RIEMANN SURFACES, PLANE ALGEBRAIC CURVES AND THEIR PERIOD MATRICES

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Abstract. The aim of this paper is to present theoretical basis for computing a representation of a compact Riemann surface as an algebraic plane curve and to compute a numerical approximation for its period matrix. We will describe a program Cars ([3]) that can be used to define Riemann surfaces for computations. Cars allows one also to perform the Fenchel–Nielsen twist and other deformations on Riemann surfaces.

Almost all theoretical results presented here are well known in classical complex analysis and algebraic geometry. The contribution of the present paper is the design of an algorithm which is based on the classical results and computes first an approximation of a polynomial representing a given compact Riemann surface as a plane algebraic curve and further computes an approximation for a period matrix of this curve. This algorithm thus solves an important problem in the general case. This problem was first solved, in the case of symmetric Riemann surfaces, in [15].

1. Introduction

A compact Riemann surface is a projective complex algebraic curve. Any Riemann surface of genus $g > 1$ can be embedded into a complex projective space $\mathbb{P}^{3g-4}(\mathbb{C})$ via the bicanonical map. The image of the Riemann surface in $\mathbb{P}^{3g-4}(\mathbb{C})$ is a complex algebraic curve of degree $4g-4$ defined by a set of homogeneous polynomials. A projection of this algebraic curve onto a generic plane in $\mathbb{P}^{3g-4}(\mathbb{C})$ is a (singular) plane algebraic curve defined by one homogeneous polynomial of degree at most $4g-4$.

In this paper we will derive a method for computing an approximation of a polynomial that corresponds, in the way described above, to a given compact Riemann surface. This is based on the basis for holomorphic 2–forms of a Riemann surface given by S. Wolpert in [18].

In his forthcoming thesis, Pekka Smolander studies this embedding in more detail deriving estimates for the errors of the numercial approximations to be used.

For the reader's convenience we recall here the constructions related to the bicanonical map, Poincáre series and the equations of the image of a Riemann surface under a bicanonical map.

Symmetric Riemann surfaces form a special case in that for such Riemann surfaces we can actually compute numerically the canonical map. Therefore symmetric

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Riemann surfaces, except for the hyperelliptic ones, of genus g, $g > 2$, can be realized as curves in $\mathbb{P}^{g-1}(\mathbb{C})$ instead of $\mathbb{P}^{3g-4}(\mathbb{C})$. Their degree is also generally lower. The computation of the canonical map is based on the considerations of Seppälä ([15]) which rely on a paper of Clifford Earle and Albert Marden ([4]).

The canonical map cannot[1] be used in the case of hyperelliptic Riemann surfaces (because it's not an embedding) nor in the case of general compact Riemann surfaces (because we cannot approximate it numerically). In these cases we use the bicanonical map.

To this end we review, in a subsequent chapter, results of Wolpert, Kra, Maskit and others, concerning the construction of a basis for holomorphic 2 forms on a Riemann surface. This can be done with the Poincaré series and there are no problems with the convergence. Using this bicanonical map we then express our Riemann surface as an algebraic curve in a projective space.

2. Representing Riemann surfaces for computations by a computer

We adopt the most classical approach to Riemann surfaces: a compact orientable Riemann surface is simply a sphere with a certain number g of handles attached to it.

'Handle' is a torus from which an open disk has been removed. It is a Riemann surface of genus 1 with 1 boundary component. In order to attch a handle to a sphere it is necessary to delete an open disk, from the sphere, and then glue the handle to the sphere along the boundary of this deleted disk.

Therefore, in order to define a Riemann surface of genus g with n boundary components, it is necessary to build first a sphere with many $(g+n)$ holes. Such a sphere can be obtained by taking $n+g-2$ spheres with three holes and gluing them together, in a suitable way, along certain boundary components.

So we construct complicated Riemann surfaces in the above way from the following elementary parts:

Y–pieces: These are spheres with three boundary components. They are often referred to as *pairs of pants.*

Q–pieces: These are the handles, i. e., torii with one boundary component.

The above Riemann surfaces are the most elementary hyperbolic Riemann surfaces. A hyperbolic Riemann surface X has the unit disk as its universal cover. The hyperbolic metric of X comes from the hyperbolic metric of the unit disk via the representation $X = D/G$ where G is a Fuchsian group.

If $X = D/G$ is a hyperbolic Riemann surface with boundary components, then:

- G is a freely generated group.
- The hyperbolic metric of (the interior of) X, obtained from that of the unit disk D, is complete. In particular the boundary curves of X have infinite length in this metric.

A Fuchsian group G_Y corresponding to a Y–piece Y is freely generated by two hyperbolic Möbius transformations g and h whose axes do not intersect. The axes of the the transformations g and h cover closed geodesic curves homotopic to two of the three boundary curves of Y. Provided that the orientations are suitably chosen,

[1]An alternative method for finding equations for hyperelliptic Riemann surfaces has been presented in [14]. This uses the canonical mapping which can be used to get the hyperelliptic projection and thus can be used to approximate an equation for a hyperelliptic Riemann surface.

the axis of $g \circ h$ covers the geodesic curve freely homotopic to the third boundary component.

It is well known that the lengths of the geodesic curves freely homotopic to the boundary curves determine the hyperbolic metric of the Y–piece in question (see e.g. [16]). The lengths of these geodesic curves freely homotopic to the boundary curves can, furthermore, be freely chosen. Therefore the Teichmüller space of a Y–piece is simply $(\mathbb{R}_+)^3$.

A Q–piece Q can be expressed as $Q = D/G_Q$ where G_Q is a group generated by two hyperbolic Möbius transformations g and h with intersecting axes. Here again G_Q is a free group. The commutator of g and h, $c = g \circ h^{-1} \circ g^{-1} \circ h$, is a hyperbolic Möbius transformation, whose axis covers a closed geodesic freely homotopic to the boundary curve of Q (provided that the orientations are suitably chosen).

Using the above described Y–pieces $Y = D/G_Y$ and Q–pieces $Q = D/G_Q$ we build Riemann surfaces in the following way.

Let W_1 and W_2 be Y– or Q–pieces. Assume that the geodesic curves α_j, homotopic to a boundary curve of W_j, $j = 1, 2$, have the same length. Delete, from W_1 and from W_2, the infinite cylinders bounded by the geodesics α_j and by the respective boundary curves. Next identify α_1 with α_2 using constant speed. This procedure can be repeated to get any Riemann surface of any finite type.

Cars ([3]) is a program which can be used to define random Y–pieces and Q–pieces and glue them together in the way described above. The program keeps in mind the generators of the Fuchsian groups corresponding to all the above pieces. The program allows the user to deform the resulting Riemann surface by Dehn twists along the gluing geodesics. This deformation can be conveniently done by a mouse. That program has many other functionalities that will be described in part later and more completely elsewhere[2].

Let now $G_j = \langle g_j, h_j \rangle$ be the Fuchsian group corresponding the a Q–piece Q_j in the above construction, $j = 1, \ldots, g$. The program Cars produces, especially, a list containing these Möbius transformations g_j and h_j. By the construction, it is immediate that the axes of the transformations g_j and h_j cover curves freely homotopic to generators of the first homology group of the resulting Riemann surface. Therefore, when constructing Riemann surfaces by the program Cars we get, in addition, a homology basis for the Riemann surface just constructed. This is the homology basis that will be used later to compute a period matrix for the Riemann surface.

3. Real curves and symmetric Riemann surfaces

Symmetric Riemann surfaces form a special class that can be treated separately. For non–hyperelliptic symmetric Riemann surfaces we may form the canonical map and find a representation as an algebraic plane curve of a *lower degree* than in the general case. For the computation of a period matrix of a symmetric Riemann surface we also have a more precise algorithm ([15]) than that of this paper (which works in the general case). Therefore we review, in this section, some basic results concerning symmetric Riemann surfaces.

Geometrically a smooth real algebraic curve C is a compact Riemann surface X together with an antiholomorphic involution $\sigma : X \to X$. This involution is induced by the complex conjugation. A Riemann surface X which has an antiholomorphic

[2]For a short manual of Cars, see http://klein.math.fsu.edu/~seppala/Symbolic/Software/index.html

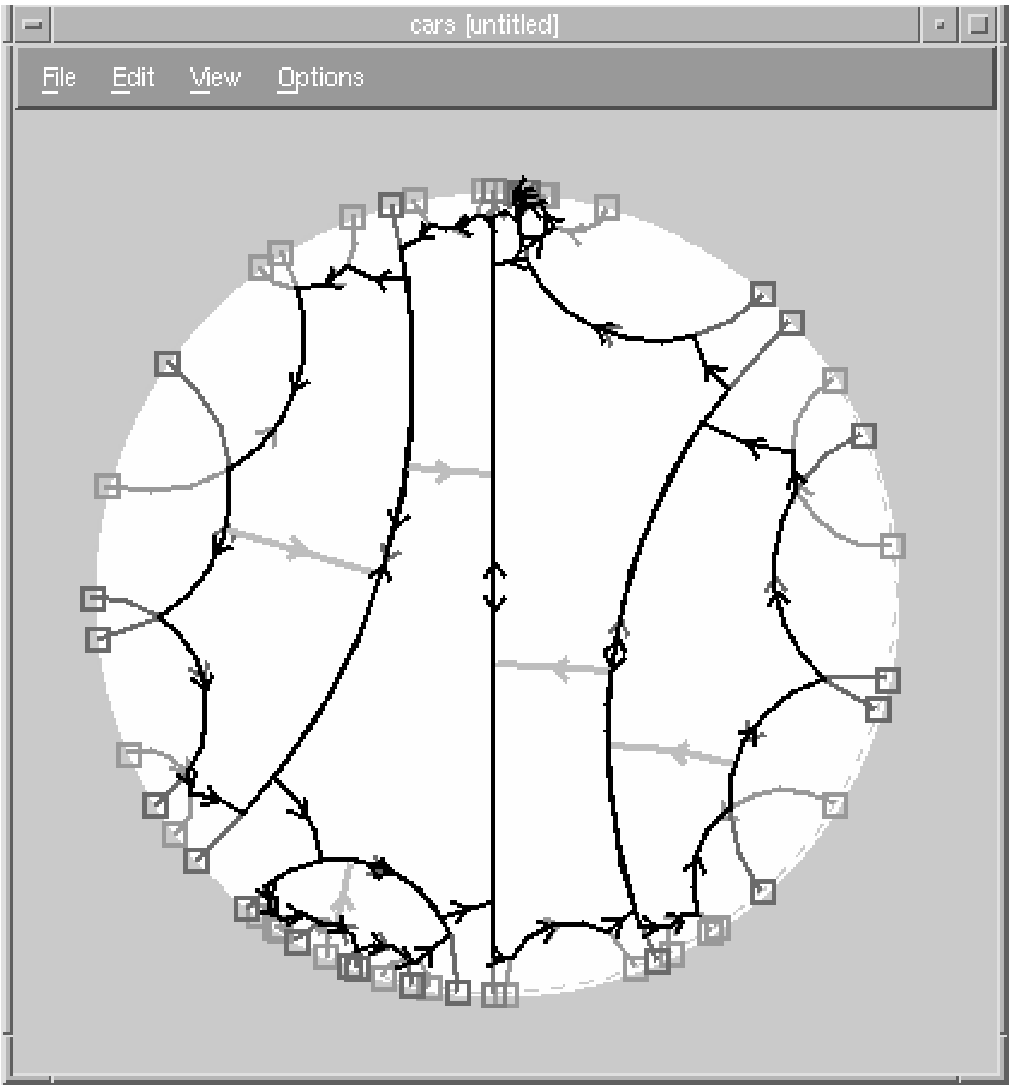

FIGURE 1. This figure shows a display of CARS of a genus four Riemann surface that has been obtained by first gluing two Y–pieces together; the black arc on the vertical diagonal of the unit disk is the "waist" geodesic along which gluing has been performed. Then four Q–pieces has been added to fill the holes. The union of the polygons bounded by the black geodesic arcs forms in this case a fundamental domain for the action of the corresponding group.

involution will be called *symmetric*. Any symmetric compact Riemann surface is a smooth real algebraic curve. This correspondence between real algebraic curves and symmetric Riemann surfaces was observed already by Felix Klein ([8]) and has recently been investigated in detail by many authors.

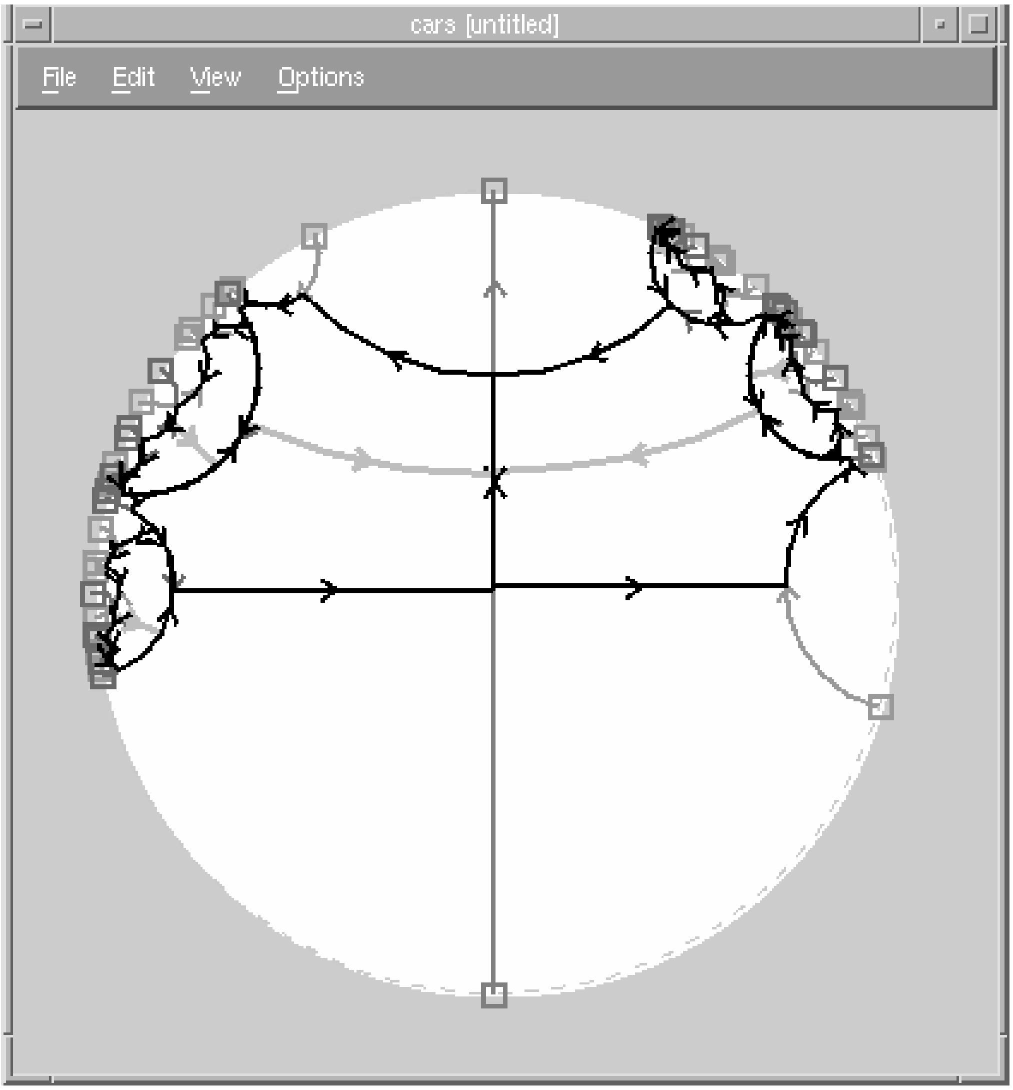

FIGURE 2. This shows the Riemann surface of Figure 1 after a deformatin which has made the "waist" geodesic much shorter. in addition a Fenchel–Nielsen twist has been performed along the "waist". The result is that all closed geodesics intersecting the "waist" (and not homotopic to a multiple of it) have become much longer. This is a simple consequence of a well known property of hyperbolic metrics, often referred to as a Marguillis theorem, and clearly displayed in these figures.

For our applications it is convenient to divide real algebraic curves or symmetric Riemann surfaces (X, σ) into three classes:

- Curves without real points. They are symmetric Riemann surfaces (X, σ) for which the involution σ does not have fixed points.

- Orthosymmetric real curves are symmetric Riemann surfaces (X,σ) for which $X \setminus X_\sigma$ has two components. Here X_σ denotes the fixed–point set of $\sigma : X \to X$.
- Diasymmetric real curves are symmetric Riemann surfaces (X,σ) such that X_σ is non–empty and $X \setminus X_\sigma$ has only one component.

Sometimes real curves without real points are considered to be special cases of orthosymmetric Riemann surfaces. It is, however, more convenient for us to consider them as a separate class.

Let (X,σ) be an orthosymmetric real curve and let Y be a component of $X \setminus X_\sigma$. It is immediate that Y is a (non–compact) Riemann surface with ideal boundary components which correspond to the fixed–point set X_σ of σ. Recall that components of the fixed–point set of an antiholomorphic involution are always simple closed geodesic curves on X. Therefore, the Riemann surface Y is the interior of a bordered Riemann surface $\overline{Y}$.

We have assumed that the genus of X is at least 2. Then Y can be expressed as D/G where G is a freely generated group of Möbius transformations acting in the unit disk D. The group G is of the second kind, i.e., the domain $\Omega(G)$ of discontinuity of G is connected and symmetric with respect the unit circle ∂D. The reflection in ∂D is an antiholomorphic involution mapping $\Omega(G)$ onto itself. Call this mapping $\tilde{\sigma}$. Then $\tilde{\sigma}(z) = 1/\overline{z}$.

It follows that $X = \Omega(G)/G$ and the reflection $\tilde{\sigma} : \Omega(G) \to \Omega(G)$ induces the symmetry $\sigma : X \to X$. This is the representation that we will use to study orthosymmetric real curves.

Assume now that (X,σ) is diasymmetric. Consider the quotient $Y = X/\langle\sigma\rangle$. Since (X,σ) is diasymmetric, Y is a non–orientable Klein surface. Since (X,σ) has real points, i.e., since $\sigma : X \to X$ has fixed points, Y is a bordered Klein surface. Let Y^o be the orientable double covering of the non–orientable Klein surface Y. The Klein surface Y^o is an orientable bordered Klein surface having an antiholomorphic involution $\tau : Y^o \to Y^o$ such that $Y = Y^o/\langle\tau\rangle$ where $\langle\tau\rangle$ is the group generated by τ.

Let X^S be the Schottky double of Y^o. It is obtained by gluing Y^o and its mirror image (complex conjugate) together along the boundary curves. The involution $\tau : Y^o \to Y^o$ induces an antiholomorphic involution $\tau : X^S \to X^S$ and $X = X^S/\langle\tau\rangle$. This is obvious by the construction. For more details about the Schottky double and other double coverings see e.g. [17, Sections 3.2 – 3.4].

It is important that X^S is now an orthosymmetric Riemann surface. It can, therefore be expressed as $X^S = \Omega(G)/G$ for a Fuchsian group G of the second kind. This orthosymmetric Riemann surface has an additional symmetry induced by the symmetry τ of the orientable double covering Y^o. This symmetry acts on X^S and $X = X^S/\langle\tau\rangle$. This construction allows us to apply the methods, that we will first develop for orthosymmetric Riemann surfaces, to the diasymmetric case.

4. Poincaré series

Assume that a Riemann surface X is given in terms of its Fuchsian group G. In the classical case, G acts in the unit disk D, and $X = D/G$. For practical purposes, we often consider other kinds of representations for X in which we have $X = \Omega(G)/G$, where $\Omega(G)$ is the domain of discontinuity of G. In these cases the domain of discontinuity of G is connected but not simply connected.

In all of the above cases, holomorphic m–forms (or m–differentials) of X are holomorphic functions defined either in the unit disk (if $X = D/G$) or in the domain of discontinuity of G (if $X = \Omega(G)/G$) satisfying, at all points z:

$$\omega(g(z))\left(\frac{\partial g(z)}{\partial z}\right)^m = \omega(z),\ \forall g \in G. \tag{1}$$

We use the notation

$$D^m(G) = \{\phi \text{ a } m\text{–differential} \mid \int_{\Omega/G} |\phi|\lambda^{2-m} < \infty\}$$

for the space of integrable holomorphic m–forms of the group G. Here λ is the tensor of the hyperbolic metric of the Riemann surface $\Omega(G)/G$.

We are mainly interested in holomorphic 2–forms and 1–forms.

Let G be a Fuchsian group acting in the unit disk D and let $f : D \to \hat{\mathbb{C}}$ be a meromorphic function. Let $m \geq 1$ be an integer. Consider *the Poincaré series*

$$\Theta^m(f)(z) = \sum_{g\in G} f(g(z))\left(\frac{\partial g(z)}{\partial z}\right)^m. \tag{2}$$

If this series converges, then its limit is a meromorphic function ω satisfying equation (1). Use the notation $\Theta(f)(z) = \Theta^1(f)(z)$.

The series (2) were introduced by H. Poincaré ([13]) in 1884, who, in that paper, proved that these Poincaré series do converge if $m \geq 2$. Poincaré series can be used to approximate holomorphic or meromorphic m–forms of any Riemann surface, $m > 1$.

Assume that a symmetric Riemann surface X is given in the form $X = \Omega(G)/G$ where G is a Fuchsian group of the *second kind* acting in the unit disk. This kind of a representation is always possible if the the fixed–point set X_σ of the symmetry $\sigma : X \to X$ of X is not empty and $X \setminus X_\sigma$ has two components.

The fact that G is of the second kind means that the limit set of G is of measure 0. This fact can be used to show that in this case, for suitable rational functions f, $\Theta(f)$ is a holomorphic one form of G. We have, in fact, the following result.

Lemma 1 (Earle and Marden [4, Corollary 1, page 208]). *Assume that the group G is freely generated by the hyperbolic Möbius transformations $g_1, g_2, \ldots, g_p$ mapping the unit disk onto itself and that G is of the second type. For $j = 1, 2, \ldots, p$ let*

$$h_j(z) = \frac{\overline{\zeta}_j}{1 - \overline{\zeta}_j z}, \quad \zeta_j = g_j(0), \quad 1 \leq j \leq p. \tag{3}$$

Then the Poincaré series $\Theta(h_j)$ converge to holomorphic 1-forms of the group G and $\{\Theta(h_j) \mid 1 \leq j \leq p\}$ is a basis for holomorphic differentials on $X = \Omega(G)/G$.

The case of higher order forms has been under intensive study for several years. The problem of finding a generating set, in terms of Poincaré series, for higher order holomorphic forms was solved in the general case in [9] by I. Kra.

For our purposes it is enough to find generators for the space of holomorphic 2–forms of a Fuchsian group G. We will use the construction of S. Wolpert [18].

Assume that G is a Fuchsian group of the first kind acting in the unit disk D in such a way that D/G is a compact Riemann surface of genus p. Let $g \in G$ be a (primitive in G) Möbius transformation covering the homotopy class of a simple closed geodesic curve $\gamma \in D/G$. Assume that

$$g(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1. \tag{4}$$

Let

$$\nu(g) = (\mathrm{tr}^2 g - 4)^{1/2} = \sqrt{k_g} - \frac{1}{\sqrt{k_g}} = ((a+d)^2 - 4)^{1/2}. \tag{5}$$

Here k_g is the multiplier of the hyperbolic Möbius transformation g and $\mathrm{tr}^2 g$ is the square of the trace of a matrix, of determinant $+1$, corresponding to the Möbius transformation g. Observe that such a matrix is well–defined only up to sign. The *square* of the trace is, however, always well–defined.

Using this notation define

$$\omega_g = \nu(g)(cz^2 + (d-a)z - b)^{-2}. \tag{6}$$

The importance of the formula (6) lies in the fact that

$$(\omega_g \circ g)\left(\frac{\partial g}{\partial z}\right)^2 = \omega_g \tag{7}$$

as can be verified by a direct computation. Therefore ω_g is a holomorphic 2–form of the cyclic group $\langle g \rangle$. This motivates one to look at the *Petersson–Poincaré series*

$$\theta^*_\gamma = \sum_{h \in \langle g \rangle \backslash G} (\omega_g \circ h)\left(\frac{\partial h}{\partial z}\right)^2 \tag{8}$$

associated to the simple closed geodesic curve γ.

Wolpert has shown the following result [18, Theorem 3.7, page 521]:

Theorem 2. *Let $\alpha_1, \ldots, \alpha_{3p-3}$ be a decomposition of the Riemann surface D/G into Y–pieces (i.e., into pairs of pants, $\{\alpha_j\}$ is a maximal set of non–intersecting simple closed curves). Then the Petersson–Poincaré series $\theta^*_{\alpha_j}$ converge, their limits are holomorphic 2–forms of the group G, and $\{\theta^*_{\alpha_j}\}$ is a basis (over $\mathbb{C}$) for the space of holomorphic 2–forms of the group G.*

We will use this basis in our numerical computations.

In certain applications it may be necessary to study higher order pluricanonical mappings. That can be done numerically as well. For completeness, we recall here briefly the result of Kra which allows one to approximate higher order pluricanonical mappings numerically.

Consider the space of holomorphic m forms, $2 \le m \le 2g-2$. Let $a_1, \ldots, a_{2m-1}$ be distinct fixed points of hyperbolic elements of G. Let $L(G)$ denote the limit set of G.

We define a family of rational functions $f(z,\zeta)$ with variable ζ for $z \in L(G) \setminus \{a_1, \ldots, a_{2m-1}\}$ by

$$f(z,\zeta) = -\frac{1}{2\pi}\frac{1}{\zeta - z}\prod_{j=1}^{2m-1}\frac{z-a_j}{\zeta - a_j}.$$

We set

$$\phi_m(z,\zeta) = \sum_{g \in G} f(z, g(\zeta))\left(\frac{\partial g(\zeta)}{\partial \zeta}\right)^m. \tag{9}$$

For these functions we get the following theorem [9, Theorem 1].

Theorem 3. *The Poincaré series (9) converge, their limits are holomorphic m–forms of the group G and it is possible to choose points $b_1, \ldots, b_d$ from the limit set of G in such a way that the forms $\phi_m(b_1, \cdot), \ldots, \phi_m(b_d, \cdot)$ form a basis for holomorphic m–forms of the group G.*

5. Mapping of Riemann surfaces into the projective plane

Every compact Riemann surface X of genus g can be embedded into the complex projective space $\mathbb{P}^{3g-4}(\mathbb{C})$ by the *bicanonical map.* For the convenience of the reader we recall its construction here.

The bicanonical map uses holomorphic 2–forms. It is a special case of the more general m–canonical (pluricanonical) map which is formed as follows.

The space $D^m(X)$ of holomorphic m–forms of X is a complex vector space of dimension $d = (2m-1)(g-1)$. Let $\eta_1, \ldots, \eta_d$ be a basis for $D^m(X)$. Consider a point $p \in X$. The elements η_j are not functions defined on X but differentials. For each local coordinate (U, z)[3] each η_j associates a holomorphic function $\eta_j^z : U \to \mathbb{C}$. The values of these functions at the point p depend on the choice of the local coordinate (U, z). However, the *quotients* of these values do not depend on the choice of (U, z). We obtain, therefore, a well defined mapping

$$\phi_{mK} : X \to \mathbb{P}^{d-1}(\mathbb{C}),\ p \mapsto (\eta_1^z(p), \ldots, \eta_d^z(p)). \tag{10}$$

Mapping (10) is the *pluricanonical map of X.* For $m = 2$, ϕ_{2K} is the *bicanonical map* and for $m = 1$, ϕ_K is the *canonical map.*

The image of X under the bicanonical mapping ((10) for $m = 2$) is a non–singular complex algebraic curve whose degree equals twice the number of zeros of any holomorphic 1–form. Hence the degree is $4g - 4$.

The projection of $\phi_{2K}(X) \subset \mathbb{P}^{3g-4}(\mathbb{C})$ onto a generic plane in $\mathbb{P}^{3g-4}(\mathbb{C})$ is a (singular) plane curve of degree at most $4g - 4$. It is defined by one homogeneous polynomial of degree at most $4g - 4$. The statement about the degree follows from the fact that, in projections, the degree of an algebraic curve does not grow.

We propose here a method that can be used, in the general case, to compute an equation for the image of $\phi_{2K}(X)$ under a projection into plane. Our main problem is how to choose this 'generic' plane in $\mathbb{P}^{3g-4}(\mathbb{C})$. That will be done in the following way.

We assume that X is a compact Riemann surface and express it in the form $X = D/G$ where G is a Fuchsian group of the second kind acting in the unit disk and $\Omega(G)$ is the domain of discontinuity of G. Let $\omega_1, \ldots, \omega_{3g-3}$ be a basis for $D^2(X)$ given by Theorem 3. By means of the Poincaré series (9) we can approximate values of these base differentials as precisely as we want.

Let $n_1^j, n_2^j, \ldots, n_{3g-3}^j$, $j = 1, 2, 3$, be three sets of random integers. Form the following three quadratic differentials

$$\eta_j = \sum_k n_k^j \omega_k,\ \ j = 1, 2, 3. \tag{11}$$

[3] Here U is an open set of X containing p and $z : U \to z(U)$ is a homeomorphism onto an open subset of the complex plane.

Assume that they are linearly independent. If they are not linearly independent, then choose another set of random integers n_m^j in such a way that the differentials η_j in (11) are linearly independent. Now form the mapping

$$\phi : X \to \mathbb{P}^2(\mathbb{C}),\ X \ni p \mapsto (\eta_1(p), \eta_2(p), \eta_3(p)). \tag{12}$$

Theorem 4. *Assume that X is a riemann surface of genus g, $g > 2$. For a generic choice of the differentials η_1, η_2, η_3, the image of X under the mapping (12) is an algebraic curve of degree at most $4g-4$ with at most ordinary quadratic singularities.*

This result is based on a special case of well–known classical results. For reader's convenience we will sketch a proof below. We would like to observe, at this point, that there is another way to form a basis for quadratic differentials that is due to Petri ([12] or [1, Page 127]).

In the case of non–hyperelliptic symmetric Riemann surfaces we may use Lemma 1 and study the canonical map instead of the bicanonical map. This gives us, for the given Riemann surface, an equation that is of lower degree than that of the above construction. By the results of [15] we have also some control over the inaccuracies of our approximations.

Let $\phi_{2K} :\longrightarrow \mathbb{P}^{\nVdash\eth-\not\trianglerighteq}$ be the bicanonical map defined by the basis $(\eta_1, \ldots, \eta_{3g-3})$. For $g > 2$ this map is an embedding. In particular $\phi_{2K}(X)$ is a smooth curve.

If V is the projective subspace of $\mathbb{P}^{\ltimes}$ defined by $x_0 = \cdots = x_m = 0$, $m < n$, then one can define the projection from V of $\mathbb{P}^{\ltimes} - \mathbb{V}$ onto $\mathbb{P}^{\gtrdot}$ by

$$(x_0, \ldots, x_m, x_{m+1}, \ldots, x_n) \longmapsto (x_0, \ldots, x_m)\,.$$

Choosing another set of homogeneous coordinates for $\mathbb{P}^{\ltimes}$ we can define the projection from any projective subspace.

Proof of Theorem 4 is based on the following well–known result (see e.g. [7, pp. 310–314].

Lemma 5. *There exists a non empty open set U in $\mathbf{G}_{(\mathbf{3g-7,3g-4})}$ (the space of projective subspaces of dimension $3g-7$ in $\mathbb{P}^{\nVdash\eth-\not\trianglerighteq}$) such that if $V \in U$ then,*

- *the projection from V of $\phi_{2K}(X)$ into $\mathbb{P}^{\nvDash}$ is well defined.*
- *the image of $\phi_{2K}(X)$ under this projection is a curve with at most ordinary quadratic singularities.*

In general this lemma is formulated by saying that if C is a smooth curve in $\mathbb{P}^{\ltimes}$ then there exists a non empty open set U_n such that the projection of C onto $\mathbb{P}^{\ltimes-\nVdash}$ from a point in U is a smooth curve if $n > 3$ and a curve with at most ordinary quadratic singularities if $n = 3$. Now it is easy to check that composing $m+1$ successive projections from points is the same as projecting from a projective subspace of dimension m. Now let U_n be as above and define $U_{n-1} \subset \mathbb{P}^{\ltimes-\nVdash}$ in a similar fashion. Choosing a point $p_0 \in U_n$ and a point $p_1 \in U_{n-1}$ defines a line in $\mathbb{P}^{\ltimes}$ (the pull-back of p_{n-1} by the projection from p_n). The existence of U_n and U_{n-1} implies the existence of an open set in the space of lines in $\mathbb{P}^{\ltimes}$. Obviously this argument generalizes and proves the existence of the open set U. This proves Lemma 5

The open set U of Lemma 5 is in general not easy to compute but to find a space V in U we can do the following. Let $\{a_1, \ldots, a_k\}$ be the coefficients of the polynomials defining $\phi_{2K}(X)$ and let $\{b_0, \ldots, b_{k'}\}$ be the coefficients of the linear equations defining V. If the family $\{a_1, \dots, a_k, b_0, \dots, b_{k'}\}$ is algebraically independent over $\mathbb{Q}$ then V is generic, in the sense of Weil, with respect to $\phi_{2K}(X)$. This implies that V will be in U. We can reformulate this in a less formal way, by saying that if we choose V "generic" then the conclusions of Lemma 5 will hold. Of course for a computer generic makes no sense but if we replace "generic" by "sufficiently general" then we can get a reasonably good approximation by taking a random V. This can be achieved in the following way. Let $\{\eta_k\}$ be the quadratic differentials we have obtained in the preceding section and let η_i, $1 \leq i \leq 3$, be random linear combinations of the η's then the curve in $\mathbb{P}^{\not\models}$ defined by (η_1, η_2, η_3) will be the projection of $\phi_{2K}(X)$ from a random V. These remarks prove Theorem 4.

6. An equation for a Riemann surface

In this section we put the pieces of the previous sections together to produce an algorithmic way to compute an equation for a Riemann surface. This works in the general case of Riemann surfaces of genus g, $g > 2$. It is *probabilistic* in the sense that it depends on choosing a sufficiently generic projection.

Assume that a Riemann surface X of genus > 2 is given either as $X = D/G$, where G is of the first kind, or as $X = \Omega(G)/G$, where G is of the second kind and X is a symmetric, non–hyperelliptic Riemann surface.

We proceed as follows.

1. First map X,using either the canonical map or the bicanonical map, to a projective space $\mathbb{P}^n(\mathbb{C})$ for suitable n. In both cases call this map $\phi : X \to \mathbb{P}^n(\mathbb{C})$.
   (a) If X is a non–hyperelliptic symmetric Riemann surface of genus g, then use the canonical mapping provided by the basis for holomorphic 1–forms given in Lemma 1.
   (b) If X is not symmetric or if it is hyperelliptic, then use the basis for holomorphic 2–forms given in Theorem 2.
2. Project down to the projective plane $\mathbb{P}^2(\mathbb{C})$ using Theorem 4. Let $\pi : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ denote the projection.
3. In the general case, $\pi \circ \phi(X) \subset \mathbb{P}^2(\mathbb{C})$ is an algebraic curve with simple singularities. The degree of this curve is at most $d = 4g - 4$.
4. A homogeneous polynomial of degree d in three variables has
$$m = (d+1)(d+2)/2$$
coefficients. Fit such a polynomial to go through $\pi \circ \phi(X)$ by determining $2m$ 'random' points of $\pi \circ \phi(X)$. This leads to solving a set of $2m$ linear equations in m unknowns. Using the singular value decomposition described in the next section, we compute the best polynomial to fit through these points.

7. Holomorphic differentials of an algebraic plane curve

An explicit way to compute holomorphic differentials of an algebraic plane curve is known since the XIX century (Nöther). In the case of curves whose singularities are at most ordinary double points, the problem becomes much simpler and a direct algorithm can be given based only on linear algebra.

To explain the algorithm we are going to propose, we will need to recall some facts.

Let C be a plane algebraic curve defined by a homogeneous equation

$$\tilde{f}(x,y,z)=0.$$

Assume that C has only ordinary quadratic singularities, then an *adjoint* of C is an algebraic plane curve, not necessarily irreducible, which passes through all the singular points of C. In particular, if C is smooth, then any curve is an adjoint of C.

Consider the affine part of $\mathbb{P}^2(\mathbb{C})$ defined by $z \neq 0$. Let (x,y) be the corresponding affine coordinates and $f(x,y)=0$ the corresponding affine equation for C. The classical result is:

Theorem 6. *The holomorphic differentials on C are of the form*

$$\frac{\varphi(x,y)dx}{\frac{\partial f}{\partial y}}, \tag{13}$$

where $\varphi(x,y)=0$ is the equation of an adjoint to C of degree $\leq \deg f-3$.

For a proof see e.g. [2, P. 841].

Hence, in order to find a basis for the space of holomorphic differentials on C, all we have to do is to find a basis of the space of adjoint polynomials, i.e., polynomials φ such that $\varphi(x,y)=0$ is an adjoint curve to $f(x,y)=0$ of degree $\leq \deg f-3$.

We can also assume that the given curve has no singularities at $z=0$. If this were the case we can perform the following coordinate transformation:

1. Find first a line not going through any of the singular points.
2. Do a projective transformation sending this line to $z=0$.

Under our hypothesis that $f(x,y)$ has only ordinary double points, the ideal of adjoint polynomials $\mathcal{I}$ coincides with the jacobian ideal of f which defines the singular locus:

$$\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y), f(x,y)\right). \tag{14}$$

We need to find a basis for the polynomials of degree $\leq \deg f-3$ in the adjoint ideal. If we let $n=\deg f$ then our adjoint ideal can be generated by three polynomials of degree $n-1$, namely :

$$\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y), nf(x,y)-x\frac{\partial f}{\partial x}(x,y)-y\frac{\partial f}{\partial y}(x,y)\right). \tag{15}$$

and the ideal has no solutions at infinity.

Under these hypotheses the following result holds:

Proposition 1. *Let $J=(f_1,f_2,f_3)$ be the jacobian ideal of a curve defined by a polynomial f(x,y) of degree n. Assume that J has no solutions at the infinity and the degree of f_i is $n-1$. Let $D=3(n-2)+1$. Then for every $g \in J$ with* $\mathrm{degree}(g) \leq D$ *there exist polynomials a_i such that*

- $g=\sum_i a_i f_i$
- $\mathrm{degree}(a_i f_i) \leq D$ *i.e.* $\mathrm{degree}(a_i) \leq 2n-4$

Proof. [5] and [11].

If we denote with $J_D=\{p \in J \mid \mathrm{degree}(p) \leq D\}$, the previous proposition allows one to describe a set of generators for it; our aim is to extract from this set of generators a basis for the holomorphic differentials. We have to take into account the fact that the coefficients of the polynomial describing the curve are only "approximately" known, but we can take advantage of additional information such as the genus and the number of singularities of our curve.

We fix on the set of polynomials of degree up to $D = 3(n-2)+1$ a monomial basis P_D, consisting of $N = \binom{D+2}{2}$ elements. We can suppose these elements ordered in increasing degree with respect to a total degree ordering.

We use the generators of J_D to construct the matrix M_D which represents the linear map which takes any three polynomials (a_1, a_2, a_3), each of degee up to $D-(n-1)$, to $\sum_i a_i f_i$.

The matrix M_D has N rows and three blocks of $\binom{2n-2}{2}$ columns and it is defined in the following way: each block corresponds to one of the generators of the ideal J and its i–th column contains the coordinates (with respect to the monomial basis of P_D previously chosen) of the product of the considered generator by the i–th monomial (with degree up to $D-(n-1)$). This matrix was introduced by Lazard in [10] and used for solving system of polynomial equations. In particular the number of solutions of a polynomial system is equal to the difference between the number of rows and the rank of its corresponding matrix.

If we assume the curve has only ordinary double points, then the jacobian ideal has a simple zero at each double point. Thus the number of solutions of the jacobian ideal is the number of singular points of the curve. Using the genus formula for plane curves with ordinary double points, we can compute the number of singular points and thus the rank of the matrix M_D.

Corollary 1. *Let J be the jacobian ideal of $f(x,y)=0$, curve of genus g, and M_D the matrix associated to J. If n is the* $\mathrm{degree}(f)$ *and f has δ ordinary double points (so $\delta = \frac{(n-1)(n-2)}{2} - g$) then* $\mathrm{rank}(M_D(J)) = N - \delta$ *where N is the number of rows of M_D.*

Since all of our polynomials are only "approximately" given, we need to use a numerically stable algorithm and one which is able to solve the "nearby" more singular problem. If we start with perturbed coefficients and try to treat them as exact, we will have "lost" our singular points and thus changed the genus of the curve, "nearby" to the given curve is the actual curve whose holomorphic differentials we are trying to determine. For this purpose we have chosen to base our algorithm on the singular value decomposition construction. After constructing a matrix which represents all multiples of our generators up to the degree of our bound, the problem of finding the elements of degree $\leq n-3$ can be reduced to the problem of computing the null space of the submatrix representing the terms of degree $> n-3$.

We are now able to describe how to find a basis for J_{n-3}. We compute U_D and V_D orthogonal matrices and S diagonal such that $M_D = U_D S {V_D}^T$ is the singular value decomposition of M_D, [6]. For our purposes the main property of this decomposition is that the diagonal entries of S, $\sigma_1, \ldots, \sigma_N$, are such that $\sigma_k \geq \sigma_{k+1}$ and each σ_k is the 2–norm distance to the nearest matrix of rank strictly less than k. So if our choices and hypothesis were satisfied, we will find $\sigma_k \approx 0$ for $k > rank M_D = N - \delta = r$. Before proceeding we impose that $\sigma_k = 0$ for $k > r$, i.e. that the ranks is precisely r. We can verify our assumption that the curve has only ordinary double points by checking σ_{r+1} is small relative to σ_r, (**check 1** in the algorithm given below).

Also, from the singular value decomposition we can recover a basis for J_D: since $M_D V_D = U_D S$, the first r columns of U_D are the coordinates of a basis for J_D. Let us call B the submatrix of U_D consisting of the first r columns.

Our aim is to produce a basis for J_{n-3}, i.e. the subspace of J_D of degree at most $n-3$. Let $Q = \binom{n-1}{2}$ be the number of monomials of degree at most $n-3$. We look for linear combinations of the columns of B with the last $N-Q$ entries equal to zero. We partition the matrix B into two matrices calling its first Q rows, B_Q and its last $N-Q$ rows, B_{N-Q}. If we compute an SVD for $B_{N-Q} = UTV^T$ we have to impose that $rank(M_D) - g = r - g = rank(B_{N-Q})$. We again check that the computed singular values are consistent with this rank assignment, (**check 2** in the algorithm given below).

At this point if we select the last g columns of V, call them W, we have $B_{N-Q}W = 0$ so , the columns of the matrix $B_Q W$ are the coordinates of a basis for J_{n-3}.

We can summarize the previous discussion in the following algorithm:

Algorithm:

- **Input:** $f(x,y) \in R[x,y]$ polynomial of degree n, corresponding to a curve C of genus g with only ordinary quadratic singularities, and not singular at infinity.
- **Output:** A basis for the holomorphic differentials of C.
- **Step 1:** Inizialization
  $\delta := \frac{(n-1)(n-2)}{2} - g$, the number of singular points of C
  $J :=$ the jacobian ideal of f
  $D := 3(n-2)+1$ the bound for the degree of the representation of a given polynomial in terms of a set of generators of J
  $N = \binom{D+2}{2}$ the number of monomials in two variables of degree up to D
  $M_D :=$ Lazard's Matrix of J with N rows and $3(n-1)(2n-3)$ columns
  $r := N - \delta$, rank of M_D
  $Q := \binom{n-1}{2}$ the number of monomials in 2 variables of degree up to $n-3$
- **Step 2:**
  $(U_D, S, V_D) :=$ Singular Value Decomposition of M_D
  Check 1: Compare the "expected" rank with size of the singular values.
- **Step 3:** $B :=$ matrix consisting of the first r columns of U_D, partition it into two submatrix B_Q and B_{N-Q}
- **Step 4:** $(U, T, V) :=$ Singular Value Decomposition of B_{N-Q}
  Check 2: Compare rank B_{N-Q} with $r - g$
- **Step 5:** $W :=$ matrix Consisting of the last g columns of V
- **Return:** $B_Q W =$ coordinates (with respect to P_D) of the special adjoints of C.

Example. Consider the approximate curve: $f = 4y^4 + 17x^2y^2 - 19.57241479y^2 +1.3078y^2x + 4x^4 + 5.2312x^3 - 18.28965916x^2 - 5.2312x + 15.28965916$
Since $n = 4$, we have that $D = 3(n-2)+1 = 7$. The number of monomials of degree at most 7 is $N = \binom{D+2}{2} = 36$. The number of columns in each of our three blocks is the number of monomials of degree at most $2n-4 = 4$ which is $\binom{6}{2} = 15$. Thus our matrix M_D will have 36 rows and $3 \times 15 = 45$ columns. This curve is an approximation to a curve of genus two and thus has $\frac{(4-1)(4-2)}{2} - 2 = 1$ singular point. We need to impose that the rank of M_D should be $36 - 1 = 35$. The singular values for M_D are:

[145.240116769802, 142.3483360156971, 138.4051357899331,
137.85838580267691, 107.92997252322522, 101.73489738240572,
100.85055517539476, 98.275085626940864, 95.599803213284105,
92.979203049269515, 91.667583311635511, 87.388875284493949,
82.084551344358147, 71.466206332955082, 65.103790813146333,
64.170342986758357, 60.12703118386866, 57.610626034190155,
57.122327818474155, 50.062253889010123, 49.945860804547635,
24.376716864448156, 21.57351682708407, 17.028407910134916,
16.835519791397523, 16.213158394929785, 12.779409591972886,
11.465354614276119, 10.191933115097552, 6.4302241437740824,
4.1233572523901749, 3.265016592149979, 2.1434305926157369,
0.98823269770440114, 0.9074280426361615, 7.653479560680403$E-6$]
and indeed we find that $\sigma_{36} = 7.653479560680403E-6 \approx 0.0$ is very small relative to $\sigma_{35} = 0.9074280426361615$, which agrees with our assertion $rank(M_D) = 35$.

We call B the matrix obtained from the first 35 columns of the U matrix from the SVD of M_D and partition it into B_Q and B_{N-Q} where $Q = \binom{(4-3)+2}{2} = 3$.

We expect the column null space of B_{N-Q} to have the same dimension as the genus of our curve, i.e. 2. So we must impose that the rank is $35 - 2 = 33$.

The singular value spectrum for B_{N-Q} is:
[1.0000000000000004, 1.0000000000000004, 1.0000000000000002,
1.0000000000000002, 1.0000000000000002, 1.0000000000000002,
1.0000000000000002, 1.0000000000000002, 1.0000000000000002,
1.0000000000000002, 1.0000000000000002, 1.0000000000000002,
1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 0.99999999999999989, 0.99999999999999989,
0.99999999999999989, 0.99999999999999978, 0.99999999999999978,
0.99999999999999967, 0.99999999999999967, 0.99999999999999967,
0.99999999999999944, 0.99999999999999944, 0.99999999999999944,
0.99999999999999933, 0.99999999999999933, 0.26360625700904228,
2.5692155270289153$E-11$, 0.0]
in this case $\sigma_{33} \approx .26$ while $\sigma_{34} = 2.5692155270289153E-11 \approx 0.0$, which agrees with our assertion $rank(B_{N-Q}) = 33$.

Let W be the last two columns of the V matrix of the singular value decomposition of B_{N-Q}. The product $B_Q W$ represents the coordinates of the special adjoint polynomials:

$$\begin{bmatrix} -0.76390966168862817 & -8.7146632417057773E-14 \\ 0.64532319714912301 & -2.7480475025175656E-13 \\ -1.1087064319047189E-13 & -1.0000000000000004 \end{bmatrix}$$

Since the rows of this matrix are indexed by $1, x, y$, this represents the two polynomials $\{0.64532319714912301x - 0.76390966168862817, -y\}$ or equivalently $\{x - 1.1837629037099404, y\}$. Hence a basis for the space of holomorphic differentials is given by

$$\frac{(x-1.1837629037099404)dx}{16y^3+34x^2y+2.6156xy-39.14482958y} \text{ and } \frac{ydx}{16y^3+34x^2y+2.6156xy-39.14482958y} \tag{16}$$

This is the correct number of generators for holomorphic differentials of C, since the genus of C (or of the normalization of C) is

$$\frac{(4-2)(4-1)}{2} - 1 = 2. \tag{17}$$

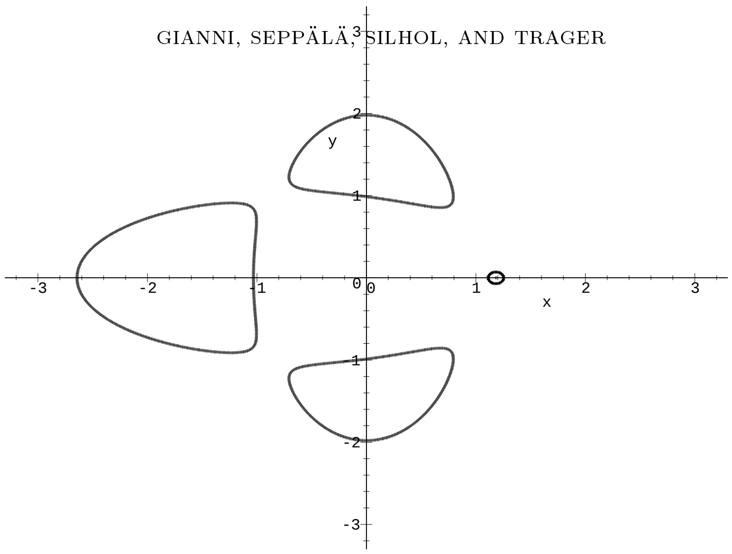

FIGURE 3. The three ovals represent the real part of the approximate curve f of the example. This curve approximates a curve having a double point at $x = 1.1837629037099404$, $y = 0$. The coordinates for this singular point can be read from the numerators of the holomorphic differentials (16). This point is the center–point of the small circle on the right hand side of the three ovals. The singularity would disappear if we were to treat the equation as exact. This is a general phenomena and because of such inaccuracies, the singular points cannot be traced by plotting the curves numerically. One needs stronger symbolic methods like the ones presented here.

8. Approximation of period matrices of compact Riemann surfaces

The above considerations provide theoretical bases for numerical approximation of period matrices of Riemann compact Riemann surfaces. We proceed as follows.

8.1. **Defining Riemann surfaces for computations.** We use the considerations of Section 2 and the program CARS to build compact genus p Riemann surfaces from $p-2$ Y–pieces and from p Q–pieces. The program makes a list of the generators of the Y–pieces and the Q–pieces in question and builds a "random" Fuchsian group G, acting either in the unit disk D in such a way that $X = D/G$ is a "random" compact Riemann surface of genus p.

Observe:

- The generators of the Q–pieces give us immediately a canonical homology basis. This is the homology basis that will be used in the computation of a period matrix.
- The generators of the Y–pieces together with one of the generators of each of the Q–pieces give us a partition on the Riemann surface in question. This

is the partition that we use in Theorem 2 to compute numerically a basis for the space of the quadratic differentials of the Riemann surface.

The program CARS allows one to define "random" Riemann surfaces of any genus. The word random in the above is in quotation marks because the Riemann surface is not truly random. Its parameters are only as good approximations of true random numbers as what is possible using a computer.

It is also possible to build Riemann surfaces with given parameters using the program CARS, that is one option offered to the user. In trying to understand Riemann surfaces it is usually not a single Riemann surface that one is interested in. One rather wants to see what happens to Riemann surfaces as we deform them. This is possible by the program CARS. One can perform Fenchel–Nielsen twists in a very convenient manner by a mouse. It is also possible to change lenghts of some of the geodesic curves along which one makes twist deformations. In this way one can actually walk around in the Teichmüller space of compact Riemann surfaces and see, on the computer screen, how the group gets deformed.

8.2. **Period matrices.** After having defined a Riemann surface $X = D/G$ by the program CARS as described above, we first use the considerations of Section 5 to approximate numerically the bicanonical mapping $\phi_{2K} : X \to \mathbb{P}^{3p-4}(\mathbb{C})$.

Let $\pi : \mathbb{P}^{3p-4}(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ be the projection defined in terms of the differentials of Theorem 4. Then $\pi \circ \phi_{2K}(X)$ is a plane algebraic curve of degree $4p - 4$ (at most).

Compute an equation for the plane algebraic curve $\pi \circ \phi_{2K}(X)$ by using the considerations of Section 6.

Next use the considerations of Section 7 to find a basis for holomorphic 1-forms of the algebraic curve $\pi \circ \phi_{2K}(X)$. Call this basis $\{\omega_1, \ldots, \omega_g\}$.

As the Riemann surface X was defined for computations, also a canonical homology basis was automatically defined. Call the simple closed curves corresponding to this basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$. For our computations these curves are expressed as arcs on the axes of the hyperbolic Möbius transformations corresponding to each of the Q–pieces that were used in the construction. So they are explicitly given.

The final step is to compute, numerically, the integrals

$$\int_{\phi_{2K}(\alpha_j)} \omega_i \quad \text{and} \quad \int_{\phi_{2K}(\beta_j)} \omega_i \tag{18}$$

which are the entries of a period matrix of the Riemann surface X.

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