The number of Simply-connected Trivalent 2-dimensional Stratifolds

J. C. Gómez-Larrañaga* F. González-Acuña† Wolfgang Heil‡

Dedicated to Sergey Antonyan on the occasion of his 65th birthday

Abstract

We describe a method for counting the number of 1-connected trivalent 2-stratifolds with a given number of singular curves and 2-manifold components.

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1 Introduction

Observations in data analysis suggest that the points in a naturally-occurring dataset tend to cluster near a manifold with singularities. In particular, for dimension 2, these manifolds with singularities are 2-stratifolds and occur in the study of the energy landscape of cyclo-octane [7], with a systematic application of local topological methods described in [10], the study of boundary singularities produced by the motion of soap films [2], and in organizing data [1]. A systematic study of trivalent 2-stratifolds was begun in [3]. Whereas closed 2-manifolds are classified by their fundamental groups, this is far from true for 2-stratifolds. In fact, for any given 2-stratifold there are infinitely many others with the same fundamental group. The question arises whether one can effectively construct all of the 2-stratifolds that have a given fundamental group.

A 2-stratifold is essentially determined by its associated bi-colored labeled graph and a presentation for its fundamental group can be read off from the labeled graph. Thus the question arises when a labeled graph determines a simply connected 2-stratifold. In [3] an algorithm on the labeled graph was developed for determining whether the graph determines a simply connected 2-stratifold and in [4] we obtained a complete classification of all trivalent labeled graphs that represent simply connected 2-stratifolds. Then

*Centro de Investigación en Matemáticas, A.P. 402, Guanajuato 36000, Gto. México. jcarlos@cimat.mx

†Instituto de Matemáticas, UNAM, 62210 Cuernavaca, Morelos, México and Centro de Investigación en Matemáticas, A.P. 402, Guanajuato 36000, Gto. México. fico@math.unam.mx

‡Department of Mathematics, Florida State University, Tallahasee, FL 32306, USA. heil@math.fsu.edu

in [5] we developed three operations on labeled graphs that will construct recursively from a single vertex all trivalent graphs that represent 1-connected 2-stratifolds. A referee of that paper asked whether it is possible to compute the number of all such labeled graphs for a given number of vertices. The purpose of the present paper is to describe a method that leads to such computations. Our approach is based on the classification theorem in [4].

A different approach, based on the operations developed in [5] is used by M. Hernández-Ketchul and J. Rodriguez-Viorato [6], who wrote a Python program that is capable of computing and printing in linear time all the distinct trivalent graphs associated to 1-connected 2-stratifolds up to 11 white vertices.

In section 2 we recall the definitions of a 2-stratifold and its associated linear graph, providing the necessary details needed for the statement of the classification theorem for trivalent 1-connected graphs. In section 3 we describe the general method for constructing the graphs corresponding to 1-connected trivalent 2-stratifolds from generating trees and skeletons, which leads to a method for counting the number of these graphs in terms of the number of black vertices of degree 3 and the number of white vertices. In section 4 we use this approach to give explicit formulas for the case of 1 black vertex of degree 3. Finally in section 5 we give a specific example to show how to compute the number of all graphs with 7 white vertices corresponding to trivalent 1-connected 2-stratifolds.

2 2-stratifolds and 2-stratifold graphs.

A 2-*stratifold* is a compact, Hausdorff space X that contains a closed (possibly disconnected) 1-manifold $X^{(1)}$ as a closed subspace with the following property: Each point $x \in X^{(1)}$ has a neighborhood $U(x)$ homeomorphic to $CL\times\mathbb{R}$, where CL is the open cone on L for some finite set $L = \{p_1, \ldots, p_d\}$ of cardinality $d > 2$ and $X - X^{(1)}$ is a (possibly disconnected) 2-manifold. By identifying $U(x)$ with $CL\times\mathbb{R}$, we call $Cp_1\times\mathbb{R}, \ldots, Cp_d\times\mathbb{R}$ the *sheets* at x.

X can be obtained as a quotient space of a disjoint collection of circles $X^{(1)}$ and a disjoint collection W of compact 2-manifolds by attaching W to $X^{(1)}$ under the attaching map ψ, where $\psi : \partial W \to X^{(1)}$ is a covering map such that $|\psi^{-1}(x)| > 2$ for every $x \in X^{(1)}$ as in figure 1. With suitable orientations, for a component C of ∂W the covering map $\psi_{|C} : C \to B \subset X^{(1)}$ is of the form $\psi(z) = z^r$, for some $r > 0$.

We associate to a given 2- stratifold $(X, X^{(1)})$ an associated bi-colored labeled graph $\Gamma = \Gamma(X, X^{(1)})$ as follows:

For each component B of $X^{(1)}$ choose a black vertex b, for each component W_i of W choose a white vertex w_i, for each component C of ∂W choose an edge c. Connect w_i to b by the edge c if $\psi(C) \subset B$.

We label the white vertices of the graph Γ by assigning to w the genus g of W (here we use Neumann's [8] convention of assigning negative genus g to nonorientable surfaces). We label an edge c by r, where r is the degree of the covering map $\psi_{|C} : C \to B$.

We say that a white vertex w has genus 0, instead of saying that the component W corresponding to w has genus 0. To simplify our figures of graphs Γ, if there is no label displayed on a white vertex w, it is understood that the label is 0.

Thus every 2-stratifold X determines uniquely a bi-colored labeled graph. Conversely, a given bi-colored labeled tree Γ determines uniquely a 2-stratifold X.

The association of the graph Γ_X to the stratifold X_Γ transforms geometrical and algebraic properties of X_Γ into combinatorial properties of the bi-colored graph.

Notation. If Γ is a bi-colored labeled graph corresponding to the 2-stratifold X we let $X_\Gamma = X$ and $\Gamma_X = \Gamma$. An example is given in Figure 1.

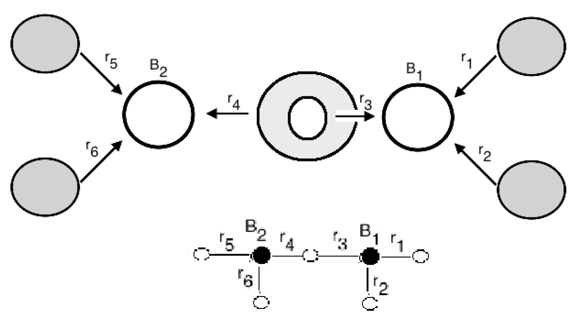

Figure 1: X_Γ and Γ_X

The fundamental group $\pi_1(X_\Gamma)$ can be computed from the bicolored graph Γ_X (see [5]). In particular, if Γ_X is a tree and all white vertices of Γ_X have genus 0 (i.e. correspond to punctured 2-spheres of X_Γ), then a presentation of $\pi_1(X_\Gamma)$ is obtained as follows:
Each black vertex b of Γ_X contributes a generator, also denoted by b, of $\pi_1(X)$.
Each white vertex w incident to edges $c_1, \ldots, c_p$ yields generators, also denoted by $c_1, \ldots, c_p$ and a relation $c_1 \cdots c_p = 1$.
Each edge c_i of Γ_X between w and b with label $m \geq 1$ yiels a relation $b^m = c_i$.

The 2-stratifold X is called *trivalent* if every point $x \in X^{(1)}$ has a neighborhood consisting of three sheets. We do not call a 2-manifold (i.e when $X^{(1)} = \emptyset$) trivalent. In terms of the associated graph $\Gamma = \Gamma_X$ this means that every black vertex is incident to either one edge of label 3, or two edges one of label 1 and one of label 2, or three edges, each of label 1.

In [4] we obtained a classification theorem of simply connected trivalent 2-stratifolds. We first review the terms used in this theorem.

(1) A $(2,1)$-*collapsible tree* is a bi-colored tree constructed as follows:
Start with a rooted tree T (which may consist of only one vertex) with root r (a vertex of T), color with white and label 0 the vertices of T, take the

barycentric subdivision $sd(T)$ of T, color with black the new vertices (the barycenters of the edges of T) and finally label an edge e of $sd(T)$ with 2 (resp. 1) if the distance from e to the root r is even (resp. odd). (We allow a one-vertex tree (with white vertex) as a $(2,1)$- collapsible tree).

(2) The *reduced subgraph* $R(\Gamma)$ is defined for a bi-colored labeled tree Γ for which the components of $\Gamma - st(\mathbb{B})$ are $(2,1)$-collapsible trees. Here $\mathbb{B}$ denotes the union of all the black vertices of degree 3 of Γ and $st(\mathbb{B})$ denotes the open star of $\mathbb{B}$ in Γ. The reduced subgraph $R(\Gamma)$ is the graph obtained from $St(\mathbb{B})$ (the closed star of $\mathbb{B}$) by attaching to each white vertex w of $St(\mathbb{B})$ that is not a root, a $b12$-tree as in Figure 2, such that the terminal edge has label 2.

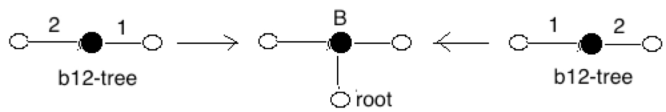

Figure 2: Attaching $b12$ trees

(3) A *horned tree* is a bi-colored tree constructed as follows:
Start with a tree T that has at least two edges and all of whose nonterminal vertices have degree 3. Color a vertex of T white (resp. black) if it has degree 1 (resp. 3). Trisect the terminal edges of T and bisect the nonterminal edges, obtaining the graph H_T. Color the additional vertices v so that H_T is bi-colored, that is, v is colored black if v is a neighbor of a terminal vertex of H_T and white otherwise. Then label the edges such that every terminal edge has label 2, every nonterminal edge has label 1.

We can now state the classification theorem of [4]:

Theorem 1. *Let X_Γ be a trivalent 2-stratifold with associated graph Γ_X. Let $\mathbb{B}$ denote the union of all the black vertices of degree 3 of Γ and $st(\mathbb{B})$ denote the open star of $\mathbb{B}$ in Γ.*
Then X_Γ is simply connected if and only if Γ_X is a tree with all white vertices of genus 0 and all terminal vertices white. such that the components of $\Gamma - st(\mathbb{B})$ are $(2,1)$-collapsible trees and the reduced graph $R(\Gamma)$ contains no horned tree.

3 Skeletons

Let X_Γ be a 2-stratifold whose associated graph Γ_X has n white vertices and b black vertices of degree 3. We say that Γ_X is trivalent 1-connected if X_Γ is trivalent 1-connected.

We count the number of trivalent 1-connected graphs Γ_X for a given number n of white vertices by first counting those for a given number b of black vertices of degree 3. For such given b, the possible Γ_X are obtained from the "skeleton graphs" (defined below) that correspond to the reduced subgraphs in Theorem 1.

Generating trees. For a given $b \geq 0$, a generating tree is an unlabeled tree with exactly b black vertices and all white vertices (if any) of degree ≥ 3.

Skeletons. To a generating tree T we assign a skeleton T_S as follows: Subdivide each edge that is incident to two black vertices and color the new vertices white. Attach edges to each black vertex such that in the resulting tree T_S each black vertex has degree 3 and all terminal vertices are white. To the white vertices $w_1, \ldots, w_k$ of T_S assign labels $T(a_1), \ldots, T(a_k)$, where a_i is an integer ≥ 1 $(1 \leq i \leq k)$
.

Figure 3 (resp. Figure 4) shows all generating trees and their skeletons for $b = 0, 1, 2, 3$ (resp. $b = 4$).

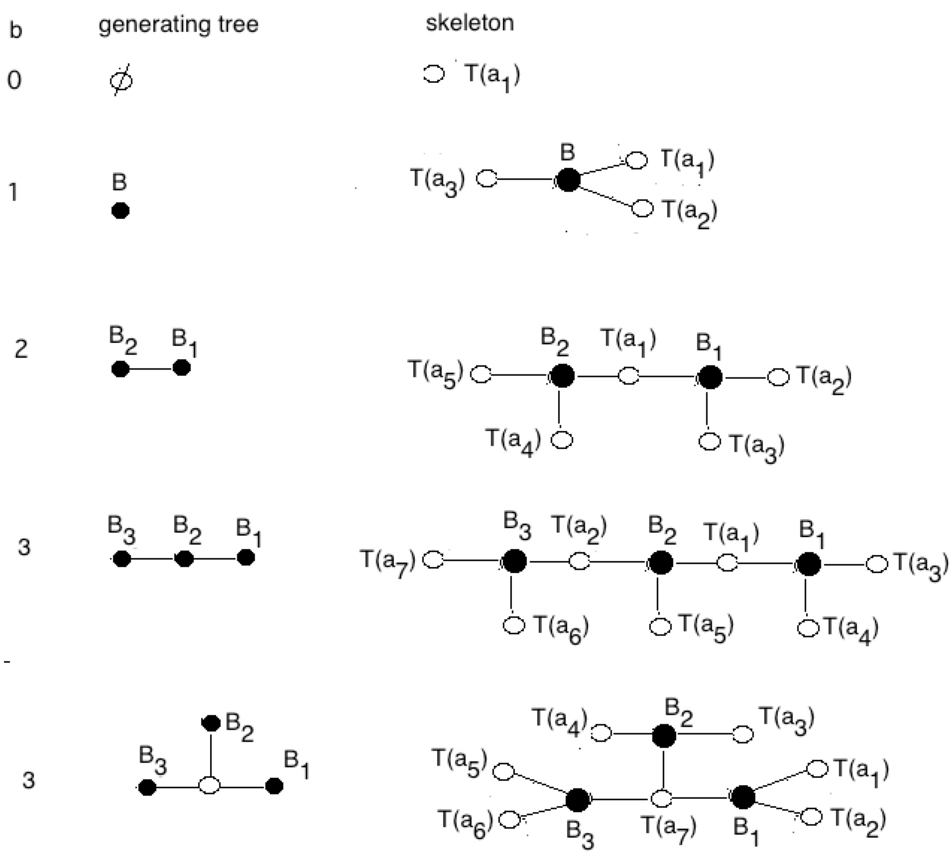

Figure 3: generating trees and skeletons for b=1,2,3

Rooted trees. A rooted tree (T, r) is a tree T with one distinguished vertex r, called the root of T.

Bi-rooted trees. A bi-rooted tree (T, m, r) is a tree T with two distinguished vertices; one called the mark m and the other one called the root r. We allow $m = r$, in which case one has a rooted tree.

d**-rooted trees**. For $d \geq 3$, a d-rooted tree $(T, m_1, \ldots, m_{d-1}; r)$ is a tree T with d distinguished vertices: $d-1$ marks $m_1, \ldots, m_{d-1}$ and one root r. We allow $m_i = r$, for some i, $1 \leq i \leq d-1$, but $m_i \neq m_j$ for $i \neq j$.

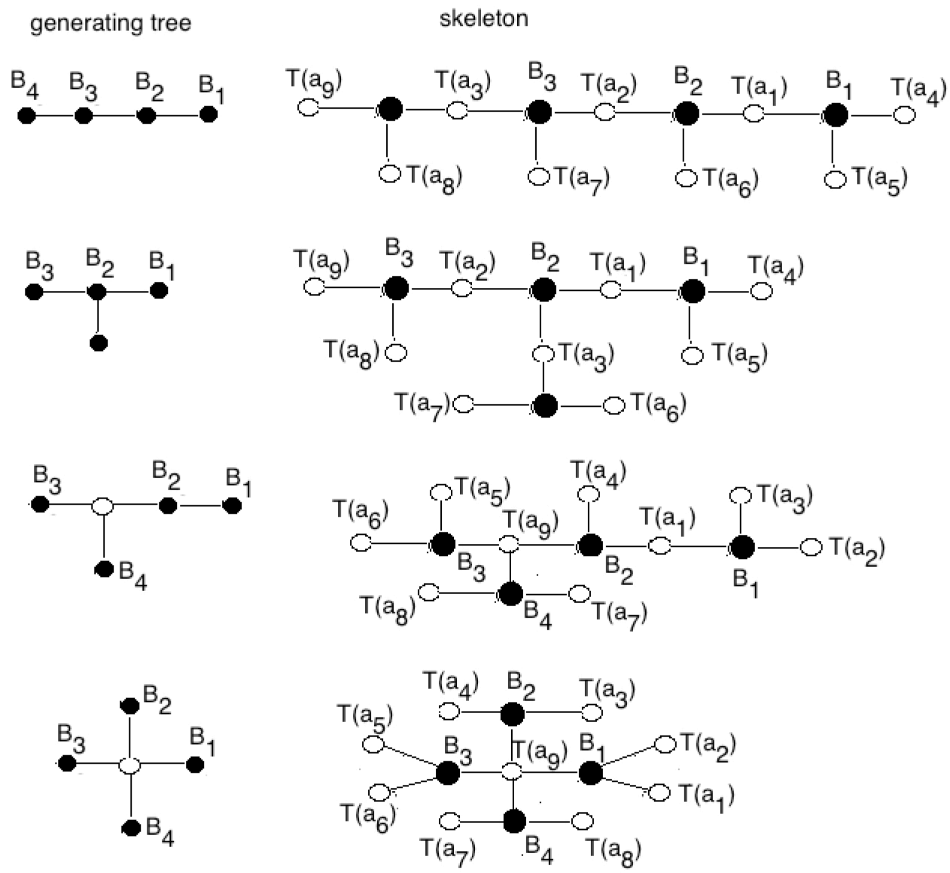

Figure 4: generating trees and skeletons for b=4

An isomorphism between bi-rooted trees (T, m, r), (T', m', r') (resp. d-rooted trees $(T, m_1, \ldots, m_{d-1}; r)$, $(T', m'_1, \ldots, m'_{d-1}; r')$) s a tree isomorphism $f : T \to T'$ such that $f(r) = r'$ and $f(m) = m'$ (resp. $f(m_i) = m'_i$ for $i = 1, \ldots d-1$)

Lemma 1. *There is a* $1-1$*-correspondence between* $(2,1)$*-collapsible trees and rooted trees.*

Proof. Color the vertices of the rooted tree white and bisect all edges. The new vertices are colored black. In the resulting tree Γ assign label 2 (resp. label 1) to an edge that has even (resp. odd) distance to the root. Then Γ is a $(2,1)$-collapsible tree. □

We now use the term *rooted tree* also for the associated $(2,1)$-collapsible tree.

By Theorem 1 every 1-connected trivalent graph $\Gamma = \Gamma_X$ is obtained from $St(\mathbb{B})$ by attaching $(2,1)$-collapsible trees to the white vertices of $St(\mathbb{B})$.

If $St(\mathbb{B})$ is connected and Γ has b black vertices of degree 3 and n white vertices, then Γ is obtained from a skeleton (with b black vertices) by attaching to each white vertex labeled $T(a_i)$ a $(2,1)$-collapsible tree having a_i

white vertices such that the attachment is along the mark of the corresponding bi-rooted tree. Furthermore $n = a_1 + \cdots + a_k$, where k is the number of white vertices of the skeleton. (If the generating tree has no white vertices, then $k = 2b + 1$). The symmetry group of the skeleton acts on the set of all these Γ's and to avoid repetitions we must only count the elements in the orbits of this action. This needs to be done in such a way so that the resulting bi-colored trees do not contain horned trees.

If $St(\mathbb{B})$ is not connected then Γ is obtained from a skeleton by first splitting some white non-terminal vertices. For example, if $b = 2$, the skeleton splits into two cases, depending on whether $St(\mathbb{B})$ is connected or disconnected, see Figure 5. In the disconnected case the vertex of degree 2 splits into two vertices and we must also consider, for a given partition $n = a_1 + a_2 + a_3 + a_4$, the number of attachments of tri-rooted trees with a_1 white vertices to these two vertices along two marks.

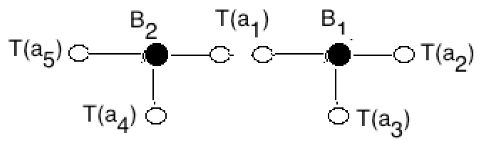

Figure 5: case $b = 2$ disconnected

Similarly for $b \geq 2$ the skeleton splits into several cases and one must count the number of possible attachments of k-rooted trees for $1 \leq k \leq d+1$.

4 Number of trivalent graphs with at most one black vertex of degree 3

In this section we develop explicit formulas for the number of 1-connected trivalent graphs with n white vertices and one black vertex of degree 3.

Definition 1. *R_n is the number of (unlabeled) rooted trees with n (white) vertices.*
M_a is the number of (isomorphism classes of) bi-rooted trees with exactly a vertices.
$U_a = M_a - R_a$ is the number of bi-rooted trees with a vertices where the mark m is different from the root r.

The values of R_n for $n \leq 30$ can be found [9].

Case $b = 0$. Here Γ_X is a $(2,1)$-collapsible tree. By lemma 1 the number of distinct 1-connected trivalent graphs Γ_X is R_n.

Case $b = 1$. Here Γ_X is obtained from a $b111$-tree (a tree with one black vertex of degree 3 and 3 white vertices and all edges labeled 1) by identifying each white vertex v_i of $b111$ with a white vertex of a $(2,1)$-collapsible tree T_i $(i = 1, 2, 3)$ such that the reduced subgraph $R(\Gamma)$ of Γ_X is not a horned tree. This is the case if and only if at at least one of the v_i's is attached to a root of T_i.

In the skeleton graph for $b = 1$ let v_i be the white vertex with label $T(a_i)$. Here $T(a_i)$ is a bi-rooted tree with a_i vertices and the vertex of $T(a_i)$ marked m_i is identified with the vertex v_i of the $b111$-graph. The (white) edges of the bi-rooted tree $T(a_i)$ are then bisected, with the resulting vertices colored black. An edge in the bisected tree receives label 2 (resp. 1) if its distance to the corresponding root r_i is even (resp. odd).

If Γ_X has n white vertices we have $a_1 + a_2 + a_3 = n$ and in order to count all non-isomorphic graphs with n white vertices we have, by symmetry of $b111$, exactly one of the three cases S, I, E, below:

(i) S (scalene): $a_1 > a_2 > a_3$

(ii) I (isosceles): $a_1 \neq a_2$, $a_2 = a_3$

(iii) E (equilateral): $a_1 = a_2 = a_3$. (This occurs only when $n = 3k$ for some integer k)

In each of the three cases let $n = a_1 + a_2 + a_3$ be a given partition. We count the number of distinct trivalent 1-connected graphs with 1 black vertex of degree 3 and n white vertices.

(i) S_n: There are M_{a_i} ways of attaching a birooted tree $T(a_i)$ with a_i vertices to v_i, so there are $M_{a_1}M_{a_2}M_{a_3}$ ways of producing "scalene (a_1, a_2, a_3)" trivalent trees. However, some of these are not 1-connected because they contain horned subtrees. So we need to subtract the number of attachments where all three vertices v_i are attached to T_i's along non-roots i.e. along marks m_i different from the roots r_i. The number of these is $U_{a_1}U_{a_2}U_{a_3}$. Therefore:

(i) The number of distinct trivalent 1-connected graphs is
$M_{a_1}M_{a_2}M_{a_3} - U_{a_1}U_{a_2}U_{a_3}$.

An example is shown in Figure 6 for the case $(a_1, a_2, a_3) = (3, 2, 1)$.

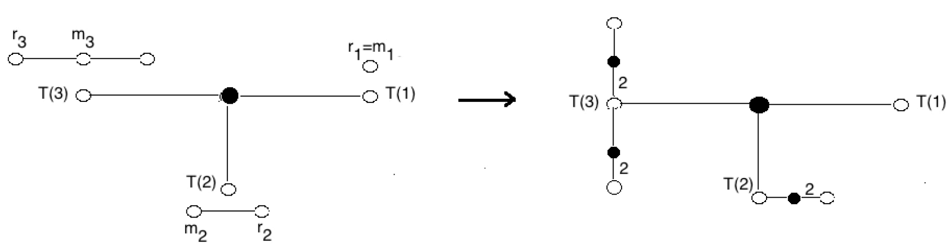

Figure 6: Obtaining a Γ_X from the skeleton for $b = 1$

(ii) I_n: Let $a_1 \neq a := a_2 = a_3$. There are M_{a_1} ways to attach a birooted tree $T(a_1)$ with a_1 vertices to v_1. Let $S_1, \ldots, S_{M_a}$ be the distinct birooted trees with a vertices. By symmetry, attaching S_i to v_2 and S_j to v_3 produces the same (isomorphic) result as attaching S_i to v_3 and S_j to v_2. Therefore the number of distinct graphs obtained is the number of triples

$\{T(a_1), S_i, S_j\}$ with $M_a \geq i \geq j \geq 1$. To obtain the graphs corresponding to 1-connected 2-stratifolds we need to disregard the cases that give horned subtrees. Therefore from Lemma 2 below we obtain

(ii) The number of distinct isosceles trivalent 1-connected graphs is $M_{a_1} C(M_a + 1, 2) - U_{a_1} C(U_a + 1, 2)$.

(iii) E_n: Let $a := a_1 = a_2 = a_3$. Let $S_1, \ldots, S_{M_a}$ be the distinct bi-rooted trees with a vertices. By symmetry, an attachment of (S_i, S_j, S_k) to (v_1, v_2, v_3) yields isomorphic graphs if the indices i, j, k are permuted. Therefore the number of distinct graphs obtained is the number of attachments of (S_i, S_j, S_k) to (v_1, v_2, v_3) with $M_a \geq i \geq j \geq k \geq 1$. Subtracting the cases that lead to horned subtrees and using Lemma 2 we obtain:

(iii) The number of distinct equilateral trivalent 1-connected graphs with 1 black vertex of degree 3 and n white vertices is

$$\begin{cases} C(M_a + 2, 3) - C(U_a + 2, 3) & \text{if } n \text{ is divisible by 3,} \\ 0 & \text{otherwise.} \end{cases}$$

Summing up we obtain the following Theorem.

Theorem 2. *The number of distinct trivalent 1-connected 2-stratifold graphs with 1 black vertex of degree 3 and n white vertices is $S_n + I_n + E_n$.*
Here $S_n = \sum(M_{a_1} M_{a_2} M_{a_3} - U_{a_1} U_{a_2} U_{a_3})$, where the sum is over $a_1 > a_2 > a_3$ and $a_1 + a_2 + a_3 = n$
$I_n = \sum(M_{a_1} C(M_a + 1, 2) - U_{a_1} C(U_a + 1, 2))$, where the sum is over $a_1 \neq a$, $a_1 + 2a = n$

$$E_n = \begin{cases} C(M_a + 2, 3) - C(U_a + 2, 3) & \textit{if } 3 \textit{ divides } n \textit{ and } 3a = n, \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 2. *Let $m \geq 1$ and let $K = \{(k_1, \ldots, k_r) \in \mathbb{Z}^r \mid m \geq k_r \cdots \geq k_2 \geq k_1 \geq 1\}$. Then the cardinality of K is $C(m + r - 1, r)$.*

Here $C(p, q)$ is the binomial coefficient $p!/q!(p - q)!$.

Proof. An element of K is a non-increasing function $k : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, m\}$, where $k(i) = k_i$. Let $\#k^{-1}(i)$ be the cardinality $k^{-1}(i)$ and denote the m-vector $k^{-1} = (\#k^{-1}(1), \#k^{-1}(2), \ldots, \#k^{-1}(m))$ by $\#k^{-1}(1) \,|\, \#k^{-1}(2) \,|\, \ldots \,|\, \#k^{-1}(m)$ (with $m - 1$ dividing bars).
From this m-vector delete $\#k^{-1}(i)$ if $\#k^{-1}(i) = 0$ and replace $\#k^{-1}(i)$ by n asterisks $*$ if $\#k^{-1}(i) = n$ to get a string of $|$'s and $*$'s.
For example if $m = 8$, $r = 6$ and $k = (k_1, \ldots, k_6) = (1, 4, 4, 7, 7, 7)$, $k^{-1} = 1|0|0|2|0|0|3|0 \leftrightarrow *||| **||| *** |$.

This gives a bijection from the set of non-increasing functions $k : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, m\}$ to the set of all strings of length $m + r - 1$ on the symbols $|$ and $*$ with exactly r asterisks $*$. □

5 An example for $n = 7$

In this example we show how to compute the number of 1-connected 2-stratifold graphs with $n = 7$ white vertices. First we list a few values of R_n,

M_n, U_n.

n	R_n	M_n	U_n
1	1	1	0
2	1	2	1
3	2	5	3
4	4	13	9
5	9	35	26
6	20	95	75
7	48	256	208

$R_n =$ number of rooted trees with n vertices

$M_n =$ number of bi-rooted trees with n vertices

$U_n =$ number of bi-rooted trees with n vertices and root different from the mark

The table below shows how to compute the number of 1-connected Γ_X with $n = 7$ white vertices. Here b denotes the number of black vertices of degree 3. The total number of non-homeomorphic X_Γ corresponding to graphs with $n = 7$ vertices is 167.

			total number cases
$b = 0$	$R_7 = 48$		48
$b = 1$	$S_7 = M_4M_2M_1 - U_4U_2U_1$		26
	$I_7 = M_5C(M_1 + 1, 2) - U_5C(U_1 + 1, 2)$	$=35 \cdot 1 - 25 \cdot 0$	35
	$+M_3C(M_2 + 1, 2) - U_3C(U_2 + 1, 2)$	$+5 \cdot 3 - 3 \cdot 1$	12
	$+M_1C(M_3 + 1, 2) - U_1C(U_3 + 1, 2)$	$+1 \cdot 15 - 0 \cdot 6$	15
	$E_7 = 0$		
$b = 2$	$St(\mathbb{B})$ connected: v_0, v_1, v_2 vertices of $St(B_1)$		
	v_0, v_3, v_4 vertices of $St(B_2)$		
	3 cases for middle vertex v_0: $a_0 = 3, 2, 1$:		
	$a_0 = 3$ $a_1 = a_2 = a_3 = a_4 = 1$	M_3	5
	$a_0 = 2$ $a_1 = 2,\ a_2 = a_3 = a_4 = 1$	M_2M_2	4
	$a_0 = 1$ $a_1 = 3,\ a_2 = a_3 = a_4 = 1$	M_3	5
	$a_1 = 2,\ a_2 = 2, a_3 = a_4 = 1$	$C(M_2 + 1, 2)$	3
	$a_1 = 2,\ a_3 = 2, a_2 = a_4 = 1$	$C(M_2 + 1, 2)$	3
	$St(\mathbb{B})$ disconnected: v_0, v_1, v_2 vertices of $St(B_1)$		
	v_0', v_3, v_4 vertices of $St(B_2)$		
	may assume tri-rooted tree is attached		
	between v_0 and v_0'. Let $a = a_0 + a_0' \geq 2$		
	$a = 2,\ a_1 = 2, a_3 = a_4 = 1$	M_2M_2	4
	$a = 3,\ a_1 = a_3 = a_4 = 1$	5	5
$b = 3$	linear case	1	1
	star case	1	1
	Total cases for $b = 0, 1, 2, 3$		167

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