models of V is a morphism $\mathfrak{V}_1 \longrightarrow \mathfrak{V}_2$ whose induced morphism $\mathfrak{V}_1^{\mathrm{an}} \longrightarrow \mathfrak{V}_2^{\mathrm{an}}$ of K-analytic spaces commutes with the isomorphisms $\mathfrak{V}_1^{\mathrm{an}} \xrightarrow{\sim} V$ and $\mathfrak{V}_2^{\mathrm{an}} \xrightarrow{\sim} V$.

2.5. **Quasi-compact adic spaces are limits of formal models.** Let $\mathfrak{X}$ be an admissible formal R-scheme. We write $\mathfrak{X}^{\mathrm{ad}}$ for the adic space associated to $\mathfrak{X}^{\mathrm{an}}$, and refer to this as the *adic Raynaud fiber* of $\mathfrak{X}$. It comes with a *specialization morphism*

$$\mathrm{sp}_{\mathfrak{X}} \colon (\mathfrak{X}^{\mathrm{ad}}, \mathscr{O}_{\mathfrak{X}^{\mathrm{ad}}}^{\circ}) \longrightarrow (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$$

of locally topologically ringed spaces over $\mathrm{Spec}\, K$. Here, $\mathscr{O}_{\mathfrak{X}^{\mathrm{ad}}}^{\circ} \subset \mathscr{O}_{\mathfrak{X}^{\mathrm{ad}}}$ denotes the subsheaf of power bounded analytic functions on $\mathfrak{X}^{\mathrm{ad}}$.

When $\mathfrak{X}^{\mathrm{ad}}$ is quasi-compact, these specialization morphisms induce a natural isomorphism

$$(\mathfrak{X}^{\mathrm{ad}}, \mathscr{O}_{X^{\mathrm{ad}}}^{\circ}) \xrightarrow{\sim} \varprojlim \mathfrak{X}'$$

in the category of locally toplogically ringed space, where the inverse limit is taken over the category of formal models $\mathfrak{X}'$ of $\mathfrak{X}^{\mathrm{an}}$ [Sch12, Theorem 2.22].

3. Adic tropicalization

We now define the adic tropicalization of a subscheme of a toric variety. Its underlying set is the disjoint union of all initial degenerations. This is equipped with the structure of a locally topologically ringed space, via an indentification with the inverse limit of all formal models associated to admissible polyhedral covers of the ordinary extended tropicalization.

3.1. **Tropicalization.** We briefly recall the basic notion of the ordinary *extended tropicalization*, hereafter referred to as *tropicalization*, for subschemes of toric varieties. See [Pay09, §3] for further details. Let Y_{Σ} be the toric variety over K associated to a fan Σ in $N_{\mathbb{R}}$. Each cone $\sigma \in \Sigma$ corresponds to an affine torus-invariant open subvariety $U_{\sigma} \subset Y_{\Sigma}$, whose coordinate ring is the semigroup ring $K[S_{\sigma}]$ generated over K by the semigroup S_{σ} of characters of the dense torus that extend to regular functions on U_{σ}. We equip $\mathbb{R} \sqcup \{\infty\}$ with the topology that makes the exponential map $\mathbb{R} \sqcup \{\infty\} \longrightarrow \mathbb{R}_{\geq 0}$, given by $x \mapsto e^{-x}$, a homeomorphism. The tropicalization of U_{σ} is the space of semigroup homomorphisms

$$\mathrm{Trop}(U_{\sigma}) = \mathrm{Hom}(S_{\sigma}, \mathbb{R} \sqcup \{\infty\}),$$

with the topology induced by that of $\mathbb{R} \sqcup \{\infty\}$.

Just as U_{σ} decomposes as a disjoint union of torus orbits O_{τ} corresponding to the faces $\tau \preceq \sigma$, the tropicalization $\mathrm{Trop}(U_{\sigma})$ decomposes as a disjoint union of real vector spaces

$$\mathrm{Trop}(U_{\sigma}) = \bigsqcup_{\tau \preceq \sigma} N_{\mathbb{R}}/\mathrm{span}(\tau).$$

Here each $N_{\mathbb{R}}/\mathrm{span}(\tau)$ is canonically identified with the tropicalization $\mathrm{Trop}(O_{\tau})$ of the open torus orbit $O_{\tau} \subset U_{\sigma}$ corresponding to τ. Each inclusion of faces $\tau \preceq \sigma$ induces open immersions $U_{\tau} \subset U_{\sigma}$

and $\text{Trop}(U_\tau) \subset \text{Trop}(U_\sigma)$. The tropicalization $\text{Trop}(Y_\Sigma)$ is obtained by gluing $\{\text{Trop}(U_\sigma)\}_{\sigma \in \Sigma}$ along the open immersions $\text{Trop}(U_\tau) \subset \text{Trop}(U_\sigma)$, just as the toric variety Y_Σ is obtained by gluing $\{U_\sigma\}_{\sigma \in \Sigma}$ along the open immersions $U_\tau \subset U_\sigma$. The natural tropicalization maps on affine open toric subvarieties glue to give a proper continuous surjection

$$\text{Trop}\colon Y_\Sigma^{\text{an}} \longrightarrow\!\!\!\rightarrow \text{Trop}(Y_\Sigma).$$

Now, and for the remainder of the paper, we fix a separated scheme X of finite type over K. Let $\iota\colon X \hookrightarrow Y_\Sigma$ be a closed embedding in a toric variety. Then the tropicalization

$$\text{Trop}(X, \iota) \subset \text{Trop}(Y_\Sigma)$$

is the image of the closed subset $\iota(X)^{\text{an}} \subset Y_\Sigma^{\text{an}}$ under Trop.

Let $\mathfrak{a} \subset K[M]$ denote the ideal cutting out $\iota(X) \cap T$, where T denotes the dense torus in Y_Σ. Every point $v \in N_\mathbb{R}$ has an associated R-scheme

$$\text{Spec } R[M]^v / (\mathfrak{a} \cap R[M]^v), \tag{1}$$

where

$$R[M]^v := \left\{ \sum_{u \in M} a_u \chi^u \in K[M] : \text{val}(a_u) + \langle u, v \rangle \geq 0 \text{ for all } u \in M \right\}.$$

The special fiber of (1) is the *initial degeneration of X at v*, denoted $\text{in}_v(X)$; it is nonempty if and only if $v \in \text{Trop}(X, \iota)$.

More generally, for any $w \in \text{Trop}(Y_\Sigma)$, there is a unique cone $\sigma \in \Sigma$ such that w lies in the tropicalization $\text{Trop}(O_\sigma) \cong N_\mathbb{R}/\text{span}(\sigma)$ of the open torus orbit $O_\sigma \subset Y_\Sigma$. In this case, $M_\sigma := \text{span}(\sigma)^\perp \cap M$ is dual to the lattice $\text{span}(\sigma) \cap N$, and we define

$$R[M_\sigma]^w := \left\{ \sum_{u \in M_\sigma} a_u \chi^u \in K[M_\sigma] : \text{val}(a_u) + \langle u, w \rangle \geq 0 \text{ for all } u \in M_\sigma \right\}.$$

The *initial degeneration of X at w*, denoted $\text{in}_w(\iota(X) \cap O_\sigma)$, is the special fiber of the scheme $\text{Spec } R[M_\sigma]^w / (\mathfrak{a} \cap R[M_\sigma]^w)$. For each cone $\sigma \in \Sigma$, the intersection of $\text{Trop}(X, \iota)$ with $\text{Trop}(O_\sigma)$ is a finite polyhedral complex that parametrizes weight vectors on monomials in M_σ such that $\text{in}_w(\iota(X) \cap O_\sigma)$ is nonempty. For simplicity, given $\iota\colon X \hookrightarrow Y_\Sigma$, $\sigma \in \Sigma$, and $w \in N_\mathbb{R}/\text{span}(\sigma)$, we sometimes write $\text{in}_w(X)$ for $\text{in}_w(\iota(X) \cap O_\sigma)$. With this notation,

$$\text{Trop}(X, \iota) = \{w \in \text{Trop}(Y_\Sigma) : \text{in}_w(X) \neq \emptyset\}.$$

3.2. **Admissible polyhedral covers.** Let $\Gamma \subset \mathbb{R}$ be the value group of K. As in the previous section, we consider the toric variety Y_Σ over K associated to a fan Σ in $N_\mathbb{R}$.

Let $P \subset N_\mathbb{R}$ be a polyhedron, the intersection of finitely many closed halfspaces. Then the *recession cone σ_P* is the closed polyhedral cone given by

$$\sigma_P = \{v \in N_\mathbb{R} : v + P \subset P\}.$$

Equivalently, the cone σ_P is obtained by taking the closure of the cone over $P \times \{1\}$ in $N_\mathbb{R} \times \mathbb{R}$ and intersecting with $N_\mathbb{R} \times \{0\}$.

A polyhedron $P \subset N_\mathbb{R}$ is (Γ, Σ)-*admissible* if its recession cone σ_P is in Σ, and P itself can be expressed as an intersection of finitely many halfspaces

$$P = \{v \in N_\mathbb{R} : \langle u_i, v \rangle \geqslant \gamma_i, \text{ for } 1 \leq i \leq n\},$$

with u_i in the character lattice M and γ_i in Γ. Note that any (Γ, Σ)-admissible polyhedron is pointed, i.e., its minimal faces have dimension zero, since the tail cone is in the fan Σ, which is a collection of pointed cones.

An *extended (Γ, Σ)-admissible polyhedron* is the closure $\overline{P}$ in $\text{Trop}(Y_\Sigma)$ of a pointed (Γ, Σ)-admissible polyhedron P in $N_\mathbb{R}$. An *extended (Γ, Σ)-admissible polyhedral complex* Δ is a locally

finite collection of extended (Γ, Σ)-admissible polyhedra, which we refer to as the *faces* of the complex, whose intersections with $N_{\mathbb{R}}$ form a polyhedral complex. The *support* $|\Delta|$ of Δ is the union of its faces, and Δ is *complete* if $|\Delta| = \mathrm{Trop}(Y_\Sigma)$. Note that we require an extended (Γ, Σ)-admissible complex to be locally finite not only at points in the dense subset $N_{\mathbb{R}}$, but at every point $w \in |\Delta|$. See also Remark 3.3.1, below.

Remark 3.2.1. Recall that Γ-admissible fans in $N_{\mathbb{R}} \times \mathbb{R}_{\geqslant 0}$, in the sense of [Gub13, §7], correspond naturally and bijectively with toric schemes over the valuation ring $R \subset K$. The basic construction is recalled in §3.3 below. For now, note that extended (Γ, Σ)-admissible polyhedra correspond naturally and bijectively with Γ-admissible cones in $N_{\mathbb{R}} \times \mathbb{R}_{\geqslant 0}$ that meet $N_{\mathbb{R}} \times \mathbb{R}_{>0}$ and whose intersection with $N_{\mathbb{R}} \times \{0\}$ is a face of Σ. This correspondence takes an extended (Γ, Σ)-admissible polyhedron $\overline{P}$ to the closure of the cone over $P \times \{1\}$. Similarly, locally finite extended (Γ, Σ)-admissible polyhedral complexes correspond naturally and bijectively with the locally finite Γ-admissible fans in $N_{\mathbb{R}} \times \mathbb{R}_{\geqslant 0}$ whose restriction to $N_{\mathbb{R}} \times \{0\}$ is Σ. The latter correspond, in turn, with locally finite type toric schemes over the valuation ring whose general fiber is Y_Σ.

We will use the following lemma, on existence of complete complexes that simultaneously refine any finite collection of admissible polyhedra, in the proofs of Theorems 1.2 and 4.1.3. It is an immediate consequence of Friedenberg's completion theorem [Fri19], which says that, for any finite extended (Γ, Σ)-admissible polyhedral complex Δ', there is a complete, locally finite extended (Γ, Σ)-admissible polyhedral complex Δ that contains Δ' as a subcomplex.

Lemma 3.2.2. *Let $\overline{P}_1, \ldots, \overline{P}_n$ be extended (Γ, Σ)-admissible polyhedra. Then there is a complete, locally finite extended (Γ, Σ)-admissible polyhedral complex Δ such that each $\overline{P}_i$ is a union of faces of Δ, for $1 \leq i \leq n$.*

Proof. By Friendenberg's completion theorem, for each i there is a complete locally finite extended (Γ, Σ)-admissible complex Δ_i that contains $\overline{P}_i$ as a face. Then we can take Δ to be the smallest common refinement of $\Delta_1, \ldots, \Delta_n$. $\qquad\square$

3.3. **Gubler models of toric varieties.** If $\overline{P} \subset \mathrm{Trop}(Y_\Sigma)$ is an extended (Γ, Σ)-admissible polyhedron, then $\mathrm{Trop}^{-1}(\overline{P})$ is an affinoid analytic domain in Y_Σ^{an}, and is canonically realized as the Raynaud fiber of the formal completion of an integral model of the affine open subvariety $U_{\sigma_P} \subset Y_\Sigma$, as we now explain. Let $R[M]^P$ denote the R-algebra

$$R[M]^P \;=\; \left\{ \sum_{u \in M} a_u \, \chi^u \in K[M] \;:\; \mathrm{val}(a_u) + \langle u, v \rangle \geq 0 \;\; \text{for all } v \in P \right\}, \qquad (2)$$

and let $\mathscr{U}_P$ denote the R-scheme $\mathscr{U}_P = \mathrm{Spec}\, R[M]^P$. By [Gub13, Propositions 6.6 and 6.10], $\mathscr{U}_P$ is a normal scheme flat over R with generic fiber $(\mathscr{U}_P)_K := \mathscr{U}_P \otimes_R K$ naturally isomorphic to the affine toric variety associated to the recession cone σ_P:

$$(\mathscr{U}_P)_K \;\cong\; U_{\sigma_P}.$$

Let $\mathfrak{U}_P$ denote the formal R-scheme obtained as the completion of $\mathscr{U}_P$ along its special fiber. Since K is algebraically closed, the value group Γ is divisible, and hence the formal scheme $\mathfrak{U}_P$ is admissible [Gub13, Proposition 6.7].

Let $\mathfrak{U}_P^{\mathrm{an}}$ denote the Raynaud fiber of $\mathfrak{U}_P$. Then the Raynaud fiber $\mathfrak{U}_P^{\mathrm{an}}$ is naturally identified with an affinoid domain inside the Berkovich analytification $U_{\sigma_P}^{\mathrm{an}}$ [Gub13, §4.13]. Specifically,

$$\mathfrak{U}_P^{\mathrm{an}} \;\cong\; \left\{ x \in U_{\sigma_P}^{\mathrm{an}} : |f(x)| \leq 1 \text{ for all } f \in K[S_{\sigma_P}] \right\}.$$

Applying [Gub13, Lemma 6.21] to each orbit $O_\sigma \subset Y_\Sigma$, for $\sigma \in \Sigma$, we see that this affinoid domain is the inverse image of $\overline{P} \subset \mathrm{Trop}(Y_\Sigma)$ under the tropicalization map,

$$\mathfrak{U}_P^{\mathrm{an}} = \mathrm{Trop}^{-1}(\overline{P}). \qquad (3)$$

These models of affinoid domains associated to extended (Γ, Σ)-admissible polyhedra glue together naturally, as follows. Each inclusion of a face of an extended (Γ, Σ)-admissible polyhedron $\overline{Q} \subset \overline{P}$ induces a Zariski open embedding of R-schemes $\mathscr{U}_Q \subset \mathscr{U}_P$ and a Zariski open embedding of formal R-schemes $\mathfrak{U}_Q \subset \mathfrak{U}_P$. If Δ is an extended (Γ, Σ)-admissible polyhedral complex, gluing along these open embeddings naturally produces a flat R-scheme $\mathscr{Y}_\Delta$ and an admissible formal R-scheme $\mathfrak{Y}_\Delta$, respectively. By construction, $\mathfrak{Y}_\Delta$ is the formal completion of $\mathscr{Y}_\Delta$ along its special fiber. Furthermore, the Raynaud fiber $\mathfrak{Y}_\Delta^{\mathrm{an}}$ is the preimage of the support $|\Delta| \subset \mathrm{Trop}(Y_\Sigma)$ under the tropicalization map, i.e.,

$$\mathfrak{Y}_\Delta^{\mathrm{an}} = \mathrm{Trop}^{-1}(|\Delta|). \tag{4}$$

Remark 3.3.1. In [Gub13], such algebraic and formal models are considered only for finite complexes Δ. Even for an arbitrary (infinite) complex Δ, one may construct the affine R-schemes $\mathscr{U}_\mathscr{P}$, glue along the open embeddings corresponding to inclusions of faces to construct $\mathscr{Y}_\Delta$, and formally complete along the special fiber to obtain $\mathfrak{Y}_\Delta$. One must take care, however, in comparing the Raynaud fiber $\mathfrak{Y}_\Delta^{\mathrm{an}}$ with $\mathrm{Trop}^{-1}(|\Delta|)$; indeed, Example 3.3.2 shows that these two analytic spaces are not isomorphic in general when Δ is not locally finite. Nevertheless, the standard constructions of $\mathfrak{Y}_\Delta^{\mathrm{an}}$, its natural map to $\mathrm{Trop}^{-1}(|\Delta|)$, and the proof that this map is an isomorphism when Δ is finite are all local; they extend verbatim to the case where Δ is locally finite.

Example 3.3.2. We briefly sketch an example of a (Γ, Σ)-admissible complex that is not locally finite, for which the analogue of (4) does not hold. Let $\Gamma = \mathbb{Q}$, $N = \mathbb{Z}$, and $\Sigma = \{0\}$. Consider the complex Δ whose maximal faces are the point 0 together with the intervals $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ for positive integers n. Then $|\Delta| = [0, 1]$, but Δ is not locally finite at 0. The Raynaud fiber $\mathfrak{Y}_\Delta^{\mathrm{an}}$ is the disjoint union of $\mathrm{Trop}^{-1}(0)$ and $\mathrm{Trop}^{-1}((0, 1])$, which is disconnected, and not isomorphic to $\mathrm{Trop}^{-1}(|\Delta|)$.

3.4. **Gubler models of closed subvarieties.** Let $\iota \colon X \hookrightarrow Y_\Sigma$ be the inclusion of a closed subscheme, and let Δ be an extended (Γ, Σ)-admissible polyhedral complex in $\mathrm{Trop}(Y_\Sigma)$. Then, for each face $\overline{P}$ of Δ, we obtain an R-model $\mathscr{X}_P$ of $X \cap U_{\sigma_P}$ as the closure of X in the toric scheme $\mathscr{U}_P$. Gluing along the inclusions of faces in Δ, we obtain an R-model $\mathscr{X}_\Delta$ of X, which we refer to as the *(algebraic) Gubler model of X associated to the pair (ι, Δ)*.

Completing each R-scheme $\mathscr{X}_P$ along its special fiber, we obtain an admissible formal R-scheme $\mathfrak{X}_P$. Gluing these formal schemes along the inclusions in Δ, we obtain an admissible formal R-scheme $\mathfrak{X}_\Delta$ isomorphic to the completion of $\mathscr{X}_\Delta$ along its special fiber, whose Raynaud fiber $\mathfrak{X}_\Delta^{\mathrm{an}}$ is the preimage of $|\Delta|$ in X^{an}. In particular, $\mathfrak{X}_\Delta$ is a formal model of X^{an} if and only if $|\Delta|$ contains $\mathrm{Trop}(X, \iota)$. In this case, we say that Δ *covers* $\mathrm{Trop}(X, \iota)$ and refer to $\mathfrak{X}_\Delta$ as the *formal Gubler model of X^{an} associated to the pair (ι, Δ)*. When we want to stress the role of the closed embedding $\iota : X \hookrightarrow Y_\Sigma$ in the construction of the formal Gubler model $\mathfrak{X}_\Delta$, we write $\mathfrak{X}_{(\iota, \Delta)}$.

3.5. **Adic tropicalization as an inverse limit of formal models.** Let Δ and Δ' be extended (Γ, Σ)-admissible polyhedral complexes that cover $\mathrm{Trop}(X, \iota)$. We say that Δ' *refines* Δ if each face $\overline{P}'$ of Δ' is contained in some face $\overline{P}$ of Δ. The induced maps $\mathfrak{X}_{P'} \to \mathfrak{X}_P$ glue to produce a morphism of formal R-schemes $\mathfrak{X}_{\Delta'} \longrightarrow \mathfrak{X}_\Delta$. This gives a functor from the category of extended (Γ, Σ)-admissible complexes that cover $\mathrm{Trop}(X, \iota)$, in which the morphisms are refinements, to the category of admissible formal models of X^{an}.

Definition 3.5.1. *The* adic tropicalization *of the closed embedding $\iota : X \hookrightarrow Y_\Sigma$ is the locally topologically ringed space*

$$\left(\mathfrak{Trop}(X, \iota), \mathscr{O}_{\mathfrak{Trop}(X, \iota)}\right) := \varprojlim(\mathfrak{X}_\Delta, \mathscr{O}_{\mathfrak{X}_\Delta}),$$

where the limit is taken over all models associated to extended (Γ, Σ)-admissible polyhedral complexes that cover $\mathrm{Trop}(X, \iota)$ and all morphisms induced by refinements.

When no confusion seems possible, we denote both the adic tropicalization of ι and its underlying topological space simply by $\mathfrak{Trop}(X, \iota)$.

Remark 3.5.2. For a detailed example of the adic tropicalization of a line in $\mathbb{P}^2$, with accompanying figures, see [Fos16, Example 3.5.6].

Consider the *category of toric embeddings of* X, whose objects are closed embeddings $\iota : X \hookrightarrow Y_\Sigma$ into toric varieties, and whose morphisms are commutative diagrams

$$
\begin{array}{ccc}
 & & Y_\Sigma \\
 & \overset{\iota}{\nearrow} & \downarrow f \\
X & \underset{\iota'}{\searrow} & \\
 & & Y_{\Sigma'}
\end{array}
$$

induced by a toric morphism $\phi : \Sigma \longrightarrow \Sigma'$ of fans. Given such a diagram, let Δ' be an extended (Γ, Σ)-admissible polyhedral complex that covers $\mathrm{Trop}(X, \iota')$. Note that we can refine any extended (Γ, Σ)-admissible polyhedral complex Δ that covers $\mathrm{Trop}(X, \iota)$ to obtain a cover Δ^+ such that the induced morphism $\mathrm{Trop}(f) : \mathrm{Trop}(Y_\Sigma) \longrightarrow \mathrm{Trop}(Y_{\Sigma'})$ maps each $\overline{P} \in \Delta^+$ into some $\overline{P}' \in \Delta'$. There are then induced morphisms of algebraic and admissible formal R-schemes

$$\mathscr{X}_P \to \mathscr{X}_{P'} \quad \text{and} \quad \mathfrak{X}_P \to \mathfrak{X}_{P'},$$

which glue to give morphisms of algebraic and formal Gubler models

$$\mathscr{X}_\Delta \to \mathscr{X}_{\Delta'} \quad \text{and} \quad \mathfrak{X}_\Delta \to \mathfrak{X}_{\Delta'}.$$

From this it follows that each morphism f of toric embeddings induces a morphism of adic tropicalizations,

$$\mathfrak{Trop}(X, f) : \mathfrak{Trop}(X, \iota) \longrightarrow \mathfrak{Trop}(X, \iota'),$$

in the category of locally topologically ringed spaces; hence, adic tropicalization is a functor from toric embeddings to locally topologically ringed spaces.

3.6. **Adic tropicalization as a union of initial degenerations.** We now show that the disjoint union of the initial degenerations $\mathrm{in}_w(X)$, for $w \in \mathrm{Trop}(X, \iota)$, is naturally identified with the underlying set of the adic tropicalization $\mathfrak{Trop}(X, \iota)$.

Lemma 3.6.1. *For each point $w \in \mathrm{Trop}(X, \iota)$, not necessarily Γ-rational, there is a natural isomorphism of k-schemes*

$$\mathrm{in}_w(X) \overset{\sim}{\longrightarrow} \varprojlim_{P \ni w} (\mathfrak{X}_P)_s.$$

Proof. Consider first the special case where w is in the dense open subset $N_\mathbb{R} \subset \mathrm{Trop}(Y_\Sigma)$. For each (Γ, Σ)-admissible polyhedron P that contains w, we have $R[M]^P \subset R[M]^w$. Furthermore, from (2) we see that, for each $f \in K[M]$ there is a (Γ, Σ)-polyhedron P_f (possibly empty) such that $f \in R[M]^w$ if and only if $w \in P_f$. It follows that $R[M]^w = \bigcup_{P \ni w} R[M]^P$. Passing to the special fibers of the associated R-schemes, we conclude that $\mathrm{in}_w X \cong \varprojlim_{P \ni w} (\mathfrak{X}_P)_s$, as required.

For the general case, consider $w \in \mathrm{Trop}(O_\sigma)$, for some $\sigma \in \Sigma$. Let $\overline{P}$ be an extended (Γ, Σ)-admissible polyhedron that contains w. Then the recession cone of P contains σ, and $\overline{P} \cap \mathrm{Trop}(O_\sigma)$ is the image of P under the projection $N_\mathbb{R} \longrightarrow N_\mathbb{R}/\mathrm{span}(\sigma)$. Because each linear function in $u \in M_\sigma$ is constant on the fibers of this projection, we obtain inclusions $R[M_\sigma]^{\overline{P} \cap \mathrm{Trop}(O_\sigma)} \hookrightarrow R[M]^P$, and

$$\bigcup_{\overline{P} \ni w} R[M_\sigma]^{\overline{P} \cap \mathrm{Trop}(O_\sigma)} \longhookrightarrow \bigcup_{\overline{P} \ni w} R[M]^P. \tag{5}$$

We must show that (5) becomes an isomorphism over the residue field k. As a first step, suppose $a_u \chi^u$ is a nonzero monomial in $\bigcup_{\overline{P} \ni w} R[M]^P$ and $u \notin M_\sigma$. We claim that $a_u \chi^u$ vanishes modulo

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the maximal ideal $\mathfrak{m} \subset R$. To see this, note that the linear function u must be strictly positive on the interior of σ, which contains a Γ-rational point v'. Let $P' = P + v'$. Then $w \in \overline{P}'$ and $a_u \chi^u \in \mathfrak{m} R[M]^{P'}$. Hence $a_u \chi^u$ vanishes modulo $\mathfrak{m}$, as claimed. The remainder of the argument is similar to the previous case. $\qquad \square$

Proposition 3.6.2. *There is a natural bijection*

$$\bigsqcup_{w \in \mathrm{Trop}(X, \iota)} \left| \mathrm{in}_w(X) \right| \xrightarrow{\sim} \left| \mathfrak{Trop}(X, \iota) \right| \tag{6}$$

between the set underlying the adic tropicalization $\mathfrak{Trop}(X, \iota)$ *and the disjoint union of all initial degenerations of* X *at all points on* $\mathrm{Trop}(X, \iota)$.

Proof. To construct the map (6), it suffices to construct a map $\left| \mathrm{in}_w(X) \right| \longrightarrow \varprojlim_\Delta \left| \mathfrak{X}_\Delta \right|$, where the limit is over (Γ, Σ)-admissible covers of $\mathrm{Trop}(X, \iota)$. Each such cover contains a face $\overline{P}$ that contains w, and the existence of this map then follows from Lemma 3.6.1.

By [Gub13, Proposition 8.8], if $\overline{P}$ and $\overline{Q}$ are disjoint faces of Δ, then $\mathfrak{X}_P$ and $\mathfrak{X}_Q$ are disjoint in $\mathfrak{X}_\Delta$. It follows that the map (6) is injective. To see that (6) is surjective, observe that again by [Gub13, Proposition 8.8] any point p in the inverse limit $\varprojlim_\Delta \left| \mathfrak{X}_\Delta \right|$ lies in $\mathfrak{X}_{P_i}$ for some nested decreasing sequence of admissible polyhedra P_i in $\mathrm{Trop}(Y_\Sigma)$ such that $\bigcap P_i$ is the single point w. Then p will lie in the image of $\left| \mathrm{in}_w(X) \right|$. $\qquad \square$

4. The adic limit theorem

In this section, we prove Theorem 1.1, showing that the adic analytification of a subscheme of a toric variety is recovered as the inverse limit of the adic tropicalizations of any system of toric embeddings that satisfies the condition (†) from the introduction. The proof is not as direct as those of the tropical limit theorems in [Pay09, FGP14]. We first prove a strong cofinality statement for Gubler models of a projective scheme (Theorem 4.1.3), and then deduce that adic analytifications of certain affinoid domains in analytifications of affine schemes can be obtained as inverse limits of algebraizable formal models (Corollary 4.1.4). Theorem 1.1 then follows easily.

4.1. **Cofinality of Gubler models for affinoid domains.** Recall that an integral model of X is an R-scheme of finite type $\mathscr{X}$ together with an isomorphism $\mathscr{X} \times \mathrm{Spec}\, K \cong X$. A formal scheme $\mathfrak{X}$ is *algebraizable* if it is isomorphic to the formal completion of an integral model. By construction, any formal Gubler model is algebraizable.

Lemma 4.1.1. *If X is projective, then the algebraizable formal models are cofinal in the inverse system of all formal models of X^{an}.*

Proof. Let $\mathfrak{X}$ be an arbitrary formal model of X^{an}. We must show that there is an algebraizable formal model that dominates $\mathfrak{X}$. Fix an embedding of X in the projective space $\mathbb{P}_K^n$, and let $\mathfrak{X}'$ be the formal completion of the closure of X in $\mathbb{P}_R^n$. Because X is projective, the adic space X^{ad} is quasi-compact, and therefore every admissible formal model of X^{ad} is *quasi-paracompact* in the sense of [Bos14, §8.2, Definition 12]. By Raynaud's theorem [Bos14, §8.4, Theorem 3], this implies that there is an admissible formal blowup $\mathfrak{X}'' \to \mathfrak{X}'$ such that $\mathfrak{X}''$ admits a morphism to $\mathfrak{X}$. The center of this admissible formal blowup is a coherent ideal sheaf on $\mathfrak{X}'$ and, since $\mathfrak{X}'$ is projective over $\mathrm{Spf}\, R$, this coherent sheaf is algebraizable [Abb10, Theorem 2.13.8]. Then $\mathfrak{X}''$ is an algebraizable formal model of X^{an} that dominates $\mathfrak{X}$, and the lemma follows. $\qquad \square$

We now state and prove a technical theorem about the Gubler models $\mathfrak{X}_\Delta$ of projective schemes.

Definition 4.1.2. *We will say that a formal Gubler model $\mathfrak{X}_\Delta$ of X is* adapted *to an analytic domain $V \subset X^{\mathrm{an}}$ if there is a subcomplex $\Delta' \subset \Delta$ such that V is the Raynaud fiber $\mathfrak{X}_{\Delta'}^{\mathrm{an}} \subset \mathfrak{X}_\Delta^{\mathrm{an}}$.*

Equivalently, $\mathfrak{X}_\Delta$ is adapted to V if $V = \mathrm{Trop}^{-1}(|\Delta'|)$ for some subcomplex Δ' of Δ.

Theorem 4.1.3. *Let X be projective over K, let $U \subset X$ be an affine open subscheme, and let $\{g_1, \ldots, g_n\} \subset K[U]$ be a set of generators for the coordinate ring. Consider the affinoid domain*

$$V = \big\{ x \in U^{\mathrm{an}} \ : \ |g_\ell(x)| \leq 1, \ for \ 1 \leq \ell \leq n \big\}.$$

Then the formal Gubler models $\mathfrak{X}_{(\iota, \Delta)}$ adapted to V such that U is the preimage under ι of a torus invariant affine open subvariety are cofinal in the inverse system of all formal models of X^{an}.

Proof. By Lemma 4.1.1, the algebraizable models of X are cofinal in the inverse system of all formal models. Thus it suffices to show that every algebraizable model is dominated by a formal Gubler model $\mathfrak{X}_{(\iota, \Delta)}$ that is adapted to V, such that U is the preimage under ι of a torus invariant affine open subvariety.

Fix an affine open cover $X = U_1 \cup \cdots \cup U_r$, with $U = U_1$. Let $\mathscr{X}$ be an algebraic integral model of X, with $\mathfrak{X}$ its formal completion. Cover $\mathscr{X}$ by finitely many affine opens. After refinement, we may assume that this cover consists of affine opens labeled $\mathscr{U}_{ij}$ such that, for each i and j, the generic fiber $\mathscr{U}_{ij} \times \mathrm{Spec}\, K$ is a distinguished affine open $D(f_{ij}) \subset U_i$ for some $f_{ij} \in K[U_i]$. For each i and j, choose a presentation

$$\mathscr{U}_{ij} \ = \ \mathrm{Spec}\, R[x_{ij1}, \ldots, x_{ijs}]\big/ \mathfrak{a}_{ij},$$

where $x_{ijk} = y_{ijk}/f_{ij}^{m_k}$, for some $y_{ijk} \in K[U_i]$ and $m_k \in \mathbb{Z}_{\geq 0}$.

By [FGP14, Theorem 4.2], the inverse system of all toric embeddings of X satisfies ($\dagger$) with respect to any affine open cover, so we can choose a closed embedding $\iota : X \hookrightarrow Y_\Sigma$ such that

  (i) each U_i is the preimage of a torus invariant affine open U_{σ_i}, for $1 \leq i \leq r$,
  (ii) each of the functions f_{ij} and y_{ijk} is the pullback of a character that is regular on U_{σ_i}, and
  (iii) each of the functions g_ℓ appearing in the statement of the theorem is the pullback of a character that is regular on U_{σ_1}, for $1 \leq \ell \leq n$.

We will construct an admissible polyhedral cover Δ of $\mathrm{Trop}(X, \iota)$ such that the associated Gubler model $\mathfrak{X}_\Delta$ is adapted to V and dominates $\mathfrak{X}$ by finding admissible polyhedra P and P_{ij} such that $V = \mathfrak{X}_P^{\mathrm{an}}$ and $(\widehat{\mathscr{U}_{ij}})^{\mathrm{an}} = \mathfrak{X}_{P_{ij}}^{\mathrm{an}}$ and then applying Lemma 3.2.2. The details are as follows.

Let M be the character lattice of the dense torus in Y_Σ. Choose $u_1, \ldots, u_n$ in M such that g_ℓ is the pullback of the character χ^{u_ℓ}, which is regular on U_{σ_1}. If these do not generate $K[U_{\sigma_1}]$, then choose additional characters $\chi^{u_{n+1}}, \ldots, \chi^{u_{n'}}$ so that $\{\chi^{u_i} : 1 \leq i \leq n'\}$ does generate. Let a_i be the minimum of the continuous function $\mathrm{val}\, \chi^{u_i}$ on the compact space V. Let $\overline{P}$ be the extended (Γ, Σ)-admissible polyhedron in $\mathrm{Trop}(Y_\Sigma)$ given by the closure of

$$P = \{ v \in N_\mathbb{R} \ : \ \langle u_i, v \rangle \leq a_i \ for \ 1 \leq i \leq n' \}.$$

Then $V = \mathrm{Trop}^{-1}(\overline{P})$, by construction. By (3), it follows that V is the Raynaud fiber $\mathfrak{X}_P^{\mathrm{an}}$.

Next, choose u_{ij} and u_{ijk} in M such that f_{ij} and y_{ijk} are the pullbacks of the characters $\chi^{u_{ij}}$ and $\chi^{u_{ijk}}$, respectively, each of which is regular on U_{σ_i}. Recall that the distinguished affine open $D(f_{ij}) \subset U_i$ is the general fiber of $\mathscr{U}_{ij}$, and hence is the preimage of the torus invariant affine open $U_{\sigma_{ij}} \subset U_{\sigma_i}$ on which $\chi^{u_{ij}}$ is invertible. In particular, the characters $\chi^{u_{ijk} - m_k u_{ij}}$ are regular on $U_{\sigma_{ij}}$. As in the previous paragraph, we can choose additional characters to generate $K[U_{\sigma_{ij}}]$ and the valuations of all of these characters achieve their minima on the compact affinoid domain $(\widehat{\mathscr{U}_{ij}})^{\mathrm{an}}$. Let $\overline{P}_{ij}$ be the closure in $\mathrm{Trop}(Y_\Sigma)$ of the polyhedron in $N_\mathbb{R}$ on which the linear functions corresponding to each of these characters is bounded below by the respective minima of valuations on $(\widehat{\mathscr{U}_{ij}})^{\mathrm{an}}$. Then P_{ij} is (Γ, Σ)-admissible by construction, and there is a natural morphism

$$\varphi_{ij} : \mathfrak{X}_{P_{ij}} \to \mathfrak{X},$$

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inducing an identification $\mathfrak{X}_{P_{ij}}^{\mathrm{an}} \xrightarrow{\sim} (\widehat{\mathscr{U}_{ij}})^{\mathrm{an}}$ on the Raynaud fibers. Note also that $\{(\widehat{\mathscr{U}_{ij}})^{\mathrm{an}}\}$ covers X^{an} and hence $\{\overline{P}_{ij}\}$ covers $\mathrm{Trop}(X, \iota)$.

By Lemma 3.2.2, there is an extended (Γ, Σ)-admissible polyhedral complex Δ such that $\overline{P}$ and each of the $\{\overline{P}_{ij}\}$ is a union of faces of Δ. Moreover, since the $\{\overline{P}_{ij}\}$ cover $\mathrm{Trop}(X, \iota)$, we may assume that each face of Δ is contained in some $\overline{P}_{ij}$, without changing the Gubler model $\mathfrak{X}_\Delta$. Then each inclusion $\overline{Q} \subset \overline{P}_{ij}$, for $Q \in \Delta$ induces morphisms of formal schemes

$$\mathfrak{X}_Q \longrightarrow \mathfrak{X}_{P_{ij}} \xrightarrow{\varphi_{ij}} \mathfrak{X},$$

which glue together to give $\mathfrak{X}_\Delta \to \mathfrak{X}$, a morphism of formal models of X^{an}, as required. $\qquad\square$

Corollary 4.1.4. *Let U be an affine scheme of finite type over K, with $\{g_1, \ldots, g_n\}$ a generating set for the coordinate ring $K[U]$. Let $V \subset U^{\mathrm{an}}$ be the affinoid domain*

$$V = \{x \in U^{\mathrm{an}} : |g_\ell(x)| \leq 1 \ \ for \ \ 1 \leq \ell \leq n\},$$

and let $\mathcal{S}_V$ be the inverse system consisting of all formal models of V of the form $\mathfrak{U}_{(\iota, \Delta)}$, where ι is a closed embedding of U into an affine toric variety. Then the pair $(V^{\mathrm{ad}}, \mathscr{O}_{V^{\mathrm{ad}}}^\circ)$ is naturally isomorphic to $\varprojlim_{\mathcal{S}_V} (\mathfrak{U}_{(\iota, \Delta)}, \mathscr{O}_{V^{\mathrm{ad}}}^\circ)$.

Proof. Choose a projective compactification X of U. Recall that X^{ad} is the projective limit of the inverse system of formal models of X^{an}. By Theorem 4.1.3, the formal Gubler models $\mathfrak{X}_{(\iota, \Delta)}$ that are adapted to V and such that U is the preimage under ι of a torus invariant affine open subvariety are cofinal in the inverse system of all formal models of X^{an}. Hence V^{ad} is isomorphic to the inverse limit of formal models $\mathfrak{X}_{\Delta'}$ of V that appear as formal Zariski opens $\mathfrak{X}_{\Delta'} \subset \mathfrak{X}_{(\iota, \Delta)}$ arising from subcomplexes $\Delta' \subset \Delta$, as in Definition 4.1.2. We claim that each of these formal models is dominated by a model in $\mathcal{S}_V$. To see this, choose such a model, and refine Δ so that all of the recession cones of the faces in the induced refinement Δ'' of Δ' are faces of σ. Then $\mathfrak{X}_{\Delta''}$ is in $\mathcal{S}_V$ and dominates $\mathfrak{X}_{\Delta'}$, as required. $\qquad\square$

Proof of Theorem 1.1. Let X be a separated finite type K-scheme, and let $\mathcal{S}$ be a system of toric embeddings of X satisfying condition (†) of §1 with respect to an affine open cover $X = U_1 \cup \cdots \cup U_r$. We want to show that the induced map

$$(X^{\mathrm{ad}}, \mathscr{O}_{X^{\mathrm{ad}}}^\circ) \longrightarrow \varprojlim_{\mathcal{S}} (\mathfrak{Trop}(X, \iota), \mathscr{O}_{\mathfrak{Trop}(X,\iota)}) \tag{7}$$

is an isomorphism of locally topologically ringed spaces.

Consider a single affine open $U = U_i$ in our chosen cover. Choose a generating set $g_1, \ldots, g_n$ for the coordinate ring $K[U_i]$. Let $V \subset U^{\mathrm{an}}$ be the affinoid domain

$$V = \{x \in U^{\mathrm{an}} : |g_\ell(x)| \leq 1, \ \text{for } 1 \leq \ell \leq n\}.$$

By Corollary 4.1.4, the ringed space $(V^{\mathrm{ad}}, \mathscr{O}_{V^{\mathrm{ad}}}^\circ)$ is naturally isomorphic to the inverse limit of its formal models arising as $\mathfrak{U}_{(\jmath, \Delta_0)}$, where $\jmath : U \hookrightarrow U_\tau$ is a closed embedding into an affine toric variety and Δ_0 is an extended (Γ, τ)-admissible polyhedral complex such that $\mathrm{Trop}^{-1}(|\Delta_0|) = V$. (Here, we use τ also to denote the fan consisting of its faces.) We fix one such model $\mathfrak{U}_{(\jmath, \Delta_0)}$.

Replacing the generating set $\{g_1, \ldots, g_n\}$ in the description above by $\{ag_1, \ldots, ag_n\}$, for $a \in K^*$ with $|a| \ll 1$, we can cover U^{an} by affinoid domains of this form. Since the analogous statement holds for each of the affine opens $U_1, \ldots, U_r$ and these cover X, to prove the theorem it will suffice to show that there is a Gubler model of X associated to an embedding $\iota \in \mathcal{S}$ that is adapted to V, and such that the induced formal model of V dominates $\mathfrak{U}_{(\jmath, \Delta_0)}$.

Let $h_1, \ldots, h_s$ be the respective pullbacks to U of characters $\chi^{u_1}, \ldots, \chi^{u_s}$ that generate $K[U_\tau]$. By condition (†) there is an embedding $\iota : X \hookrightarrow Y_\Sigma$ in $\mathcal{S}$ such that $U = \iota^{-1}(U_\sigma)$ for some torus

invariant affine open $U_\sigma \subset Y_\Sigma$ and $h_1, \ldots, h_s$ are the pullbacks of characters that are regular on U_σ. By construction, there is a natural toric morphism $\pi \colon U_\sigma \to U_\tau$ that commutes with the embeddings of U.

Let $\overline{P}_0$ be a face of Δ_0. The preimage $\overline{Q}_0 = \mathrm{Trop}(\pi)^{-1}(\overline{P}_0)$ in $\mathrm{Trop}(U_\sigma)$ is not necessarily (Γ, σ)-admissible, since the recession cone of Q_0 is not necessarily a face of σ. Note, however, that the recession cone of Q_0 is the preimage of a face of τ, and hence cut out by some supporting hyperplanes of the preimage of τ. Since σ maps into τ, these hyperplanes pullback to supporting hyperplanes of σ. Moreover, since V is compact, the continuous functions $\mathrm{val}\,\chi^{u_i}$ achieve their respective minima a_i. Let Q be the polyhedron obtained by intersecting Q_0 with the halfspaces $\langle u_i, v \rangle \geqslant a_i$, for $1 \leq i \leq s$. Then Q is (Γ, σ)-admissible, $\mathrm{Trop}^{-1}(\overline{Q}) \subset X^{\mathrm{an}}$ is equal to $\mathrm{Trop}^{-1}(\overline{P})$, and there is a naturally induced morphism of formal models $\mathfrak{U}_Q \to \mathfrak{U}_P$.

Performing the above procedure for each face of Δ_0, we arrive at a collection of extended (Γ, Σ)-admissible polyhedra $\overline{Q}_1, \ldots, \overline{Q}_r$ with morphisms of formal models $\mathfrak{U}_{Q_i} \to \mathfrak{U}_{\Delta_0}$ and

$$V = \mathrm{Trop}^{-1}(\overline{Q}_1 \cup \cdots \cup \overline{Q}_r).$$

By Lemma 3.2.2, there is an extended (Γ, Σ)-admissible complex Δ that covers $\mathrm{Trop}(X, \iota)$ such that each $\overline{Q}_i$ is a union of faces of Δ. Then the Gubler model $\mathfrak{X}_\Delta$ is adapted to V and the induced formal model dominates $\mathfrak{U}_{\Delta_0}$, as required. $\qquad\square$

4.2. Gubler models of compact analytic domains.
We now extend our results on cofinality of Gubler models from projective schemes to compact analytic domains in subschemes of toric varieties, proving Theorem 1.2. We begin with an algebraic lemma.

Lemma 4.2.1. *Let $\mathfrak{X}$ be an admissible formal R-scheme and let $g \in \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ be a function whose restriction to the adic fiber is invertible in $\Gamma(\mathfrak{X}^{\mathrm{ad}}, \mathscr{O}^\circ_{X^{ad}})$. Then $g^{-1} \in \mathscr{O}_{\mathfrak{X}}$.*

Proof. It suffices to consider the case where $\mathfrak{X} = \mathrm{Spf}\,A$ is affine. Since the restriction of g to the adic fiber is invertible in $\Gamma(\mathfrak{X}^{\mathrm{ad}}, \mathscr{O}^\circ_{X^{ad}})$, its inverse must be power bounded, i.e., we have $g^{-1} \in A^\circ_K$. It remains to show that g^{-1} is in the subring $A \subset A^\circ_K$.

By [GRW17, Proposition 2.12], the power bounded ring A°_K is an integral extension of A. Therefore, g^{-1} must satisfy an identity $(g^{-1})^n + c_{n-1}(g^{-1})^{n-1} + \cdots + c_0 = 0$ with coefficients $c_i \in A$. Multiplying by g^{n-1} gives $g^{-1} = -c_{n-1} - \cdots - c_0 g^{n-1}$. In particular, g^{-1} is in A, as required. $\quad\square$

This lemma allows us to construct formal Gubler models that are adapted to certain rational domains, as follows.

Proposition 4.2.2. *Let X be a K-variety, let $\mathcal{S}$ be an inverse system of toric embeddings of X satisfying ($\dagger$), and let $\mathfrak{X}_\Delta$ be the formal Gubler model associated to a pair (ι, Δ), for ι in $\mathcal{S}$. Let $P \in \Delta$ be an admissible polyhedron and let $\mathfrak{U}_P \langle \frac{1}{g} \rangle$ be the distinguished formal open subscheme of $\mathfrak{U}_P$ associated to some $g \in \widehat{R[M]^P}$. Then there is a formal Gubler model $\mathfrak{X}_{\Delta'}$ of X that is adapted to $\mathfrak{U}_P \langle \frac{1}{g} \rangle^{\mathrm{an}}$ and dominates $\mathfrak{X}_\Delta$.*

Proof. Fix a finite set of monomial generators $\{\chi^{u_1}, \ldots, \chi^{u_n}\}$ on $\mathscr{U}_P = \mathrm{Spec}\,K[M]^P$. Then $\{\chi^{u_1}, \ldots, \chi^{u_n}, \frac{1}{g}\}$ is a set of topological generators on $\mathfrak{U}_P \langle \frac{1}{g} \rangle$. Define $U := (\mathscr{U}_P)_K$, the affine algebraic generic fiber of the algebraic R-model $\mathscr{U}_P$. Because $\mathfrak{U}_P \langle \frac{1}{g} \rangle^{\mathrm{ad}}$ is a quasicompact subset of U^{ad}, we can replace the analytic function $\frac{1}{g}$ on $\mathfrak{U}_P \langle \frac{1}{g} \rangle^{\mathrm{ad}}$ with a rational function f_0 on U such that $|g^{-1}(x)| \leq 1$ if and only if $|f_0(x)| \leq 1$ for all $x \in \mathfrak{U}_P \langle \frac{1}{g} \rangle^{\mathrm{ad}}$.

Using condition ($\dagger$), fix a closed embedding $\iota : X \hookrightarrow Y_\Sigma$ such that U is the pullback of a torus invariant affine open U_σ, and such that f_0 is the pullback of a character on Y_Σ expressed as the ratio of two characters that are regular on U_σ. Then $\mathfrak{U}_P \langle \frac{1}{g} \rangle^{\mathrm{ad}}$ becomes the preimage of an admissible

polyhedron $\overline{P}'$ under the composition $X^{\mathrm{ad}} \to X^{\mathrm{an}} \to \mathrm{Trop}(X, \iota)$. Applying Lemma 3.2.2, we obtain a locally finite (Γ, Σ)-admissible polyhedral complex Υ such that both $\overline{P}'$ and each face of Δ is a union of faces of Υ, and such that Υ covers $\mathrm{Trop}(X, \iota)$. We then have a morphism of admissible formal R-models $\mathfrak{X}_\Upsilon \longrightarrow \mathfrak{X}_\Delta$.

Let $\Upsilon' \subset \Upsilon$ be the subcomplex consisting of all polyhedra in P'. By construction, we have an isomorphism of ringed spaces

$$\left(\mathfrak{X}_{\Upsilon'}^{\mathrm{ad}}, \mathscr{O}_{\mathfrak{X}_{\Upsilon'}^{\mathrm{ad}}}^{\circ}\right) \xrightarrow{\sim} \left(\mathfrak{U}_P\langle\tfrac{1}{g}\rangle^{\mathrm{ad}}, \mathscr{O}_{\mathfrak{U}_P\langle\tfrac{1}{g}\rangle^{\mathrm{ad}}}^{\circ}\right). \tag{8}$$

To see that (8) extends to a morphism of admissible formal R-models $\mathfrak{X}_{\Upsilon'} \longrightarrow \mathfrak{U}_P\langle\tfrac{1}{g}\rangle$, it suffices to show that each of the topological generators $\chi^{u_1}, \ldots, \chi^{u_n}, \tfrac{1}{g}$ on $\mathfrak{U}_P\langle\tfrac{1}{g}\rangle$ is in $\mathscr{O}_{\mathfrak{X}_{\Upsilon'}}$. Now χ^{u_i} is in $\mathscr{O}_{\mathfrak{X}_{\Upsilon'}}$ by construction, and g is invertible in $\mathscr{O}_{\mathfrak{X}_{\Upsilon'}}$. By Lemma 4.2.1, this implies that g^{-1} is also in $\mathscr{O}_{\mathfrak{X}_{\Upsilon'}}$, as required. $\qquad\square$

Lemma 4.2.3. *Let $\mathcal{S}$ be an inverse system of toric embeddings of X that satisfies* (†)*, and let $V \subset X^{\mathrm{an}}$ be a compact analytic domain with formal model $\mathfrak{V}$. For any point $x \in V^{\mathrm{ad}}$, there exists an embedding $\iota : X \hookrightarrow Y_\Sigma$ in $\mathcal{S}$, an admissible polyhedral cover Δ_x of $\mathrm{Trop}(X, \iota)$, a polyhedron $P_x \in \Delta_x$, and a function g_x on $\mathfrak{U}_{P_x}$ such that the distinguished open $\mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle$ comes with a morphism $\mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle \longrightarrow \mathfrak{V}$ whose adic Raynaud fiber is the inclusion of an analytic domain containing x.*

Proof. For each $x \in \mathfrak{V}^{\mathrm{ad}}$, choose an affine open subscheme $\mathfrak{V}' = \mathrm{Spf}\, A' \subset \mathfrak{V}$ such that $x \in (\mathfrak{V}')^{\mathrm{ad}}$. Fix a finite set of topological generators $\{h_1, \ldots, h_\ell\}$ of A'. Note that all generators h_i, $1 \le i \le \ell$, lie in the power bounded stalk $\mathscr{O}_{\mathfrak{V}^{\mathrm{ad}}, x}^{\circ}$. By Theorem 1.1, our hypothesis that $\mathcal{S}$ is an inverse system of toric embeddings of X satisfying (†) implies that we have a neighborhood basis of x formed by adic fibers of distinguished affine opens $\mathfrak{U}_P\langle\tfrac{1}{g}\rangle$, where $\overline{P}$ is a face of an extended (Γ, Σ)-admissible complex giving rise to a Gubler model of X^{an}. Thus, because we have only finitely many h_i's, we can find a single admissible formal R-scheme $\mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle$ inside some formal Gubler model $\mathfrak{X}_{\Delta_x}$ of X associated to an admissible pair (ι, Δ_x), for ι in $\mathcal{S}$, such that $x \in \mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle^{\mathrm{ad}}$, and such that each h_i is a function on $\mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle$, for $1 \le i \le \ell$. By construction, this comes with a morphism $\mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle \longrightarrow \mathfrak{V}$, such that x is in the image of the adic Raynaud fiber of this morphism. $\qquad\square$

Proof of Theorem 1.2. For each $x \in V^{\mathrm{ad}}$, construct $\mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle \longrightarrow \mathfrak{V}$ as in Lemma 4.2.3. The fact that V^{ad} is quasicompact implies that we can pass to a finite cover of V^{ad} by affinoid domains $\mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle^{\mathrm{ad}} \subset V^{\mathrm{ad}}$. By Proposition 4.2.2, each $\mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle$ has a second associated Gubler model $\mathfrak{X}_{\Upsilon_x}$ of X with dominating morphism $\mathfrak{X}_{\Upsilon_x} \longrightarrow \mathfrak{X}_{\Delta_x}$, and containing a subcomplex $\Upsilon'_x \subset \Upsilon_x$ with restriction $\mathfrak{X}_{\Upsilon'_x} \longrightarrow \mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle$ that induces an isomorphism

$$\mathfrak{X}_{\Upsilon'_x}^{\mathrm{ad}} \xrightarrow{\sim} \mathfrak{U}_{P_x}\langle\tfrac{1}{g_x}\rangle^{\mathrm{ad}}.$$

Because the system of Gubler models is cofiltered, there exists a single formal Gubler model $\mathfrak{X}_\Upsilon$ dominating all $\mathfrak{X}_{\Upsilon'_x}$ simultaneously. Let $\Upsilon' \subset \Upsilon$ denote the subcomplex consisting of all polyhedra mapping into Υ'_x for some x. Then by construction, we have a morphism $\mathfrak{X}_{\Upsilon'} \longrightarrow \mathfrak{V}$ that restricts to an isomorphism of adic fibers $\mathfrak{X}_{\Upsilon'}^{\mathrm{ad}} \xrightarrow{\sim} V^{\mathrm{ad}}$. $\qquad\square$

5. The limit theorem for structure sheaves on Berkovich spaces

Recall that a subset $V \subset X^{\mathrm{an}}$ is an *analytic domain* if each point $x \in V$ has a neighborhood of the form $V_1 \cup \cdots \cup V_n$, where each $V_i \subset V$ is affinoid, and $x \in V_1 \cap \cdots \cap V_n$. A cover of an analytic domain V by analytic subdomains V_i is *admissible* if every point $x \in V$ has a neighborhood which is a finite union $V_1 \cup \cdots \cup V_n$, where each V_i is an analytic domain in the cover, and $x \in V_1 \cap \cdots \cap V_n$. We write X_G^{an} for X^{an} equipped with this G-topology. Note that open subsets are analytic domains, so

the G-topology refines the ordinary topology. See [Ber90, Ber93] for the G-topology on Berkovich spaces, as well as [BGR84, §9.1] for further details on Grothendieck topologies in general, including the technical notion of *slightly finer* Grothendieck topologies, which appears in the discussion below.

Note that sheaves on X_G^{an} are determined by their values on affinoid domains and restriction maps for inclusions of affinoid subdomains. The value of the structure sheaf $\mathscr{O}(X_G)$ on an affinoid domain $\mathscr{M}(A)$ is simply the affinoid algebra A, and the restriction maps to affinoid subdomains are the usual ones.

5.1. **The tropical G-topology.** We now describe an analogous G-topology on tropicalizations. Let $\iota : X \hookrightarrow Y_\Sigma$ be a closed embedding into a toric variety over K. A *polyhedral domain* in $\mathrm{Trop}(X, \iota)$ is the intersection with a (Γ, Σ)-admissible polyhedron. These are the analogues of affinoid domains in X^{an}. Then a *tropical domain* W is a subset of $\mathrm{Trop}(X, \iota)$ in which every point $x \in W$ has a neighborhood which is a finite union $W_1 \cup \cdots \cup W_n$, where each $W_i \subset W$ is a polyhedral domain and $x \in W_1 \cap \cdots \cap W_n$. A cover of a tropical domain W by tropical subdomains W_i is *admissible* if every point $x \in W$ has a neighborhood which is a finite union $W_1 \cup \cdots \cup W_n$, where each W_i is a tropical domain in the cover, and $x \in W_1 \cap \cdots \cap W_n$. We write $\mathrm{Trop}(X, \iota)_G$ for the tropicalization equipped with this G-topology. Just as a sheaf on X_G^{an} is determined by its values on affinoid domains and the restriction maps between them, a sheaf on $\mathrm{Trop}(X, \iota)_G$ is determined by its values on polyhedral domains and the restriction maps between them.

Note that the preimage of a polyhedral (resp. tropical) domain in $\mathrm{Trop}(X, \iota)$ is an affinoid (resp. analytic) domain in X^{an}. Hence the pullback of an admissible cover of $\mathrm{Trop}(X, \iota)$ is an admissible cover of X^{an}, so the tropicalization map

$$X_G^{\mathrm{an}} \longrightarrow \mathrm{Trop}(X, \iota)_G$$

is continuous not only in the ordinary topologies, but also with respect to the G-topologies on both sides. The projections

$$\mathrm{Trop}(X, \iota')_G \longrightarrow \mathrm{Trop}(X, \iota)_G$$

induced by morphisms of toric embeddings are likewise continuous with respect to the G-topologies.

The set-theoretic identification $X^{\mathrm{an}} = \varprojlim \mathrm{Trop}(X, \iota)$, together with the G-topologies on the tropicalizations, induces a G-topology on X^{an}, which we call the *tropical topology*. It is the coarsest G-topology on X^{an} with respect to which each of the projections $X^{\mathrm{an}} \to \mathrm{Trop}(X, \iota)_G$ is continuous.

Proposition 5.1.1. *The G-topology on X^{an} is slightly finer than the tropical topology on X^{an}.*

The following lemma is a key step in the proof of the proposition.

Lemma 5.1.2. *Every analytic domain $V \subset X^{\mathrm{an}}$ admits an admissible cover by analytic domains that are admissible in the tropical topology.*

Proof. Choose an admissible cover $V = \bigcup \mathscr{M}(K \otimes_R A_i)$ by affinoid domains, where each A_i is an admissible R-algebra. It suffices to prove that each $\mathscr{M}(K \otimes_R A_i)$ has an admissible tropical cover. We now use the same strategy that we used in the proof of Theorem 1.2.

Let $\mathfrak{V} = \mathrm{Spf}\, A_i$. For each $x \in \mathscr{M}(K \otimes_R A_i)^{\mathrm{ad}}$, construct $\mathfrak{U}_{P_x}\langle \frac{1}{g_x} \rangle \longrightarrow \mathfrak{V}$ as in the statement of Lemma 4.2.3, using the inverse system $\mathcal{S}$ of all closed embeddings of X into toric varieties. Recall that $\mathfrak{U}_{P_x}$ is a formal polyhedral domain in a Gubler model $\mathfrak{X}_{\Delta_x}$ of X. By Proposition 4.2.2, we can find a formal Gubler model $\mathfrak{X}_{\Delta'_x}$ of X that is adapted to $\mathfrak{U}_{P_x}\langle \frac{1}{g_x} \rangle^{\mathrm{an}}$ and dominates $\mathfrak{X}_{\Delta_x}$. Since $\mathscr{M}(K \otimes_R A_i)^{\mathrm{ad}}$ is quasicompact, we can choose a finite collection of points $x \in \mathscr{M}(K \otimes_R A_i)$ such that the adic open subspaces $\mathfrak{U}_{P_x}\langle \frac{1}{g_x} \rangle^{\mathrm{ad}}$ cover $\mathscr{M}(K \otimes_R A_i)^{\mathrm{ad}}$, and thus the analytic domains $\mathfrak{U}_{P_x}\langle \frac{1}{g_x} \rangle^{\mathrm{an}}$ cover $\mathscr{M}(K \otimes_R A_i)^{\mathrm{an}}$. Choose a single Gubler model $\mathfrak{X}_\Upsilon$ dominating the models $\mathfrak{X}_{\Delta'_x}$ for this finite collection of points $x \in \mathscr{M}(K \otimes_R A_i)^{\mathrm{ad}}$. Let $\Upsilon' \subset \Upsilon$ be the subcomplex consisting of all

polyhedra $Q \in \Upsilon$ such that $\mathfrak{X}_Q^{\mathrm{an}}$ lies in one of the analytic domains $\mathfrak{U}_{P_x}\langle \frac{1}{g_x} \rangle^{\mathrm{an}}$. Then the collection $\{ \mathfrak{X}_Q^{\mathrm{an}} : Q \in \Upsilon' \}$ is an admissible tropical cover of $\mathscr{M}(K \otimes_R A_i)$. $\hspace{1cm} \square$

Proof of Proposition 5.1.1. We need to check criteria (i) through (iii) of [BGR84, Definition 9.1.2.1]. To see that (i) holds, observe that every admissible open in the tropical topology is an affinoid and hence an analytic domain. Since every tropical admissible cover is G-admissible, this implies that the G-topology is finer than the tropical topology on X^{an}. Criterion (ii) follows immediately from Lemma 5.1.2. To verify (iii), we need to show that any G-covering of a tropical domain has a tropical refinement. Let V be a tropical domain, with $\{V_i\}_{i \in I}$ a G-admissible cover by analytic domains. Then each point $x \in V$ has a neighborhood in V which is a finite union of analytic domains $V_{i1} \cup \cdots \cup V_{ir}$, where each V_{ij} contains x. By Lemma 5.1.2, each of the analytic domains V_{ij} admits an admissible tropical cover. This shows that every point in V has a neighborhood which is a finite union of tropical domains, each contained in one of the analytic domains V_i. All of these tropical domains together form an admissible tropical cover of V that refines the G-admissible cover $\{V_i\}$, as required. $\hspace{1cm} \square$

Sheaves in a Grothendieck topology extend uniquely to sheaves in slightly finer Grothendieck topologies [BGR84, Proposition 9.2.3.1], so sheaves on X_G^{an} are determined by their values on tropical domains and the restriction maps between them.

There is a natural structure sheaf $\mathscr{O}^{\mathrm{trop}}$ in the tropical topology on X^{an}, which records the structure sheaves of all formal Gubler models associated to admissible subdivisions of $\mathrm{Trop}(X, \iota)$, as follows: Let $\overline{P}$ be the closure of a Γ-admissible polyhedron with recession cone in Σ, and let $V \subset X^{\mathrm{an}}$ be the preimage of $\overline{P}$ under the tropicalization map $X^{\mathrm{an}} \to \mathrm{Trop}(X, \iota)$. By §3.4, we have $V = \mathfrak{X}_P^{\mathrm{an}}$. Then we define

$$\mathscr{O}^{\mathrm{trop}}(V) := \text{ coordinate ring of } \mathfrak{X}_P.$$

Note that the global analytic functions on V are $K[V] = K \otimes_R \mathscr{O}^{\mathrm{trop}}(U)$.

Corollary 5.1.3. *The analytic structure sheaf $\mathscr{O}_{X_G^{\mathrm{an}}}$ is the unique sheaf in the G-topology that extends the sheaf $K \otimes_R \mathscr{O}^{\mathrm{trop}}$ in the tropical topology.* $\hspace{1cm} \square$

Remark 5.1.4. Fix a pair of K-varieties X and Y. Because morphisms $X^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$ are exactly morphisms of ringed spaces that are locally dual to bounded morphisms of affinoids in the G-topologies on X^{an} and Y^{an}, Proposition 5.1.1 and its Corollary 5.1.3 imply that $\mathrm{Hom}_K(X^{\mathrm{an}}, Y^{\mathrm{an}})$ is determined locally in the tropical topologies on X and Y. In this sense, we can recover the entire category of analytifications using the tropical topologies on analytifications.

Proof of Theorem 1.3. Let $X_{\mathrm{trop}}^{\mathrm{an}}$ denote X^{an} equipped with its tropical topology. By [BGR84, Proposition 9.2.3.1], restriction to the tropical topology induces an equivalence of topoi

$$\mathrm{Sh}(X_G^{\mathrm{an}}) \overset{\sim}{\longrightarrow} \mathrm{Sh}(X_{\mathrm{trop}}^{\mathrm{an}}). \tag{9}$$

The sheaf $\mathscr{O}_{X_G^{\mathrm{an}}}$ is a ring object in the topos $\mathrm{Sh}(X_G^{\mathrm{an}})$, and the sheaf $\mathscr{O}^{\mathrm{trop}}$ is a ring object in the topos $\mathrm{Sh}(X_{\mathrm{trop}}^{\mathrm{an}})$. By definition, each gives its respective topos the structure of a ringed topos. By Corollary 5.1.3, the equivalence (9) takes

$$\mathscr{O}_{X_G^{\mathrm{an}}} \longmapsto K \otimes_R \mathscr{O}^{\mathrm{trop}},$$

and is thus an equivalence of ringed topoi.

The fact that the functor (9) is restriction to a slightly coarser topology implies that at each point $x \in X^{\mathrm{an}}$, we have an isomorphism of stalks

$$\mathscr{O}_{X_G^{\mathrm{an}}, x} \cong \mathscr{O}_x^{\mathrm{trop}}.$$

Because $\mathscr{O}_{X_G^{\mathrm{an}}}$ is a sheaf of local rings, this makes (9) an equivalence of locally ringed topoi. $\hspace{1cm} \square$

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