Differential Forms on Modular Curves $\mathbb{H}/\Gamma(k)$

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In this paper we continue the investigation of the geometric properties of modular curves using theta functions with rational characteristics started in [FK2] and [FKK]. In those papers, theta functions with rational characteristics were used to construct $SL_2(Z_k)$ equivariant mapping $\mathbb{H}/\Gamma(k) \to \mathbb{CP}^{\frac{k-3}{2}}$. Moreover, quotients of modular curves were also included in [BK]. Recently, Farkas and Kra used the theory developed in the cited papers to give new proofs of Ramanujan's congruences and discover some new ones (see [FK3] for details). In the present paper, we construct differentials on holomorphic curves $\mathbb{H}^2/\Gamma(k)$ for various k using the functions from [FKK]. We use these to obtain partial information about gap sequences that we believe wasn't known before. In some cases we will also construct half-canonical classes (i.e., forms of weight 1 that correspond to half-canonical class). The structure of the paper is as follows. In the first section, we will briefly review the theory from [FKK] and we will prove an explicit transformation formula for the relevant functions. Then we will give a general way of constructing differential forms using this formula. In the second paragraph we will look at particular examples of k and will show what kind of information one can get about modular curves using the theorem in §1.

§1. Preliminaries

We assume that the reader is familiar with the basic notion and structure of modular curves. Here we will review the basics of theta functions and main results of [FKK].

Definition. Given $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in \mathbb{R}^2$, $\tau \in \mathbb{H}$, $z \in \mathbb{C}$ we define

$$\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp 2\pi i\left\{\frac{1}{2}\left(n+\frac{\varepsilon}{2}\right)^2\tau + \left(n+\frac{\varepsilon}{2}\right)\left(z+\frac{\varepsilon'}{2}\right)\right\}.$$

The series are uniformly and absolutely convergent on compact subsets of $\mathbb{C}\times\mathbb{H}$.

The main property we need is the transformation formula for theta functions.

Transformation Formula. (For the proof, see [FK1].) For any $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in \mathbb{R}^2$ and any element $\gamma = \left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]$ of $SL_2(\mathbb{Z})$ we have

$$\frac{\exp \pi i\left\{\frac{-cz^2}{c\tau+d}\right\}\Theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)}{\Theta\left[\begin{smallmatrix}a\varepsilon+c\varepsilon'-ac\\ b\varepsilon+d\varepsilon'-bd\end{smallmatrix}\right](z,\tau)} = K\left(\left[\begin{matrix}\varepsilon\\ \varepsilon'\end{matrix}\right],\gamma\right)(c\tau+d)^{\frac{1}{2}}. \qquad (*)$$

$K(\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right],\gamma)$ is a proportionality factor depending on $\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right]$ and γ. (We will write it explicitly for our special case later.)

When $\varepsilon = \frac{m}{k}$ and $\varepsilon' = \frac{m'}{k'}$, k is an odd prime and m, m' are odd integers. [FK1] showed that $\Theta\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right](z,\tau)$ is invariant under the action of γ provided $\gamma \in \Gamma(k)$ (i.e., $\gamma \equiv I \bmod k$). This observation leads to the construction of modular form in [FK2]. In [FKK] the functions of the form $\Theta\left[\begin{smallmatrix}\frac{m}{k}\\ 1\end{smallmatrix}\right](0,k\tau)$ were considered , k is any odd number and m is an odd number and $1 \le m \le k-1$. The following theorem was shown in [FKK].

Theorem 2. *The mapping*

$$\tau \to \left(\Theta\left[\begin{matrix}\frac{1}{k}\\ 1\end{matrix}\right](0,k\tau),\Theta\left[\begin{matrix}\frac{3}{k}\\ 1\end{matrix}\right](0,k\tau)\cdots\Theta\left[\begin{matrix}\frac{k-2}{k}\\ 1\end{matrix}\right](0,k\tau)\right)$$

from $\mathbb{H} \to \mathbb{C}^{\frac{k-1}{2}}$ *induces a mapping* $\mathbb{H}/\Gamma(k) \to \mathbb{CP}^{\frac{k-3}{2}}$ *that is* $SL_2(\mathbb{Z})/\Gamma(k) = PSL_2(Z_k)$ *equivariant.*

The proof was based on the following "cheap trick".

$$\Theta\left[\begin{matrix}\frac{m}{k}\\ 1\end{matrix}\right]\left(0,k\cdot\frac{a\tau+b}{c\tau+d}\right) = \Theta\left[\begin{matrix}\frac{m}{k}\\ 1\end{matrix}\right]\left(0,\frac{ak\tau+kb}{\frac{c}{k}k\tau+d}\right) = \Theta\left[\begin{matrix}\frac{m}{k}\\ 1\end{matrix}\right]\left(0,\begin{pmatrix}a & b\\ c & d\end{pmatrix}\circ k\tau\right).$$

Then the formula (*) was used to show that

$$\frac{\Theta\left[\begin{smallmatrix}\frac{m}{k}\\ 1\end{smallmatrix}\right](0,k\tau)}{\Theta\left[\begin{smallmatrix}\frac{m'}{k}\\ 1\end{smallmatrix}\right](0,k\tau)}$$

is a function on the surface $\mathbb{H}/\Gamma(k)$ and we were done. In order to use $\Theta\left[\begin{smallmatrix}\frac{m}{k}\\ 1\end{smallmatrix}\right](0,k\tau)$ to construct holomorphic differentials on the surface $\mathbb{H}/\Gamma(k)$ we must have more precise information about the factor of transformation; i.e., we must give an explicit formula for $u_k[\gamma]$, $u_k[\gamma]$ satisfies

$$\Theta\left[\begin{matrix}\frac{m}{k}\\ 1\end{matrix}\right](0,k\gamma\tau) = u_k[\gamma](c\tau+d)^{\frac{1}{2}}\Theta\left[\begin{matrix}\frac{m}{k}\\ 1\end{matrix}\right](0,k\tau)$$

for $\gamma \in \Gamma(k)$. We use the $\eta(\tau)$ that is invariant under the entire group $SL_2(\mathbb{Z})$. (In our notation $\Theta[{\frac{1}{3} \atop 1}](0,3\tau)$ see [FKK] for more details.)

This formula is written and proved in [KQ] and we will bring it here for the convenience of the reader.

Let $v_\eta(\gamma)$ denote the multiplier system for the $\eta(\tau)$; i.e., $\eta(\gamma\tau) = v_\eta(\gamma)\cdot(c\tau+d)^{\frac{1}{2}}\eta(\tau)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, v_η is given by the following formula.

$$v_\eta(\gamma) = \begin{cases} \left(\frac{d}{c}\right)^* \exp\left\{\frac{\pi i}{12}[(a+d)c - bd(c^2-1) - 3c]\right\}, & c \text{ odd} \\ \left(\frac{c}{d}\right)^* \exp\left\{\frac{\pi i}{12}[(a+d)c - bd(c^2-1) + 3d - 3 - 3cd]\right\}, & c \text{ even} \end{cases}$$

and $(-)^*$ and $(-)_*$ are defined as follows.

$$\left(\frac{c}{d}\right)^* = \left(\frac{c}{|d|}\right) \quad \text{and} \left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right)(-1)^{\frac{\text{sign } c-1}{2}\cdot\frac{\text{sign } d-1}{2}}$$

when $(c,d)=1$ and d is odd and $(-)$ is the usual Jacobi symbol. Also the definition is completed by putting $(\frac{0}{\pm 1})^* = 1$ and $(\frac{0}{1})_* = 1$ and $(\frac{0}{-1})_* = 1$. This formula is proved in [Kn] pp. 51–62 and we will not repeat the proof here.

As a corollary for this theorem we state the transformation formula for the function $\Theta[{\frac{m}{k} \atop 1}](0,k\tau)$ and $m = 2\ell - 1$.

Theorem 3. *Let $u_k[\gamma]$ represent the factor of proportionality in the formula for $\Theta[{\frac{2\ell-1}{k} \atop 1}](0,k\tau)$; i.e., $\Theta[{\frac{2\ell-1}{k} \atop 1}](0,k\gamma(\tau)) = u_k[\gamma](c\tau+d)^{\frac{1}{2}}\Theta[{\frac{m}{k} \atop 1}](0,k\tau)$. Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma \equiv I \bmod k$.*

$$u_k[\gamma] = \begin{cases} \left(\frac{d}{\frac{c}{k}}\right)^* \exp\left\{\frac{\pi i}{4}\left[(a+d)\frac{c}{k} - kbd\left(\frac{c^2}{k^2}-1\right) - \frac{3c}{k}\right]\right\}, & c \text{ odd} \\ \left(\frac{\frac{c}{k}}{d}\right)^* \exp\left\{\frac{\pi i}{4}\left[(a+d)\frac{c}{k} - kbd\left(\frac{c^2}{k^2}-1\right) - 3d - 3 - \frac{3c}{k}d\right]\right\}, & c \text{ even} \end{cases}$$

Proof. The proof consists of the following steps.

Step 1. We can look at the functions $\Theta[{\frac{2\ell-1}{k} \atop 1}](kz,k\tau)$, $\ell = 0,1,\ldots,k$. From the Fourier expansion is easily seen that these functions are linearly independent. See [KO] for details. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma \in \Gamma(k)$ then we can write:

$$\Theta\begin{bmatrix} \frac{2\ell-1}{k} \\ 1 \end{bmatrix}\left(k\frac{z}{c\tau+d}, k\frac{a\tau+b}{c\tau+d}\right) = \Theta\begin{bmatrix} \frac{2\ell-1}{k} \\ 1 \end{bmatrix}\left(\frac{kz}{\frac{c}{k}k\tau+d}, \frac{ak\tau+kb}{\frac{c}{k}k\tau+d}\right)$$

or else

$$\Theta\begin{bmatrix}\frac{2\ell-1}{k}\\ 1\end{bmatrix}\left((kz)\circ\begin{pmatrix} a & kb\\ \frac{c}{k} & d\end{pmatrix},\begin{pmatrix} a & kb\\ \frac{c}{k} & d\end{pmatrix}\circ(k\tau)\right).$$

However, by the theory developed in [FK2] we can rewrite the last expression as

$$K\left(\begin{bmatrix}\frac{2\ell-1}{k}\\ 1\end{bmatrix},\gamma_k\right)(c\tau+d)^{\frac{1}{2}}\exp\pi i\left(-\frac{kcz^2}{c\tau+d}\right)\Theta\begin{bmatrix}\frac{2\ell-1}{1}\\ 1\end{bmatrix}(kz,k\tau)$$

$\gamma_k = (\begin{smallmatrix} a & kb\\ \frac{c}{k} & d\end{smallmatrix})$ and $K([\begin{smallmatrix}\frac{2\ell-1}{k}\\ 1\end{smallmatrix}],\gamma_k)$ is the proportionality factor from the beginning. We see that $u_k[\gamma]$ is equal to $K([\begin{smallmatrix}\frac{2\ell-1}{k}\\ 1\end{smallmatrix}],\gamma_k)$ $1\le \ell\le \frac{k-1}{2}$ (set $z=0$ in the last formulas). We conclude that

$$K\left(\begin{bmatrix}\frac{2\ell-1}{k}\\ 1\end{bmatrix},\gamma_k\right)=u_k[\gamma]$$

also for $\ell=1,\ldots,2k$. Thus we can write

$$\Theta\begin{bmatrix}\frac{2\ell-1}{k}\\ 1\end{bmatrix}(0,k\gamma(\tau))=K\left(\begin{bmatrix}1\\ 1\end{bmatrix},\gamma_k\right)(c\tau+d)^{\frac{1}{2}}\Theta\begin{bmatrix}\frac{2\ell-1}{k}\\ 1\end{bmatrix}(0,k\tau).$$

Therefore all is left to compute is $K([\begin{smallmatrix}1\\ 1\end{smallmatrix}],\gamma_k)$. We now use the fact that

$$c\eta^3(\tau)=\Theta'[\begin{smallmatrix}1\\ 1\end{smallmatrix}](0,\tau).$$

Thus it is easily seen (derive (*) and plug $z=0$ in it) that $K([\begin{smallmatrix}1\\ 1\end{smallmatrix}],\gamma_k)=v_\eta(\gamma_k)^3$. From which the result is immediate (using the formula for $v_\eta(\gamma)$). The main benefit of the explicit formula is that it gives us a tool to construct holomorphic differentials on $\mathbb{H}/\Gamma(k)$. More precisely we will have the following theorem:

Theorem 4. *Let $k=pq$ (p,q are odd primes) and suppose the following condition is satisfied. We can find r,m s.t. $r+m=4$ and $mq+r=0 \bmod 8$. Then*

$$\prod_{i\equiv 1}^{m}\Theta\begin{bmatrix}\frac{2\ell_i-1}{p}\\ 1\end{bmatrix}(0,p\tau)\prod_{j\equiv 1}^{r}\Theta\begin{bmatrix}\frac{2m_j-1}{k}\\ 1\end{bmatrix}(0,k\tau)\qquad (**)$$

are holomorphic differentials on $\mathbb{H}/\Gamma(k)$.

Proof. First we show that the character is trivial. When we are looking for $\gamma\in\Gamma(k)$ we remember that $\Gamma(k)\subset\Gamma(p)$. Applying Theorem 3 we get the result by straightforward calculation (left to the reader). Using the condition on r,m we now check that the series are indeed define holomorphic differentials. Since $\prod_{i\equiv 1}^{m}\Theta[\begin{smallmatrix}\frac{2\ell_i-1}{p}\\ 1\end{smallmatrix}](0,p\tau)\prod_{j\equiv 1}^{r}\Theta[\begin{smallmatrix}\frac{2m_j-1}{k}\\ 1\end{smallmatrix}](0,k\tau)$ are nowhere vanishing inside $\mathbb{H}$, we just need to verify the assertion at the cusps. But

$$Span\langle\prod_{i\equiv 1}^{m}\Theta[\begin{matrix}\frac{2\ell_i-1}{p}\\ 1\end{matrix}](0,p\tau)\prod_{j\equiv 1}^{r}\Theta[\begin{matrix}\frac{2m_j-1}{k}\\ 1\end{matrix}](0,k\tau)\rangle$$

is $SL_2(\mathbb{Z})$ invariant from the theory developed in [FKK] so we can consider the order of these series at ∞ (Since the group $SL_2(\mathbb{Z})$ is transitive on the cusps. Computing the orders of the series at ∞ we are using coordinate $e^{\frac{2\pi i\tau}{pq}}$ thus $\mathrm{ord}_\infty\Theta[\begin{smallmatrix}\frac{2\ell_i-1}{p}\\ 1\end{smallmatrix}](0,p\tau)=\frac{(2\ell_i-1)^2q}{8}$ and $\mathrm{ord}_\infty\Theta[\begin{smallmatrix}\frac{2m_j-1}{k}\\ 1\end{smallmatrix}]=\frac{(2m_j-1)^2}{8}$. Thus the order of the series evaluated at ∞ is

$$\sum_{i=1}^{m}\frac{(2\ell_i-1)^2}{8}q+\sum_{j=1}^{r}\frac{(2m_j-1)^2}{8}.$$

But

$$\sum_{i=1}^{m}(2\ell_i-1)^2q+\sum_{j=1}^{r}(2m_j-1)^2\equiv mq+r \bmod 8$$

since $mq+r\equiv 0 \bmod 8$ we have holomorphic differentials on $\mathbb{H}^2/\Gamma(k)$.

In the next paragraph we are going to look at specific cases in order to see what kind of information the series above can provide is within the surface $\mathbb{H}^2/\Gamma(k)$.

§2. Examples

Let us look at the most elementary case where we can apply our theta series namely we look at the case where $k=7 \bmod 8$, k prime. In this case we conclude that

$$\Theta'[\begin{matrix}1\\ 1\end{matrix}](0,\tau)\Theta[\begin{matrix}\frac{2\ell-1}{k}\\ 1\end{matrix}](0,k\tau)$$

are differentials on $\mathbb{H}^2/\Gamma(k)$. (Although the case isn't covered by the main theorem from §1. The proof is the same.) The mapping $\mathbb{H}/\Gamma(k)\to\mathbb{CP}^{\frac{k-3}{2}}$ which is induced by

$$\tau\mapsto\left(\Theta'\begin{bmatrix}1\\ 1\end{bmatrix}(0,\tau)\Theta\begin{bmatrix}\frac{1}{k}\\ 1\end{bmatrix}(0,k\tau),\ldots,\Theta'\begin{bmatrix}1\\ 1\end{bmatrix}(0,\tau)\Theta\begin{bmatrix}\frac{k-2}{k}\\ 1\end{bmatrix}(0,k\tau)\right)$$

is projectively the same mapping defined in [FKK]. Since $SL_2(Z_k)$ acts on

$$Span\left(\Theta\begin{bmatrix}\frac{1}{k}\\ 1\end{bmatrix}(0,k\tau,\ldots,\Theta\begin{bmatrix}\frac{k-2}{k}\\ 1\end{bmatrix}(0,k\tau)\right).$$

We immediately obtain that we have an $SL_2(Z_k)$ equivariant embedding of subspace of the space of differentials into $\mathbb{CP}^{\frac{k-3}{2}}$. Also $SL_2(Z_k)$ clearly acts linearly on

$$\Theta'[{1 \atop 1}](0,\tau)\Theta[{\frac{2\ell-1}{k} \atop 1}](0,k\tau).$$

We thus summarize in the following.

Proposition 2.1. *For $k = 7 \bmod 8$ the mapping defined in [FKK] can be regarded as a part of the canonical embedding of the curve $\mathbb{H}^2/\Gamma(k)$. Further $SL_2(Z_k)$ acts linearly on the space $\Theta'[{1 \atop 1}](0,\tau)\Theta[{\frac{2\ell-1}{k} \atop 1}](0,k\tau)$, thus we obtain a $\frac{k-1}{2}$ irreducible representation of $SL_2(Z_k)$ as a subspace of the space of the diffrentials.*

We can also produce an explicit canonical divisor for this curve: note that for each characteristic $[{\frac{2\ell-1}{k} \atop 1}]$ we can associate a cusp (see [FK2]). Choose the cusp ∞ to be associated with $[{\frac{1}{k} \atop 1}]$ and then sending the characteristic $[{\frac{2\ell-1}{k} \atop 1}]$ to the cusp represented by the rational number $\frac{2\ell-1}{k}$. Now if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(k)$ s.t. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\infty = \frac{2\ell-1}{k}$ then the action on the function $\Theta[{\frac{2\ell-1}{k} \atop 1}](0,k\tau)$ of such matrix will take $\Theta[{\frac{1}{k} \atop 1}](0,k\tau)$ to $\Theta[{\frac{2\ell-1}{k} \atop 1}](0,k\tau)$. Thus we conclude $\mathrm{ord}_{k\frac{2\ell-1}{k}}\Theta[{\frac{1}{k} \atop 1}](0,k\tau) = \frac{(2\ell-1)^2}{8}$ by [FKK]. Since the order of $\Theta'[{\frac{1}{k} \atop 1}](0,k\tau)$ is $\frac{k}{8}$ at all the cusps we conclude that

$$\mathrm{ord}_\infty\Theta\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)\Theta\begin{bmatrix}\frac{2\ell-1}{k}\\1\end{bmatrix}(0,k\tau) = \frac{k+(2\ell-1)^2-8}{8}.$$

The same calculation will give that in the other cusps the divisor is $\frac{k+1-8}{8}$. We can now state the following proposition.

Proposition 2.3. *The divisor*

$$\sum_{\ell\neq 1} P_{\frac{2\ell-1}{k}}^{\frac{(2\ell-1)^2+k-8}{8}} \sum P_i^{\frac{k+1-8}{8}}$$

is a canonical divisor. (P_i are the cusps that aren't equal to the cusps corresponding to $\frac{2\ell-1}{k}$.)

Example. Suppose that $k = 7$ then we recover the fact that $P_{\frac{5}{7}}^3 P_{\frac{3}{7}}$ is a divisor of a differential on $\mathbb{H}/\Gamma(7)$.

Remark. It is verified that degree of the divisor in the proposition is precisely $2g-2$ where g is the genus of $\mathbb{H}/\Gamma(k)$.

We can construct also a differential for the curve $X_0(k^2)$ using the following observation. Let

$$\Gamma(k,k)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\,\middle|\, b\equiv c\equiv 0 \bmod k\right\}.$$

We will have a natural mapping $\Gamma_0(k^2)$ into $\Gamma(k,k)$ given by $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto\begin{pmatrix} a & kb\\ \frac{c}{k} & d\end{pmatrix}$ where $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(k^2)$ and this we have a natural isomorphism $\mathbb{H}/\Gamma(k,k)\cong\mathbb{H}/\Gamma_0(k^2)$. But $\Gamma(k,k)/\Gamma(k)\simeq\mathbb{Z}_{\frac{k-1}{2}}$. (A cyclic Abelian group with $\frac{k-1}{2}$ elements.) Moreover $\Gamma(k,k)$ acts on the functions $\Theta'[{}^{1}_{1}](0,\tau)\Theta[{}^{\frac{2\ell-1}{k}}_{1}](0,k\tau)$ by a cyclic permutation of order $\frac{k-1}{2}$. (This follows from the fact that $\Theta'[{}^{1}_{1}](0,\tau)$ is invariant under $SL_2(\mathbb{Z})$ and the general theory of the action of $SL_2(Z_k)$ on $\Theta[{}^{\frac{2\ell-1}{k}}_{1}](0,k\tau)$ developed in [FKK].) Thus there exists $c_i\in\mathbb{C}$ s.t. $\sum c_\ell\Theta'[{}^{1}_{1}](0,\tau)\Theta[{}^{\frac{2\ell-1}{k}}_{1}](0,k\tau)$ is an eigenfunction of eigenvalue 1 for $\Gamma(k,k)$. It thus will be clearly a differential for $\Gamma(k,k)$ and thus for $\Gamma_0(k^2)$. Set $X_0(k,k)=\mathbb{H}^0/\Gamma(k,k)$.

Example. The curve $X_0(7,7)=\mathbb{H}/\Gamma(7,7)$ is of genus 1. The linear combination in this case $\frac{1}{3}(\Theta'[{}^{1}_{1}](\Theta[{}^{\frac{1}{7}}_{1}](0,7\tau)+\Theta[{}^{\frac{3}{7}}_{1}](0,7\tau)+\Theta[{}^{\frac{5}{7}}_{1}](0,7\tau))$ is going to give the unique non vanishing differential on this curve. (Up to a constant)

Example 2.1. We consider now the next case when $k=p^2$ and $p\equiv 3 \bmod 8$, p odd prime number. Then the series $\Theta[{}^{\frac{2\ell_j-1}{p}}_{1}](0,p\tau)\Theta[{}^{\frac{2\ell_i-1}{p}}_{1}](0,p\tau)\Theta[{}^{\frac{2m_j-1}{p^2}}_{1}](0,p^2\tau)\Theta[{}^{\frac{2m_j-1}{p^2}}_{1}](0,p^2\tau)$ $1\le\ell_i\le p$ and $1\le m_j\le p^2$ are holomorphic differential forms for the group $\mathbb{H}/\Gamma(p^2)$. (We use the theorem from §1 where $q=p$.) The total number of such series is

$$\binom{\frac{p^2-1}{2}}{2}\binom{\frac{p-1}{2}}{2}+\binom{\frac{p^2-1}{2}}{2}\frac{p-1}{2}+\binom{\frac{p-1}{2}}{2}\frac{p^2-1}{2}+\frac{p^2-1}{2}\frac{p-1}{2}.$$

The genus of $X(p^2)$ is on the other hand given by

$$\frac{1}{24}p^4(p^2-1)-\frac{1}{4}p^2(p^2-1).$$

Therefore at least asymptotically the number of holomorphic differentials constructed by us is $C\cdot g$ where C is a certain constant.

Let us compute the order of

$$\Theta\begin{bmatrix}\frac{2\ell_i-1}{p}\\ 1\end{bmatrix}(0,p\tau)\Theta\begin{bmatrix}\frac{2\ell_j-1}{p}\\ 1\end{bmatrix}(0,p\tau)\Theta\begin{bmatrix}\frac{2m_h-1}{p^2}\\ 1\end{bmatrix}(0,p^2\tau)\Theta\begin{bmatrix}\frac{2m_n-1}{p^2}\\ 1\end{bmatrix}(0,p^2\tau)$$

at ∞. Thus

$$\mathrm{ord}_\infty\left(\Theta\begin{bmatrix}\frac{2\ell_i-1}{p}\\ 1\end{bmatrix}(0,p\tau)\Theta\begin{bmatrix}\frac{2\ell_j-1}{p}\\ 1\end{bmatrix}(0,p\tau)\Theta\begin{bmatrix}\frac{2m_h-1}{p^2}\\ 1\end{bmatrix}(0,p^2\tau)\Theta\begin{bmatrix}\frac{2m_n-1}{p^2}\\ 1\end{bmatrix}(0,p^2\tau)\right)$$

is:

$$\frac{(2\ell_i-1)^2p}{8}+\frac{(2\ell_j-1)^2p}{8}+\frac{(2m_n-1)^2}{8}+\frac{(2m_h-1)^2}{8}-1$$

$$=p\left[\frac{(2\ell_i-1)^2+(2\ell_j-1)^2}{8}\right]+\frac{(2m_h-1)^2+(2m_h-1)^2}{8}-1.$$

By [FK1, p. 81] we conclude that the numbers

$$p\left[\frac{(2\ell_i-1)^2+(2\ell_j-1)^2}{8}\right]+\frac{(2m_h-1)^2+(2m_h-1)^2}{8}$$

are "gaps" in the Weierstrass gap sequence for ∞, hence the following proposition.

Proposition 2.5. *There is no function of degree*

$$k\left[\frac{(2\ell_i-1)^2+(2\ell_j-1)^2}{8}\right]+\frac{(2m_h-1)^2+(2m_n-1)^2}{8}$$

that has the only pole only at ∞ on $X(p^2)$.

We can also look on forms of weight 1 for our curves $X_0(p^2)$. Thus the modular forms $\Theta[\begin{smallmatrix}\frac{2\ell_i-1}{p}\\ 1\end{smallmatrix}](0,p\tau)\Theta[\begin{smallmatrix}\frac{2\ell_j-1}{p^2}\\ 1\end{smallmatrix}](0,p^2\tau)$ is a weight 1 for the group $\Gamma(p^2)$ s.t. its square is a differential. We obtain the following lemma.

Lemma. *For $k\equiv 3 \bmod 8$ there exist a half canonical class κ invariant under the action of $SL_2(p^2)$ on the curve $\mathbb{H}/\Gamma(p^2)$. Furthermore $\dim H^0(\kappa)\geq\frac{p-1}{2}\frac{p^2-1}{2}$ so there exist a multidimensional theta function that vanishes at least to the order $\frac{p-1}{2}\frac{p^2-1}{2}$ on the curve.*

Proof. The statement about $SL_2(p^2)$ follows from the fact that

$$Span\left\langle\Theta\begin{bmatrix}\frac{2\ell_i-1}{p}\\ 1\end{bmatrix}(0,p\tau)\Theta\begin{bmatrix}\frac{2\ell_j-1}{p^2}\\ 1\end{bmatrix}(0,p^2\tau)\right\rangle$$

is $SL_2(p^2)$ invariant. The second statement follows from the fact that there are no square relations between theta functions. The last statement follows from Riemann's theta divisor theorem [FK3, p. 298].

This theorem also enables us to construct modular forms of weight 1 on some other curves. Let us look first at the curve $X_0(p^4)$. Then the curve $X_0(p^4) \cong X_0(p^2,p^2)$. because we can replace $\Gamma_0(p^4)$ by the subgroup $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \bmod p^2\}$. The group $\Gamma(p^2,p^2)/\Gamma(p^2)$ acts on the series

$$\Theta\left[\begin{array}{c}\frac{2\ell_i-1}{p}\\ 1\end{array}\right](0,p\tau)\Theta\left[\begin{array}{c}\frac{2m_h-1}{p^2}\\ 1\end{array}\right](0,p^2\tau).$$

On the first factor it acts through the fact that $\Gamma(p^2,p^2) \subset \Gamma(p,p)$ and on the second one we have an action with 2 orbits: the first comes from the characteristics $\Theta\left[\begin{array}{c}\frac{2m_h-1}{k^2}\\ 1\end{array}\right](0,p^2\tau)$ and $(2m_h-1,p)=1$ and the other orbit is $\Theta\left[\begin{array}{c}\frac{2\ell_i-1}{p}\\ 1\end{array}\right](0,p^2\tau)$. At any rate diagonalizing we will get modular forms of weight 1 for the curves $X_0(p^4)$.

Example. For $k=9$ it was shown [KQ] that $\eta^2\Theta\left[\begin{array}{c}\frac{2\ell-1}{9}\\ 1\end{array}\right](0,9\tau)\Theta\left[\begin{array}{c}\frac{2m-1}{9}\\ 1\end{array}\right](0,9\tau)$ is a basis for differentials forms for $\mathbb{H}/\Gamma(9)$. Then we will have that $\eta\Theta\left[\begin{array}{c}\frac{2\ell-1}{9}\\ 1\end{array}\right](0,9\tau)$ is a theta characteristic moreover by [KQ] $\dim H^0(\mathbb{H}/\Gamma(9),\kappa)=4$ so this is an even characteristic. (In fact the characteristic is naturally associated with the mapping $X(9)/\langle\tau\to\tau+3\rangle$ since the mapping $\varphi : X(9)\to X(9)/\tau\to\tau+3$ is a $3:1$ Galois map. Taking the $\sqrt{d\varphi}$ will give us a half canonical class that will be correspond to theta characteristicsκ.) The group $\Gamma(9,9)/\Gamma(9)$ is a cyclic group of order 3. It preserves the function $\Theta\left[\begin{array}{c}\frac{1}{3}\\ 1\end{array}\right](0,3\tau)$ and permutes the functions $\Theta\left[\begin{array}{c}\frac{1}{9}\\ 1\end{array}\right](0,9\tau)$, $\Theta\left[\begin{array}{c}\frac{5}{9}\\ 1\end{array}\right](0,9\tau)$ and $\Theta\left[\begin{array}{c}\frac{7}{9}\\ 1\end{array}\right](0,9\tau)$. Thus we have 2 modular forms we can construct on $X_0(9,9)$ of weight 1. a calculation yields that $\frac{1}{3}(\Theta\left[\begin{array}{c}\frac{1}{9}\\ 1\end{array}\right](0,9\tau)+\Theta\left[\begin{array}{c}\frac{5}{9}\\ 1\end{array}\right](0,9\tau)+\Theta\left[\begin{array}{c}\frac{7}{9}\\ 1\end{array}\right](0,9\tau))$ is theta characteristics for the curve of $X_0(81)\simeq X_0(9,9)$ that is a curve of genus 4. We see that $\dim H^0(X_0(81),\kappa')=2$ (κ' is a theta characteristics on $X_0(81)$).

Remark. It will be of interest to find the corresponding Galois representations for the half canonical class on $X_0(81)$.

Conclusion

In the first paragraph we constructed differential forms for $\mathbb{H}/\Gamma(pq)$ p,q are prime odd numbers. We gave two examples of how we can apply these objects to study differential forms on the curves above and also on the curves $X_0(k^r)$.

We hope that these series will open some new routes to obtain information on these curves that play so important role in number theory.

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