MINIMALITY OF THE ACTION ON THE UNIVERSAL CIRCLE OF UNIFORM FOLIATIONS

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ABSTRACT. Given a uniform foliation by Gromov hyperbolic leaves on a 3-manifold, we show that the action of the fundamental group on the universal circle is minimal and transitive on pairs of different points. We also prove two other results: we prove that general uniform Reebless foliations are $\mathbb{R}$-covered and we give a new description of the universal circle of $\mathbb{R}$-covered foliations with Gromov hyperbolic leaves in terms of the JSJ decomposition of M.

1. INTRODUCTION

Consider a Reebless foliation $\mathcal{F}$ on a closed 3-manifold M without spherical or projective plane leaves. This implies that the universal cover $\widetilde{M}$ of M is homeomorphic to $\mathbb{R}^3$ [Pal] and that every leaf of $\mathcal{F}$ is a properly embedded plane [Nov]. We denote by $\widetilde{\mathcal{F}}$ to the lift of $\mathcal{F}$ to $\widetilde{M}$.

A specific class of foliations are those called *uniform* which means that in the universal cover, any two leaves are at finite Hausdorff distance from each other. See section 2.2 for the several variations of the definition of uniform foliations. Fibrations over the circle are one obvious example. Much more generally, slitherings, introduced by Thurston in [Th] (see also [Ca_4]) are examples of such foliations. In addition from any slithering example one can construct other examples of uniform foliations by blowing up some leaves into foliated interval bundles. All of these examples of uniform foliations are what is called $\mathbb{R}$-covered. Recall that a foliation is $\mathbb{R}$-covered if the *leaf space* $\mathcal{L}_{\mathcal{F}} = \widetilde{M}/_{\widetilde{\mathcal{F}}}$ of $\widetilde{\mathcal{F}}$ is homeomorphic to $\mathbb{R}$. We first prove:

Theorem 1.1. *A uniform Reebless foliation in a closed* 3*-manifold* M *is* $\mathbb{R}$*-covered.*

This result has no restriction on the intrinsic metric in the leaves.

Theorem 1.1 implies that all Reebless uniform foliations are obtained from either slithering foliations or blow ups of slithering foliations as explained in [Ca_4, Construction 9.14 and Theorem 9.15] using a result of Thurston [Th, Theorem 2.7]. The proof implicitly uses the fact that the foliation is $\mathbb{R}$-covered. We provide a proof of the $\mathbb{R}$-covered property here.

The requirement of Reebless in Theorem 1.1 is not superfluous: any foliation in the 3-sphere S^3 (or any closed M^3 with finite fundamental group)

Key words and phrases. 3-manifold topology, foliations, group actions.

S.F was partially supported by Simons Foundation grant number 637554. R. P. was partially supported by CSIC 618, FCE-1-2017-1-135352. This work was completed while R.P. was serving as a Von Neumann fellow at IAS, funded by the Minerva Research Foundation Membership Fund and NSF DMS-1638352.

is uniform, however none are $\mathbb{R}$-covered, because they have Reeb components. To prove Theorem 1.1 it is enough to show that the leaf space is Hausdorff (see e.g. [Ca_4, CC], a quick account of the most relevant material is presented in §2.1).

Some uniform foliations are quite special, for example linear foliations in $\mathbb{T}^3$ or in nilmanifolds. These foliations have leaves that are parabolic. But for most uniform foliations, one can apply a beautiful result of Candel [Can] to see that there is a metric on M making each leaf negatively curved (see e.g. [FP_1, §5.1] for a specific statement).

For our next result we will consider the following setting: $\mathcal{F}$ will be a uniform foliation on a closed Riemannian 3-manifold M such that the metric restricted to each leaf of $\mathcal{F}$ is Gromov hyperbolic (in particular, it has to be Reebless since the torus does not admit a negatively curved metric). We will call such foliations *uniform hyperbolic* foliations.

For such foliations, one can consider, for each leaf $L \in \widetilde{\mathcal{F}}$ the circle at infinity $S^1(L)$ defined as the set of geodesic rays up to being a finite Hausdorff distance apart (see §2.4). The fact that the foliation is $\mathbb{R}$-covered is very useful to define a *universal circle* S^1_{univ} which is essentially a canonical way to identify all the $S^1(L)$ as one varies $L \in \widetilde{\mathcal{F}}$. The precise definition will be given in §2.5. See also [Th, Ca_4, Ca_1, Fen_2, CD, Fra] among other places where universal circles are defined in even more general situations.

Our main result is the following:

Theorem 1.2. *Let $\mathcal{F}$ be a uniform hyperbolic foliation on a 3-manifold M. Then, the fundamental group $\pi_1(M)$ acts minimally on the universal circle S^1_{univ}. Moreover, the diagonal action on pairs of different points of S^1_{univ} has dense orbits.*

This result extends a very well known result about actions of hyperbolic groups on their Gromov boundary (see [Gr, §8.2]) and complements well with [Ca_1, Lemma 5.2.2] which is stated for non-uniform $\mathbb{R}$-covered foliations. The result was motivated by some applications to partially hyperbolic dynamics (it will be used in [FP_2]). We hope this result may have independent interest or find other applications.

Some proofs of intermediate steps are simpler if one restricts to the case of atoroidal 3-manifolds where one has transverse pseudo-Anosov flows that helps understanding the action on the universal circle ([Th, Ca_1, Fen_2]).

When the manifold has a non-trivial JSJ decomposition, the proof includes a careful study of the intersection between leaves of the foliation and the pieces of the JSJ decomposition. This results in a new way to look at the universal circle that may be of independent interest and holds for general (both uniform and non uniform) $\mathbb{R}$-covered foliations. See Proposition 4.9.

Because of our applications, at the end of the paper we explain how the results hold also for *branching foliations*, which are a technical object featuring often in partially hyperbolic dynamics.

2. Preliminaries

2.1. **Reebless foliations.** We will be mainly concerned with Reebless foliations in this article. See [Ca_4, §4] for a broad introduction.

A Reeb component is a foliation of the solid torus, so that the boundary is a leaf. In addition all the leaves in the interior are planes and spiral or limit towards the boundary. There is a circle worth of leaves in the interior. By an abuse of terminology we also consider Reeb component a quotient of this, which may be a foliation of a solid Klein bottle. If a foliation by surfaces $\mathcal{F}$ in a closed 3-manifold M does not have *Reeb components* it follows from a celebrated result of Novikov [Nov] that when lifted to the universal cover, the foliation is made of simply connected leaves and the *leaf space* $\mathcal{L}_{\mathcal{F}} = \widetilde{M}/_{\widetilde{\mathcal{F}}}$ is a simply connected (possibly non-Hausdorff) one-dimensional manifold. If there is a leaf of $\widetilde{\mathcal{F}}$ which is a sphere or a projective plane, it follows that the foliation $\widetilde{\mathcal{F}}$ is equivalent to the trivial foliation by spheres in $S^2 \times \mathbb{R}$. If there are no projective space or spherical leaves of $\mathcal{F}$ then a result of Palmeira [Pal] implies that $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$. We refer the reader to [CC, Ca_4] for a broad treatment, we will assume some familiarity with the theory of foliations.

We will not be to precise about regularity of our foliations. Everything works for foliations of class $C^{0,1+}$ as defined in [CC] (i.e. continuous with C^1 leaves tangent to a continuous distribution). Thanks to [Ca_2] in view of the nature of our result, this is a quite general assumption.

To show that a foliation is $\mathbb{R}$-covered, it is enough to show that its leaf space is Hausdorff (see e.g. [Fen_2, Lemma 2.2]).

A *taut foliation* is a foliation such that every leaf intersects a closed transversal. Notice that taut foliations must be Reebless. Another relevant result about foliations in 3-manifolds is the following (see [Gab] or [CC, Theorem II.9.5.5]):

Theorem 2.1 (Roussarie-Gabai)**.** *Let $\mathcal{F}$ be a taut foliation in a 3-manifold M and let $T \subset M$ be an embedded incompressible torus or Klein bottle. Then, T can be isotoped to be either a leaf of $\mathcal{F}$ or in general position with respect to $\mathcal{F}$. In particular in the second case the induced foliation by $\mathcal{F}$ in T does not have singularities. If $\mathcal{F}$ is taut one can isotope T to be either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$.*

2.2. **Uniform foliations.** In this paper we will mainly concentrate in the following class of foliations.

Definition 2.2. *Let $\mathcal{F}$ be a foliation in a manifold M. We say that $\mathcal{F}$ is uniform, if for any two leaves L, F of the lifted foliation $\widetilde{\mathcal{F}}$ to $\widetilde{M}$, then the Hausdorff distance between L and F is finite.*

There have been several forms of the definition of uniform foliations, which we review here. Our definition is the weakest or most general possible. In his seminal article [Th, Definition 2.1], Thurston originally defined uniform foliation as a codimension one foliation in any dimension satisfying Definition 2.2 and so that in addition any closed transversal is not null homotopic. Calegari [Ca_1, Definition 2.1.5] or [Ca_4, Definition 9.13], defined uniform for codimension one foliations in 3-manifolds M satisfying Definition 2.2 and so that the foliation is also taut. The first author [Fen_2, Definition 2.4] defined uniform for codimension foliations in 3-manifolds satisfying Definition 2.2.

Note that Definition 2.2 does not require M to be 3-dimensional or $\mathcal{F}$ codimension one, but we will restrict to this case in this paper.

Thurston [Th] remarks on the connection of the uniform property with the Reebless condition for codimension one foliations in 3-manifolds. After [Th, Definition 2.1] it is stated that if a foliation verifies that every closed transversal is not-nullhomotopic then there are no Reeb components. This is true if one additionally assumes that the foliation is uniform, and we prove this in § 3.1.

2.3. **JSJ decompositions.** We refer the reader to [Hat] for a more complete account on this.

What we will use is that every irreducible[1] 3-manifold M admits a canonical collection (unique up to isotopy) of embedded incompressible tori and Klein bottles $T_1, \ldots, T_k$ such that if we cut M along the tori/Klein bottles, each *piece* (i.e. connected component of the complement) is either atoroidal or Seifert. We will exclude the case where M is a torus bundle up to finite cover since in this case there can be a unique piece which is Seifert but its fibration may not be not unique up to isotopy – for example in $\mathbb{T}^3$. When $k \geqslant 1$ and M is not a torus bundle up to finite cover, we say that M has a *non-trivial JSJ decomposition*.

Remark 2.3. We will abuse terminology use and refer to $T_1, \ldots, T_k$ as the JSJ tori, even though some components may be Klein bottles.

Let M be a irreducible 3-manifold with non-trivial JSJ decomposition and let $M_1, \ldots, M_n$ the pieces of its JSJ decomposition (i.e. the connected components of $M \backslash \{T_1, \ldots, T_k\}$, notice that it could be that $n = 1$ even if $k \geqslant 1$). In $\widetilde{M}$ we consider all connected components of the lifts $\widetilde{M}_i^j$ of each M_i.

It turns out that the following holds:

Proposition 2.4. *The graph consisting of vertices in each of the $\widetilde{M}_i^j$ and edges between vertices sharing a boundary is an infinite tree $\mathcal{T}$. Moreover, the fundamental group acts naturally on $\mathcal{T}$ and for every element $\gamma \in \pi_1(M)$ the set of fixed points of γ in $\mathcal{T}$ has diameter*[2] *at most* 2.

Proof. The fact that it is a tree follows directly because the lift of a JSJ torus to the universal cover is a properly embedded plane which separates $\widetilde{M}$ in exactly two connected components and this forbids the graph to have closed loops. This also implies that each M_i has infinitely many lifts; to see this, notice that if a boundary torus of M_i has infinitely many lifts in some $\widetilde{M}_i^{j_0}$ then clearly there must be infinitely many $\widetilde{M}_i^j$ because there should be at least one lift of M_i in each complementary region of the torus in $\widetilde{M}$ not containing $\widetilde{M}_i^{j_0}$. If all boundary torus have finitely many lifts, then the deck transformations that fix $\widetilde{M}_i^{j_0}$ are a finite extension of $\mathbb{Z} \oplus \mathbb{Z}$ so there must be infinitely many deck transformations moving $\widetilde{M}_i^{j_0}$ pairwise disjointly (notice that finite extensions of $\mathbb{Z} \oplus \mathbb{Z}$ cannot be the fundamental group of a closed, irreducible 3-manifold by homological reasons). Indeed, this implies

[1]Note that if there is a Reebless foliation in M, then $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$, hence M is irreducible.

[2]We are using the standard metric on a graph making each edge have length equal to 1.

that each $\widetilde{M}_i^j$ has infinitely many boundary components (this uses the fact that M_i cannot be $\mathbb{T}^2 \times (0,1)$ as our definition of having non-trivial JSJ decomposition which expressly excludes the case of torus bundles).

The fact that the set of fixed points of a deck transformation is a compact set in $\mathcal{T}$ follows the strategy the proof of [BFFP, Lemma A.3]. We sketch the main points for completeness. As in [BFFP, Appendix A] we will call the components of the lifts of tori in the JSJ collection *walls*.

We let $\gamma \in \pi_1(M)$ be a deck transformation. We first notice that if M_i is an atoroidal piece then if $\widetilde{M}_i^j$ is fixed by γ then at most one wall of $\widetilde{M}_i^j$ can be invariant under γ. Otherwise one gets a π_1-injective annulus in M_i with boundary in boundary of M_i which is not homotopic rel boundary to boundary of M_i. Using this annulus and annuli in boundary components of M_i one can piece together a π_1-injective torus or Klein bottle in M_i which is not homotopic to the boundary, contradicting that M_i is atoroidal.

Now, if M_i is a Seifert piece, then we claim that if $\widetilde{M}_i^j$ is fixed by γ then by a similar argument we see that if more than one wall is fixed, then γ must belong to the center of $\pi_1(M_i)$ (i.e. the element generated by the fibers of the Seifert fibering) in which case, γ cannot belong to the center of the Seifert pieces that are adjacent to $\widetilde{M}_i^j$.

This shows that any connected component of the fixed point set of γ has diameter at most two. But since $\mathcal{T}$ is a tree and γ acts by isometries, the fixed point set is connected. This concludes. ☐

It is simple to show:

Proposition 2.5. *If $\mathcal{F}$ is a foliation by hyperbolic leaves on a closed manifold M then it is horizontal in every Seifert piece.*

By horizontal we mean that up to isotopy of the Seifert fibration we can choose it to be everywhere transverse to all leaves of the foliation.

Proof. To see this, it is enough to work on the piece and apply [Brit] (see also [Ca_4, §4.10]). Notice that if there is a vertical sublamination then its leaves cannot be hyperbolic. ☐

2.4. **Boundaries at infinity.** Let X be a negatively curved complete space with curvature bounded from below and above. See [Gr, Led] for general references.

For such a space we define a boundary at infinity $\partial_\infty X$ defined as the equivalence relation of geodesic rays up to being at a bounded distance (see [Led, §I]). When X is a surface, the negative curvature implies that if X is simply connected then it is homeomorphic to $\mathbb{D}^2$ and one can identify the boundary $\partial_\infty X$ with the circle of directions $T^1_x X$ at any point $x \in X$. So, for simply connected surfaces of negative curvature, we denote the boundary at infinity as $S^1(X) = \partial_\infty X$.

The metric in $S^1(X)$ is only well defined up to Hölder equivalence since it is intended to be an invariant under quasi-isometries. For our purposes, it will be convenient to choose a special metric on $S^1(X)$ called the *visual metric*. For this, we fix a point $x_0 \in X$ and we measure the length of an interval $I \subset S^1(X)$ by looking at the angle formed by the interval in $T^1_{x_0} X$ of vectors whose geodesic ray starting at x_0 lands in a point of I. This is clearly dependent on the point, but we will always explicit the point we are considering.

2.5. **Universal circles.** In this section will review the construction of the universal circle for an $\mathbb{R}$-covered foliation $\mathcal{F}$ on a closed 3-manifold M so that it admits a metric which restricts in each leaf to a negatively curved surface with curvature[3] close to -1.

Denote by $\widetilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\widetilde{M}$, the universal cover of M. For each $L \in \widetilde{\mathcal{F}}$ we define $S^1(L)$ to be the boundary at infinity of L, which is well defined thanks to the fact that L is negatively curved. First there is the cylinder at infinity $\mathcal{A}$ which is the union of the $S^1(L)$ for L leaf in $\widetilde{\mathcal{F}}$. The topology in $\mathcal{A}$ is given by: given x in $\widetilde{M}$, let τ be a small transversal to $\widetilde{\mathcal{F}}$ through x. For every point y of τ, y is in $L \in \widetilde{F}$. For every v in the unit tangent bundle of L at y, let γ_v be the geodesic ray starting at y with direction v. The ideal point z_v of γ_v is a point in $S^1(L)$. It is well known that since L has negative curvature, the map $v \to z_v$ is a homeomorphism. In the same way one defines a map

$$\eta : T^1\widetilde{\mathcal{F}}|\tau \quad \to \quad B_\tau \;=\; \bigcup_{L\cap\tau\neq\emptyset} S^1(L)$$

which is the map $v \to z_v$ for any y in τ. Put a topology in B_τ so that this map is a homeomorphism. Do this for a $\pi_1(M)$ invariant collection of transversals with union intersecting every $L \in \widetilde{\mathcal{F}}$. In [Fen$_2$] it is proved that the topology in the intersection of subsets of $\mathcal{A}$ is well defined. This makes $\mathcal{A}$ into an open annulus, and $\pi_1(M)$ acts by homeomorphisms on this. In addition there is a topology on $\widetilde{M} \cup \mathcal{A}$ making it homeomorphic to $\mathbb{D}^2 \times \mathbb{R}$ and so that each $L \cup S^1(L)$ corresponds to $\mathbb{D}^2 \times \{t\}$ for some t. Again $\pi_1(M)$ acts by homeomorphisms on this topology. We now describe the universal circle of $\mathcal{F}$.

2.5.1. *Case of $\mathcal{F}$ uniform.* We denote, for $L, F \in \widetilde{\mathcal{F}}$ a map $\tau_{L,F} : L \cup S^1(L) \to F \cup S^1(F)$ which has the following properties:

- $\tau_{L,F}|_L$ is a quasi-isometry with constant $c > 1$ depending only on the Hausdorff distance between L and F,
- $\tau_{L,F}|_{S^1(L)}$ is a homeomorphism,
- $\tau_{F,G} \circ \tau_{L,F}|_{S^1(L)} = \tau_{L,G}|_{S^1(L)}$.

See [Th, §5] or [Ca$_1$, Corollary 5.3.16] or [Fen$_2$, Proposition 3.4]. Roughly the construction of such a collection of maps $\tau_{L,F}$ is as follows: Recall that a *quasi-isometry* of constant $c > 1$ is a map $\phi : L \to F$ so that $c^{-1}d_L(x, y) - c < d_F(\phi(x), \phi(y)) < cd_L(x, y) + c$. Given L, F, the Hausdorff distance between them is $a_0 > 0$. Given any x in F there is y in L with $d(x, y) < a_0 + 1$. Let $\tau_{L,F}(x) = y$. This map is well defined up to an error a_1, with a_1 depending only on a_0, see [Fen$_2$, §3].

[3]For foliations, [Can] provides a metric of curvature exactly -1, but since we want to apply this result in a slightly more general case (that is of branching foliations), we only use that the curvature is uniformly close to -1. Notice also that the metric constructed by [Can] may be only C^0 transversally to leaves, and since we are concerned only with quasi-isometric properties of leaves, it is more than fine to have just negative curvature. See also [Th, §4].

The map $\tau_{L,F}$ is a quasi-isometry, so it extends to $L \cup S^1(L)$, and it is a homemorphism into its image restricted to $S^1(L)$.

Recall that a quasigeodesic is a quasi-isometry from $\mathbb{Z}$ or $\mathbb{R}$ into $\widetilde{M}$. Since the map $\tau_{L,F}|_L$ is a quasi-isometry it takes quasigeodesics in F to quasi-geodesics in L. It is easy to see that $x \in S^1(L) \sim y \in S^1(F)$ if and only if a quasigeodesic α in L with ideal point x is a finite Hausdorff distance from a quasigeodesic in F with ideal point y.

The universal circle of $\mathcal{F}$ is then defined as the circle S^1_{univ} which is $\mathcal{A}/_\sim$ where $x \in S^1(L) \sim y \in S^1(F)$ if $y = \tau_{L,F}(x)$. Notice that it is easy to see that S^1_{univ} can be identified with $S^1(L)$ for every $L \in \widetilde{\mathcal{F}}$, so one can think of S^1_{univ} as a cannonical way to identify all boundaries at infinity of leaves.

The fundamental group $\pi_1(M)$ acts on S^1_{univ} by homeomorphisms. This is because any γ in $\pi_1(M)$ sends pairs of quasigeodesics in leaves which are a finite Hausdorff distance apart to like pairs in $\gamma(L), \gamma(F)$.

Remark 2.6. Let γ in $\pi_1(M)$ and L a leaf of $\widetilde{\mathcal{F}}$. The action of $\gamma \in \pi_1(M)$ on S^1_{univ} can be represented by an action on $S^1(L)$ identifying $S^1(L) \cong S^1_{univ}$ and so the action is obtained by the deck transformation composed with $\tau_{\gamma L,L}$. We denote the action of γ on $S^1(L)$ obtained as $\tau_{\gamma L,L} \circ \gamma$ by $\rho(\gamma)$. To see that this is well defined one needs to check that this is independent of the choice of the leaf L. For uniform foliations this is quite simple since the actions inside the leaves L and F differ by a uniform quasi-isometry which then induces a conjugacy of the actions at infinity by the same $\tau_{L,F}$.

2.5.2. *Case of $\mathcal{F}$ not uniform.* We refer to [Fen$_2$]. In this case there are no compact leaves of $\mathcal{F}$ [Fen$_2$, Lemma 2.5], and $\mathcal{F}$ has a unique minimal set $\mathcal{L}$ [Fen$_2$, Proposition 2.6]. Each complementary component of $\mathcal{L}$ is a $[0,1]$-bundle and $\mathcal{F}$ can be collapsed to produce a minimal foliation [Fen$_2$, Proposition 2.6]. Hence one can assume that $\mathcal{F}$ is minimal. There is also a canonical collapsing between the cylinders at infinity.

So assume that $\mathcal{F}$ is minimal. In [Fen$_2$, §3] it is proved that for any L, F in $\widetilde{\mathcal{F}}$ there is a dense set of directions between them which is a contracting direction between them. This means the following: Fix x in L. There is a dense set of points B in $S^1(L)$ so that for any y in B if γ is the geodesic ray in L starting in L and with ideal point y, then γ is asymptotic to F (and hence to any leaf in between L, F). Asymptotic means that distance between γ and F goes to 0 as points escape in γ. For any E between L, F there is a geodesic ray in E asymptotic to γ. This defines an ideal point in $S^1(E)$. The union of these ideal points over such E is a continuous curve in $\mathcal{A}$. The union of these for all y in B is a dense set in the subset $\mathcal{D}$ of $\mathcal{A}$ between $S^1(L)$ and $S^1(L)$. This extends uniquely to a foliation in $\mathcal{D}$ by intervals, each interval intersects a circle at infinity once and only once. One iterates this procedure making L, F escape compact sets of the leaf space in opposite directions. This defines a foliation in $\mathcal{A}$ by vertical lines, each intersecting a circle at infinity once and only once.

The universal circle of $\mathcal{F}$ is the quotient $\mathcal{A}/\sim$ where is the equivalence relation of being in the same leaf of the vertical foliation. The group of deck transformations $\pi_1(M)$ acts by homeomorphisms preserving the vertical foliation in $\mathcal{A}$. This is because it sends the contracting directions as above to contracting directions. Hence $\pi_1(M)$ acts by homeomorphisms on the universal circle S^1_{univ}. Both the vertical foliation and the universal circle pull back to the original foliation before collapsing complementary regions of the minimal set $\mathcal{L}$.

The existence of a the universal circle is much more general. It exists for every foliation with Gromov hyperbolic leaves [CD]. In addition in [CD] a universal circle is constructed for every tight essential lamination. See [Ca_4] for more on this theory.

3. Uniform foliations: proof of Theorem 1.1

In this section we prove Theorem 1.1. We first discuss in §3.1 the Reebless assumption. This subsection is independent of the proof of Theorem 1.1 and can be safely skipped. The results in §3.2 hold in more generality than the case of uniform foliations and could be of independent interest.

As explained in §2.1, Theorem 1.1 is immediate if the foliation has spherical or projective plane leaves by the Reeb stability theorem which implies in that case that up to finite cover the foliation is the trivial foliation by spheres in $S^2 \times S^1$. So in this section **we will assume throughout that leaves of $\mathcal{F}$ are not spheres or projective planes.**

3.1. Some remarks on the Reebless assumption.

It can certainly be the case that a foliation with Reeb components is uniform yet not $\mathbb{R}$-covered. Indeed, if M has finite fundamental group, any foliation in M has Reeb components by Novikov's theorem [Nov] while the universal cover is compact, so the foliation is uniform. Notice that a Reeb component has non-Hausdorff leaf space: every neighborhood of the boundary leaf contains all the leaves of the interior of the solid torus as these all accumulate the boundary. Foliations of closed 3-manifolds with finite fundamental group are all examples of uniform non-$\mathbb{R}$-covered foliations:

Question 1. *If $\mathcal{F}$ is uniform in M with infinite fundamental group, does it follow that $\mathcal{F}$ is Reebless?*

We don't know how to prove this in all generality, however we can prove the following intermediate fact.

Lemma 3.1. *If $\mathcal{F}$ a foliation in M has a Reeb component and it is uniform then every leaf in the universal cover has compact closure. In particular, any non-torsion element $\gamma \in \pi_1(M)$ acts freely on the leaf space $\mathcal{L}_{\mathcal{F}} = \widetilde{M}/_{\widetilde{\mathcal{F}}}$.*

Proof. Let us first assume that the fundamental group of the boundary tori of the Reeb component does not map to 0 in the fundamental group $\pi_1(M)$ of M. This implies that the Reeb torus lifts to a Reeb cilinder where leaves accumulate on one end of the cilinder. Let γ represent the deck transformation associated with the core of the Reeb component. Assume that the basepoint p is in a lift $\tilde{\gamma}$. We also think of γ as a deck transformation. Then in $\widetilde{M}$, $d(\gamma^n p, p) \to \infty$. One can see this in the Cayley graph of $\pi_1(M)$ with an edge metric. The Cayley graph is quasi-isometric to the universal cover $\widetilde{M}$. In the Cayley graph there are finitely many elements in the ball of any radius and this implies that $d(\gamma^n p, p) \to \infty$. In particular this implies that the leaves inside of the cilinder are not a bounded distance away from the cilinder, so the foliation is not uniform.

Now, assume that the Reeb component lifts to $\widetilde{M}$, so there are compact leaves of $\widetilde{\mathcal{F}}$. It follows that every leaf $L \in \widetilde{\mathcal{F}}$ is a finite Hausdorff distance from a compact leaf. In particular L is bounded, therefore its closure is compact. □

In particular this implies that if $\mathcal{F}$ is uniform and every closed transversal is no null homotopic then $\mathcal{F}$ is Reebless, as announced in subsection 2.2.

3.2. **Lifts of leaves at a bounded distance.** In this section we prove some general results about Reebless foliations. We will use them in the next subsection to prove Theorem 1.1.

Recall that the leaf space $\mathcal{L}_{\mathcal{F}} = \widetilde{M}/_{\widetilde{\mathcal{F}}}$ in this case is a simply connected 1-dimensional manifold which is possibly non-Hausdorff. This is because for every leaf $L \in \widetilde{\mathcal{F}}$ if t is a transversal (i.e. a curve transverse to $\widetilde{\mathcal{F}}$ homeomorphic to an open interval and intersecting L) it holds that t intersects each leaf of $\widetilde{\mathcal{F}}$ at most once (cf. §2.1).

As explained in §2.1, when $\mathcal{F}$ is not $\mathbb{R}$-covered, there *non-separated* leaves of $\widetilde{\mathcal{F}}$: that is, leaves $L, F \in \widetilde{\mathcal{F}}$ so that for every transversals t_L, t_F to respectively L and F one has a leaf $E \in \widetilde{\mathcal{F}}$ which intersects both t_L and t_F. Notice that if $L, F \in \widetilde{\mathcal{F}}$ are distinct non-separated leaves, then they cannot intersect a common foliation chart, so the distance between points in one leaf to the other leaf is bounded from below.

We give some more definitions. We refer the reader to [BFFP, Section 3 and Appendix B] for a broader introduction with similar notation. We can assume that the foliation is transversally oriented by going to a double cover and this makes no problem in our results since we are working in $\widetilde{M}$. Given two leaves $L, F \in \widetilde{\mathcal{F}}$ we call the *region between L and F* to the intersection of the complementary regions to L and F in $\widetilde{M}$ which contain respectively F and L.

Remark 3.2. If L and F are non-separated then no transversal to L can intersect F. Otherwise any leaf intersecting the transversal between L, F would separate F from L.

The following general result holds.

Proposition 3.3. *Let $\mathcal{F}$ be a Reebless foliation of a closed 3-manifold M. Assume that there are distinct leaves $L, F \in \widetilde{\mathcal{F}}$ a finite Hausdorff distance from which other, and which are non separated from each other. Then, both L and F project into compact leaves in M.*

Proof. Let A and B be the projections of L and F respectively to M. Assume that A is not compact. Hence there is a sequence of points p_i in A such that $p_i \to p$ so that p_i are not in the same plaques of a local chart around p. Without loss of generality we can assume that the sequence p_i is strictly monotone in the plaques of the chart.

One can lift the points p_i to points $x_i \in L$ and consider $\gamma_i \in \pi_1(M)$ so that $\gamma_i x_i \to x_0$ a lift of p. Let L_0 be the leaf of $\widetilde{\mathcal{F}}$ through x_0. The fact that the points p_i converge to p in different local leaves implies that $\gamma_i L$ are pairwise distinct leaves of $\widetilde{\mathcal{F}}$ – as transversals intersect a leaf only once in $\widetilde{M}$. It is exactly this property that we will show produces a contradiction.

Denote by $R > 0$ a bound of the Hausdorff distance between L and F. One can choose points $y_i \in F$ so that $d(y_i, x_i) < R+1$. Up to a subsequence, we can assume that $\gamma_i y_i \to y_0 \in F_0 \in \widetilde{\mathcal{F}}$. Notice that $L_0 \neq F_0$ for otherwise one could fix a curve in L_0 from x_0 to y_0 and that would lift to nearby leaves, giving that $\gamma_i L$ and $\gamma_i F$ intersect the same transversal for large i which is impossible since L is non-separated from F (cf. Remark 3.2).

Now, pick transversals t_{x_0} and t_{y_0} to the leaves L_0 and F_0 through x_0 and y_0 respectively. For large i it follows that the plaques through $\gamma_i x_i$ and $\gamma_i y_i$ intersect t_{x_0} and t_{y_0} respectively. and so t_{x_0} is a transversal to $\gamma_i L$ and t_{y_0} a transversal to $\gamma_i F$. Since $\gamma_i L$ and $\gamma_i F$ are non-separated it follows that there are leaves intersecting both t_{x_0} and t_{y_0} which implies that L_0 and F_0 are non-separated from each other.

Assume first that $\gamma_i L$ does not belong to the region between L_0 and F_0.

In this case L_0 separates $\gamma_i L$ from $\gamma_i F$ which is a contradiction since they are non-separated. In fact, if one considers a transversal t to $\gamma_i L$ which is contained in the component of $\widetilde{M} \backslash L_0$ not containing F_0 it follows that every leaf intersecting t must remain in this component while $\gamma_i F$ must intersect a small transversal to F_0 so belong to a different connected component of $\widetilde{M} \backslash L_0$ showing that $\gamma_i L$ and $\gamma_i F$ cannot be non-separated. The same works for $\gamma_i F$.

Suppose now that both $\gamma_i L$ and $\gamma_i F$ belong to the region between L_0 and F_0. Recall that $(\gamma_i L)$ converges to L_0 and now $\gamma_i L$ is in the complementary component of L_0 containing F_0. In particular the sequence $(\gamma_i L)$ also converges to F_0. Hence for i big $\gamma_i L$ intersects t_{F_0}. Since for i big the leaf $\gamma_i F$ also intersects t_{F_0} this would show that $\gamma_i F, \gamma_i L$ intersect a common transversal contradiction to F, L not separated from each other.

In other words, what these arguments really show is that the assumption that $\gamma_i L$ are all distinct leads to a contradiction.

This finishes the proof of the proposition. □

The following result also holds in great generality. Notice that even if we assumed that $\mathcal{F}$ is uniform, the result is not immediate since a priori we don't know if the region between two leaves has to be contained in a neighborhood of one of the leaves, this is indeed what we show here for leaves which project into compact surfaces. Given a leaf L of $\widetilde{\mathcal{F}}$, let Γ_L be the subgroup of deck transformations fixing L, in other words, the stabilizer of L in $\pi_1(M)$. Notice that $\pi(L) = L/\Gamma_L$.

Proposition 3.4. *Let $\mathcal{F}$ be a transversely oriented, Reebless foliation of a closed 3-manifold. Let $L, F \in \widetilde{\mathcal{F}}$ leaves at bounded Hausdorff distance whose projection to M are compact surfaces. Let $\widetilde{N}$ be the region between L and F. Then $\widetilde{N}$ projects to a compact $[0,1]$-bundle in $\widetilde{M}/\Gamma_L$.*

Proof. Notice that γF is at bounded Hausdorff distance from L for every $\gamma \in \Gamma_L$ since deck transformations are isometries and $\gamma L = L$. As F projects into a compact surface, it follows that the orbit of F by $\pi_1(M)$ is a closed subset of $\widetilde{M}$.

Let $R > 0$ be the Hausdorff distance between L and F and consider a closed ball B of radius $R+1$ centred at a point $x_0 \in L$. After covering it with finitely many foliation charts, by compactness one sees that only finitely many translates of F can intersect B. Since every translate of F by some element of Γ_L must intersect B, this implies that the action of Γ_L in F has finitely many translates of F. One deduces that the stabilizer of F in Γ_L is a finite index subgroup of Γ_L.

The symmetric argument says that Γ_F has a finite index subgroup fixing L. We deduce that $\Gamma = \Gamma_F \cap \Gamma_L$ is finite index in both Γ_F and Γ_L.

Consider the quotient $\widetilde{M}/_\Gamma$ of $\widetilde{M}$ by the group Γ. It follows that both L and F project to compact leaves in $\widetilde{M}/_\Gamma$.

The region $\widetilde{N}$ between L, F projects to a compact 3-manifold with boundary N_Γ in $\widetilde{M}/_\Gamma$, whose boundaries are the quotients A and B of L and F. Moreover, their fundamental group surjects into the fundamental group of N_Γ. In addition N_Γ is irreducible. It follows (see [Hem, Theorem 10.2]) that N_Γ is homeomorphic to $A \times [0,1]$ and $A \times \{1\}$ corresponds to B. Projecting to $\widetilde{M}/\Gamma_L$ one gets that $\widetilde{N}$ also projects to an $[0,1]$-bundle with a boundary a leaf homeomorphic to $C = L/_{\Gamma_L}$. This uses that $\mathcal{F}$ is transversely oriented. □

We need one additional result.

Proposition 3.5. *Suppose that $\mathcal{F}$ is a Reebless foliation in $N = \mathbb{T}^2 \times [0,1]$, so that each boundary component is a leaf of $\mathcal{F}$. Suppose that in $\widetilde{N}$, the boundary leaves $E = \widetilde{\mathbb{T}}^2 \times \{0\}$ and $G = \widetilde{\mathbb{T}}^2 \times \{1\}$ are not separated from each other in the leaf space of $\widetilde{\mathcal{F}}$. Then $\mathcal{F}$ is not a uniform foliation.*

Proof. Given a leaf L in the interior of $\widetilde{N}$, we will show it cannot be a finite Hausdorff distance from either one of the boundary leaves. Since N is a product there is $b_0 > 0$ so that $\widetilde{N}$ is contained in the neighborhood of size b_0 of E, and likewise for G. We will show that E cannot be in a bounded neighborhood of any such L as above.

Lifting to a double cover if necessary we can assume that $\mathcal{F}$ is transversely orientable.

Since $\mathcal{F}$ is Reebless the fundamental group of leaves injects in $\pi_1(N)$, so the leaves are either planes, annuli or tori. If there is a compact leaf in the interior of N, then its fundamental group injects in $\mathbb{Z}^2 = \pi_1(N)$. so it is a torus, and hence it is isotopic to $\mathbb{T}^2 \times \{0\}$. It lifts to a leaf Z in $\widetilde{N}$ which separates E from G, contradiction. So the leaves in the interior of N are only planes and annuli.

Let $A = \mathbb{T}^2 \times \{0\}$, $B = \mathbb{T}^2 \times \{1\}$. We look at the holonomy of $\mathcal{F}$ along a boundary leaf, say A. We want to find an element of $\pi_1(A)$ with contracting holonomy. Fix x a basepoint in A, let τ be a small transversal to $\mathcal{F}$ at x. Let α represent a simple closed curve in A not null homotopic. If either α or α^{-1} has contracting holonomy, that is the element we want. Otherwise there are p_i in τ converging to x so that α holonomy fixes p_i. Fix i, let C be the leaf through p_i. Then C is an annulus. Let now β another simple closed curve which generates $\pi_1(\mathbb{T}^2)$ together with α. If holonomy of β fixes p_i also then C is in fact a compact leaf, but in the interior of N, which we showed it is not possible. So replacing β by its inverse, the holonomy image of p_i under β is closer to x. If the iterates converge to x, then β is the desired element. Otherwise the iterates converge to y not x, and the leaf through y is compact, again a contradiction.

Let then α be a simple closed curve in A with contracting holonomy. We think of α also as a deck transformation. Then α fixes E.

Fix a point y in E and a transversal τ. Since holonomy of the foliation $\mathcal{F}$ is contracting in the α direction this means that $\alpha^{-1}(L)$ intersects τ and in a point closer to E. The contracting holonomy means that the sequence $(\alpha^n(L))$ converges to E as $n \to -\infty$. In fact this is an if and only if property: if there is L intersecting τ so that $(\alpha^n(L))$ converges to E as $n \to -\infty$, then α has contracting holonomy.

But α also preserves G. Since E, G non separated from each other, and $(\alpha^n(L))$ converges to E, it follows that $(\alpha^n(L))$ also converges to G when n converges to minus infinity. By the if and only if characterization above, this implies the following: If β is a simple closed curve in B freely homotopic to α then the holonomy of $\mathcal{F}$ along β is contracting as well.

We proceed with the proof of the proposition. We consider a model of N as $\mathbb{T}^2 \times [0,1]$ so that $\widetilde{N}$ is homeomorphic to $\mathbb{R}^2 \times [0,1]$ with coordinates (a,b,c) and any deck transformation acts as $(\theta(a,b),c)$, where θ is a translation of $\mathbb{R}^2$. In that way we can choose coordinates so that $\alpha(a,b,c) = ((a,b)+(1,0),c)$.

Suppose now that E is in a neighborhood of size a_0 of L. For any n there is a point p_n in L which is $< a_0$ distant from $(-n(1,0),0)$.

Claim 3.6. *Given $\epsilon > 0$, there is $a_1 > 0$ so that any point in a leaf U of $\widetilde{\mathcal{F}}$, it is less than a_1 along U from a point ϵ from E or G.*

Proof. Suppose not. Project to N, we get bigger and bigger sets in leaves which avoid an ϵ neighborhood of the boundary. Taking a limit we find a leaf V of $\mathcal{F}$ avoiding an ϵ neighborhood of the boundary. The closure of V is a lamination in N disjoint from the boundary. It is an essential lamination W. Double N to get a Seifert fibered space, W is still an essential lamination. By Brittenham's result [Brit], W has a sublamination that is either vertical or horizontal in the double of $\mathbb{T}^2 \times I$. If W is vertical it would have to intersect a boundary component of N. This is a horizontal $\mathbb{T}^2$ in the double manifold. This is a contradiction. Suppose that W is horizontal. It is also contained in $\mathbb{T}^2 \times I$, hence a "topmost" leaf would have to be compact, hence a torus. This is contained in the interior of N, again a contradiction. This proves the claim. □

We fix $\epsilon > 0$ so that the foliation $\mathcal{F}$ restricted to the ϵ neighborhood of the boundary of N is entirely described by the holonomy maps. Let $a_1 > 0$ given by the claim. So given n, there is q_n in L, which is less than a_1 along L from p_n and q_n is ϵ away from the boundary. Hence $q_n = ((-n,0)+v_n,t_n)$ where v_n is bounded under n and $|t_n| < \varepsilon$. Up to subsequence we assume that all v_{n_i} are very close to v_0 (projection to N all in a fixed foliated chart).

Now apply the holonomy of α^n to q_n. Since q_n is ϵ close to the boundary and the holonomy of α is contracting in the neighborhood of size ϵ of both A and B it follows that the holonomy image of q_{n_i} is (v_{n_i}, t_{n_i}) where t_{n_i}

is either arbitrarily close to 0 or to 1. None is either 0 or 1 as L is in the interior of N. They are all points in L, and this contradicts that L cannot intersect a transversal more than once.

This contradiction proves that the assumption that E is at a bounded distance from L is impossible. Hence the foliation $\mathcal{F}$ is not uniform. $\square$

3.3. **Proof of Theorem 1.1.** Now we are ready to prove Theorem 1.1. Let $\mathcal{F}$ be a uniform Reebless foliation on M. We want to show that $\mathcal{F}$ is $\mathbb{R}$-covered, so we assume by contradiction that there are leaves L and F of $\widetilde{\mathcal{F}}$ which are non-separated in the leaf space $\mathcal{L}_{\mathcal{F}} = \widetilde{M}/_{\widetilde{\mathcal{F}}}$ of $\mathcal{F}$. Up to a double cover we may assume that $\mathcal{F}$ is transversely oriented.

Proposition 3.3 implies that both L and F project to compact surfaces in M. Let Γ_L be the stabilizer of L in $\pi_1(M)$. Proposition 3.4 shows that the region $\widetilde{N}$ between L and F projects to a compact $[0,1]$-bundle W in $\widetilde{M}/\Gamma_L$, with one boundary L/Γ_L.

Suppose that there is a deck translate $\beta(L)$ of L or F inside $\widetilde{N}$. It projects to a surface in $\widetilde{M}/\Gamma_L$ contained in the $[0,1]$-bundle W. Since $\pi(L)$ is compact in M, then $H = \beta(L)/_{\Gamma_L}$ is also compact. Since H is π_1-injective in W it follows that H is isotopic in W to a boundary component. Lifting to $\widetilde{M}$ this implies that $\beta(L)$ separates F from L, contradicting that they are non separated.

Let $A = \pi(L)$. Suppose that there is a closed transversal to $\mathcal{F}$ through A. Lift to $\widetilde{M}$, with the transversal intersecting L and entering $\widetilde{N}$. It cannot exit $\widetilde{N}$ as F, L do not intersect a closed transversal. Hence this produces a deck translate of L inside $\widetilde{N}$ which we just proved cannot happen. Hence there are no closed transversal through either A or $B = \pi(F)$.

On the other hand suppose there are E_i converging to $F \cup L$ so that $\pi(E_i)$ is compact. For i big enough $\pi(E_i)$ is isotopic to A, and hence E_i separates F from L, contradiction. Hence $\pi(E_i)$ is non compact and there are transversals through $\pi(E_i)$ for i big enough. It follows that the region between A and B is a dead end component, see [Ca$_4$, Definition 4.27]. By [Ca$_4$, Lemma 4.28], A, B are two sided tori or Klein bottles. Lifting to a double cover we can assume that both A, B are tori.

It can be that $A = B$, but in any case $\widetilde{N}$ projects in $\widetilde{M}/\Gamma_L$ to a compact submanifold homeomorphic to $\mathbb{T}^2 \times [0,1]$.

We can now apply Proposition 3.5. Let G be a leaf in $\widetilde{N}$. By Proposition 3.5 it follows that L is not a bounded distance from G in $\widetilde{N}$. Suppose that this does not happen in $\widetilde{M}$. Then there are points p_i in L which are $> i$ distant from G along path distance in $\widetilde{N}$, but a bounded distance in $\widetilde{M}$ from q'_i in G. Notice that q'_i is a bounded distance in $\widetilde{N}$ from q_i in L – just follow along the lift of the I-bundle structure to $\widetilde{N}$. If one uses the parametrization (a, b, c) as in Proposition 3.5 one can assume up to moving them boundedly in L, that p_i, q_i have all coordinates integers and the last coordinate 0. Consider a generating set of $\pi_1(M)$ which includes 2 generators of the torus A. Then p_i, q_i are vertices of the Cayley graph. Modulo deck transformations sending p_i back to a base point, it follows that q_i is a bounded neighborhood of the origin. So only finitely many elements of $\pi_1(M)$ are allowed. It follows that q_i is a bounded distance from p_i along L. This is a contradiction.

This completes the proof of Theorem 1.1.

4. Universal circles and JSJ trees

In this section we will show that for $\mathbb{R}$-covered foliations (uniform or not) one can recover the universal circle from the JSJ decomposition of the manifold (cf. Proposition 4.9), if the manifold has a non trivial JSJ decomposition. This will allow us to prove Proposition 4.10 that we will need in the proof of Theorem 1.2. Proposition 4.10 states that the action of the fundamental group on the universal circle does not have fixed points which is certainly a fact that needs to be established if one desires to obtain minimality of the action.

Consider an $\mathbb{R}$-covered foliation $\mathcal{F}$ by leaves with curvature uniformly close to -1 on a closed 3-manifold M, so that M has non trivial JSJ decomposition. In particular the leaves are Gromov hyperbolic. If $\mathcal{F}$ is not taut, then there are dead end components, see [Ca$_4$, Definition 4.27]. In particular there are either tori or Klein bottle leaves. This is disallowed by $\mathcal{F}$ having Gromov hyperbolic leaves. Hence $\mathcal{F}$ is taut.

We will consider that M is orientable and $\mathcal{F}$ transversely orientable. The only difference in the non-orietable case is that in the JSJ decomposition we also have to consider Klein bottles. These Klein bottles lift to embedded tori in some cover of M. Then all the results follow with the same proofs.

4.1. **The trace of JSJ tori in the universal circle.** Let $M_1, \dots M_k$ be the pieces of its JSJ decomposition. Let T be a torus of the JSJ decomposition. In this section we show Proposition 4.4 which states that one can associate to each lift of a tori of the JSJ decomposition some points in the universal circle.

We first need the following lemma that puts (after isotopy) the JSJ tori in general position.

Lemma 4.1. *Any lift $\widetilde{T}$ to $\widetilde{M}$ intersects every leaf of $\widetilde{\mathcal{F}}$. In addition one can isotope T so that $\widetilde{T}$ intersects every leaf of $\widetilde{\mathcal{F}}$ in a single component, and so that the foliation induced by $\mathcal{F}$ in T has no Reeb components.*

Proof. Let $G = \mathbb{Z}^2$ be the isotropy group of $\widetilde{T}$. The set of $\widetilde{\mathcal{F}}$ leaves intersected by $\widetilde{T}$ is connected. If this set is not the whole leaf space, it is a non trivial interval in the leaf space. Let F be an endpoint. Since the leaf space is homeomorphic to $\mathbb{R}$, it follows that G preserves F. So $\pi_1(\pi(F))$ has a $\mathbb{Z}^2$ subgroup and the projection $\pi(F)$ is therefore a torus or Klein bottle. This contradicts that the leaves of $\mathcal{F}$ are Gromov hyperbolic.

Since $\mathcal{F}$ is taut, by Theorem 2.1 we can isotope T to be either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$. The first option is disallowed because of Gromov hyperbolic leaves. Hence assume that T is transverse to $\mathcal{F}$, let $\mathcal{G}$ be the induced foliation in T.

Claim 4.2. *It is possible to isotope T so that $\mathcal{G}$ has no* Reeb annuli*.*

Proof. A Reeb annulus is a foliation of the annulus so that boundaries are leaves, all other leaves spiral toward the boundary leaves, and there is no transversal arc intersecting both boundary leaves. Suppose that $\mathcal{G}$ has a Reeb annulus A. The two boundary leaves of A lift to curves in $\widetilde{M}$, contained in leaves of $\widetilde{\mathcal{F}}$ which are non separated from each other. This is because of the Reeb annulus, so in $\widetilde{A}$ the boundary infinite lines are non separated from each other. Since the foliation is $\mathbb{R}$-covered, the two leaves of $\widetilde{\mathcal{F}}$ containing these infinite lines α, β are the same leaf L. Since $\pi(\alpha), \pi(\beta)$ are freely homotopic in T, then α, β are a bounded distance from each other in $\widetilde{M}$. We now use a fact of $\mathbb{R}$-covered foliations: for any $a_0 > 0$, there is $a_1 > 0$, so that if two points x, y in a leaf F of $\widetilde{\mathcal{F}}$ are less that a_0 in $\widetilde{M}$, then they are less than a_1 in L, see [Fen$_2$, Proposition 3.4]. This holds only for $\mathbb{R}$-covered foliations. Hence α, β are a bounded distance from each other in L. It now follows that $\pi(\alpha), \pi(\beta)$ are isotopic closed curves in $\pi(L)$ and bound an annulus B in $\pi(L)$. The interior of B cannot intersect A, because any interior leaf of $\mathcal{G}$ in A limits to the boundary of A, and A, B are transverse to each other. Hence $A \cup B$ is a torus. This torus is not π_1 injective because one can produce an essential arc across A together with one across B to yield a closed curve which is null homotopic. One can easily see this as $\widetilde{B}$ is contained in the fixed leaf L, and $\widetilde{A}$ has both boundaries in L. Hence $A \cup B$ is compressible and there is a compressing disk D intersecting $A \cup B$ only in the boundary. Cutting $A \cup B$ along D, produces a sphere. Since M is irreducible, this sphere bounds a ball. Gluing back together one sees that $A \cup B$ bounds a solid torus.

What we proved is that B is isotopic to A in M. So then one can isotope A across the solid torus to the other side of B and eliminate this Reeb annulus in $\mathcal{G}$. Doing this finitely many times eliminates all Reeb annuli in $\mathcal{G}$. This proves the claim. See also [Ca$_1$, Theorem 5.3.13] for a similar statement. $\square$

Since there are no Reeb annuli in $\mathcal{G}$, it follows that $\mathcal{F}$ intersects T in a foliation uniformly equivalent to a linear foliation of the two dimensional torus. In particular any two leaves of $\widetilde{\mathcal{G}}$ are connected by a tranversal to $\widetilde{\mathcal{G}}$, hence a transversal to $\widetilde{\mathcal{F}}$ as well. It follows that any leaf F of $\widetilde{\mathcal{F}}$ intersects $\widetilde{T}$ in a single component.

This finishes the proof of the lemma. $\square$

Remark 4.3. The reason we choose the definition of non-trivial JSJ decomposition is to exclude *Sol* and *Nil* geometries for which some of the arguments do not work. These cases are not problematic to us and can be dealt with separately, and in a different way. A good thing about manifolds with non-trivial JSJ decomposition under our definition is that the tori of the decompositons are quasi-isometrically embedded: the map between the universal covers is a quasi-isometric embedding. This follows from [KL, Theorem 1.1] (see also [Ng, Section 3.1]). In particular when lifted to $\widetilde{M}$, every quasigeodesic in the lift of the torus lifts maps to a quasigeodesic in $\widetilde{M}$.

Let T be a torus of the JSJ decomposition, put in good position as in Lemma 4.1. Let $\mathcal{G}$ be the induced foliation by $\mathcal{F}$ in T. Given L leaf of $\widetilde{\mathcal{F}}$, and a lift $\widetilde{T}$ of T, then by Remark 4.3, the curve $L \cap \widetilde{T}$ is a quasigeodesic of $\widetilde{M}$. It is also a leaf of $\widetilde{\mathcal{G}}$. Since it is a quasigeodesic in $\widetilde{M}$, then it is necessarily also a quasigeodesic in L, with ideal points $a_L(\widetilde{T}), b_L(\widetilde{T})$ in $S^1(L)$. Orient the foliation $\mathcal{G}$ so that $b_L(\widetilde{T})$ corresponds to the forward direction in $\mathcal{G}$. Varying the leaf, produces corresponding ideal points $a_F(\widetilde{T}), b_F(\widetilde{T})$ in $S^1(F)$ for any F leaf of $\widetilde{\mathcal{F}}$.

Proposition 4.4. *The collection $\{b_F(\widetilde{T})\}$ as F varies over leaves of $\widetilde{\mathcal{F}}$ is a leaf of the vertical foliation in the cylinder at infinity $\mathcal{A}$. Equivalently, the point $\{b_F(\widetilde{T})\}$ is well defined in S^1_{univ} and independent of the leaf F.*

Proof. We will fix a lift $\widetilde{T}$ of some torus T of the JSJ decomposition. So, we will not include the reference to $\widetilde{T}$ in the notation.

Suppose first that $\mathcal{F}$ is uniform. Let α_L be the intersection of L and $\widetilde{T}$, that is a leaf of $\widetilde{\mathcal{G}}$. For any L, F leaves of $\widetilde{\mathcal{F}}$, the curves α_L, α_F are a bounded distance from each other in $\widetilde{T}$ − since there are no Reeb annuli in $\mathcal{G}$. It follows that α_L, α_F are a bounded distance from each other in $\widetilde{M}$. By the remark above, α_L is a quasigeodesic in L, hence, the ray β_L defining b_L is a bounded distance in L from a geodesic ray in L. Since $\mathcal{F}$ is uniform, this ray in L is a bounded distance from a geodesic ray in F defining $\tau_{L,F}(b_L)$. But β_L is a bounded distance from a corresponding ray β_F of α_F (same direction given by the foliation $\mathcal{G}$). This is bounded distance in $\widetilde{M}$. Hence β_F is a bounded distance in $\widetilde{M}$ from the geodesic ray defining $\tau_{L,F}(b_L)$. Since $\mathcal{F}$ is $\mathbb{R}$-covered, this again implies that β_F is a bounded distance from this geodesic ray in F. In particular the ideal point of β_F is $\tau_{L,F}(b_L)$. But by definition the ideal point of β_F is b_F. Hence $b_F = \tau_{L,F}(b_L)$. This proves the proposition in this case.

Suppose now that $\mathcal{F}$ is not uniform. By the description in §2.5.2 we can assume that $\mathcal{F}$ is minimal. Hence for any L, F in $\widetilde{\mathcal{F}}$ there is a dense set of directions in $S^1(L)$ which are asymptotic to F.

Fix a transversal τ in T. Lift this to a transversal $\tilde{\tau}$ in $\widetilde{T}$. For any L intersecting $\widetilde{T}$, let $x_L = \tilde{\tau} \cap L$. Let r_L be the geodesic ray in L starting at x_L and with ideal point b_L. As L varies the corresponding rays β_L in $\widetilde{\mathcal{G}}$ are boundedly close to each other in $\widetilde{T}$ and hence in $\widetilde{M}$. Hence the same happens for the geodesic rays r_L as L varies. It follows that the ideal points of β_L vary continuously with L. Hence the functions a_L, b_L from the leaf space into $\mathcal{A}$ are continuous.

Suppose that for some L, F, then $\tau_{L,F}(b_L) \neq b_F$. Since the set of contracting directions between L and F is dense in $S^1(L)$ and b_E varies continuously with E, it follows that there is some E between L, F so that b_E corresponds to a direction in E which is contracting with both L and F. Hence the ray β_E in $E \cap \widetilde{G}$ is asymptotic to a curve in L. This implies that in $\widetilde{T}$, the curve β_E is asymptotic to a curve in $\widetilde{T} \cap L$. But this can only be β_L − as $\widetilde{T} \cap L$ is a single curve and has a ray β_L corresponding to that direction. In particular this implies that $b_E = \tau_{L,E}(b_L)$. The same holds for the pair E, F. By the equivariance of the maps $\tau_{L,F}$, it now follows that $\tau_{L,F}(b_L) = b_F$.

This finishes the proof of the proposition. ∎

4.2. **JSJ universal circles.** Our setup has an $\mathbb{R}$-covered foliation $\mathcal{F}$ by leaves with curvature very close to -1 in M with non trivial JSJ decomposition. If T is a torus in the JSJ decomposition we use Lemma 4.1 and isotope T to be transverse to $\mathcal{F}$ and so that the induced foliation in T does not have any Reeb annuli.

Recall that in Proposition 2.4 we introduced the JSJ tree $\mathcal{T}$ of M. Let $T_1, ..., T_k$ be the tori in the JSJ decomposition. The fundamental group naturally acts on the tree $\mathcal{T}$. The tree $\mathcal{T}$ is infinite and in general not locally compact: there are infinitely many edges adjoining any given vertex. We observe that if M has a trivial JSJ decomposition, that is, M is either Seifert or atoroidal, then the object constructed above would be a single point. We now consider the case that M has an $\mathbb{R}$-covered foliation.

Let $W = \pi^{-1}(T_1 \cup \ldots \cup T_k)$. In other words a component of W is an arbitrary lift $\widetilde{T}$ of one of the JSJ tori.

Lemma 4.5. *Suppose that M has a non-trivial JSJ decomposition and $\mathcal{F}$ is an $\mathbb{R}$-covered foliation by leaves with curvature very close to -1.*

Then the JSJ tree $\mathcal{T}$ has an embedding into the plane well defined up to isotopy. This determines a well defined circular ordering on the set of ends of $\mathcal{T}$. A deck transformation either preserves the circular ordering, or reverses the circular ordering on the set of ends.

Proof. The curvature condition implies that $\mathcal{F}$ is Reebless.

Hence the leaves of $\widetilde{\mathcal{F}}$ are properly embedded planes in $\widetilde{M}$.

First fix a leaf F of $\widetilde{\mathcal{F}}$. Lemma 4.1 shows that any lift $\widetilde{T}$ of a JSJ torus intersects F in a single component. This component is a quasigeodesic in F. For each vertex y of $\mathcal{T}$, associated to a component V of $\widetilde{M} - W$, it has at least two edges adjoining it, let $\widetilde{T}$ be one of them. Since $\widetilde{T}$ intersects F transversely, then V also intersects F. In addition since any lift $\widetilde{T}'$ of a JSJ torus separates $\widetilde{M}$, and each such lift intersects F in a single component, it also follows that V also intersects F in a single component. Choose a point p_V in $V \cap F$ representing the vertex y of $\mathcal{T}$. It $\widetilde{T}$ is an edge of $\mathcal{T}$ adjoining components V, Z of $\widetilde{M} - W$, choose an embedded arc connecting p_V to p_Z, and intersecting $\widetilde{T}$ in a single point. This represents an embedding of the edge $\widetilde{T}$ of $\mathcal{T}$ into F. In this way we construct an embedding of $\mathcal{T}$ into F. The choices of the points p_V are well defined up to isotopy in $V \cap F$. The choices of the embedded arcs are also well defined up to isotopy. Therefore the embedding of $\mathcal{T}$ into F is well defined up to isotopy. Fix one such embedding and call $\mathcal{T}_F$ the image tree in F.

Now if L is another leaf of $\widetilde{\mathcal{F}}$, then the same reasoning applies. Notice that if V, Z components of $\widetilde{M} - W$ define and edge $\widetilde{T}$, then $V \cap L, Z \cap L$ are adjoining in L along $\widetilde{T} \cap L$ just as in F. In addition the circular ordering around a vertex is also the same whether considering it wrt to F or to L. It follows that the embeddings of $\mathcal{T}$ in F and L are isomorphic, preserving the circular ordering at the corresponding vertices.

It follows that the embedding in the plane is well defined up to isotopy. This induces a circular ordering in the set of ends of $\mathcal{T}$.

If γ is a deck transformation, and F a leaf of $\widetilde{\mathcal{F}}$, then γ also induces a homeomorphism of the embedding of $\mathcal{T}$ in F: given V components of

$\widetilde{M} - W$, then $\gamma(V)$ also intersects F in a single component, and likewise for $\widetilde{T}$ component of W. This produces the required homeomorphism of the tree $\mathcal{T}$. In addition this homeomorphism is induced by a homeomorphism between F and $\gamma(F)$, which can be either orientation preserving or reversing. It follows that this homeomorphism either preserves the circular ordering of the ends of $\mathcal{T}$ or reverses it. $\square$

Remark 4.6. We emphasize some facts proved in this lemma: if V is a component of $\widetilde{M} - W$, and F is a leaf of $\widetilde{\mathcal{F}}$, then V intersects F and in a single component (cf. Lemma 4.1). Similarly if $\widetilde{T}$ is a component of W then $\widetilde{T}$ intersects F in a single component. Therefore the trees $\mathcal{T}$ and $\mathcal{T}_F$ are canonically isomorphic. In particular if F, L are leaves of $\widetilde{\mathcal{F}}$, then $\mathcal{T}_F, \mathcal{T}_L$ are canonically isomorphic, with the circular order of the edges at any vertex preserved by the isomorphism (see also Proposition 4.4).

We produced a set with a circular order and a group action so that each group element either preserves the circular order or reverses. Given these properties, a circle with an induced action can be created. This procedure from set with circular order and group action to action on a circle was developed by Calegari and Dunfield in [CD]. We refer to [CD, Theorem 3.2] for specific details. Here we will only briefly describe the construction of the circle with the induced action.

Since the set of ends is cyclically ordered there is an embedding of the set of ends into a circle preserving the circular order. First take the closure of the image of the set of ends. If the tree were locally finite (finitely many edges at any vertex), then the set of ends would be order complete, and the image is a closed subset of the circle.. The fundamental group still acts on the closure. There may be gaps in the image. Now collapse every closure of a complementary interval (that is a gap) to a point, producing a circle S^1_{JSJ}, called the *JSJ universal circle* of $\mathcal{F}$. Deck transformations either preserve or reverse the circular ordering so induce homeomorphisms of the circle that either preverse or reverse orientation.

Remark 4.7. The JSJ universal circle depends on the foliation $\mathcal{F}$: given a different $\mathbb{R}$-covered foliation $\mathcal{F}_1$, it may induce a different circular ordering of the edges at a given vertex of the tree $\mathcal{T}$. This will produce a different circular order on the set of ends of $\mathcal{T}$ and hence a different JSJ universal circle. The tree $\mathcal{T}$ is the same and so are its ends. But the the set of edges around a vertex in $\mathcal{T}$ does not come with a natural circular order. This is the information that the $\mathbb{R}$-covered foliation is providing. And different $\mathbb{R}$-covered foliations may give different such circular orders.

Let T be a π_1-injective torus in M, put in good position as in Lemma 4.1. Given F leaf of $\widetilde{\mathcal{F}}$, we define the lamination $\mathcal{G}_F$ whose leaves are the intersections of lifts $\widetilde{T}$ of T with F. In fact $\mathcal{G}_F$ also depends on T, but for notational simplicity we omit this dependence.

Lemma 4.8. *For each π_1-injective torus T of M and for each F leaf of $\widetilde{\mathcal{F}}$, then the set of ideal points of leaves of $\mathcal{G}_F$ is dense in $S^1(F)$. In addition for any non degenerate interval J of $S^1(F)$ there are leaves of $\mathcal{G}_F$ with both ideal points in J.*

Proof. Suppose the first property is not true, let T, F failing it. Then there is a non trivial interval I in $S^1(F)$ which is disjoint from the ideal points of of $\mathcal{G}_F$. Since the curves in $\mathcal{G}_F$ are uniform quasigeodesics in F they are a uniform bounded distance from geodesics in F. Hence up to considering a subinterval, it follows that I bounds a half plane P in F which is disjoint from $\mathcal{G}_F$. Therefore there are disks D_i with radius converging to infinity disjoint from $\mathcal{G}_F$. Up to taking subsequences and deck transformations g_i, then $g_i(D_i)$ converges to a full leaf L which is disjoint from $\mathcal{G}_L$. But this is impossible since any $\widetilde{T}$ lift of T intersects every leaf of $\widetilde{\mathcal{F}}$. This proves the first property of the lemma.

Now suppose that J is a non generate interval so that no leaf of $\mathcal{G}_F$ has both ideal points in J. Let x be an iterior point of J. Let x_i a sequence of distinct points in J converging monotonically to x. There are leaves c_i of $\mathcal{G}_F$ with an ideal point arbitrarily close to x_i. Since the x_i are distinct we can choose the c_i to be distinct as well. The other endpoints of c_i are not J, hence at least $a_1 > 0$ from the first endpoint of c_i which is arbitrarily close to x. Since the c_i are uniform quasigeodesics, then up to subsequence we may assume that c_i converges to a quasigeodesic c. But then different c_i, c_j have points that are arbitrarily close to each other. This is a contradiction: if C, C' are different lifts of JSJ tori, then they cannot have points less than $a_2 > 0$ for some constant a_2. This finishes the proof of the lemma. □

We can now prove the following proposition that gives a different way to think about the universal circle of a foliation in terms of the JSJ universal circle.

Proposition 4.9. *Suppose that M has a non-trivial JSJ decomposition and $\mathcal{F}$ is an $\mathbb{R}$-covered foliation with almost hyperbolic leaves. Then there is a canonical homeomorphism between the universal circle S^1_{univ} of $\mathcal{F}$ and the JSJ universal circle S^1_{JSJ} of $\mathcal{F}$. This homeomorphisms is equivariant under deck transformations.*

Proof. For simplicity fix a leaf F of $\widetilde{\mathcal{F}}$. The universal circle of $\mathcal{F}$ is canonically identitied with $S^1(F)$. The JSJ universal circle can be obtained from the intersections with F. What we will prove is that considering F, both of these are canonically homeomorphic.

Let $\mathcal{T}_F$ be the embedded tree in F which is the homeomorphic image of $\mathcal{T}$. Fix a basepoint p in $\mathcal{T}_F$. Let $\mathcal{B}$ be the set of ends of $\mathcal{T}_F$. Since $\mathcal{T}_F$ is a tree it is easy to see that each end is uniquely associated to embedded rays in $\mathcal{T}_F$ starting at p. Let e be an end in $\mathcal{B}$ associated to a ray α in $\mathcal{T}_F$, which is also an embedded ray in F. Then α keeps intersecting lifts C_i of one of the JSJ tori, let $c_i = C_i \cap F$. Recall that c_i is a quasigeodesic with uniform constants, so globally a_0 from a geodesic in F. Any two lifts C, C' of JSJ tori have a minimum separation between them. Hence the corresponding $C \cap F, C' \cap F$ also have a minimum separation between them. Therefore the geodesics associated to c_i also escape in F and they define a unique ideal point in $S^1(F)$ which we call $f(e)$. This defines a map f from the set of ends $\mathcal{B}$ to $S^1(F)$.

Given appropriate orientations on $S^1(F)$ and the circular order on the set of ends of $\mathcal{T}_F$, it follows that the map f preserves this circular order.

In particular as one goes around once in the circular order of the ends of $\mathcal{T}_F$, then one also goes around once in $S^1(F)$. By Lemma 4.8, for each non degenerate interval J in $S^1(F)$ there is a leaf c of $\mathcal{L}_F$ with both ideal points in J. Hence any end e of $\mathcal{T}_F$ which is associated with a path in the tree $\mathcal{T}_F$ which crosses c will have $f(e)$ in J. It follows that the image of f is dense in $S^1(F)$.

Recall the construction of the construction of the JSJ universal circle S^1_{JSJ} of $\mathcal{F}$: we map the set of ends $\mathcal{B}$ to a circle S^1 preserving the circular order, take the closure and then collapse the gaps.

By the first step we can think of $\mathcal{B}$ as a subset of S^1. Let H be the closure in S^1 of the image of f. Since f preserves circular order it induces a map f_1 from H into $S^1(F)$. This map is weakly monotone. Since the image of $\mathcal{B}$ under f is dense in $S^1(F)$ it follows that given the endpoints of a gap of H they have the same image in $S^1(F)$ under f_1. This implies that f_1 induces a map f_* from the JSJ universal circle S^1_{JSJ} of $\mathcal{F}$ to $S^1(F)$.

Finally by the same reasoning if two points have the same image under f_1 then they have to be boundary points of a gap of H in S^1. This implies that f_* is a homeomorphism.

Any deck transformation γ permutes the lifts of JSJ tori and components of $\widetilde{M} - W$. It sends infinite embedded paths in the tree $\mathcal{T}_F$ to infinite paths in the tree $\mathcal{T}_{\gamma(F)}$. The tree $\mathcal{T}_{\gamma(F)}$ is canonically homeomorphic to the tree $\mathcal{T}_F$ and this identification is compatible with the identifications of $S^1(\gamma(F))$ and $S^1(F)$. It follows that the homeomorphisms f_* are equivariant. This finishes the proof of the proposition. $\square$

4.3. **Moving points in the universal circle.** The following property will be important for the proof of Theorem 1.2.

Proposition 4.10. *If $\mathcal{F}$ is a uniform $\mathbb{R}$-covered foliation by hyperbolic leaves and $\xi \in S^1_{univ}$ then there is $\gamma \in \pi_1(M)$ such that $\gamma(\xi) \neq \xi$.*

Proof. We first treat the case where the JSJ decomposition of M is trivial. If M is Seifert with hyperbolic base, the universal circle is identified with the boundary of the universal cover of the base. The base is a hyperbolic surface S. If δ is a generator of the center of $\pi_1(M)$ then $\pi_1(M)/ < \delta >$ is isomorphic to a closed surface group $(\pi_1(S))$ and acts on the boundary $\partial\widetilde{S}$. The stabilizer of each point in $\partial\widetilde{S}$ is at most infinite cyclic. The deck transformation δ acts by the identity on the universal circle of the foliation. It now follows that the stabilizer of a point of the universal circle is at most a $\mathbb{Z}\oplus\mathbb{Z}$ subgroup. By homological reasons $\mathbb{Z}\oplus\mathbb{Z}$ cannot be the fundamental group of a 3-manifold in our conditions [Hem]. This finishes the proof in the Seifert case.

If M is atoroidal then it is hyperbolic[4] and we then assume $\widetilde{M} = \mathbb{H}^3$. In this case we show that the stabilizer of ξ is at most infinite cyclic. Suppose that γ is in the stabilizer of ξ. Let F a leaf of $\widetilde{\mathcal{F}}$ and ξ_F be the ideal point of $S^1(F)$ associated to ξ. Thurston [Th] proved that the embedding $F \to \widetilde{M}$

[4]This follows from Perelmans' geometrization theorem. We do not need the full force of geometrization here, it is enough to know that atoroidal manifolds have fundamental group which is Gromov hyperbolic [GabKa], see also [Ca_4, Corollary 9.32].

extends to a continuous map $F \cup S^1(F) \to \widetilde{M} \cup S^2_\infty$ where S^2_∞ is the boundary $\partial_\infty \widetilde{M} = \partial_\infty \mathbb{H}^3$ (cf. § 2.4). Let p be the image of ξ_F under this extended map. Let β be a geodesic ray in F with ideal point ξ_F. Then $\gamma(\beta)$ is a geodesic ray in $\gamma(F)$. Since $\gamma(\xi) = \xi$, and $\mathcal{F}$ is uniform, it follows that $\gamma(\beta)$ has a subray which is a bounded distance from β. In $\widetilde{M} \cup S^2_\infty$ the image of β limits to p. Since $\gamma(\beta)$ has a subray a bounded distance from a ray of β, it follows that $\gamma(p)$ is equal to p. Hence γ is in the stabilizer of p. But it is well known that the stabilizer of a point in S^2_∞ is at most cyclic. This finishes the proof in the atoroidal case.

Foliations in manifolds with (virtually) solvable fundamental group are classified and cannot be uniform $\mathbb{R}$-covered with hyperbolic leaves (see [Pla] or [HP, Appendix B] for the C^0-case). In fact the result does not work for manifolds with (virtually) solvable fundamental group. So the remaining case to be analyzed in the proof is is when M has a non-trivial JSJ decomposition in our sense (which excludes being a torus bundle up to a finite cover).

Now we consider the case that the JSJ decomposition of M is not trivial. Let $\mathcal{T}$ be the tree of lifts of the pieces of the JSJ decomposition as in Proposition 2.4. Fix F a leaf of $\widetilde{\mathcal{F}}$. Recall from the proof of Lemma 4.8 that the following holds: for any lift $\widetilde{M}_{i_0}^{j_0}$ of a piece M_{i_0} of the JSJ decomposition of M, it intersects F in a single component. Let ξ_F be the point of $S^1(F)$ corresponding to ξ, and τ_F a different point of $S^1(F)$. Lemma 4.8 implies that there are distinct lifts $\widetilde{M}_i^j$, $j \in \mathbb{N}$, of M_i fixed, so that $\widetilde{M}_i^j \cap F$ has a boundary component c_j with both endpoints arbitrarily near τ_F and so that the collection $\widetilde{M}_i^j$ is nested with j. In particular for j big, the closure of $F \cap \widetilde{M}_i^j$ as well as all the adjacent vertices of $\mathcal{T}_F$ are far from ξ_F. Pick a deck transformation γ which fixes $\widetilde{M}_i^j$. It follows that γ must move all the vertices of $\mathcal{T}$ which are not adjacent to $\widetilde{M}_i^j$. Since the closure of $\widetilde{M}_i^j \cap F$ in $F \cup S^1(F)$ separates ξ_F from the closure of $\widetilde{M}_i^k \cap F$ in $S^1(F)$ for $k > j$, it follows that γ does not fix ξ. This concludes the proof of the proposition. $\square$

Remark 4.11. One can give a different proof of Proposition 4.10 using different machinery that we chose not to present in detail. Indeed, if there is a global fixed point ξ in the universal circle of a uniform $\mathbb{R}$-covered foliation by hyperbolic leaves, then the one-dimensional foliation by geodesics in each leaf landing as a geodesic fan on ξ is equivariant and therefore descends to a one-dimensional foliation (which if chosen to be tangent to a unit vector field defines a flow) in M. By an argument in [Ca_3] (see the proof of [Ca_3, Theorem 5.5.8]) this flow is (topologically) Anosov[5] for which $\mathcal{F}$ is the weak stable foliation. This is impossible since the flow would be $\mathbb{R}$-covered and not a suspension (because the center stable foliation is uniform). This flow also does not have periodic orbits freely homotopic to their inverses, because the orbits always point in the direction of ξ. This contradicts what is proved in [Fen_1, Bar].

[5]Or at least semiconjugate to it.

5. Proof of Theorem 1.2

We fix in $\pi_1(M)$ a finite symmetric set of generators $\mathcal{S}$ and denote by $|\gamma|$ the word length of γ with respect to $\mathcal{S}$. We will be concerned with sequences going to infinity, so the choice of $\mathcal{S}$ is irrelevant.

Theorem 1.2 concerns the action of $\pi_1(M)$ on the universal circle S^1_{univ}. The universal circle is canonically homeomorphic to $S^1(L)$ for any L leaf of $\widetilde{\mathcal{F}}$. By Remark 2.6, in order to prove Theorem 1.2, it is equivalent to consider the action of $\pi_1(M)$ on $S^1(L)$. So fix a leaf $L \in \widetilde{\mathcal{F}}$ and denote by

$$\rho(\gamma) : L \cup S^1(L) \to L \cup S^1(L), \qquad \rho(\gamma)(x) = \tau_{\gamma L, L} \circ \gamma$$

the induced action on $S^1(L)$. Again via the identification with the universal circle S^1_{univ} this is exactly the action defined in S^1_{univ} in Remark 2.6. In this way ρ is a group homomorphism from $\pi_1(M)$ into $\mathrm{Homeo}_+(S^1(L))$.

Fix a point $x_0 \in L$. The point x_0 allows us to define a *visual measure* (cf. §2.4) in $S^1(L)$ that we will also fix.

The first important property is the following:

Lemma 5.1. *Given a compact interval $\mathcal{I} \subset \mathcal{L}_{\mathcal{F}} = \widetilde{M}/_{\widetilde{\mathcal{F}}}$ containing L we have that if $\gamma_n \in \pi_1(M)$ verifies that $\gamma_n L \in \mathcal{I}$ and $|\gamma_n| \to \infty$, then it holds that for every $x \in L$ we have*

$$d_L(x, \rho(\gamma_n)x) \to \infty.$$

In particular, given $C \subset L$ compact, there is $K > 0$ such that if $|\gamma| > K$ and $\gamma L \in \mathcal{I}$ then $\rho(\gamma)C \cap C = \emptyset$.

Proof. Fix a compact fundamental domain Y of M in $\widetilde{M}$. For a given $R > 0$ there is a bounded set $G \subset \widetilde{M}$ which consists of the points z in leaves $F \in \mathcal{I}$ such that $\tau_{F,L}(z) \in B_R(x)$ where $B_R(x)$ denotes the ball of radius R in L. The set G is bounded because the quasi-isometry constants of $\tau_{F,L}|_L$ depend only on the Hausdorff distance between F and L. Since $F \in \mathcal{I}$ the Hausdorff distance is bounded. Now, one can cover C by finitely many fundamental domains, implying that if γ verifies that $\gamma L \in \mathcal{I}$ and $|\gamma|$ is sufficiently large, then $\rho(\gamma)x$ cannot be in $B_R(x)$. This completes the first part of the Lemma.

For the second statement, notice that estimates are uniform, so by compactness one gets the statement. $\square$

This allows us to show the following:

Lemma 5.2. *For every finite interval $\mathcal{I} \subset \mathcal{L}_{\mathcal{F}}$ containing L and $\varepsilon > 0$ there is $K > 0$ such that if $|\gamma| > K$ and $\gamma L \in \mathcal{I}$ we have that there are disjoint intervals I_γ, J_γ of length (for the visual measure) smaller than ε and such that*

$$\rho(\gamma)(S^1 \backslash I_\gamma) \subset J_\gamma.$$

In particular the fixed points of $\rho(\gamma)$ are in $I_\gamma \cup J_\gamma$.

Proof. Given the finite interval $\mathcal{I}$ there exists a uniform constant $c > 1$ so that for every $F \in \mathcal{I}$ the map $\tau_{F,L} : F \to L$ is a quasi-isometry with constant c. It follows that the image by $\rho(\gamma)$ of a geodesic is a c-quasigeodesic whenever $\gamma L \in \mathcal{I}$. Notice that $\tau_{F,L}|_L$ is not necessarily continuous, so $\tau_{F,L}|_L(c)$

not necessarily a continuous curve. But the quasi-isometry inequalities still hold.

Let $C \subset L$ be a compact set containing x_0 with the property that every quasi-geodesic in L with constants bounded by c which does not intersect C verifies that its visual measure is smaller than $\varepsilon/2$.

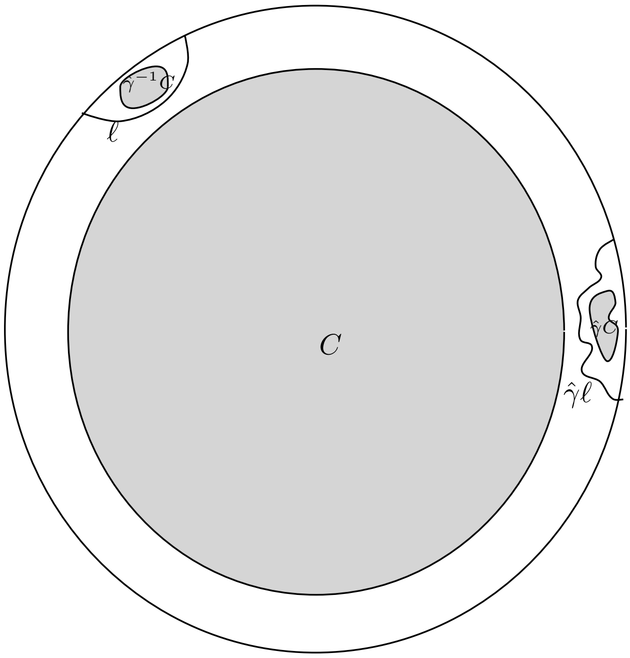

FIGURE 1. Depiction of the ingredients of the proof of Lemma 5.2. Here $\hat{\gamma} := \rho(\gamma)$.

Now, we can apply Lemma 5.1 to find K such that if γ verifies that $\gamma L \in \mathcal{I}$ and $|\gamma| > K$ then one has that $\rho(\gamma^{-1})C \cap C = \emptyset$. By choosing K a bit larger, one can assume that there is a geodesic ℓ in L which separates $\rho(\gamma^{-1})C$ from C (see figure 1). This allows to define I_γ as the (shortest) interval determined by the endpoints of ℓ (i.e. the one so that $I_\gamma \cup l \subset L \cup S^1(L)$ leaves C on the outside) and J_γ to the (shortest) interval joining the endpoints of the quasigeodesic $\rho(\gamma)(\ell)$.

□

Now we are in condition to prove minimality of the action:

Proposition 5.3. *The action of $\pi_1(M)$ on S^1_{univ} is minimal. In particular, given $\xi \in S^1(L)$ and an open interval $U \subset S^1(L)$ there exists $\gamma \in \pi_1(M)$ such that $\rho(\gamma)(\xi) \in U$.*

Proof. We choose a point ξ in $S^1(L)$ and an open set $U \subset S^1(L)$. Pick $\eta \in \pi_1(M)$ so that $\rho(\eta)\xi \neq \xi$ (cf. Proposition 4.10).

Fix T a compact fundamental domain of M in $\widetilde{M}$. Every other fundamental domain will be a translate of T by a deck transformation. Let $D = \text{diam}(T)$, which is also the diameter of any translate of T. Let $\mathcal{I} \subset \mathcal{L}_{\mathcal{F}}$

be a compact interval around L such that the union $\bigcup_{F\in\mathcal{I}} F$ contains the neighborhood of size $2D$ of the leaf L. Notice that this interval can be chosen thanks to the fact that $\mathcal{F}$ is $\mathbb{R}$-covered and uniform.

Choose points $\xi_1 \neq \xi_2$ in the interior of U and take fundamental domains T_1^n, T_2^n of M in $\widetilde{M}$ such that they intersect L in points very close to ξ_1 and ξ_2 respectively, more precisely, such that the intersection $T_i^n \cap L$ is non empty and $T_i^n \cap L \to \xi_i$ in $L \cup S^1(L)$. See figure 2.

Now, we can choose γ_n so that $\gamma_n(T_1^n) = T_2^n$. Since the diameter of T_i^n is fixed, $T_i^n \cap L \to \xi_i$ and $\xi_1 \neq \xi_2$ then as $n \to \infty$,

$$\text{for any} \quad y_1^n \in T_1^n,\ y_2^n \in T_2^n, \quad d(y_1^n, y_2^n) \to \infty.$$

It follows that $|\gamma_n| \to \infty$. Also $\gamma_n L \in \mathcal{I}$ so that Lemma 5.2 applies. Let $I_{\gamma_n}, J_{\gamma_n}$ be the intervals provided by Lemma 5.2. We choose $\varepsilon > 0$ small so that the 2ε-neighborhood of both ξ_1 and ξ_2 in $S^1(L)$ is contained in U. Suppose that, that there are arbitrarily large n so that neither I_{γ_n} or J_{γ_n} is contained in U. If not then up to subsequence and cutting subintervals of U of size ϵ on both sides, we may assume that both I_{γ_n} and J_{γ_n} are disjoint from U. In particular $\rho(\gamma_n)(U)$ is disjoint from U. But we can take a geodesic c_n with both endpoints arbitrarily close to ξ_1 (and so contained in U), and c intersecting $T_1^n \cap L$. The image $\rho(\gamma_n)(c)$ is a uniform quasigeodesic intersecting T_2^n so will have one endpoint in U which is a contradiction.

Therefore up to a subsquence and replacing U by a slightly smaller open set, it follows that either I_{γ_n} or J_{γ_n} is contained in U. Up to taking γ_n^{-1} we can assume that $J_{\gamma_n} \subset U$.

If necessary choose ε smaller so that the distance in $S^1(L)$ from ξ to $\rho(\eta)(\xi)$ is bigger than 10ε.

Assume first that $\xi \notin I_{\gamma_n}$ for arbitrarily large n. In this case, one concludes since $\rho(\gamma_n)\xi \in J_{\gamma_n} \subset U$ as desired. If $\xi \in I_{\gamma_n}$ for all large n, then by the choice of ε it follows that $\rho(\eta)(\xi) \notin I_{\gamma_n}$ for large enough n. This implies that $\rho(\gamma_n\eta)(\xi) \in J_{\gamma_n} \subset U$ completing the proof of the proposition. □

We devote the rest of the section to the proof of transitivity in pairs of points. First, we show that we can find attractor/repeller configurations in any pair of open sets.

Lemma 5.4. *For every U, V open intervals in $S^1(L)$ there is $\gamma \in \pi_1(M)$ such that $\rho(\gamma)(S^1(L) \backslash U) \subset V$.*

Proof. Consider a sufficiently large compact interval $\mathcal{I} \subset \mathcal{L}_{\mathcal{F}}$ as in the proof of Proposition 5.3 so that the union of its leaves contains a neighborhood of size larger than the diameter of a fundamental domain around L.

As in the proof of Proposition 5.3, it is possible to construct a sequence $\gamma_n \in \pi_1(M)$ such that $|\gamma_n| \to \infty$ and such that the neighborhoods I_{γ_n} and J_{γ_n} verify (up to taking a subsequence) that $I_{\gamma_n} \to \xi_1$ and $J_{\gamma_n} \to \xi_2$ where it could be that $\xi_1 = \xi_2$. This is just taking very large elements that move a fundamental domain intersecting L into other fundamental domain intersecting L and applying Lemma 5.2.

Now, using Proposition 5.3 we choose η_1 and η_2 in $\pi_1(M)$ so that $\eta_1(\xi_1) \in U$ and $\eta_2(\xi_2) \in V$. It follows that for sufficiently large n the deck transformation $\beta_n = \eta_2 \circ \gamma_n \circ \eta_1^{-1}$ verifies that $\rho(\beta_n)(S^1 \backslash U) \subset V$.

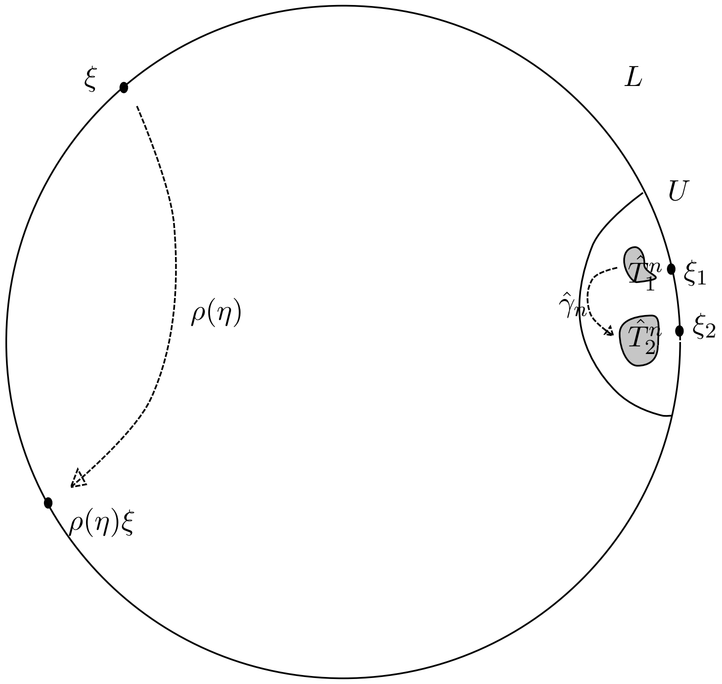

FIGURE 2. Depiction of the ingredients of the proof of Proposition 5.3. Here $\hat{\gamma}_n = \rho(\gamma_n)$ and $\hat{T}_i^n = T_i^n \cap L$.

To see this, notice that $\rho(\eta_1^{-1})(U)$ contains I_{γ_n} for suficiently large n because $\eta_1(\xi_1) \in U$. Similarly, if n is large enough, then $\eta_2(J_{\gamma_n})$ is contained in V. Since $\rho(\gamma_n)(S^1(L)\backslash I_{\gamma_n}) \subset J_{\gamma_n}$, this completes the proof. □

To complete the proof of Theorem 1.2 it is enough to show:

Proposition 5.5. *Given open intervals $U_1, V_1 \subset S^1(L)$ and $U_2, V_2 \subset S^1(L)$ there exists $\gamma \in \pi_1(M)$ such that $\rho(\gamma)U_1 \cap U_2 \neq \varnothing$ and $\rho(\gamma)V_1 \cap V_2 \neq \varnothing$. In particular, there exists a pair $\xi_1 \neq \xi_2 \in S^1_{univ}$ whose $\pi_1(M)$-orbit is dense in $S^1_{univ} \times S^1_{univ}\backslash\{diagonal\}$.*

Proof. By reducing the intervals we can assume without loss of generality that the four intervals U_1, U_2, V_1, V_2 are disjoint.

Apply Lemma 5.4 to find deck transformations γ and η which verify that $\rho(\gamma)(S^1(L)\backslash U_1) \subset V_2$ and $\rho(\eta)(S^1(L)\backslash \rho(\gamma)V_1) \subset U_2$.

Now, the transformation $\eta\gamma$ is the desired one. Indeed,

$$\rho(\gamma)U_1 \cap \rho(\gamma)V_1 = \varnothing, \quad \text{or} \quad \rho(\gamma)U_1 \subset S^1(L)\backslash\rho(\gamma)V_1,$$

which implies that $\rho(\eta\gamma)U_1 \subset U_2$. In addition

$$V_2 \subset \rho(\eta\gamma)V_1, \quad \text{because} \quad \rho(\eta\gamma)V_1 = \rho(\eta)\rho(\gamma)V_1 \supset S^1(L)\backslash U_2 \supset V_2.$$

The existence of dense orbits is standard. Indeed, pick a countable basis $\{U_n\}$ of intervals generating the topology of $S^1(L)$. The set $A_{n,m}$ of pairs of different points ξ_1, ξ_2 such that there exists $\gamma \in \pi_1(M)$ such that $\rho(\gamma)\xi_1 \in U_n$

and $\rho(\gamma)\xi_2 \in U_m$ is clearly open and it is dense because of what we just proved. Then, the intersection $\bigcap_{n,m} A_{n,m}$ is a residual subset by Baire's category theorem and the orbit of points in $A_{n,m}$ is always dense in $S^1_{univ} \times S^1_{univ}$. $\square$

6. Branching foliations

In this section we just point out that all our results work in the setting of branching foliations as they appear in the study of partially hyperbolic dynamics. These objects were introduced by Burago-Ivanov [BI]. We give here a definition that excludes a priori the existence of Reeb component like objects.

A *branching foliation* $\mathcal{F}_{bran}$ in a 3-manifold M is a collection of immersed surfaces (tangent to a continuous distribution) called leaves with the following properties. If $\widetilde{\mathcal{F}}_{bran}$ is the lift of the collection to $\widetilde{M}$ then:

- Each leaf L of $\widetilde{\mathcal{F}}_{bran}$ is a properly embedded plane in $\widetilde{M}$ and separates $\widetilde{M}$ in two open regions $L^{\oplus}$ and $L^{\ominus}$. Denote $L^+ = L \cup L^{\oplus}$ and $L^- = L \cup L^{\ominus}$.
- Every point in $\widetilde{M}$ belongs to at least one leaf $L \in \widetilde{\mathcal{F}}_{bran}$.
- The leaves do not topologically cross. That is, given two leaves L and F of $\widetilde{\mathcal{F}}_{bran}$ we have that $F \subset L^+$ or $F \subset L^-$.
- Given a sequence of points $x_n \to x \in \widetilde{M}$ and leaves L_n with $x_n \in L_n$ it follows that through x there is a leaf $L \in \widetilde{\mathcal{F}}_{bran}$ which is the uniform limit in compact parts of L_n.

In [BFFP, §10] a careful study of the properties of these objects is performed, including a study of the *leaf space* associated to such a branching foliation. In particular, it makes perfect sense to talk about uniform branching foliations and $\mathbb{R}$-covered ones. Moreover, in the partially hyperbolic setting there exists foliations in M that approach the center stable and center unstable branching foliations. In this setting this can be used to have in general situations a metric in M which gives curvature arbitrarily close to -1 to all leaves of $\mathcal{F}$. In this setting, one can define a universal circle as one does for general foliations.

All arguments performed in this note thus hold for branching foliations. We state the result in this context for future use.

Theorem 6.1. *Let $\mathcal{F}$ be a uniform branching foliation. Then, it is $\mathbb{R}$-covered. Moreover, if M admits a metric making every leaf negatively curved, then the action of $\pi_1(M)$ is minimal in the universal circle S^1_{univ} and moreover it acts transitively in pairs of points of S^1_{univ}.*

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