Noname manuscript No.
(will be inserted by the editor)

Continuous limits of random walks over point processes and self-excited Black-Scholes models

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This version: October 4, 2019 / Received: date / Accepted: date

Abstract We modify the classical one-dimensional random walk by letting the time of each step be determined by a random point process, such as a self-exciting Hawkes process. We do not require the random walk have independent increments or satisfy the Markov property. In a suitable rescaled limit as space and time step sizes tend to zero, we show, as a generalization of Donsker's theorem, that the random walk weakly converges to a time-changed Brownian motion, where the time change is the compensator of the original counting process.

As an application, we view stock price changes as determined by random arrival times in a limit order book. For a stock price process driven by the limiting time-changed Brownian motion, we establish conditions under which European option payoffs are attainable as the terminal value of a self-financing strategy in the stock and a bond, and establish a unique no-arbitrage pricing formula. For a European call option we obtain an explicit formula parametrized by the integrated intensity of arrival times over the life of the option.

The first author is partially supported by a Simons Foundation Collaboration Grant for Mathematicians.

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Keywords Random walks · Time-changed Brownian motion · Point process · Limit order book

Mathematics Subject Classification (2010) 60G50 · 60G44 · 91G20

1 Introduction

The two most basic perspectives for modeling the stochastic evolution of an asset price process are derived from either a discrete time random walk or a continuous time Brownian motion. The connection between them is that the random walk, suitably scaled in space and time, weakly converges to Brownian motion as the increment size tends to zero. Exponentiating the random walk gives either the familiar binomial tree model or the geometric Brownian motion and the Black-Scholes-Merton model for option pricing.

For all their virtues, it is well known that these models are too homogeneous to be very good descriptions of the typical behavior of stock price processes. Rare events are too rare, volatility is too constant, and memory is too short. Instead, observed "stylized facts" of asset prices include fat tailed distributions of returns, heteroscedasticity, uncorrelated but not independent increments, and self-exciting behavior. Efforts to address these observations in the continuous-time framework include generalizing Brownian motion to Lévy processes (yielding fat tail distributions), and introducing explicit stochastic volatility as a separate source of randomness (such as with the Heston model).

In this paper we are motivated instead by the literature on dynamics of the limit order book that underlies stock price formation. From this perspective, stock price movements are the result of limit and market order arrivals collected in an electronic limit order book. Order arrival times are random and described by point processes, most notably in recent literature by "self-exciting" Hawkes processes, e.g. [18], [1], [4], [32], [15]. It is these order arrivals that explain movements of the quoted market price. In this view, price heteroscedasticity is explained endogenously by the variability in arrival rates of orders arriving in the order book.

We will consider a generalized random walk for which the i.i.d. space increments (e.g. log price changes) occur at random times given by a simple point process and associated counting process, such as a Hawkes process, rather than at the usual deterministic times. By natural scaling in space and time, we show this generalized random walk weakly converges to a time-changed Brownian motion, where the time change is the compensator of the original counting process. With mild assumptions, the resulting time-changed Brownian motion is a continuous square integrable martingale with respect to a suitable filtration, and has uncorrelated, but not necessarily independent, increments. The limiting continuous time process can serve as the basis for a Black-Scholes style option pricing model that incorporates the characteristics of the chosen point process describing the "excitability" of order arrivals in the limit order book underlying the asset price. This affords a perspective on a heteroscedastic stock price model for option pricing in which the fluctuating volatility is intrinsically connected to the underlying order book determining prices. When the point process is a homogeneous Poisson process with unit intensity, we recover the usual Brownian motion in the limit.

We next briefly summarize our main results, with more complete statements later. We say that a simple counting process N is *regular* if the compensator Λ

of N is *continuous, strictly increasing, and* $\Lambda(\infty) = \infty$. This is a mild assumption, and includes most examples of interest. In order to properly scale the intensity of a counting process, we show that for any regular counting process N with compensator Λ and for any integer $n \geq 1$ there exists a regular counting process N^n with compensator $\Lambda^n = n\Lambda$.

If we are given an i.i.d. sequence $\{\epsilon_i : i \geq 1\}$ of random variables with $E[\epsilon_i] = 0$ and $Var(\epsilon_i) = \sigma^2 < \infty$ and independent of N^n for all n, we may define a sequence of rescaled random walks over N by

$$\mathcal{S}_N^n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{N^n(t)} \epsilon_i.$$

These random walks over N can be compared to the standard sequence of rescaled random walks over deterministic (integer) times

$$\mathcal{W}^n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \epsilon_i,$$

that are well-known (Donsker's Theorem [5]) to converge weakly in the Skorokhod topology to the process $\sigma\mathcal{B}$, where $\mathcal{B}$ is standard Brownian motion.

A similar fact is true for the random walks $\mathcal{S}_N^n$ over N: we show, as a generalized version of Donsker's theorem, that the sequence $\{\mathcal{S}_N^n : n \geq 1\}$ of stochastic processes defined above converges weakly to the scaled, time-changed Brownian motion $\sigma\mathcal{B} \circ \Lambda$. Section 5 presents a natural generalization to sequences of random walks with a double sequence ϵ_i^n of jump random variables.

The time change Λ is only assumed to be the compensator of a regular counting process, and so need not be a Levy subordinator, or even Markov. This opens the door to "self-exciting" point processes such as those often used to model order arrivals in the high-frequency limit order book.

Theorems 3, 4, and 5 establish various useful properties of $\mathcal{B}(\Lambda(t))$: it is a continuous square integrable martingale; its quadratic variation is Λ; it is a standard Brownian motion if and only if N is Poisson with unit intensity; it has uncorrelated increments, but independent increments if and only if Λ is deterministic.

As an application, recall that the Black-Scholes stock price model is a rescaled limit of the binomial tree option pricing model. But instead of branching at the jump times of the deterministic counting process $\{\lfloor t \rfloor : t \geq 0\}$, we may substitute any regular counting process N, for example a Hawkes process. The resulting limiting stock price process becomes

$$S(t) = S_0 \exp(\sigma\mathcal{B}(\Lambda(t)) + \mu\Lambda(t)),$$

where $\mathcal{B}$ is a standard Brownian motion. If we take the numeraire as

$$B_t = \exp(rt),$$

we have a generalized Black-Scholes model that is flexible enough to display most of the stylized facts of stock price returns. For certain contingent claims X paying off at the maturity time T, and conditional on the value of $\Lambda(T)$, we show X

is attainable (may be replicated by a self-financing strategy) and obtain a no-arbitrage pricing formula of the form

$$V_t(X) = E_Q[\frac{B_t}{B_T} X \mid \mathcal{F}_t]$$

where Q is an equivalent martingale measure for the discounted stock. This is stated with precise hypotheses as Theorem 7 in section 6.

As an application of the pricing formula, we can give a unique conditional price (equation (7)) of a vanilla European call option, conditional on the value $T' = \Lambda(T)$ of the integrated intensity over the life of the option. The formula turns out to be the classical Black-Scholes formula, but with the strike price K replaced by $e^{r(T'-T)}K$, and the maturity T replaced by T'. The resulting call price is an increasing function of T', which can be considered a "market activity" parameter similar in effect to the volatility.

In the literature, financial applications of time-changed Brownian motion (and, more generally, time-changed Lévy processes) have been extensively studied, especially by Peter Carr and collaborators, for example in [8], [9], [10], [11], and also others such as [19], and goes back to [12] and [23]. The main difference with our work is that in those cases the time change process is an a priori ingredient in the model, whereas in this work the time change is a derived feature arising out of the scaled limit of discrete random walks toward a continuous process. Typically in the literature attention is directed toward Markov time change processes such as an integrated affine process ([8], [11], [19]) due to the need to compute the Laplace transform of the time change. In [9], Carr and Lee can consider an arbitrary continuous time change process because the swap price they study is independent of the time change. In this paper we restrict attention to independent continuous time changes, but are most interested as an application in the non-Markov case of the integrated intensity of a Hawkes process. (However, none of our results assume any process is Hawkes.) In all cases our time change comes from the compensator of a point process that provides the jump times of an underlying discrete random walk, whose rescaled limit is our time-changed Brownian motion.

Another significant direction in the literature has been the study of option pricing for discrete time models motivated by the binomial tree model, e.g. [13], [22]. In [27] the authors examine the weak convergence of discrete models where the jumps are general Bernoulli random variables, and study the corresponding convergence of option prices. They also consider a subclass of random jump times defined by N_n equally spaces points in the interval $[0, t]$, where N_n is a sequence of independent integer-values random variables. This framework is generalized somewhat by Prigent [25], in which the arrival times remain equally spaced but there is a more general finite probability space Ω_n on which the n-th model is defined. The comprehensive book [25] also surveys a variety of papers examining various versions of binomial models with special forms of randomized time steps, such as [14] in which the time steps are derived from Poisson processes. Jacod and Shiryaev [20] develop some quite general convergence theorems that imply Donsker's theorem, but restrict attention to semi-martingales with independent increments. These settings do not subsume our framework of i.i.d. jumps over regular counting processes, in which the jump variables need not be finite valued and the counting processes may be self-exciting.

Prigent [25], with many related references, is primarily focused on the general question of whether option prices for discrete models converge to corresponding prices for the continuous-time weak limits. This is a subtle topic we have not addressed in this paper.

Prigent [26] studies the class of risk-neutral measures for a quite general discrete market model where the log return is defined by marked point processes. Here and elsewhere, an option price is usually thought of as the conditional expectation of a discounted payoff with respect to a risk-neutral measure. This yields an arbitrage-free option price, but it is not unique when the payoff is not attainable.

In the present paper our perspective is somewhat different. We are considering more general random walks than in [27], but we don't examine the option prices for the discrete models directly, but rather show weak convergence to a continuous time model and price options there. Moreover, rather than try to characterize the class of risk-neutral measures, we focus on attainable claims, which have unique option prices. This perspective is motivated by the desire to treat potentially non-Markov point processes describing arrival times that have recently attracted attention for modeling the dynamics of the limit order book.

The remainder of the paper is organized as follows. In section 2 we describe random walks over continuous time point processes and their rescalings. Section 3 describes the weak limit of the rescaled random walks as a time-changed Brownian motion. Properties of the rescaled limit are discussed in section 4. Section 5 describes a generalization of the rescaled sequence of random walks in which the i.i.d. distribution of jumps need not be fixed along the rescaling sequence. Pinned processes are introduced in section 6. The pricing of terminal-time-payoff options is discussed in section 7 and we establish the option pricing formula that holds in this framework. Most proofs appear in the Appendix.

2 Discrete time random walks

In this section we define some terms and describe a class of random walks over counting processes suited to our purposes. For background, see for example [17], [25], and [31].

2.1 Point processes, counting processes, and compensators

We will be considering only simple, non-explosive point processes on $[0,\infty)$, i.e. sequences $\{T_n\}$ of $[0,\infty)$-valued random variables on a probability space $(\Omega, \mathbb{F}, P)$ such that $T_0 = 0$, $T_n < T_{n+1}$ a.s. for all n, and $\lim_{n\to\infty} T_n = +\infty$. For such a point process $\{T_n\}$, the corresponding counting process is

$$N(t) = \sum_{n\geq 1} 1_{(T_n \leq t)}$$

with $N(0) = 0$. The natural filtration $\mathcal{F}_t^N = \sigma(N_s : s \leq t)$ of N is automatically right continuous ([24], I.25). Assuming $E[N(t)] < \infty$ for all t, the Doob-Meyer decomposition (e.g. [16]) gives us a unique, cadlag $\mathcal{F}^N$-predictable process Λ, called the *compensator of* N, such that $\Lambda(0) = 0$ a.s., $E\big[\Lambda(t)\big] < \infty$ for all t, and $N(t) - \Lambda(t)$ is a cadlag $\mathcal{F}^N$-martingale.

In this paper we restrict attention to the (large) class of regular counting processes, defined as follows.

Definition 1 A simple, nonexplosive counting process N is *regular* if:

1. $E[N(t)] < \infty$ for all t,
2. the compensator Λ of N is continuous and strictly increasing, and
3. $\Lambda(\infty) = \infty$.

2.2 Random walks

The classical one-dimensional random walk $\mathcal{W}(t)$ is a piecewise constant cadlag stochastic process defined by

$$\mathcal{W}(t) = \sum_{i=1}^{\lfloor t \rfloor} \epsilon_i,$$

where $\{\epsilon_i : i \geq 0\}$ is an i.i.d. sequence of random variables and we assume $E[\epsilon_i] = 0$ and $Var(\epsilon_i) = \sigma^2 < \infty$.

Instead of jumps restricted to integer times $t = 1, 2, 3, \ldots$ of the deterministic counting process $\lfloor t \rfloor$, we can consider random walks with discontinuities at the random jump times of a regular counting process N. We then define

$$\mathcal{S}_N(t) = \mathcal{W}(N(t)) = \sum_{i=1}^{N(t)} \epsilon_i.$$

For the process $\mathcal{W}$, if we rescale space by $1/\sqrt{n}$ and event frequency by n, we obtain the rescaled random walk

$$\mathcal{W}^n(t) = \frac{1}{\sqrt{n}} \mathcal{W}(nt) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \epsilon_i,$$

with jumps occurring at integer multiples of $1/n$. This is the scaling needed for convergence in the limit to Brownian motion.

If we want to find the scaling limit for the process $\mathcal{S}_N$, we need a natural scaling for the counting process N that corresponds to $\lfloor nt \rfloor$; we need to scale the jump arrival frequency or *intensity* of N. When Λ is absolutely continuous, the intensity λ is defined by $\Lambda(t) = \int_0^t \lambda_s \, ds$, and indicates the rate per unit time of event arrivals. Scaling the event arrival rate by a factor n is equivalent to multiplying the intensity λ by n. This is the same as multiplying Λ by n, and the latter makes sense even when an intensity doesn't exist. Therefore we need a scaled version N^n of N such that the compensator of N^n is $n\Lambda$.

Theorem 1 *Let $N = \{N(t) : t \geq 0\}$ be a regular counting process. For any $n \geq 1$, there exists a counting process N^n with compensator $\Lambda^n = n\Lambda$.*

In particular, we may take $N^n(t) = N(\tau_t^n)$, where $\tau_t^n = \inf\{s : \Lambda(s) > n\Lambda(t)\}$.

We may then define the corresponding scaled random walk defined by a regular counting process $N(t)$ to be

$$\mathcal{S}_N^n(t) = \frac{1}{\sqrt{n}}\mathcal{W}(N^n(t)) = \frac{1}{\sqrt{n}}\sum_{i=1}^{N^n(t)} \epsilon_i.$$

Of course $\mathcal{W}^1 = \mathcal{W}$ and $\mathcal{S}_N^1 = \mathcal{S}_N$.

Like the rescaled classical random walk, the random walk $\mathcal{S}_N^n$ is a martingale with respect to the natural filtration and we have explicit expressions for the variance and covariances in terms of N.

Lemma 1 *For each $n = 1, 2, \dots$ and for all $s, t \geq 0$ we have:*

1. $E\big[\mathcal{S}_N^n(t)\big] = 0.$
2. $Cov\big(\mathcal{S}_N^n(s), \mathcal{S}_N^n(t)\big) = E\big[\mathcal{S}_N^n(s)\mathcal{S}_N^n(t)\big] = \frac{\sigma^2}{n}E\Big[N^n(s\wedge t)\Big] = \sigma^2 E\Big[N(s\wedge t)\Big]$, *where $s \wedge t = \min\{s,t\}$.*
3. $Var\big(\mathcal{S}_N^n(t)\big) = E\big[\big(\mathcal{S}_N^n(t)\big)^2\big] = \frac{\sigma^2}{n}E\Big[N^n(t)\Big] = \sigma^2 E\Big[N(t)\Big].$
4. *For each $n \geq 1$, $\mathcal{S}_N^n = \{\mathcal{S}_N^n(t) : t \geq 0\}$ is a cadlag martingale with respect to the history $\sigma(\mathcal{S}_N^n(u) : 0 \leq u \leq t)$ of $\mathcal{S}_N^n$.*

3 Weak convergence of the rescaled limit

In this section we consider weak convergence of rescaled random walks as the scaling factor n tends to infinity. In the case of the classical random walks $\mathcal{W}^n(t)$, the limit is Brownian motion (this is known as Donsker's theorem). For the random walks $\mathcal{S}^n$, we obtain a time changed Brownian motion. A detailed reference on weak convergence in this context is the book of Prigent [25].

We denote by D_∞ the space of cadlag real-valued functions on $[0, \infty)$, and d_∞ the Skorokhod metric on D_∞, for which the metric space (D_∞, d_∞) is separable and complete (see e.g. [5] for definitions). We will be considering weak convergence of sequences in D_∞ relative to the Skorokhod topology.

Let $(\Omega, \mathbb{F}, P)$ be a standard Wiener probability space on which are defined (i) standard Brownian motion $\mathcal{B}(t)$, (ii) the i.i.d. sequence of random variables $\{\epsilon_i\}$ with mean 0 and variance σ^2, and (iii) the independent regular counting process N with compensator Λ. Then $\mathcal{B} \circ \Lambda$, defined by $(\mathcal{B} \circ \Lambda)(t) = \mathcal{B}(\Lambda(t))$, is a random element of the metric space (D_∞, d_∞), and so are $\mathcal{W}^n$ and $\mathcal{S}_N^n$ for each n.

We may now restate theorem 2 with more precision:

Theorem 2 *Given a regular counting process N with compensator Λ, the sequence $\{\mathcal{S}_N^n : n \geq 1\}$ of rescaled random walks over N converges weakly in (D_∞, d_∞) to the time-changed Brownian motion $\sigma\mathcal{B} \circ \Lambda$. Moreover, $\mathcal{B}$ and Λ are independent.*

We remark that Λ need not be a subordinator, so the limit $\sigma\mathcal{B} \circ \Lambda$, while continuous, need not be a Lévy process, or even Markov. Theorem 2 is, in spirit, a generalization of Donsker's theorem, except that the counting process $\lfloor nt \rfloor$ is not regular (having a discontinuous compensator). The Donsker case is recovered when N is chosen to be a homogeneous Poisson process with unit intensity, in which case the compensator is $\Lambda(t) = t$ and the weak limit is then $\sigma\mathcal{B}(t)$, a multiple of standard Brownian motion.

4 Properties of the rescaled limit

The process $\mathcal{B} \circ \Lambda$ depends on N because Λ depends on N. To emphasis that dependence, we will use the notation

$$\mathcal{B}^N = \mathcal{B} \circ \Lambda, \text{ i.e. } \mathcal{B}^N(t) = \mathcal{B}(\Lambda(t)).$$

To set notation, let $\{\mathcal{F}_t^N\}$ be the filtration of the history of N, $\mathcal{F}_t^N = \sigma(N_u : u \leq t)$. Recalling that $\Lambda(t)$ is strictly increasing and therefore the functional inverse Λ^{-1} is well-defined, let $\{\mathcal{H}_t\}$ be the filtration defined by

$$\mathcal{H}_t = \sigma(\mathcal{B}(r), \Lambda^{-1}(u) : r \leq t, u \leq t).$$

By the usual augmentation if necessary, we assume that these filtrations satisfy the usual conditions.

Definition 2 (Revuz and Yor, Ch. V, [28]) Let $\mathcal{T} = \{\mathcal{T}_t : t \geq 0\}$ be a right-continuous filtration. A *$\mathcal{T}$-time-change* C is a family C_t, $t \geq 0$, of $\mathcal{T}$-stopping times such that the maps $s \to C_s$ are a.s. non-decreasing and right-continuous.

Given a time-change C, a process X is said to be *C-continuous* if X is constant on each interval $[C_{t-}, C_t]$.

Lemma 2 (Revuz and Yor [28]) *Let $\mathcal{T}$ be a right-continuous filtration, C be an a.s. finite $\mathcal{T}$-time-change, and X be a continous $\mathcal{T}$-local martingale. If X is C-continuous, then X_C is a continuous local martingale with respect to the filtration $\{\mathcal{T}_{C_t} : t \geq 0\}$.*

Now we can take the first steps in understanding the properties of $\mathcal{B}^N$.

Lemma 3 *For each $t \geq 0$, $\Lambda(t)$ is an $\mathcal{H}$-stopping time and $\{\Lambda(t) : t \geq 0\}$ is an a.s. finite $\mathcal{H}$-time-change.*

Proof. For each $t \geq 0$, $\Lambda(t)$ is an $\mathcal{H}$-stopping time: for any $s \geq 0$, $\Lambda^{-1}(s)$ is $\mathcal{H}_s$-measurable, hence $\{\Lambda(t) \leq s\} = \{\Lambda^{-1}(s) \geq t\} \in \mathcal{H}_s$. Since $\Lambda(t)$ is continuous and strictly increasing in t, this means that Λ is an a.s. finite $\mathcal{H}$-time-change.

Lemma 4 *$\mathcal{B}^N$ is a continuous local martingale with respect to the filtration $\mathcal{H}_\Lambda = \{\mathcal{H}_{\Lambda_t} : t \geq 0\}$.*

Proof. Apply Lemma 2 with $\mathcal{T} = \mathcal{H}$, $C = \Lambda$, and $X = \mathcal{B}$. It is easy to see that $\mathcal{B}$ is a continuous $\mathcal{H}$-martingale, and hence a continuous $\mathcal{H}$-local martingale. It remains to see that $\mathcal{B}$ is Λ-continuous, which is immediate by continuity of Λ. □

For processes X, Y, denote the quadratic covariation of X and Y by $[X, Y]$. The proofs of the following basic facts about $\mathcal{B}^N$ appear in the Appendix.

Theorem 3 *For all $t \geq 0$,*

1. $[\mathcal{B}^N, \mathcal{B}^N](t) = \Lambda(t)$.
2. $[\Lambda, \Lambda](t) = 0$.
3. $[\mathcal{B}^N, \Lambda](t) = 0$.

Corollary 1 *The following are equivalent:*

1. *$\mathcal{B}^N$ is a standard Brownian motion.*

2. $[\mathcal{B}^N, \mathcal{B}^N](t) = t$ *for all* $t \geq 0$.
3. $N(t) - t$ *is a martingale with respect to the natural filtration.*
4. N *is a Poisson process with unit intensity.*

Proof. Since by Lemma 4 $\mathcal{B}^N$ is a continuous local martingale, (1) and (2) are equivalent as a consequence of Levy's characterization of Brownian motion (see e.g. Protter [24]). (3) and (4) are equivalent by Watanabe's characterization of Poisson processes (Watanabe [30]). It remains to show that (2) and (3) are equivalent. By Theorem 3, $[\mathcal{B}^N, \mathcal{B}^N](t) = \Lambda(t)$ for all $t \geq 0$. Hence (2) holds if and only if $\Lambda(t) = t$ for all t, which is true if and only if (3) since Λ is the compensator of N. ◻

We now summarize a few more important properties of $\mathcal{B}^N$:

Theorem 4 *Let N be a regular counting process with compensator Λ, $\mathcal{B}$ denote standard Brownian motion, and $\mathcal{H}$ the filtration $\mathcal{H}_t = \sigma(\mathcal{B}(r), \Lambda^{-1}(u) : r \leq t, u \leq t)$. Then the limit $\mathcal{B}^N$ of the random walks $\mathcal{S}_N^n$ over N is a continuous square integrable martingale with respect to the filtration $\mathcal{H}_\Lambda$.*

Furthermore $(\mathcal{B}^N)^2 - \Lambda$ is a continuous martingale with respect to the filtration $\mathcal{H}_\Lambda$, and

1. $E[\mathcal{B}^N(t)] = 0$.
2. $cov(\mathcal{B}^N(s), \mathcal{B}^N(t)) = E[N(s \wedge t)]$ *for all* $s, t \geq 0$.
3. $\mathcal{B}^N$ *has uncorrelated increments.*

Although the process $\mathcal{B}^N$ has uncorrelated increments, in general (and by design) it does not have independent increments. The following theorem describes when this stronger property holds.

Theorem 5 *Let $\sigma > 0$, $m \in \mathbb{R}$. The following are equivalent:*

1. $\mathcal{B}^N$ *has independent increments.*
2. Λ *is a deterministic function.*
3. N *has independent increments.*
4. $\sigma\mathcal{B}^N + m\Lambda$ *has independent increments.*

Moreover, if any of the above happens, then, for all $s < t$, $\mathcal{B}^N(t) - \mathcal{B}^N(s)$ is normally distributed with mean zero and variance $\Lambda(t) - \Lambda(s)$.

These properties make $\mathcal{B}^N$ useful as a model for financial price processes. Financial returns time series typically have close to zero autocorrelation, but squared returns show positive autocorrelation due to typically observed heteroscedasticity and the corresponding failure of independence of returns.

5 Weak convergence of generalized rescaled random walks

We need not insist that the distribution of the jumps ϵ_i be fixed as the sequence of random walks are rescaled toward the limit $\mathcal{B}^N$. Similar results hold in the following more general situation.

Consider a double sequence of jump random variables $\{\epsilon_i^n : n \geq 1, i \geq 1\}$. We make the following standing assumptions.

There exists $m \in \mathbb{R}, \sigma, M > 0$, and $p > 2$ such that for each $n \geq 1$:

1. $\{\epsilon_i^n : i \geq 1\}$ is a sequence of i.i.d. random variables
2. $E[\epsilon_1^n] = m/n$,
3. $Var(\epsilon_1^n) = \sigma^2/n$, and
4. $||\epsilon_1^n||_p \leq M/\sqrt{n}$,

where $||.||_p$ denotes the L^p norm. Note that we do not need to assume that ϵ_i^n and ϵ_i^m are independent for $n \neq m$.

Now for a regular counting process N with compensator Λ and independent of the ϵ_i^n, we can define the generalized random walks based on this double sequence as

$$\hat{\mathcal{W}}^n(t) = \sum_{i=1}^{\lfloor nt \rfloor} \epsilon_i^n$$

and

$$\hat{\mathcal{S}}_N^n(t) = \sum_{i=1}^{N^n(t)} \epsilon_i^n.$$

Note that the previously treated random walks $\mathcal{W}^n$ and $\mathcal{S}_N^n$ are special cases of this setting in which $m = 0$ and $\epsilon_i^n = \epsilon_i/\sqrt{n}$ for all i, n, provided that the assumption $\epsilon_1 \in L^2$ is strengthened slightly to L^p for some $p > 2$ (which also holds in typical cases like normal, Bernoulli, or exponential jumps). The first part of the next theorem could be considered a simple extension of Donsker's theorem to double sequences.

Theorem 6 *For the stochastic processes $\hat{\mathcal{W}}^n(t)$, $\hat{\mathcal{S}}^n(t)$ defined above for $n \geq 1$, we have, as $n \to \infty$,*

$$\hat{\mathcal{W}}^n \Rightarrow \{mt + \sigma\mathcal{B}(t) : t \geq 0\}$$

and

$$\hat{\mathcal{S}}_N^n \Rightarrow m\Lambda + \sigma\mathcal{B}^N$$

where $\Rightarrow$ denotes weak convergence in the space (D, d) as before.

6 Pinned processes

In discussing option prices we will focus on a special class of processes called *pinned processes*, of which the Brownian bridge is an example.

Definition 3 We say a continuous random process $\Lambda(t)$ with $\Lambda(0) = 0$ is *pinned* at time T if $\Lambda(T) = T'$ for some deterministic T'.

Absolutely continuous, increasing, even self-exciting pinned processes are not hard to come by. For example, let r_t be any positive adapted RCLL process, possibly including a self-excited process such as the intensity of a Hawkes process. Let $R_t = e^{-r_t} < 1$. For any $T', c > 0$ with $c < T'/T^2$, define

$$\Lambda(t) = \frac{T'}{T}t - c(T-t)\int_0^t R_s\, ds.$$

Then $\Lambda(0) = 0$, $\Lambda(T) = T'$, $\Lambda(\infty) = \infty$, Λ is absolutely continuous, adapted to the natural filtration of r, and is increasing on $[0, \infty)$ because

$$\Lambda'(t) = \frac{T'}{T} + c \int_0^t R_s \, ds - c(T - t)R_t > \frac{T'}{T} - c(T - t) > 0.$$

The following proposition demonstrates that regular counting processes with pinned compensators are easily constructed.

Proposition 1 *Denote by $P(t)$ the homogeneous Poisson process with rate 1, and let Λ be any absolutely continuous, strictly increasing process pinned at time T. For each t, denote by $\mathcal{F}_t$ the sigma-algebra $\mathcal{F}_t^P \vee \mathcal{F}_t^{\Lambda^{-1}} = \sigma(P(s), \Lambda^{-1}(s) : s \leq t)$. Then $\Lambda(t)$ is an $\mathcal{F}$-stopping time for each t, and*

$$N(t) = P(\Lambda(t))$$

is a regular counting process on $[0, T]$ with compensator Λ with respect to $\mathcal{F}_\Lambda$.

Proof. We just need to show that $\Gamma \equiv N - \Lambda$ is an $\mathcal{F}_\Lambda$-martingale on $[0, T]$. Given $s < t \leq T$, we know that $S = \Lambda(s)$ and $T = \Lambda(t)$ are $\mathcal{F}$-stopping times bounded by $T' = \Lambda(T)$. Therefore by the optional stopping theorem (e.g. [28], II.3.2) applied to the martingale $P(t) - t$, we obtain

$$\Gamma(s) = P(\Lambda(s)) - \Lambda(s) = E[P(\Lambda(t)) - \Lambda(t) | \mathcal{F}_{\Lambda(s)}] = E[\Gamma(t) | \mathcal{F}_{\Lambda(s)}].$$

□

7 Option Pricing

The familiar Black-Scholes option pricing framework for a bond and stock price process B_t and $S(t)$, given by

$$B_t = \exp(rt), \quad S(t) = S_0 \exp(\sigma \mathcal{B}(t) + \mu t),$$

is often motivated as a limit of the exponential of a binomial random walk of the form $\mathcal{W}^n(t)$ discussed above. At short time scales, however, we tend to observe non-independent increments and some level of self-exciting behavior in the limit order book.

For example, the Hawkes process (e.g. [18], [1], [2], [15], [32] among many references) is a popular counting process to model the arrival rates of orders to the order book, and hence the jump times of the stock price process. We may take N to be a regular Hawkes process with intensity

$$\lambda_t = \alpha + \beta \int_0^t \mu(t - s) N(ds) > 0,$$

where $\alpha > 0$ and the response function $\mu(t)$ is a positive function satisfying $\int_0^\infty \mu(t) \, dt < \infty$. The (non-Markov) compensator is $\Lambda(t) = \int_0^t \lambda_t \, dt$. Then the random walks $\mathcal{S}_N^n$, $n \geq 1$, can be the basis of a model for the log stock price that will reflect the self-exciting nature of the Hawkes process. However, we need not restrict attention to Hawkes processes. When N is any regular point process with self-exciting characteristics, we can call the resulting (see Theorem 6) market model a

"Self-Excited Black-Scholes" model:

$$B_t = \exp(rt), \quad S(t) = S_0 \exp(\sigma \mathcal{B}^N(t) + \mu \Lambda(t)), \tag{1}$$

where r, σ, μ are positive constants.

This turns out to be a suitable framework for option pricing that can reflect heteroscedasticity and other non-stationary stylized facts of stock price behavior. The goal of this section is to show that in certain cases, conditional on the value $\Lambda(T)$, we can also obtain a pricing formula analogous to the standard Black-Scholes formula.

Recall the filtration $\mathcal{H}_t = \sigma(\mathcal{B}(r), \Lambda^{-1}(u) : r, u \leq t)$ with respect to which $\Lambda(t)$ is a stopping time for each t. If we define

$$\mathcal{G}_t = \mathcal{H}_{\Lambda_t},$$

then both $\mathcal{B}^N$ and Λ, hence $S(t)$, are $\mathcal{G}$-adapted processes on $(\Omega, \mathbb{F}, P)$.

It's convenient to define the additional filtrations, assumed complete:

$$\mathcal{F}_t^{\mathcal{B}} = \sigma(\mathcal{B}(s) : s \leq t) \subset \mathcal{H}_t,$$

$$\mathcal{F}_t^{\Lambda^{-1}} = \sigma(\Lambda^{-1}(s) : s \leq t) \subset \mathcal{H}_t,$$

$$\mathcal{F}_t^{\Lambda} = \sigma(\Lambda(s) : s \leq t),$$

$$\mathcal{F}_t^{S} = \sigma(S(s) : s \leq t) \subset \mathcal{G}_t.$$

To discuss the problem of option pricing, we review some standard terminology. We will say that a pair (ϕ_t, ψ_t) is a *self-financing strategy* if ϕ and ψ are $\mathcal{G}$-predictable processes such that if $V_t = \phi_t S(t) + \psi_t B_t$, then $dV_t = \phi_t dS(t) + \psi_t dB_t$. Here we interpret ϕ_t as the number of shares of stock held in a portfolio at time t, ψ_t the same for the bond, and so V_t is the time-t portfolio value.

If we fix a deterministic maturity time T, we call a random payoff $X \in \mathcal{G}_T$ at time T a *claim*. A claim X is an *attainable claim* if there exists a self-financing strategy (ϕ, ψ) with

$$X = V_T = \phi_T S(T) + \psi_T B_T,$$

i.e. the self-financing strategy replicates the claim at the terminal time. When this is the case, V_t will be the no-arbitrage price of the claim X at any time $t < T$.

In the classical Black-Scholes model with a stock and a bond, every claim is attainable (the market is complete) due to the martingale representation property for Brownian motion. In the context of the market model (1), a suitable representation property for the time-changed Brownian motion $\mathcal{B}^N$ is as far as we know not available. Instead, we will show that certain classes of claims are attainable, and in the process establish a familiar-looking conditional expectation formula for the claim price, with respect to a suitably defined measure. The remainder of this section spells out the details.

Market Assumptions:

1. $(\Omega, P, \mathbb{F})$ *is a probability space supporting a Brownian motion* $\mathcal{B}(t)$ *and an independent regular point process* N *with absolutely continuous compensator* Λ*, with* $\Lambda(t) = \int_0^t \lambda_u \, du$*, where* λ *is left-continuous with right limits and for some* $\epsilon > 0$*, for all* u*,* $\lambda_u > \epsilon$ *a.s.; and*
2. *there are two tradable assets, a stock* $S(t)$ *and bond* $B(t)$ *given by*
$$B_t = e^{rt}, \quad S(t) = S_0 \exp(\sigma \mathcal{B}^N(t) + \mu \Lambda(t)), \quad \sigma, \mu > 0, r \geq 0. \tag{2}$$
3. $X \in \mathcal{G}_T \cap L^1(\Omega, \mathbb{F}, P)$ *is a European option contract payoff at a fixed maturity* $T > 0$*.*

Theorem 7 (Pricing a European option) *Under the Market Assumptions above, suppose further that the compensator* Λ *of* N *is pinned at* T*. Let* Q *be the equivalent martingale measure defined by*

$$\frac{dQ}{dP} = \exp(-\int_0^{\Lambda(T)} \gamma_t \, d\mathcal{B}(t) - \frac{1}{2}\int_0^{\Lambda(T)} \gamma_t^2 \, dt), \tag{3}$$

where

$$\gamma_t = \frac{1}{\sigma}(\mu + \sigma^2/2 - \frac{r}{\lambda_{\Lambda^{-1}(t)}}), \quad 0 \leq t \leq \Lambda(T).$$

Then the discounted stock price $Z_t = B_t^{-1} S_t$ *(stopped at* T*) is a* $(Q, \mathcal{G})$*-martingale.*

Moreover, suppose in addition that the payoff X *is* Q*-independent of* $\mathcal{F}_T^{\Lambda}$*. Then* X *is attainable and the unique no-arbitrage price* V_t *of* X *at time* $t < T$ *is*

$$V_t = (B_t/B_T) E_Q[X|\mathcal{G}_t]. \tag{4}$$

The proof is postponed to the Appendix.

Corollary 2 *Under the Market Assumptions above, if* N *is an inhomogeneous Poisson process with positive intensity, then* X *is attainable and its unique no-arbitrage price at time* $t < T$ *is*

$$V_t = (B_t/B_T) E_Q[X|\mathcal{G}_t],$$

where the equivalent martingale measure is defined in equation (3).

Proof. An inhomogeneous Poisson process has a deterministic intensity, hence deterministic compensator Λ. The conclusion is immediate from Theorem 7. ◻

Corollary 3 *Under the Market Assumptions above, suppose further that the compensator* Λ *of the regular point process* N *is pinned at time* T*. Let* $X \in L^1$ *be the payoff of a vanilla option of the form* $X = f(S(T))$ *for some function* f*.*

Then X *is attainable and the unique no-arbitrage price* V_t *of* X *at time* $t < T$ *is*

$$V_t = (B_t/B_T) E_Q[X|\mathcal{G}_t], \tag{5}$$

where the measure Q *is the equivalent martingale measure defined by equation (3).*

Proof of Corollary 3. Let $\Lambda(T) = T'$. Since $S(T) = \exp(\sigma \mathcal{B}(\Lambda(T)) + m\Lambda(T)) = \exp(\sigma \mathcal{B}(T') + mT')$, it follows that $S(T)$, hence X, is independent of Λ and Λ^{-1}. The conclusion follows from Theorem 7. ◻

Corollary 4 *Under the assumptions of Corollary 3, suppose that X is a European-style payoff of the form $X = f(S(T))$. Then the time 0 price of the option is given by*

$$V_0 = e^{-rT} \int_{-\infty}^{\infty} f(S_0 e^{-\sigma^2 T'/2 + rT} e^{\sigma y}) p(y; \sigma^2 T') dy, \tag{6}$$

where $T' = \Lambda(T)$ and $p(y; \sigma^2 T')$ is the pdf of a normal distribution in y with mean zero and variance $\sigma^2 T'$.

In particular, if X is a European call option with strike K and maturity T, then its time-0 price is

$$V_0 = C_{BS}(\sigma, r, S_0, e^{r(T'-T)} K, T'), \tag{7}$$

where $C_{BS}(\sigma, r, S_0, K, T)$ is the standard Black-Scholes call price at strike K and maturity T.

Corollary 4 follows straightforwardly from (5) and the fact, as shown in the proof of Theorem 7, that

$$\begin{aligned} S(T) &= S_0 \exp(\sigma \tilde{\mathcal{B}}(\Lambda(T)) - (1/2)\sigma^2 \Lambda(T) + rT) \\ &= S_0 \exp(\sigma \tilde{\mathcal{B}}(T') - (1/2)\sigma^2 T' + rT), \end{aligned}$$

where $\tilde{\mathcal{B}}(T')$ is normally distributed with respect to Q with mean zero and variance T'.

Corollaries 3 and 4 raise the question of what happens when Λ is not pinned at time T, such as for a typical Hawkes process. In that case we don't expect payoffs to be attainable in general, and therefore neither do we expect there to be a unique no-arbitrage price. However, there are some things to say.

When the compensator Λ is not pinned, $\Lambda(T) = \int_0^T \lambda_s \, ds$ is a random variable representing the accumulated intensity of N up to time T. We can view equations (6) and (7) as the price conditional on $\Lambda(T)$, and therefore a scenario-based price, where the scenarios are parametrized by $\Lambda(T)$, the accumulated activity of price changes over the period $[0, T]$. These results would be relevant to stress-testing portfolio values under different possible future regimes of accumulated market activity over the life of the option.

It is straightforward to check that the expression in equation (7) is increasing in $T' = \Lambda(T)$ when the other parameters are held constant. Therefore one can think of the accumulated intensity $\Lambda(T)$ over the life of the option as playing a similar role as the volatility parameter σ in affecting the option price. The parameter $\Lambda(T)$ is a "trading activity" parameter separate from volatility but having a similar qualitative effect on the option price.

8 Conclusion

When the jump times of a one-dimensional random walk are determined by the random times of a counting process N with continuous compensator Λ, we obtain an extension of the usual case of deterministic jump times. We prove that the rescaled limit as the time step and jump size tend to zero is a time-changed Brownian motion $\mathcal{B}^N(t) = \mathcal{B}(\Lambda(t))$, where the time change is the compensator

Λ. When N is a homogeneous Poisson process with unit intensity, the limit reverts to the standard Brownian motion $\mathcal{B}(t)$.

We establish various properties of the limit process $\mathcal{B}^N$, including that it is a continuous square integrable martingale with respect to an appropriate filtration, and has independent increments if and only if the time change Λ is deterministic.

Motivated by the random jump times in the limit order book for a stock price and Theorem 6, we consider a generalized Black-Scholes model with a bond or cash account $B_t = e^{rt}$ and a stock $S(t) = S_0 \exp(\sigma \mathcal{B}^N(t) + \mu \Lambda(t))$. This market can be thought of as a continuous limit of a discrete stock price model where price changes are driven by a point process connected to the underlying limit order book activity. In this sense the market model's heteroskedastic features are derived from market-clock variations rather than imposed by an exogenously estimated stochastic volatility.

For certain classes of option payoffs X, and conditional on $\Lambda(T)$, we establish an option pricing formula in a familiar form

$$V_t = (B_t/B_T)E_Q[X|\mathcal{H}_{\Lambda(t)}],$$

where Q is the risk-neutral measure and $\mathcal{H} = \mathcal{F}^{\mathcal{B}} \vee \mathcal{F}^{\Lambda^{-1}}$. As an application, we can price a European call option with strike K and maturity T, conditional on the value $\Lambda(T) = T'$ of the integrated intensity of the counting process N over the life of the option, as

$$C_{BS}(\sigma, r, S_0, e^{r(T'-T)}K, T'),$$

where C_{BS} is the usual Black-Scholes call option price formula.

A technical obstacle to expanding the class of attainable claims is the question of whether or not, or when, $\mathcal{B}^N$ has the predictable representation property with respect to $\mathcal{H}_\Lambda$.

9 Appendix

In this Appendix we collect the previously postponed proofs.

Theorem 1: *Let $N = \{N(t) : t \geq 0\}$ be a regular counting process. For any integer $n \geq 1$, there exists a counting process N^n with $\mathcal{F}^n$-compensator $\Lambda^n = n\Lambda$ where $\mathcal{F}^n_t = \sigma\{N^n_s : 0 \leq s \leq t\}$ is the history of N^n.*

In particular, we may take $N^n(t) = N(\tau^n_t)$, where $\tau^n_t = \inf\{s : \Lambda(s) > n\Lambda(t)\}$.

Proof. Recall that we are writing $\mathcal{F}$ as the right-continuous filtration generated by the history of N, assumed complete.

Fix $n \geq 1$. Since the compensator Λ is strictly increasing and continuous, so is τ^n_t as a function of t, and $\Lambda(\tau^n_t) = n\Lambda(t)$. Next, we need to know that τ^n_t is an $\mathcal{F}$-stopping time for each t. The definition of τ^n_t implies

$$\{\tau^n_t \geq s\} = \{\Lambda(s) \leq n\Lambda(t)\},$$

so τ^n_t is an $\mathcal{F}$-stopping time provided $\{\Lambda(s) \leq n\Lambda(t)\} \in \mathcal{F}_s$ for all s. There are two cases.

Case 1: $s < t$. Then $\Lambda(s) \leq \Lambda(t) \leq n\Lambda(t)$ almost surely, hence $\{\Lambda(s) \leq n\Lambda(t)\}$ has full measure and so belongs to $\mathcal{F}_s$ by completeness of $\mathcal{F}$.

Case 2: $s \geq t$. Then $\mathcal{F}_t \subset \mathcal{F}_s$, so both Λ_s and $\Lambda(t)$ are $\mathcal{F}_s$ measurable and again $\{\Lambda(s) \leq n\Lambda(t)\} \in \mathcal{F}_s$.

Since τ_t^n is a stopping time, we may apply the Optional Stopping Theorem to the bounded stopping time $\tau_t^n \wedge u$, for any constant $u > 0$, and the $\mathcal{F}$-martingale

$$M_t = N(t) - \Lambda(t).$$

We may deduce via a monotone convergence argument as $u \to \infty$ that $\{M_{\tau_t^n}\}$ is an $\mathcal{F}_{\tau_t^n}$-martingale. □

Lemma 1: *For each $n = 1, 2, \ldots$ and for all $s, t \geq 0$ we have:*

1. $E\big[\mathcal{S}_N^n(t)\big] = 0.$
2. $Cov\big(\mathcal{S}_N^n(s), \mathcal{S}_N^n(t)\big) = E\big[\mathcal{S}_N^n(s)\mathcal{S}_N^n(t)\big] = \frac{\sigma^2}{n} E\Big[N^n(s \wedge t)\Big] = \sigma^2 E\Big[N(s \wedge t)\Big]$, *where $s \wedge t = \min\{s, t\}$.*
3. $Var\big(\mathcal{S}_N^n(t)\big) = E\big[\big(\mathcal{S}_N^n(t)\big)^2\big] = \frac{\sigma^2}{n} E\Big[N^n(t)\Big] = \sigma^2 E\Big[N(t)\Big].$
4. *For each $n \geq 1$, $\mathcal{S}_N^n = \{\mathcal{S}_N^n(t) : t \geq 0\}$ is a cadlag martingale with respect to the history $\sigma(\mathcal{S}_N^n(u) : 0 \leq u \leq t)$ of $\mathcal{S}_N^n$.*

Proof. Recall our notation $\mathcal{F}_t^n = \sigma\{N^n(s) : 0 \leq s \leq t\}$, the history of N^n.

Part (1) is immediate from the independence of N^n and $\{\epsilon_i : i \geq 1\}$, by first conditioning on $\mathcal{F}_t^n$ and using the property $E[\epsilon_i] = 0$.

Part (2) follows from a similar computation, using independence, and the facts

$$E(\epsilon_i \epsilon_j) = \sigma^2 \delta_{ij} \text{ and } E[N^n(s)] = nE[N(s)],$$

where $\delta_{ij} = 1$ if $i = j$, otherwise 0.

Part (3) is immediate from part (2).

For part (4), first we see that $\mathcal{S}_N^n(t)$ is in L^1 because, by Jensen's inequality and part (3),

$$E[|\mathcal{S}_N^n(t)|] \leq (E[(\mathcal{S}_N^n(t))^2])^{\frac{1}{2}} = (\sigma^2 E[N(t)])^{\frac{1}{2}} < \infty.$$

Now fix n and let $\mathcal{I}_t = \sigma(\mathcal{S}_N^n(u) : 0 \leq u \leq t)$. For $s \leq t$,

$$\begin{aligned}
\sqrt{n} E[\mathcal{S}_N^n(t) - \mathcal{S}_N^n(s) | \mathcal{I}_s] &= E[\sum_{i=N^n(s)+1}^{N^n(t)} \epsilon_i | \mathcal{I}_s] = E[E[\sum_{i=N^n(s)+1}^{N^n(t)} \epsilon_i | \mathcal{F}_t^n \vee \mathcal{I}_s | \mathcal{I}_s] \\
&= E[\sum_{i=N^n(s)+1}^{N^n(t)} E[\epsilon_i | \mathcal{F}_t^n \vee \mathcal{I}_s] | \mathcal{I}_s],
\end{aligned}$$

where the last equality is true because $N^n(t)$ and $N^n(s)$ are $\mathcal{F}_t^n$-measurable. Moreover, when $i > N^n(s)$, ϵ_i is independent of $\mathcal{I}_s$, hence independent of $\mathcal{F}_t^n \vee \mathcal{I}_s$. Since $E[\epsilon_i] = 0$ for all i, the terms inside the sum in the last expression are zero. This means $E[\mathcal{S}_N^n(t) - \mathcal{S}_N^n(s) | \mathcal{I}_s] = 0$, so $\mathcal{S}_N^n$ is a martingale. □

Theorem 2: *Given a regular counting process N with compensator Λ, the sequence $\{\mathcal{S}_N^n : n \geq 1\}$ of rescaled random walks over N converges weakly to the time-changed Brownian motion $\sigma\mathcal{B} \circ \Lambda$ in (D_∞, d_∞). Moreover, $\mathcal{B}$ and Λ are independent.*

Proof. Donsker's Theorem ([5]) states that the standard random walk $\mathcal{W}^n$ converges weakly to $\sigma\mathcal{B}$. We will show, in addition, that $\frac{1}{n}N^n$ converges weakly to Λ, and hence in the product topology $(\mathcal{W}^n, \frac{1}{n}N^n)$ converges weakly to $(\sigma\mathcal{B}, \Lambda)$. Since, by the Lemma below, composition is continuous in the Skorokhod topology, we will obtain

$$\mathcal{S}^n = \mathcal{W}^n \circ \frac{1}{n}N^n \Rightarrow \sigma\mathcal{B} \circ \Lambda.$$

We consider the spaces $C_a = C([0,a], R)$ and $C_\infty = C([0,\infty), R)$ of continuous real-valued functions on $[0,a]$ and $[0,\infty)$, respectively, and D_a, D_∞ the spaces of real valued functions on $[0,a]$ or $[0,\infty)$, respectively, that are right continuous with left limits.

Define further subsets

$$D_\uparrow = \{x \in D_\infty : x(0) \geq 0, x \text{ is non-decreasing}\},$$
$$C_{\uparrow\uparrow} = \{x \in C_\infty : x(0) \geq 0, x \text{ is strictly increasing}\}.$$

Lemma 5 (Theorem 13.2.1 and 2 of [31]) *The composition mapping*

$$\circ : D_\infty \times D_\uparrow \to D_\infty$$

taking (x,y) to $(x \circ y)$ is measurable, and is continuous on $(C_\infty \times D_\uparrow) \cup (D_\infty \times C_{\uparrow\uparrow})$ with respect to the Skorokhod topology.

For $a \geq 0$, define the restriction operator $r_a : D_\infty \to D_a$ by $(r_a x)(t) = x(t)$ for $t \in [0,a]$. We need the following additional lemma, which follows from theorems 16.7 and 3.1 of [5]:

Lemma 6 *Let $X_n \in D_\infty$ for $n = 1,2,3,\ldots$ and $X \in D_\infty$.*

1. *X_n converges weakly to X if and only if $r_a X_n$ converges weakly to $r_a X$ for every $a \geq 0$ such that $P(X \in \{x \in D_\infty : x(a) \neq x(a^-)\}) = 0$.*
2. *If for each $\epsilon > 0$, $P(\{d_a(r_a X, r_a X_n) < \epsilon\}) \to 1$ as $n \to \infty$, then $r_a X_n$ converges weakly to $r_a X$.*

Proof of Theorem 2, Step 1: $\frac{1}{n}N^n$ converges weakly to Λ as $n \to \infty$.

By Lemma 6 and the continuity of Λ, it suffices to establish, for any $\epsilon > 0$ and $a \geq 0$,

$$P(d_a(r_a\Lambda, r_a(\frac{1}{n}N^n)) \geq \epsilon) \to 0$$

as $n \to \infty$.

As a temporary notation, let $X^n_t = \frac{1}{n}N^n(t) - \Lambda(t) = \frac{1}{n}(N^n(t) - \Lambda^n_t)$. Since Λ^n_t is the compensator of $N^n(t)$, X^n is a cadlag martingale, so by Jensen's inquality $|X^n|$ is a non-negative submartingale. Moreover

$$d_a(r_a\Lambda, r_a(\frac{1}{n}N^n)) \leq \sup_{0\leq t\leq a} |r_a\Lambda(t) - r_a(\frac{1}{n}N^n(t))| = \sup_{0\leq t\leq a} |X^n_t|.$$

By Markov's Inequality, we also have

$$P(d_a(r_a\Lambda, r_a(\frac{1}{n}N^n)) \geq \epsilon) \leq \frac{1}{\epsilon}E[d_a(r_a\Lambda, r_a(\frac{1}{n}N^n))].$$

Hence

$$P(d_a(r_a\Lambda, r_a(\frac{1}{n}N^n)) \geq \epsilon) \leq \frac{1}{\epsilon}E[\sup_{0\leq t\leq a}|X_t^n|].$$

By Jensen's inequality and Doob's martingale inequality ([3], 2.1.5), we then have

$$\frac{1}{\epsilon}E[\sup_{0\leq t\leq a}|X_t^n|] \leq \frac{1}{\epsilon}(E[(\sup_{0\leq t\leq a}|X_t^n|)^2])^{\frac{1}{2}} = \frac{1}{\epsilon}(E[\sup_{0\leq t\leq a}(|X_t^n|)^2])^{\frac{1}{2}} \leq \frac{2}{\epsilon}(E[(X_a^n)^2])^{\frac{1}{2}}.$$

To complete Step 1, it remains only to show that for any $t \geq 0$, $E[(X_t^n)^2] \to 0$ as $n \to \infty$. First, note (e.g. Theorem 2.5.3 of [16]), that since N^n is a counting process with continuous compensator Λ^n such that $E[\Lambda^n(t)] < \infty$ for all t, we have

$$E[(N^n(t) - \Lambda^n(t))^2] = E[\Lambda^n(t)] \text{ for all } t \geq 0.$$

Therefore

$$E[(X_t^n)^2] = \frac{1}{n^2}E[\Lambda^n(t)] = \frac{1}{n}E[\Lambda(t)],$$

which tends to zero as $n \to \infty$.

Proof of Theorem 2, Step 2: $(\mathcal{W}^n, \frac{1}{n}N^n)$ converges weakly to $(\sigma\mathcal{B}, \Lambda)$ in $D_\infty \times D_\infty$ and the limiting components are independent.

This step follows from the next general lemma:

Lemma 7 *For each $n \geq 1$, let X_n and Y_n be independent random elements of the separable metric spaces (S, m) and (T, n), respectively.*

Then

(a) (X_n, Y_n) converges weakly to (X, Y) in $S \times T$ if and only if X_n converges weakly to X in S and Y_n converges weakly to Y in T, and

(b) If either condition in (a) holds, then X and Y are independent.

Part (a) of Lemma 7 follows from Theorem 11.4.4 of [31]; part (b) follows straightforwardly.

Step 2 then follows from Step 1, Donsker's Theorem, and Lemma 7.

Proof of Theorem 2, Step 3: $\mathcal{S}_N^n$ converges weakly to $\sigma\mathcal{B} \circ \Lambda$.

From Step 1, $\frac{1}{n}N^n$ converges weakly to Λ, and from Donsker's theorem $\mathcal{W}^n$ converges weakly to $\sigma\mathcal{B}$. By Step 2, we therefore have

$$(\mathcal{W}^n, \frac{1}{n}N^n) \Rightarrow (\sigma\mathcal{B}, \Lambda).$$

Since $(\sigma\mathcal{B}, \Lambda) \in C_\infty \times D_\uparrow$, Lemma 5 establishes the continuity of the composition operator $\circ$ at the point $(\sigma\mathcal{B}, \Lambda)$.

Weak convergence of the composition

$$\mathcal{S}_N^n = \mathcal{W}^n \circ (\frac{1}{n}N^n)$$

to $\sigma\mathcal{B} \circ \Lambda$ therefore follows from

Proposition 2 (Continuous Mapping Theorem, 3.4.3 of [31]) *Let (S, m) and (S', m') be metric spaces with random elements X_n converging weakly to X in (S, m). Let $g : S \to S'$ be measurable and denote by $Disc(g)$ the set of points of discontinuity of g.*

If $P(X \in Disc(g)) = 0$, then $g(X_n)$ converges weakly to $g(X)$.

$\square$

Theorem 3: *For all $t \geq 0$,*

1. $[\mathcal{B}^N, \mathcal{B}^N](t) = \Lambda(t)$.
2. $[\Lambda, \Lambda](t) = 0$.
3. $[\mathcal{B}^N, \Lambda](t) = 0$.

Proof.

Recall that $\mathcal{H}_t = \sigma(\mathcal{B}(r), \Lambda^{-1}(u) : r, u \leq t)$ and $\mathcal{G}_t = \mathcal{H}_{\Lambda(t)}$.

We need the following lemmas.

Lemma 8 (2.4 in Kobayashi [21]) *If $\mathcal{T}$ is a filtration satisfying the usual conditions, C is a finite $\mathcal{T}$-time-change, and Z is a $\mathcal{T}$-semimartingale that is C-continuous, then*

$$[Z, Z](C) = [Z(C), Z(C)].$$

Lemma 9 *Λ^{-1} is a finite $\mathcal{G}$-time-change.*

Assume Lemma 9 for the moment; we prove it below. Since Λ^{-1} is continuous, $\mathcal{B}^n$ is Λ^{-1}-continuous. Therefore by Lemma 8 we may deduce, for each t,

$$[\mathcal{B}^N, \mathcal{B}^N](\Lambda^{-1}(t)) = [\mathcal{B}^N(\Lambda^{-1}), \mathcal{B}^N(\Lambda^{-1})](t) = [\mathcal{B}, \mathcal{B}](t) = t$$

since $\mathcal{B}^N(t) = \mathcal{B}(\Lambda(t))$. Composing both sides by Λ yields part (i). A similar argument yields part (ii). Applying this argument to the continuous process $\mathcal{B}^N + \Lambda$ and applying the polarization identity gives us part (iii).

Proof of Lemma 9

Since Λ is continuous, strictly increasing, and $\Lambda(\infty) = \infty$, the same is true for Λ^{-1}, so we only need to show that $\Lambda^{-1}(t)$ is a $\mathcal{G}$-stopping time for all t.

Fix $t > 0$; we want to show that $\{\Lambda^{-1}(t) \leq s\} \in \mathcal{G}_s = \mathcal{H}_{\Lambda(s)}$ for all $s \geq 0$, which is true if and only if

$$\{\Lambda^{-1}(t) \leq s\} \cap \{\Lambda(s) \leq u\} \in \mathcal{H}_u$$

for all $u \geq 0$. We will repeatedly use the convenient fact that for any $v, w > 0$, $\{\Lambda^{-1}(v) \leq w\} = \{\Lambda(w) \geq v\}$. In particular,

$$\{\Lambda^{-1}(t) \leq s\} \cap \{\Lambda(s) \leq u\} = \{\Lambda(s) \geq t\} \cap \{\Lambda(s) \leq u\}.$$

It remains to prove that the latter belongs to $\mathcal{H}_u$.

Case 1: $u < t$. Then the event in question is empty, hence in $\mathcal{H}_u$.

Case 2: $u \geq t$. Then for all $n = 1, 2, 3, \ldots$, $u > t - \frac{1}{n}$, and so

$$\{\Lambda^{-1}(t - \frac{1}{n}) \geq s\} \in \mathcal{H}_{t-\frac{1}{n}} \subset \mathcal{H}_u.$$

Hence

$$\{\Lambda(s) < t\} = \cup_{n=1}^{\infty}\{\Lambda(s) \le t - \frac{1}{n}\} \in \mathcal{H}_u.$$

This means the complementary event $\{\Lambda(s) \ge t\}$ also belongs to $\mathcal{H}_u$. In addition,

$$\{\Lambda(s) \le u\} = \{\Lambda^{-1}(u) \ge s\} \in \mathcal{H}_u,$$

and we are done. □

Theorem 4: *Let N be a regular counting process with compensator Λ, $\mathcal{B}$ denote standard Brownian motion, and $\mathcal{H}$ the filtration $\mathcal{H}_t = \sigma(\mathcal{B}(r), \Lambda^{-1}(u) : r \le t, u \le t)$. Then the limit $\mathcal{B}^N$ of the random walks $\mathcal{S}_N^n$ over N is a continuous square integrable martingale with respect to the filtration $\mathcal{H}_\Lambda$.*

Furthermore $(\mathcal{B}^N)^2 - \Lambda$ is a continuous martingale with respect to the filtration $\mathcal{H}_\Lambda$, and

1. *$E[\mathcal{B}^N(t)] = 0$.*
2. *$cov(\mathcal{B}^N(s), \mathcal{B}^N(t)) = E[N(s \wedge t)]$ for all $s, t \ge 0$.*
3. *$\mathcal{B}^N$ has uncorrelated increments.*

Proof.

We make use of *Burkholder's Inequality* (see Protter [24], theorem IV.73): if X is a continuous local martingale, $X_0 = 0$, $2 \le p < \infty$, and T is a finite stopping time, then

$$E[(\sup_{0\le s\le T} |X_s|)^p] \le C_p E[[X,X]_T^{p/2}],$$

where $C_p = [q^p(\frac{p(p-1)}{2})]^{p/2}$ with q defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Now we know (Lemma 4) that $\mathcal{B}^N$ is a continuous local martingale with respect to $\mathcal{G} = \mathcal{H}_\Lambda$. Using Jensen's inequality, then Burkholder's inequality, then Theorem 3, we have

$$E[\sup_{0\le s\le t} |\mathcal{B}^N(s)|] \le (E[(\sup_{0\le s\le t} |\mathcal{B}^N(s)|)^2])^{\frac{1}{2}} \le 2(E[[\mathcal{B}^N, \mathcal{B}^N](t)])^{\frac{1}{2}} = 2E[\Lambda(t)]^{\frac{1}{2}} < \infty.$$

This implies that $\mathcal{B}^N$ is a martingale.

By similar argument,

$$E[(\mathcal{B}^N(t))^2] \le E[(\sup_{0\le s\le t} |\mathcal{B}^N(s)|)^2] \le 4E[[\mathcal{B}^N, \mathcal{B}^N](t)] = 4E[\Lambda(t)] < \infty, \qquad (8)$$

so $\mathcal{B}^N$ is square integrable.

Next, by standard arguments (e.g. Protter [24], Theorem II.20), the continuous process

$$(\mathcal{B}^N)^2 - \Lambda = (\mathcal{B}^N)^2 - [\mathcal{B}^N, \mathcal{B}^N] = 2\int \mathcal{B}_-^N \, d\mathcal{B}^N$$

is a local martingale. Using an argument similar to equation (8) we may deduce that $(\mathcal{B}^N)^2 - \Lambda$ is also a martingale.

It remains to prove the three numbered statements of the Theorem. Part 1 is immediate since $\mathcal{B}^N$ is a martingale and $\mathcal{B}^N(0) = 0$.

For part 2, assume $s \leq t$. By part 1, $cov(\mathcal{B}^N(s), \mathcal{B}^N(t)) = E[\mathcal{B}^N(s)\mathcal{B}^N(t)]$. By conditioning on $\mathcal{G}_s$ and using the martingale property, it is easily shown that

$$E[\mathcal{B}^N(s)\mathcal{B}^N(t)] = E[(\mathcal{B}^N(s))^2] = E[N(s)],$$

the last equality because $(\mathcal{B}^N)^2 - \Lambda$ is a martingale with initial value zero.

For part 3, we compute as follows, employing part 2. For all $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$,

$$\begin{aligned} &E[(\mathcal{B}^N(t_2) - \mathcal{B}^N(t_1))(\mathcal{B}^N(t_4) - \mathcal{B}^N(t_3))] \\ &= E[\mathcal{B}^N(t_2)\mathcal{B}^N(t_4)] - E[\mathcal{B}^N(t_2)\mathcal{B}^N(t_3)] - E[\mathcal{B}^N(t_1)\mathcal{B}^N(t_4)] + E[\mathcal{B}^N(t_1)\mathcal{B}^N(t_3)] \\ &= E[N(t_2)] - E[N(t_2)] - E[N(t_1)] + E[(N(t_1)] = 0. \end{aligned}$$

□

Theorem 5: *Let $\sigma > 0$, $m \in \mathbb{R}$. The following are equivalent:*

1. *$\mathcal{B}^N$ has independent increments.*
2. *Λ is a deterministic function.*
3. *N has independent increments.*
4. *$\sigma\mathcal{B}^N + m\Lambda$ has independent increments.*

Moreover, if any of the above happens, then, for all $s < t$, $\mathcal{B}^N(t) - \mathcal{B}^N(s)$ is normally distributed with mean zero and variance $\Lambda(t) - \Lambda(s)$.

Proof.

We first show that (1) and (2) are equivalent. Assume first that $\mathcal{B}^N$ has independent increments, meaning that for any $s \leq t$,

$$E[\mathcal{B}^N(t) - \mathcal{B}^N(s)|\mathcal{I}_s] = E[\mathcal{B}^N(t) - \mathcal{B}^N(s)]$$

where $\mathcal{I}$ is the natural filtration of $\mathcal{B}^N$. Note $\mathcal{I}_s \subset \mathcal{G}_s$ for all s, since $\mathcal{B}^N$ is a $\mathcal{G}$-martingale, hence $\mathcal{G}$-adapted.

Using the identity

$$\mathcal{B}^N(t)^2 - \mathcal{B}^N(s)^2 = (\mathcal{B}^N(t) - \mathcal{B}^N(s))^2 + 2\mathcal{B}^N(s)(\mathcal{B}^N(t) - \mathcal{B}^N(s))$$

and taking expectation conditional on $\mathcal{I}_s$, we obtain, by Theorem 4 and independence of increments,

$$\begin{aligned} E[(\mathcal{B}^N(t))^2 - (\mathcal{B}^N(s))^2|\mathcal{I}_s] &= E[(\mathcal{B}^N(t) - \mathcal{B}^N(s))^2] + 2\mathcal{B}^N(s)E[\mathcal{B}^N(t) - \mathcal{B}^N(s)] \\ &= E[(\mathcal{B}^N(t))^2] + E[(\mathcal{B}^N(s))^2] - 2E[\mathcal{B}^N(s)\mathcal{B}^N(t)] \\ &= E[N(t)] + E[N(s)] - 2E[N(s)] \\ &= E[\Lambda(t)] - E[\Lambda(s)]. \end{aligned}$$

We therefore have

$$E[(\mathcal{B}^N(t))^2 - E[\Lambda(t)]|\mathcal{I}_s] = (\mathcal{B}^N(s))^2 - E[\Lambda(s)].$$

Thus $(\mathcal{B}^N(t))^2 - E[\Lambda(t)]$, which is evidently also L^1, is an $\mathcal{I}$-martingale.

On the other hand, from Theorem 4 and since $\mathcal{I} \subset \mathcal{G}$, we know that $(\mathcal{B}^N(t))^2 - \Lambda(t)$ is an $\mathcal{I}$-martingale.

We also know that $\mathcal{B}^N$ is a square integrable martingale with respect to $\mathcal{G}$, hence $\mathcal{I}$, so that $(\mathcal{B}^N)^2$ is a non-negative $\mathcal{I}$-submartingale. By the Doob-Meyer decomposition, $(\mathcal{B}^N)^2$ has a unique compensator A ($\mathcal{I}$-predictable, increasing, right-continuous, L^1 with $A(0) = 0$) such that $(\mathcal{B}^N)^2 - A$ is an $\mathcal{I}$-martingale. Since both Λ and $E[\Lambda]$ satisfy these conditions, by uniqueness we must have $\Lambda(t) = E[\Lambda(t)]$ for all t, so Λ must be a deterministic function.

Conversely, assume that Λ is deterministic. (The idea for this part of the proof comes from the proof of Levy's Theorem in Protter [24], theorem II.39.) We wish to prove that $\mathcal{B}^N$ has independent increments; we establish the slightly stronger conclusion that, for $s < t$, $\mathcal{B}^N(t) - \mathcal{B}^N(s)$ is independent of $\mathcal{G}_s$.

Fix $u \in R$ and define the smooth function $F(x,t) = \exp(iux + \frac{u^2}{2}t)$. Define $Z_t = F(\mathcal{B}^N(t), \Lambda(t))$. Itô's formula yields

$$\begin{aligned} Z_t = 1 + iu \int_0^t Z_s \, d\mathcal{B}^N(s) + \frac{u^2}{2} \int_0^t Z_s \, d\Lambda(s) - \frac{u^2}{2} \int_0^t Z_s \, d[\mathcal{B}^N, \mathcal{B}^N](s) \\ + \frac{iu^3}{4} \int_0^t Z_s \, d[\mathcal{B}^N, \Lambda](s) + \frac{u^4}{8} \int_0^t Z_s \, d[\Lambda, \Lambda](s). \end{aligned}$$

By Theorem 3, this means

$$Z_t = 1 + iu \int_0^t Z_s \, d\mathcal{B}^N(s).$$

Hence Z is a local $\mathcal{G}$-martingale. Therefore by [24], Theorem I.51, it is also a martingale: for any $t > 0$,

$$\sup_{0 \le r \le t} |Z_r| = \sup_{0 \le r \le t} |\exp(iu\mathcal{B}^N(r) + \frac{u^2}{2}\Lambda(r))| = \sup_{0 \le r \le t} |\exp(\frac{u^2}{2}\Lambda(r))| = \exp(\frac{u^2}{2}\Lambda(t)),$$

meaning, since $\Lambda(t)$ is a deterministic finite constant, that

$$E[\sup_{0 \le r \le t} |Z_r|] = \exp(\frac{u^2}{2}\Lambda(t)) < \infty.$$

We thus have, for $s < t$,

$$E[\exp(iu\mathcal{B}^N(t) + \frac{u^2}{2}\Lambda(t))|\mathcal{G}_s] = \exp(iu\mathcal{B}^N(s) + \frac{u^2}{2}\Lambda(s))$$

and, since Λ is deterministic,

$$E[\exp(iu(\mathcal{B}^N(t) - \mathcal{B}^N(s)))|\mathcal{G}_s] = \exp(-\frac{u^2}{2}(\Lambda(t) - \Lambda(s))).$$

Since the right hand side of this last equality is deterministic, we deduce that $\mathcal{B}^N(t) - \mathcal{B}^N(s)$ is independent of $\mathcal{G}_s$, as desired.

Moreover, since now

$$E[\exp(iu(\mathcal{B}^N(t) - \mathcal{B}^N(s)))] = \exp(-\frac{u^2}{2}(\Lambda(t) - \Lambda(s)))$$

holds for all $u \in R$, and the right hand side is the moment generating function of a normal random variable, we deduce that $\mathcal{B}^N(t) - \mathcal{B}^N(s)$ is normally distributed with mean zero and variance $\Lambda(t) - \Lambda(s)$.

Next we prove that (2) and (3) are equivalent.

Assume that N has independent increments, i.e. for $s \leq t$, $N(t) - N(s)$ is independent of $\mathcal{F}_s^N$. Then

$$E[N(t) - N(s)|\mathcal{F}_s^N] = E[N(t) - N(s)] = E[\Lambda(t) - \Lambda(s)],$$

so that

$$E[N(t) - E[\Lambda(t)]|\mathcal{F}_s^N] = N(s) - E[\Lambda(s)].$$

Again by the Doob-Meyer uniqueness of the compensator, we have $\Lambda(t) = E[\Lambda(t)]$ and hence Λ is deterministic.

Conversely, if Λ is deterministic, then Bremaud [7] shows, in an extension of Watanabe's characterization theorem for Poisson processes, that N has independent increments.

Assuming (2), we now get (1) and therefore immediately (4). It remains to show that (4) implies (2).

For ease of notation, let $X_t = \sigma\mathcal{B}^N(t) + m\Lambda(t)$. Let $\mathcal{F}^X$ denote the natural filtration of X, and $\mathcal{F}^{\mathcal{B}^N}$ the natural filtration of $\mathcal{B}^N$.

If X has independent increments, then for $s < t$,

$$E[X_t - X_s|\mathcal{F}_s^X] = E[X_t - X_s] = E[m\Lambda(t) - m\Lambda(s)],$$

so $X_t - mE[\Lambda(t)]$ is an $\mathcal{F}^X$-martingale. Now since $[\mathcal{B}^N, \mathcal{B}^N]_t = \Lambda(t)$ and $[X, X]_t = \sigma^2\Lambda(t)$, we can see that Λ is adapted to both $\mathcal{F}^X$ and $\mathcal{F}^{\mathcal{B}^N}$. Hence X is adapted to $\mathcal{F}^{\mathcal{B}^N}$ and $\mathcal{B}^N$ is adapted to $\mathcal{F}^X$, which implies that $\mathcal{F}^{\mathcal{B}^N} = \mathcal{F}^X$.

As a result, $X_t - mE[\Lambda(t)]$ is an $\mathcal{F}^{\mathcal{B}^N}$-martingale. By Theorem 4, so is $X_t - m\Lambda(t)$, and hence, by subtraction, so is $\Lambda(t) - E[\Lambda(t)]$. Notice now that both $\Lambda(t)$ and $E[\Lambda(t)]$ qualify as compensators of the nonnegative submartingale $\Lambda(t)$. By the uniqueness of the Doob-Meyer decomposition, $\Lambda(t) = E[\Lambda(t)]$, and hence Λ is deterministic.

□

For Theorem 6, first recall our notation. We are considering a double sequence of jump random variables $\{\epsilon_i^n : n \geq 1, i \geq 1\}$, and we assume there exists $m \in \mathbb{R}, \sigma, M > 0$, and $p > 2$ such that for each $n \geq 1$:

1. $\{\epsilon_i^n : i \geq 1\}$ is a sequence of i.i.d. random variables
2. $E[\epsilon_1^n] = m/n$,
3. $Var(\epsilon_1^n) = \sigma^2/n$, and
4. $||\epsilon_1^n||_p \leq M/\sqrt{n}$,

where $||.||_p$ denotes the L^p norm. Note that we do not need to assume that ϵ_i^n and $\epsilon_{m,i}$ are independent for $n \neq m$. We then define

$$\hat{\mathcal{W}}^n(t) = \sum_{i=1}^{\lfloor nt \rfloor} \epsilon_i^n$$

and

$$\hat{\mathcal{S}}_N^n(t) = \sum_{i=1}^{N^n(t)} \epsilon_i^n.$$

Theorem 6: *For the stochastic processes $\hat{\mathcal{W}}^n(t)$, $\hat{\mathcal{S}}^n_N(t)$ defined above for $n \geq 1$, we have, as $n \to \infty$,*

$$\hat{\mathcal{W}}^n \Rightarrow \{mt + \sigma \mathcal{B}(t) : t \geq 0\}$$

and

$$\hat{\mathcal{S}}^n_N \Rightarrow m\Lambda + \sigma \mathcal{B}^N$$

where $\Rightarrow$ denotes weak convergence in the space (D_∞, d_∞) as before.

Proof.

We give the proof for the case $m = 0$; the general case $m \neq 0$ is a straightforward extension.

Part 1: $\hat{\mathcal{W}}^n \Rightarrow \sigma\mathcal{B}$.

For this part we employ a convergence lemma found in Whitt [31], Internet Supplement Thm 2.4.2.

First, some notation. Define the truncation function $h : R \to R$ by $h(x) = x$ for $|x| \leq 1$, $h(x) = 0$ for $|x| \geq 2$, and extended linearly to preserve continuity: $h(x) = 2 - x$ if $x \in [1, 2]$ and $-2 - x$ if $x \in [-2, -1]$. Note $|h(x)| \leq 1$ all x and h is supported in $[-2, 2]$.

Also, for convenience define $\mathcal{Z}$ to be the collection of all bounded, continuous functions $g : R \to R$ such that g vanishes on a neighborhood of 0, and there exists $y \in R$ such that $g(x) \to y$ as $x \to \pm\infty$.

Definition 4 A sequence $\{X_n\}$ of random variables is *infinitesimal* if

$$\lim_{n\to\infty} P(|X_n| > \delta) = 0 \text{ for all } \delta > 0,$$

and is *super-infinitesimal* if

$$\lim_{n\to\infty} nP(|X_n| > \delta) = 0 \text{ for all } \delta > 0,$$

Lemma 10 (Whitt, Theorem 2.4.2) *With the notation above, if the sequence $\{\epsilon^n_1\}$ is infinitesimal, then*

$\hat{\mathcal{W}}^n \Rightarrow L$, where L is a Levy process with characteristics (b, σ^2, μ)
if and only if

For h, $\mathcal{Z}$ defined above, the following three statements are true:

1. $\lim_{n\to\infty} nE[h(\epsilon^n_1)] = b$,
2. $\lim_{n\to\infty} nVar(h(\epsilon^n_1)) = \sigma^2$, *and*
3. $\lim_{n\to\infty} nE[g(\epsilon^n_1)] = \int_R g d\mu$ *for all* $g \in \mathcal{Z}$.

Since our desired limit $\sigma\mathcal{B}$ is a Levy process with characteristics $(0, \sigma^2, 0)$, we complete the proof of Part 1 by showing that $\{\epsilon^n_1\}$ is infinitesimal, and

(i) $\lim_{n\to\infty} nE[h(\epsilon^n_1)] = 0$,
(ii) $\lim_{n\to\infty} nVar(h(\epsilon^n_1)) = \sigma^2$, and
(iii) $\lim_{n\to\infty} nE[g(\epsilon^n_1)] = 0$.

Lemma 11 *The sequence $\{\epsilon^n_1\}$ is super-infinitesimal.*

The Lemma follows immediately from Markov's inequality and our assumptions $E[\epsilon_1^n] = 0, Var(\epsilon_1^n) = ||\epsilon_1^n||_2^2 = \sigma^2/n$, and $||\epsilon_1^n||_p \leq M/\sqrt{n}$ for some $\sigma, M > 0$ and $p > 2$:

$$nP(|\epsilon_1^n| > \delta) = nP(|\epsilon_1^n|^p > \delta^p) \leq \frac{n}{\delta^p} E[|\epsilon_1^n|^p] = \frac{n}{\delta^p} ||\epsilon_1^n||_p^p \leq \frac{n}{\delta^p} \frac{M^p}{n^{p/2}}.$$

Since super-infinitesimal implies infinitesimal, it remains to verify (i)-(iii). For convenience, define $A_1^n = \{|\epsilon_1^n| \leq 1\}$ and $A_2^n = \{|\epsilon_1^n| > 1\}$. Since

$$h(\epsilon_1^n) = \epsilon_1^n 1_{A_1^n} + h(\epsilon_1^n) 1_{A_2^n},$$

to prove (i) it suffices to show that $nE[\epsilon_1^n 1_{A_1^n}]$ and $nE[h(\epsilon_1^n) 1_{A_2^n}]$ both tend to zero as $n \to \infty$.

First, since $E[\epsilon_1^n] = 0$, we may apply Hölder's inequality as follows:

$$|E[\epsilon_1^n 1_{A_1^n}]| = |E[\epsilon_1^n - \epsilon_1^n 1_{A_1^n}]| \leq E[|\epsilon_1^n - \epsilon_1^n 1_{A_1^n}|] \leq E[|\epsilon_1^n|^2]^{1/2} E[|1 - 1_{A_1^n}|^2]^{1/2}.$$

Since $E[|\epsilon_1^n|^2] = Var(\epsilon_1^n) = \sigma^2/n$, the latter is equal to $[\frac{\sigma^2}{n} P(|\epsilon_1^n| > 1)]^{1/2}$. Multiplying by n,

$$n|E[\epsilon_1^n 1_{A_1^n}]| \leq \sigma [nP(|\epsilon_1^n| > 1)]^{1/2},$$

which tends to zero as $n \to \infty$, by Lemma 11 with $\delta = 1$.

For the second term, recalling $|h(x)| \leq 1$ for all x, we have

$$n|E[h(\epsilon_1^n) 1_{A_2^n}]| \leq nE[1_{A_2^n}] = nP(A_2^n) = nP(|\epsilon_1^n| > 1),$$

which tends to zero again by Lemma 11. This establishes (i).

Next is condition (ii). Using the decomposition with A_1^n and A_2^n as before, we may obtain

$$nVar(h(\epsilon_1^n)) = nE[(\epsilon_1^n)^2 1_{A_1^n}] + nE[h^2(\epsilon_1^n) 1_{A_2^n}] - nE[h(\epsilon_1^n)]^2.$$

The second two terms tend to zero as follows.

$$|nE[h^2(\epsilon_1^n) 1_{A_2^n}]| \leq nE[|h^2(\epsilon_1^n)| 1_{A_2^n}] \leq nE[1_{A_2^n}] = nP(|\epsilon_1^n| > 1)$$

tends to zero by Lemma 11. Since $|h(x)| \leq 1$ for all x, we have $(h(x))^2 \leq |h(x)|$, and hence

$$nE[h(\epsilon_1^n)]^2 \leq n|E[h(\epsilon_1^n)]|,$$

which tends to zero by part (i).

It remains to show that

$$\lim_{n \to \infty} nE[(\epsilon_1^n)^2 1_{A_1^n}] = \sigma^2.$$

Applying Hölder's inequality with $p' = p/2$ and $1/q = 1 - 1/p'$,

$$n|E[(\epsilon_1^n)^2 - (\epsilon_1^n)^2 1_{A_1^n}]| = nE[(\epsilon_1^n)^2 (1 - 1_{A_1^n})] \leq nE[((\epsilon_1^n)^2)^{p'}]^{1/p'} E[(1 - 1_{A_1^n})^q]^{1/q}.$$

The latter is equal to

$$n||\epsilon_1^n||_p^2 (P(|\epsilon_1^n| > 1))^{1/q} \leq M^2 (P(|\epsilon_1^n| > 1))^{1/q},$$

and this tends to zero, since $\{\epsilon_1^n\}$ is infinitesimal. Since $nE[(\epsilon_1^n)^2] = nVar(\epsilon_1^n) = \sigma^2$, we obtain the result, which completes (ii).

For property (iii), let $g \in \mathcal{Z}$. From the definition of $\mathcal{Z}$, there exists $r, C > 0$ such that $g = 0$ on $[-r, r]$ and $|g| \le C$ on R. Then

$$n|E[g(\epsilon_1^n)]| \le nE[|g(\epsilon_1^n)| 1_{\{|\epsilon_1^n|>r\}}] \le CnE[1_{\{|\epsilon_1^n|>r\}}] = CnP(|\epsilon_1^n| > r),$$

which tends to zero by Lemma 11.

Proof of Theorem 6, Part 2: $\hat{\mathcal{S}}_N^n \Rightarrow \sigma \mathcal{B}^N$.

From Part 1, Step 1 of the proof of Theorem 2, and Lemma 7,

$$(\hat{\mathcal{W}}^n, \frac{1}{n} N^n) \Rightarrow (\sigma \mathcal{B}, \Lambda).$$

By the Continuous Mapping Theorem and the continuity of the composition operator, as before, we obtain

$$\hat{\mathcal{W}}^n \circ \frac{1}{n} N^n \Rightarrow \sigma \mathcal{B} \circ \Lambda,$$

i.e.

$$\hat{\mathcal{S}}_N^n \Rightarrow \sigma \mathcal{B}^N \text{ in } D_\infty.$$

□

Theorem 7: *Under the Market Assumptions above, suppose further that the compensator Λ of N is pinned at T. Let Q be the equivalent martingale measure defined by*

$$\frac{dQ}{dP} = \exp(-\int_0^{\Lambda(T)} \gamma_t \, d\mathcal{B}(t) - \frac{1}{2} \int_0^{\Lambda(T)} \gamma_t^2 \, dt), \tag{9}$$

where

$$\gamma_t = \frac{1}{\sigma}(\mu + \sigma^2/2 - \frac{r}{\lambda_{\Lambda^{-1}(t)}}), \quad 0 \le t \le \Lambda(T).$$

Then the discounted stock price $Z_t = B_t^{-1} S_t$ (stopped at T) is a $(Q, \mathcal{G})$-martingale.

Moreover, suppose in addition that the payoff X is Q-independent of $\mathcal{F}_T^\Lambda$. Then X is attainable and the unique no-arbitrage price V_t of X at time $t < T$ is

$$V_t = (B_t / B_T) E_Q[X | \mathcal{G}_t]. \tag{10}$$

Proof of Theorem 7.

Given X, our goal is to construct a self-financing strategy in the stock and bond with time-t value given by V_t of Equation (10).

In this proof we use the convention that when we call a process $M(t)$ a martingale that is only defined on an interval $[0, t_0]$, we mean that the process stopped at t_0 is a martingale.

Since Λ is absolutely continuous with derivative λ bounded below by $\epsilon > 0$, the inverse function theorem tells us that

$$\Lambda^{-1}(t) = \int_0^t \bar{\lambda}_u \, du,$$

where

$$\bar{\lambda}_u = \frac{1}{\lambda_{\Lambda^{-1}(u)}}$$

is positive and bounded above by $1/\epsilon$. It follows that

$$\gamma_t = \frac{1}{\sigma}(\mu + \sigma^2/2 - r\bar{\lambda}_t)$$

is bounded and left continuous with right limits. By Girsanov's Theorem ([24], theorem III.42), the equivalent measure Q on $(\Omega, \mathbb{F})$ defined by (9) is such that, for $0 \le t \le \Lambda(T)$,

$$\begin{aligned}\tilde{\mathcal{B}}(t) &= \mathcal{B}(t) + \int_0^t \gamma_s\, ds = \mathcal{B}(t) + \frac{\mu + \sigma^2/2}{\sigma} t - \frac{r}{\sigma}\int_0^t \bar{\lambda}_u\, du \\ &= \mathcal{B}(t) + \frac{\mu + \sigma^2/2}{\sigma} t - \frac{r}{\sigma}\Lambda^{-1}(t)\end{aligned}$$

is a standard Brownian motion with respect to $(Q, \mathcal{H})$. Following our convention, we write

$$\tilde{\mathcal{B}}^N(t) = \tilde{\mathcal{B}}(\Lambda(t)) = \mathcal{B}^N(t) + \frac{\mu + \sigma^2/2}{\sigma}\Lambda(t) - \frac{r}{\sigma}t.$$

By Theorem 4, we have $\tilde{\mathcal{B}}^N(t)$ is a square integrable $(Q, \mathcal{G})$-martingale, and since $[\tilde{\mathcal{B}}^N, \tilde{\mathcal{B}}^N] = \Lambda$, therefore so is the discounted stock price

$$Z_t \equiv B_t^{-1} S(t) = S_0 \exp(\sigma \mathcal{B}^N(t) + \mu\Lambda(t) - rt) = S_0 \exp(\sigma\tilde{\mathcal{B}}^N(t) - \frac{1}{2}\sigma^2\Lambda(t)).$$

We are assuming that the option payoff X is independent of $\mathcal{F}_T^\Lambda$. This is equivalent to the independence of X from $\mathcal{F}_{\Lambda(T)}^{\Lambda^{-1}}$, since these two sigma-algebras are equal.

To construct a self-financing replicating portfolio, the difficulty is that the Predictable Representation Property (PRP, [28], V.4) enjoyed by Brownian motion does not necessarily hold for arbitrary continuous square integrable martingales like $\mathcal{B}^N$. The PRP for Brownian motion is what makes the Black-Scholes option pricing theory work.

Our strategy is to apply the PRP to the Brownian motion model, and then change variables by means of the time change Λ and the optional stopping theorem.

Recall that $\mathcal{B}^N$, Λ, and S are adapted to the filtration $\mathcal{G}_t = \mathcal{H}_{\Lambda_t}$, and $\mathcal{B}$, Λ^{-1} are adapted to $\mathcal{H}_t$.

Let

$$Y(t) = Z(\Lambda^{-1}(t)) = S_0 \exp(\sigma\tilde{\mathcal{B}}(t) - \frac{1}{2}\sigma^2 t).$$

Evidently Y is a $(Q, \mathcal{F}^{\tilde{\mathcal{B}}})$ martingale.

Now define

$$E(t) = E_Q[B_T^{-1} X | \mathcal{H}_t].$$

Since

$$\mathcal{H} = \mathcal{F}^{\mathcal{B}} \vee \mathcal{F}^{\Lambda^{-1}} = \mathcal{F}^{\tilde{\mathcal{B}}} \vee \mathcal{F}^{\Lambda^{-1}},$$

the independence of X from $\mathcal{F}_{\Lambda(T)}^{\Lambda^{-1}}$ implies $E(t) = E_Q[B_T^{-1} X | \mathcal{F}_t^{\tilde{\mathcal{B}}}]$ for $t \le \Lambda(T)$.

Since $E_Q[B_T^{-1}X|\mathcal{F}_t^{\tilde{\mathcal{B}}}]$ is a $(Q, \mathcal{F}^{\tilde{\mathcal{B}}})$-martingale, by the Brownian martingale representation property (e.g. [28], V.3), there is an $\mathcal{F}^{\tilde{\mathcal{B}}}$-predictable, hence $\mathcal{H}$-predictable, process η_t such that

$$dE(t) = \eta_t dY(t). \tag{11}$$

Next we wish to compose this equation with Λ, which is justified by the following lemma.

Lemma 12 ([21], lemma 2.3) *Let $\mathcal{H}$ be a filtration satisfying the usual conditions and X be an $\mathcal{H}$-semimartingale that is C-continuous, where C is a finite $\mathcal{H}$-time-change. Let $L(X,\mathcal{H})$ denote the class of $\mathcal{H}$-predictable processes H for which the stochastic integral $\int_0^t H_s\, dX_s$ can be constructed.*

If $H \in L(X,\mathcal{H})$, then $H_{C_{t-}} \in L(X_C, \mathcal{H}_C)$ for all t. Moreover, with probability one, for all $t \geq 0$,

$$\int_0^{C_t} H_s\, dX_s = \int_0^t H_{C_{s-}}\, dX_{C_s}.$$

Since Λ is a finite time-change with respect to $\mathcal{H}$, and letting $\phi_t = \eta_{\Lambda(t)}$, then ϕ_t is $\mathcal{H}_\Lambda$-predictable and we obtain from Equation (11) that

$$dE(\Lambda(t)) = \phi_t dZ(t). \tag{12}$$

We may now consider a portfolio holding ϕ_t shares of stock and $\psi_t = E(\Lambda(t)) - \phi_t Z(t)$ shares of the bond at time t.

The portfolio value process

$$V_t = \phi_t S(t) + \psi_t B_t = B_t E(\Lambda(t))$$

is self-financing by virtue of an easy computation using (12).

To complete the argument, the optional stopping theorem (e.g. [28], II.3) tells us that for any $t \in [0,T]$:

$$E(\Lambda(t)) = E_Q[B_T^{-1}X|\mathcal{H}_{\Lambda(t)}]. \tag{13}$$

Our portfolio strategy is therefore a replicating strategy because

$$V_T = B_T E(\Lambda(T)) = B_T E_Q[B_T^{-1}X|\mathcal{H}_{\Lambda(T)}] = X.$$

Therefore the no-arbitrage price of X at any earlier time t must be the value of the replicating portfolio

$$V_t = B_t E(\Lambda(t)) = (B_t/B_T) E_Q[X|\mathcal{H}_{\Lambda(t)}]$$

as desired. This completes the proof of Theorem 7. □

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