Hurwitz Spaces and Braid Group Representations

by

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0. Introduction

In this paper we investigate certain moduli spaces ("Hurwitz spaces") of branched covers of the Riemann sphere S^2, and representations of finite index subgroups of the spherical braid group which arise from these Hurwitz spaces. (By *spherical* braid group, we mean the group of braids in the 2-sphere; we will refer to the more classical group of braids in the plane as the *planar* braid group.)

Hurwitz spaces play an important role in realizing groups as Galois groups, a role which has been explored primarily by Fried and Volklein in [F1], [F2], [FV], [V1], [V2], etc. In these works, they have given a couple of constructions of Hurwitz spaces, examined their algebraic structure, and explored their applications to the inverse Galois problem. In section 1 of this paper we give an alternative construction of the Hurwitz spaces, exhibiting them as homogeneous spaces of $\mathrm{Aut}(S^2)$, the group of orientation preserving homeomorphisms of S^2. This point of view enables us to prove that the universal cover of a Hurwitz space is homotopy equivalent to S^3 (see the discussion just after Proposition 4), which is equivalent to showing that the Teichmuller space of a sphere with 3 or more punctures is contractible. Of course this is not a new result, but we believe our rather elementary topological proof is interesting enough to include.

In [A], Arnol'd described representations of the planar braid groups on the homology of hyperelliptic curves. Later, Magnus and Peluso [MP] analyzed these representations and related ones obtained from "generalized" hyperelliptic curves in more detail and expressed them in terms of Burau representations. (Note that the Burau representation doesn't satisfy the extra relation it would need to provide a representation of the spherical braid group.) In [F2], M. Fried used Hurwitz spaces to describe an action of certain finite index subgroups of the spherical braid group on the homology of Riemann surfaces given as branched covers of S^2 without automorphisms. (Since these subgroups have finite index, one may obtain representations of the whole spherical braid group by inducing.) In Section 1 of this paper, we describe these representations using the topological point of view developed in our construction of the Hurwitz spaces. We also show that in the case where the original branched cover has nontrivial automorphisms, instead of obtaining a representation of a subgroup of the spherical braid group, one obtains a representation of an *extension* of this subgroup, where the extension is precisely by the group of automorphisms of the original covering space of S^2.

To compute explicit examples of these representations we give, in Section 2, an algorithm that enables us to construct a homology basis of a branched cover of S^2 given its combinatorial description. Such algorithms have been described before (e.g., Tretkoff and Tretkoff [T]). We give one which is different than what we have seen in print, and seems best suited to our needs.

In Section 3, we use this algorithm to compute explicitly two examples which illustrate the ideas above –

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one with automorphisms and one without automorphisms. For the example with automorphisms, we use the classical example of hyperelliptic curves. This example was discussed by Arnol'd and by Magnus and Peluso, but they only considered the planar braid group, not the spherical one (i.e., they didn't let any non-infinite branch points pass through infinity and, if infinity was a branch point, they didn't let it move around). Since the automorphism group is Z_2 in this case, our general theory shows that one obtains a representation of a Z_2 extension of the spherical braid group. We show that, in this case, the representation does not factor through a representation of the unextended braid group. For the example without automorphisms, we compute generators of the appropriate subgroup of the braid group and compute their images under the representation as automorphisms of the homology of the covering surface. We then use a theorem of Fried to conclude that a generic Riemann surface of genus 1 can be mapped analytically to S^2 such that the resulting branched cover has the same combinatorial description as this second example.

1. A Topological Construction of Hurwitz Spaces

Let $\phi : \Sigma \rightarrow S^2$ be an n-sheeted branched cover, and let $X = \{x_1, \ldots, x_r\}$ denote the points of S^2 over which branching occurs. Suppose one chooses a new set $X' = \{x'_1, \ldots, x'_r\}$ with each x'_i close to its corresponding x_i. One may deform the original branched cover to a new one in which the branching takes place over X' instead of X, and whose structure over a neighborhood of X' corresponds precisely to the old structure over a neighborhood of X. To be a bit more precise, one may obtain the new branched cover $\phi' : \Sigma \rightarrow S^2$ by letting $\phi' = g \circ \phi$, where g is a self-homeomorphism of S^2, close to the identity, taking each x_i to x'_i. We will construct a connected moduli space H, called a *Hurwitz space*, encoding the above deformations in the sense that each point of H corresponds to an equivalence class of branched covers, where $\phi_1 : \Sigma_1 \rightarrow S^2$ is equivalent to $\phi_2 : \Sigma_2 \rightarrow S^2$ if and only if there is a homeomorphism $h : \Sigma_1 \rightarrow \Sigma_2$ such that $\phi_1 = \phi_2 \circ h$.

We will see that the construction of this Hurwitz space H is straightforward.

Next, we would like to construct a total space U which is a bundle over H with fiber Σ in which each fiber maps to S^2 by the branched cover corresponding to the point in the base space H over which that fiber lies, and all of these maps fit together to give a continuous map $\Phi : U \rightarrow S^2$. One interesting result of this construction (when it is possible!) is a natural action (up to isotopy) of $\pi_1(H)$ on Σ, and hence a representation of $\pi_1(H)$ on $H_1(\Sigma)$ (in this paper, all homology will be with integer coefficients unless otherwise indicated). It turns out that the construction of U is straightforward if we assume there are no non-trivial automorphisms (i.e., self-equivalences) of the original branched cover $\phi : \Sigma \rightarrow S^2$. However, if we start with a ϕ that admits nontrivial automorphisms, then one cannot in general construct U with the above properties and cannot obtain a natural representation of $\pi_1(H)$ on $H_1(\Sigma)$ as just described. In this case, we will show how to make a related construction which results in a representation on $H_1(\Sigma)$ of an extension of $\pi_1(H)$ by $\mathrm{Aut}(\Sigma)$. In Section 3 of this paper, we will give an example in which this extension is nontrivial and the representation doesn't factor through $\pi_1(H)$.

We now turn to the construction of H. As above, fix an n-sheeted branched cover $\phi : \Sigma \rightarrow S^2$ and let $X = \{x_1, \ldots, x_r\}$ denote the points of S^2 over which branching occurs. Let $\mathrm{Aut}(S^2)$ denote the group of orientation-preserving homeomorphisms of S^2 with the compact-open topology. The key idea in this construction is to define H as a homogeneous space of the group $\mathrm{Aut}(S^2)$. We begin with the following lemma. (We thank Allen Hatcher for showing us its proof.)

Lemma 1: *The inclusion $SO(3) \to Aut(S^2)$ is a weak homotopy equivalence (i.e., it induces an isomorphism on all homotopy groups and, hence, on all homology groups).*

Proof: Kirby and Siebenmann ([KS], p. 254) prove that $O(2) \to \mathrm{Homeo}(R^2)$ is a homotopy equivalence. Using the 5-lemma to compare the homotopy exact sequences of the fibrations

$$O(2) \to O(3) \to S^2$$

and

$$\mathrm{Homeo}(R^2) \to \mathrm{Homeo}(S^2) \to S^2$$

then proves the lemma.

Note: The stronger theorem, that $SO(3) \to Aut(S^2)$ is a homotopy equivalence, is proved by Haver in [H].

Define the "large diagonal" $\Delta \subset (S^2)^r$ by $\Delta = \{(y_1, \ldots, y_r) : y_i = y_j \text{ for some } i \neq j\}$. Let S_r denote the symmetric group and define

$$\Pi = \frac{(S^2)^r - \Delta}{S_r}.$$

Clearly, $(S^2)^r - \Delta \to \Pi$ is a covering space. Let P denote the composition

$$\mathrm{Aut}(S^2) \to (S^2)^r - \Delta \to \Pi,$$

where the first map is the evaluation $f \mapsto (f(x_1), \ldots, f(x_r))$. Define the following three subgroups of $\mathrm{Aut}(S^2)$:

$$\mathcal{G} = P^{-1}[x_1, \ldots, x_r]$$

$$\mathcal{G}_0 = \text{ the path component of } \mathcal{G} \text{ containing Id}$$

$$G = \{g \in \mathrm{Aut}(S^2) : \text{ there exists a homeomorphism } h : \Sigma \to \Sigma \text{ with } \phi = g \circ \phi \circ h\}$$

We now observe that $\mathcal{G}_0 \subseteq G \subseteq \mathcal{G}$: The second of these inclusions is easy since, by definition, if $g \in G$ then the branched covers ϕ and $g \circ \phi$ are equivalent and, hence, have the same branch locus in S^2. The first of these inclusions follows immediately from Lemma 3, below.

We define $H = \mathrm{Aut}(S^2)/G$ to be our Hurwitz space, where we are associating to each coset gG the branched cover $g \circ \phi : \Sigma \to S^2$. Note that, since $\Pi = \mathrm{Aut}(S^2)/\mathcal{G}$ and G contains the identity component of $\mathcal{G}$, the map $H = \mathrm{Aut}(S^2)/G \to \mathrm{Aut}(S^2)/\mathcal{G} = \Pi$ exhibits H as a covering space of Π. This cover is finite-sheeted since once you fix the images of the branch points there are only a finite number of n-sheeted branched covers of S^2 (up to equivalence). The fundamental group of Π is the well known r-strand braid group of S^2, which we will denote $B_r(S^2)$. Thus $\pi_1(H)$ is a finite index subgroup of $B_r(S^2)$.

We now pause to prove two lemmas involving $\mathcal{G}_0$. One of these lemmas we have already used; we will also need them in what follows.

Lemma 2: *If $r \geq 3$, then $\pi_i(\mathcal{G}_0) = 0$ for all i.*

Lemma 3: *Given $g \in \mathcal{G}_0$, there is a homeomorphism $h_g : \Sigma \to \Sigma$ such that $g \circ \phi = \phi \circ h_g$. If $r \geq 3$, there is a unique way of choosing each h_g such that $g \mapsto h_g$ defines a continuous group homomorphism $\mathcal{G}_0 \to Homeo(\Sigma)$ and such that ϕ is equivariant with respect to the resulting action of $\mathcal{G}_0$ on Σ.*

Proof of Lemma 2: During this proof, we will incorporate the number r into the notation for $\mathcal{G}_0$ and related constructions as a superscript in order to keep track of which r we are considering. Let $\bar{\mathcal{G}}^r = \{g \in \mathrm{Aut}(S^2) : g(x_i) = x_i \text{ for } 1 \leq i \leq r\}$. Clearly $\mathcal{G}_0^r$ is the identity component of $\bar{\mathcal{G}}^r$ and $\mathrm{Aut}(S^2)/\bar{\mathcal{G}}^r = (S^2)^r - \Delta$. It follows that $Q = \mathrm{Aut}(S^2)/\mathcal{G}_0^r$ is a covering space of $(S^2)^r - \Delta$. We will now show that $\pi_i(\mathcal{G}_0^r) = 0$ for all r and for $i \geq 3$. Consider the homotopy sequence of the fibration

$$\Gamma \to (S^2)^r - \Delta \to (S^2)^{r-1} - \Delta \tag{1}$$

where Γ is the complement of $r-1$ points in S^2 and, hence, homotopy equivalent to a graph. It immediately follows that $\pi_i((S^2)^r - \Delta) \to \pi_i((S^2)^{r-1} - \Delta)$ is an isomorphism for $i \geq 3$ and $r > 1$. By looking at a sequence of such maps, we see that for this range of i and r, $\pi_i((S^2)^r - \Delta) \cong \pi_i(S^2)$. Now consider the commutative diagram

$$\begin{array}{ccl}
\pi_i(\mathrm{Aut}(S^2)) & \to & \pi_i(Q) \\
 & & \downarrow \\
\uparrow & & \pi_i((S^2)^r - \Delta) \\
 & & \downarrow \\
\pi_i(SO(3)) & \to & \pi_i(S^2)
\end{array}$$

The two vertical maps on the right are isomorphisms by what we just proved and because one is induced by a covering map. The horizontal map on the bottom is an isomorphism by the fibration $S^1 \to SO(3) \to S^2$. The left-hand vertical map (induced by inclusion) is an isomorphism by Lemma 1. It follows that the top map is an isomorphism for $i \geq 3$ and for all $r \geq 1$. Applying this fact to the fibration $\mathcal{G}_0^r \to \mathrm{Aut}(S^2) \to Q$ and its long exact homotopy sequence

$$\cdots \to \pi_i(Q) \to \pi_{i-1}(\mathcal{G}_0^r) \to \pi_{i-1}(\mathrm{Aut}(S^2)) \to \pi_{i-1}(Q) \to \cdots \tag{2}$$

proves that $\pi_i(\mathcal{G}_0^r) = 0$ for $i \geq 3$ and for all $r \geq 1$. Because, in addition, we know that $\pi_2(\mathrm{Aut}(S^2)) = \pi_2(SO(3)) = 0$, it follows that $\pi_2(\mathcal{G}_0^r) = 0$. To show that, for $r \geq 3$, $\pi_1(\mathcal{G}_0^r) = 0$, we observe from the long exact sequence (2) that it will suffice to show first that $\pi_2(Q) = 0$ and secondly that $\pi_1(\mathrm{Aut}(S^2)) \to \pi_1(Q)$ is injective.

For the first, we instead show that $\pi_2((S^2)^r - \Delta) = 0$ which is equivalent since $Q \to (S^2)^r - \Delta$ is a covering map. Since $PSL(2,C)$ acts freely and transitively on $(S^2)^3 - \Delta$, it follows that $\pi_2((S^2)^3 - \Delta) \cong \pi_2(PSL(2,C)) \cong \pi_2(SO(3)) = 0$. Now assume inductively that $r \geq 4$ and $\pi_2((S^2)^{r-1} - \Delta) = 0$. The inductive step then follows immediately from the fibration (1) above and its long exact sequence.

We now need to show that $\pi_1(\mathrm{Aut}(S^2)) \to \pi_1(Q)$ is injective for $r \geq 3$. First, note that $\pi_1(\mathrm{Aut}(S^2)) \to \pi_1((S^2)^3 - \Delta)$ is injective since $\mathrm{Aut}(S^2) \simeq SO(3) \simeq (S^2)^3 - \Delta$. By applying π_1 to the commutative diagram

$$\begin{array}{ccl}
 & & Q \\
 & \nearrow & \downarrow \\
\mathrm{Aut}(S^2) & \to & (S^2)^r - \Delta \\
 & \searrow & \downarrow \\
 & & (S^2)^3 - \Delta
\end{array}$$

and using the fact that $\pi_1(\mathrm{Aut}(S^2)) \to \pi_1((S^2)^3 - \Delta)$ is an isomorphism, the desired injectivity follows for $r \geq 3$. This completes the proof of lemma 2.

Proof of Lemma 3: Let $g \in \mathcal{G}_0$. Choose a path g_t in $\mathcal{G}_0$ from the identity to g. Let $y \in \Sigma - \phi^{-1}(X)$. Let $\alpha : I \to \Sigma - \phi^{-1}(X)$ be the lift of the path $g_t(\phi(y))$ which starts at y, and define $h_g(y) = \alpha(1)$. Define h_g

to be the identity on $\phi^{-1}(X)$. Then $h_g : \Sigma \to \Sigma$ is a homeomorphism and $g \circ \phi = \phi \circ h_g$. Furthermore, if $r \geq 3$ then, since $\pi_1(\mathcal{G}_0) = 0$, any two such paths g_t would lead to homotopic paths in $S^2 - X$. Hence, for $r \geq 3$, $g \mapsto h_g$ is a well-defined homomorphism $\mathcal{G}_0 \to \mathrm{Homeo}(\Sigma)$ making ϕ equivariant. This completes the proof of Lemma 3.

Clearly, the subgroups $\mathcal{G}$, G, and $\mathcal{G}_0$ are not normal in $\mathrm{Aut}(S^2)$. However $\mathcal{G}_0$, being the path component of the identity, is normal in both G and $\mathcal{G}$. As a result, we know that the covers $Q \to H$ and $Q \to \Pi$ are regular covers, with deck groups $G/\mathcal{G}_0$ and $\mathcal{G}/\mathcal{G}_0$ respectively. The question of whether G is normal in $\mathcal{G}$ (or, equivalently, whether $H \to \Pi$ is regular) is more subtle and will depend on the original branched cover $\phi : \Sigma \to S^2$. It is fairly easy to see that if $\phi : \Sigma \to S^2$ is not regular than G cannot be normal in $\mathcal{G}$. However the converse is not true.

By considering the exact sequence (2) introduced in the proof of Lemma 2, we see that Q is homotopy equivalent to $SO(3)$ (since both are weakly homotopy equivalent to $\mathrm{Aut}(S^2)$). The group $PSL(2, C) \subset \mathrm{Aut}(S^2)$ acts on Q from the left. Since $SO(3) \to PSL(2, C) \to Q$ is a homotopy equivalence, we have:

Proposition 4: *The space $PSL(2, C)\backslash Q$ is contractible.*

Note: This is not a new result, since $PSL(2, C)\backslash Q$ is just the Teichmuller space of the r-punctured sphere.

Since $\pi_1(Q) = Z_2$, it follows that Q is not quite the universal cover of Π. We conclude that the universal cover of Q and, in fact, of all the spaces H (no matter which branched cover we began with, as long as $r \geq 3$) is homotopy equivalent to S^3, the universal cover of $SO(3)$. To understand more precisely the covering space $Q \to \Pi$, we will begin by recalling some basic facts about the braid group $B_r(S^2) = \pi_1(\Pi)$. (Our main reference for these facts is Birman's book [B].) A presentation for $B_r(S^2)$ is given by the generators $Q_1, \ldots, Q_{r-1}$ and the following relations:

$Q_i Q_j = Q_j Q_i$ when $|j - i| > 1$

$Q_i Q_{i+1} Q_i = Q_{i+1} Q_i Q_{i+1}$ for $i = 1, \ldots, r-1$

$Q_1 Q_2 \cdots Q_{r-1} Q_{r-1} Q_{r-2} \cdots Q_1 = 1.$

Note that the first two relations give the classical r-strand braid group of the disc, while the third is necessary because our braids are in S^2. This presentation may be found in [B], p. 34], but was discovered originally by Fadell and Van Buskirk [FB]. The center of this group is the subgroup of order 2 generated by $(Q_1 \cdots Q_{r-1})^r$. (For a proof, see [B], p. 154 or [G].) This generator is pictured in figure 1. If one pictures the r strands as drawn lengthwise on a single ribbon, this generator corresponds to performing a single full twist of the ribbon.

Figure 1

Proposition 5: *The covering space $Q \to \Pi$ corresponds to the center of $\pi_1(\Pi)$*

Proof: Since $\mathcal{G}_0$ is contractible, $\pi_1 Q = \pi_1 \mathrm{Aut}(S^2) = \pi_1 SO(3) = Z_2$. Furthermore, the generator of this Z_2 is a single rotation of S^2. The braid arising from this rotation is precisely the indicated generator of the center of the braid group, proving the proposition.

We now turn to the construction of the universal bundle U over H. For the remainder of this section, we assume that $r \geq 3$. It is natural to construct this bundle first over $\mathrm{Aut}(S^2)$ since H is a quotient of $\mathrm{Aut}(S^2)$. Thus, define

$$\Phi : \mathrm{Aut}(S^2) \times \Sigma \to S^2$$

by $\Phi(f, x) = f(\phi(x))$. For each $f \in \mathrm{Aut}(S^2)$, observe that $\Phi|\{f\} \times \Sigma$ is precisely the branched cover $f \circ \phi$. Note that $\mathcal{G}_0$ acts on $\mathrm{Aut}(S^2) \times \Sigma$ from the right by $(f, x) \cdot g = (f \circ g, h_g^{-1}(x))$ (where $g \mapsto h_g$ is the homomorphism defined in Lemma 3). Clearly, $(\Phi(f, x)) \cdot g = \Phi((f, x) \cdot g)$; in other words, g identifies the fiber over f with the fiber over $f \circ g$ by an equivalence between the branched covers obtained by restricting Φ to these fibers. Define

$$W = (\mathrm{Aut}(S^2) \times \Sigma)/\mathcal{G}_0;$$

then Φ induces a map (which we continue to call by the same name) $\Phi : W \to S^2$. Note that W is a fiber bundle over $Q = \mathrm{Aut}(S^2)/\mathcal{G}_0$ with fiber Σ. Because $\mathcal{G}_0$ is connected, there is a canonical homeomorphism (up to isotopy) between any two fibers of $W \to Q$, and hence a canonical isomorphism between their homology groups.

We have an action of $G/\mathcal{G}_0$ on Q with quotient H. Can this action be covered by an action of $G/\mathcal{G}_0$ on W by maps which induce equivalences between the fibers (with respect to the branched covers induced by Φ)? If so, the quotient will give us the desired space U. The answer to this question is, in general, no, but depends on the group $\mathrm{Aut}(\phi)$ of automorphisms (self-equivalences) of the branched cover $\phi : \Sigma \to S^2$. Define

$$\tilde{G} = \{(g, h) : g \in G, h : \Sigma \to \Sigma \text{ is a homeomorphism, and } g \circ \phi \circ h^{-1} = \phi\}.$$

There is an exact sequence

$$1 \to \mathrm{Aut}(\phi) \to \tilde{G} \to G \to 1.$$

Clearly, $\tilde{G}$ acts on $\mathrm{Aut}(S^2) \times \Sigma$ from the right by $(f, x) \cdot (g, h) = (f \circ g, h^{-1}(x))$.

For the remainder of this section, we assume that $r \geq 3$. Lemma 3 provides a continuous injective group homomorphism $\mathcal{G}_0 \to \tilde{G}$ defined by $g \mapsto (g, h_g)$. The image of this map is the identity component of $\tilde{G}$ and, hence, a normal subgroup of $\tilde{G}$. Hence we have the exact sequence

$$1 \to \mathrm{Aut}(\phi) \to \tilde{G}/\mathcal{G}_0 \to G/\mathcal{G}_0 \to 1. \tag{3}$$

Furthermore, the group $\tilde{G}/\mathcal{G}_0$ acts on W in a way that covers the action of $G/\mathcal{G}_0$ on Q (as the group of deck transformations of $Q \to H$). If $\mathrm{Aut}(\phi) = 1$, then $\tilde{G}/\mathcal{G}_0 = G/\mathcal{G}_0$ and $U = W/(G/\mathcal{G}_0)$ is our universal Σ-bundle over H which, in turn, yields the desired representation of $\pi_1(H)$ on $H_1(\Sigma)$. In general, however, we obtain such a universal bundle only if there is a monomorphism $G/\mathcal{G}_0 \to \tilde{G}/\mathcal{G}_0$ splitting the exact sequence (3) above.

2. The Homology of a Branched Cover: An Algorithm

Since we now wish to carry out these computations for specific examples, we will need an algorithm for calculating a basis of the fundamental group and first homology of a branched cover of S^2. Because this procedure will be used more than once in this paper, we give a fairly detailed explanation in this section.

Let S_n (i.e., the symmetric group) act on $\{1,\ldots,n\}$ from the right. Let $\{x_1,\ldots,x_r\}$ be a set of r distinct points in S^2; denote by X the disjoint union of r open discs in S^2, one centered at each x_i. Let $w_1,\ldots,w_{r-1}$ be based, pairwise disjoint (except at the basepoint) simple loops in S^2-X such that each w_i winds once counterclockwise around x_i, but encloses none of the other x_j. Hence $w_1w_2\ldots w_{r-1}$ is a loop which winds once clockwise around x_r. It follows that $\pi_1(S^2-X)=F_{r-1}=F(w_1,\ldots,w_{r-1})$.

Assume we are given a representation $\rho : F_{r-1}\to S_n$ such that $\rho(F_{r-1})$ acts transitively on $\{1,\ldots,n\}$. Denote $\rho(w_i)$ by ρ_i. Let $\phi:\Sigma_0\to S^2-X$ denote the connected covering space corresponding to the subgroup $\rho^{-1}(Stab(1))$. In the natural way, identify $\phi^{-1}(\text{basepoint})$ with the set $\{1,\ldots,n\}$. Let w_{ij} (for $1\le i\le r-1$ and $1\le j\le n$) denote the lift of w_i to Σ_0 which starts at the point j (and hence ends at the point $(j)\rho_i$). Let $\phi:\Sigma\to S^2$ denote the branched cover obtained by glueing a disjoint union of discs (one for each component of $\partial\Sigma_0$) to Σ_0, each disc being attached by a homeomorphism from its boundary to the corresponding component of the boundary of Σ_0. Then ϕ is extended to Σ as a branched cover in the obvious way, so that all branch points are contained in $\phi^{-1}\{x_1,\ldots,x_r\}$. Following Magnus, we refer to the $(r-1)$-tuple $(\rho_1,\ldots,\rho_{r-1})$ as the *signature* of the branched cover. We now give an algorithm for getting from the signature of Σ to a set of generators of $\pi_1(\Sigma)$ which is also a basis of the free abelian group $H_1(\Sigma)$.

Step 1: Denote by Γ the subset of S^2 with the following structure as a graph: its only vertex is the basepoint, and its edges are the closed loops $w_1,\ldots,w_{r-1}$. Let

$$\tilde{\Gamma}=\phi^{-1}(\Gamma)=\{1,\ldots,n\}\cup\bigcup_{i,j}w_{ij}.$$

Let T denote a maximal tree in $\tilde{\Gamma}$ (probably found using a breadth-first search) and, for each $i=1,\ldots,n$, let α_i denote the unique simple edge path in T joining 1 to i. Note: α_i should be expressed as a word in the w_{ij}. Also, if you use a breadth-first search to find T, the α_i may be recorded virtually as a by-product.

Step 2: It follows that $\pi_1(\Sigma_0,1)$ is free, with one generator u_{ij} for each $w_{ij}\notin T$. To be precise, $u_{ij}=\alpha_j w_{ij}\alpha^{-1}_{(j)\rho_i}$. Of course, thought of as homology classes, the u_{ij} form a basis of the free abelian group $H_1(\Sigma_0)$.

Step 3: Define a matrix A with $(r-1)n$ columns as follows: index the columns by the ordered pairs (i,j) where $1\le i\le r-1$ and $1\le j\le n$. Given a cycle $(i_1,\ldots,i_k)$ appearing in ρ_i, construct a corresponding row of A to have a 1 in each of the columns $(i,i_1),(i,i_2),\ldots,(i,i_k)$ and a 0 in all other columns. Do this for every cycle in every ρ_i for $1\le i\le r-1$.

Next, let $\rho_r=\rho_1\rho_2\ldots\rho_{r-1}$. For each $i=1,\ldots,n$, let

$$E_i=\{(1,i),(2,(i)\rho_1),(3,(i)\rho_1\rho_2),\ldots,(r-1,(i)\rho_1\rho_2\ldots\rho_{r-2})\}.$$

Given a cycle $C=(i_1,\ldots,i_k)$ appearing in ρ_r, let $E_C=E_{i_1}\cup E_{i_2}\cup\ldots\cup E_{i_k}$. (This is a disjoint union.) Then define a new row of A to have a 1 in column (i,j) for each $(i,j)\in E_C$ and a 0 in all other columns. In this manner define one new row of A for each cycle C appearing in ρ_r. This completes the construction of A; it has one row for each cycle in each of the permutations $\rho_1,\ldots,\rho_r$. Clearly, A is the matrix of the boundary operator from the 2-chains to the 1-chains of Σ, since each cycle in each ρ_i corresponds to the boundary of a 2-cell.

Step 4: The purpose of step 4 is to choose a subset of the set $\{u_{ij}\}$ which will actually be a basis of $H_1(\Sigma)$. Before describing the algebra, we give a geometric description of this process. First remove a 2-cell from Σ. (This corresponds to deleting one row of A.) This does not change the homology of the surface. Next perform a sequence of elementary collapses on the remaining surface as follows. Choose a 1-cell which is in the boundary of the now-punctured surface but is not in the tree T. This is a "free edge," i.e., it is contained in the boundary of only one 2-cell. The removal of this edge together with the 2-cell in whose boundary it appears constitutes an elementary collapse and does not change the homotopy type of the remaining surface. (The algebraic version of this process will be the removal of a row and a column of A.) Continue to perform elementary collapses in this manner (never collapsing edges in the tree T) until there are no remaining 2-cells. The generators corresponding to those remaining edges which are not in T are then seen to be a basis of $H_1(\Sigma)$. Meanwhile, note that each time we perform a collapse we express the generator being eliminated in terms of the remaining generators using the relation which corresponds to the boundary of the collapsed 2-cell. Thus once we are finished we have produced a basis and we have also expressed all of the generators $\{u_{ij}\}$ in terms of that basis.

We now give a strictly algebraic description of Step 4: Delete the columns of A indexed by those pairs (i, j) for which $w_{ij} \in T$. Delete one row of A (it doesn't matter which one). Call the resulting matrix B, and perform the following procedure on it. Find a column of B containing a single 1 (so that all other entries in that column are 0). Eliminate from B the row and the column containing that 1. Note that the deleted row gives a relation which enables us to express the generator corresponding to the deleted column in terms of the generators corresponding to the other 1's in the deleted row. Hence the generator corresponding to the deleted column is superfluous, which is why we delete it. (It is worth stopping to record the expression for this generator in terms of the others – this expression will be useful in the ensuing computations.) Repeat this procedure (finding a column with only one 1 and then deleting the corresponding column and a row) until there is only one row remaining. Denote by $\tilde{S}$ the set of those pairs (i, j) which correspond to the remaining columns of B. Choose a pair $(i_0, j_0) \in \tilde{S}$ such that the entry indexed by (i_0, j_0) in the remaining row of B is 1. Let $S = \tilde{S} - \{(i_0, j_0)\}$. Then the set $\{u_{ij} : (i, j) \in S\}$ generates $\pi_1(\Sigma, 1)$ (note that the basepoint is the point of the fiber labelled "1"); in fact it freely generates the fundamental group of Σ with one puncture (to be precise, of Σ minus the disc corresponding to that row of A which was deleted when producing the original B). Thought of as homology classes, this set of u_{ij} also provides a basis of the free abelian group $H_1(\Sigma)$.

3. Examples

In this section, we work out two examples of the constructions made in the first two sections. In the first example, a double branched cover of S^2 (i.e., hyperelliptic curve), there exists a nontrivial automorphism of the cover; as the theory predicts, our efforts to extract from this situation a representation of the spherical braid group don't work, and we show the reader precisely what "goes wrong." In the second example, a branched cover with no automorphisms, we run through the entire procedure outlined in sections 1 and 2, and arrive at an interesting representation of an index 12 subgroup of the spherical braid group.

Example 1. Hyperelliptic surfaces. We consider the hyper-elliptic curve for which $n = 2$ and the genus is $\frac{r-2}{2}$ where r (an even number) is the number of branch points. In this case, the moduli space H is simply $((S^2)^r - \Delta)/S_r$, instead of some nontrivial cover of it, the signature is simply $\{(12),(12),\ldots,(12)\}$, and $\pi_1(H)$ is the entire spherical braid group $B_r(S^2)$. To apply the algorithm from section 2, we choose a maximal tree consisting of the single edge w_{11}. Our algorithm then yields the $r-2$ basis elements (for $H_1(\Sigma)$) given by $u_{22} = w_1w_2, u_{32} = w_1w_3, \ldots, u_{r-1,2} = w_1w_{r-1}$, and the relations $u_{11} = u_{12} = 0$ and $u_{i1} = -u_{i2}$ for $2 \le i \le r-1$. It is then easy to compute the action of $Q_1, \ldots, Q_{r-1}$ on the basis as follows:

$$Q_1(u_{22}) = Q_1(w_1w_2) = w_2w_2^{-1}w_1w_2 = w_{21}w_{21}^{-1}w_{11}w_{22} = u_{22}$$

$$Q_1(u_{i2}) = Q_1(w_1w_i) = w_2w_i = w_{21}w_{i2} = -u_{22} + u_{i2} \text{ for } i > 2.$$

(To help the reader see what we're doing here, recall that to express a word in the w_{ij}'s in terms of the basis elements we (1) replace each w_{ij} by u_{ij}, and (2) apply the relations among the u_{ij}'s given above to express it in terms of the basis elements. In this last step, we also change from multiplicative notation to additive notation to emphasize that $H_1(\Sigma)$ is abelian.)

Similarly, we compute the action of $Q_2, \ldots, Q_{r-1}$ on the basis to obtain the matrices:

$$Q_1 \mapsto \begin{pmatrix} 1 & -1 & \ldots & \ldots & \ldots & \ldots & \ldots & -1 \\ 0 & 1 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 & 1 \end{pmatrix},$$

$$Q_i \mapsto \begin{pmatrix} I_{i-2} & 0 & 0 \\ 0 & \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} & 0 \\ 0 & 0 & I_{r-2-i} \end{pmatrix}$$

for $1 < i < r-1$ and

$$Q_{r-1} \mapsto \begin{pmatrix} & -1 \\ I_{r-3} & +1 \\ & -1 \\ \vdots & \vdots \\ 0 & +1 \end{pmatrix}.$$

The careful reader will now ask, "But how can a representation of the braid group be obtained, since there is a nontrivial automorphism group Z_2 and hence no bundle space?" To address this question, note that when calculating the action of the Q_i's on the basis elements we made a hidden assumption, since after applying the relevant braid, there are actually two ways of identifying the resulting Riemann surface with the original one; the two ways differ by the nontrivial automorphism that interchanges the two leaves. An easy exercise shows that this automorphism acts on $H_1(\Sigma)$ by $-Id$. Hence each of these matrices is only well defined up to a sign. The reader may verify by direct computation that these matrices (as computed) do satisfy the planar braid relations $Q_iQ_{i+1}Q_i = Q_{i+1}Q_iQ_{i+1}$ for all i and $Q_iQ_j = Q_jQ_i$ for $|i-j| > 1$. But when we plug them into the word $Q_1Q_2\cdots Q_{r-1}Q_{r-1}\cdots Q_1$, we obtain $-Id$; hence they do not satisfy the last relation required of the spherical braid group. Since each Q_i appears twice in this word, the situation cannot be remedied by changing the sign of one or more of these matrices. Hence our attempt to build a representation of the spherical braid group in this case is stymied, as the theory predicted it might be (because of the existence of automorphisms). To get an actual representation, we would have to use as our domain a certain Z_2-extension of the braid group as indicated in section 1. Note that the representation of the planar braid group we constructed here is identical to the one constructed by Arnol'd in [A]; it arises if we restrict the branch points from passing through ∞.

Our second example is a branched cover $\phi : \Sigma \to S^2$ with no automorphisms (i.e., no self-homeomorphisms of Σ covering the identity on S^2). We calculate the subgroup of the braid group and its representation on $H_1(\Sigma)$ as described in section 1. In order for the corresponding representation of the subgroup of the braid group to be non-trivial, we choose a covering for which $\mathrm{Aut}(\phi) = 1$ and genus(Σ) is nonzero. The example we choose is a 3-to-1 cover of S^2 branched over a set $X = \{x_1, x_2, x_3, x_4\}$ of four points with branching data over these four points given by the signature. $(\rho_1, \rho_2, \rho_3, \rho_4) = ((12), (23), (132), (132))$. To be precise, for each $i = 1, \ldots, 4$ let w_i denote a based simple closed curve in S^2 enclosing x_i but excluding the other elements of X. Then $\pi_1(S^2 - X) = \langle w_1, \ldots, w_4 | w_1 w_2 w_3 w_4 = 1\rangle = F_3 =$ the free group on any 3 of these generators. Our branched cover is defined by the homomorphism $\rho : \pi_1(S^2 - X) \to S_3$ which takes each w_i to ρ_i. By the usual Hurwitz formula, we see that genus$(\Sigma) = 1$. The fact that $\mathrm{Aut}(\phi) = 1$ for this example follows from the fact that the subgroup of S_3 generated by the ρ_i has trivial centralizer (actually, it is all of S_3 in this case).

We begin by calculating a basis for $H_1(\Sigma)$ using the procedure outlined earlier. As our maximal tree, we choose w_{11} and w_{12}. Our procedure then yields generators $w_{12}^{-1} w_{22}$ and $w_{12}^{-1} w_{32} w_{11}^{-1}$ as generators of $\pi_1(\Sigma, 1)$; the images of these elements give a basis of $H_1(\Sigma)$.

We will now construct the Hurwitz space H (resulting from Σ) as a cover of $\Pi = ((S^2)^4 - \Delta)/S_4$. Recall that $\mathcal{G} = \{f \in \mathrm{Aut}(S^2) : f(X) = X\}$ and $\mathcal{G}_0 =$ the identity component of $\mathcal{G}$. Let

$$M_4(S^2) = \mathcal{G}/\mathcal{G}_0$$

By isotopy extension, there is a (surjective) map $B_4(S^2) \to M_4(S^2)$ (extend the isotopy of X to all of S^2; then, look at the final homeomorphism of the isotopy); the kernel of this map is the center of $B_4(S^2)$, a subgroup of order 2 whose generator we described above (for a proof see [B], p. 165). Given an element Q of $B_4(S^2)$, we will use the same symbol for the corresponding element of $M_4(S^2)$. We wish to describe the "points" of H lying above $[X] \in \Pi$. Such a point will correspond to a branched cover $\phi' : \Sigma' \to S^2$. These two branched covers will be related to each other by the commutative diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{F} & \Sigma' \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{Q} & S^2 \end{array}$$

where F is a homeomorphism. We wish to describe Σ' by giving its signature $(\rho'_1, \ldots, \rho'_4)$, where $\rho'(w_i) = \rho'_i$. To do this, we note that Q induces a map $Q_* : \pi_1(S^2 - X) \to \pi_1(S^2 - X)$ and for the map F to be defined, we must have $\rho = \rho' \circ Q_*$, i.e., $\rho' = \rho \circ Q_*^{-1}$. (Of course, we could conjugate this ρ' by any element of S_3 and obtain an equivalent branched cover.) Thus to describe these branched covers, we need to write down the action of $B_4(S^2)$ (with generators $\{Q_1, Q_2, Q_3\}$) on $\pi_1(S^2 - X)$, (with generators $\{w_1, w_2, w_3, w_4\}$). This action is given by the well-known formulae:

$$Q_i(w_i) = w_{i+1}$$

$$Q_i(w_{i+1}) = w_{i+1}^{-1} w_i w_{i+1}$$

$$Q_i(w_j) = w_j \text{ for } j \neq i, i+1.$$

It follows that if, for example, $Q = Q_1 \in B_4(S^2)$, then the signature of the corresponding Σ' is given by $(\rho_1', \rho_2', \rho_3', \rho_4') = (\rho_1\rho_2\rho_1^{-1}, \rho_1, \rho_3, \rho_4)$, and similarly for the other Q_i. This gives us an action of $B_4(S^2)$ on the set of signatures (actually an action on the set of equivalence classes of signatures where two signatures are considered equivalent if they are conjugate by some element of S_3). To enumerate the points of H lying above $[X] \in \Pi$, we begin with the original signature $((12), (23), (132), (132))$ and repeatedly apply Q_1, Q_2 and Q_3, each time checking whether the resulting signature is equivalent to any of the ones we've already enumerated. Eventually we arrive at a set of signatures which is closed (up to conjugation in S_3) under application of all of the Q_i. In this case we obtain the following 12 signatures:

$$A = ((12), (23), (132), (132))$$

$$B = ((12), (123), (23), (132))$$

$$C = ((132), (12), (23), (132))$$

$$D = ((12), (12), (123), (132))$$

$$E = ((12), (123), (123), (23))$$

$$F = ((23), (132), (23), (132))$$

$$G = ((132), (12), (123), (23))$$

$$H = ((123), (23), (23), (132))$$

$$I = ((23), (132), (123), (23))$$

$$J = ((132), (132), (12), (23))$$

$$K = ((123), (23), (123), (23))$$

$$L = ((123), (132), (23), (23))$$

The reader may check that applying any Q_i to any of these signatures produces a signature which is equivalent to one of these and that no two of these are equivalent. So each Q_i acts as a permutation of these signatures. These permutations are given as follows:

$$Q_1 = (BCFH)(EGIK)$$

$$Q_2 = (ABDF)(GJKL)$$

$$Q_3 = (BEFI)(CGHK)$$

The reader may also check that these three permutations generate a transitive subgroup of S_{12}.

By identifying the 12 points of H lying above $[X] \in \Pi$, and showing how the generators of $\pi_1(\Pi)$ permute these points, we have described completely the covering space $H \to \Pi$. By the theoretical considerations of the first section, we know that there is a fiber bundle U with fiber Σ and base H, and a resulting representation of $\pi_1(H)$ on $H_1(\Sigma)$. To calculate this representation, we must first find generators of $\pi_1(H)$. We choose the "point" in H corresponding to the signature C as a basepoint. Then $\pi_1(H, C)$ is identified with the subgroup of $B_4(S^2)$ which fixes C. To identify generators of this subgroup, we use a procedure rather analogous with the procedure in section 2 we used to calculate generators of $\pi_1(\Sigma)$, although in this case we use the unbranched cover $H \to \Pi$ instead of the branched cover $\Sigma \to S^2$.

We construct a 2-complex K with one 0-cell, three 1-cells (which we call q_1, q_2, and q_3, corresponding to the generators of $B_4(S^2)$), and four 2-cells which are attached according to the relations $Q_1Q_3Q_1^{-1}Q_3^{-1}$, $Q_1Q_2Q_1Q_2^{-1}Q_1^{-1}Q_2^{-1}$, $Q_2Q_3Q_2Q_3^{-1}Q_2^{-1}Q_3^{-1}$, and $Q_1Q_2Q_3Q_3Q_2Q_1$. We form the 12-to-1 covering space $\tilde{K}$ of K corresponding to the stabilizer of the point C in the homomorphism $B_4(S^2) \to S_{12}$ given above.

Note: In order to arrange that the action of $B_4(S^2)$ on $\{A, B, C, \ldots, L\}$ is from the right instead of the left, we let each q_i act on this set as Q_i^{-1}.

We label the 0-cells of $\tilde{K}$ by $A, B, C, \ldots L$, and denote by q_{iA} the lift of q_i starting at A, by q_{iB} the lift of q_i starting at B, etc. We choose as our maximal tree in $\tilde{K}$ the set

$$T = \{q_{1B}, q_{1C}, q_{1F}, q_{2B}, q_{2D}, q_{2K}, q_{2L}, q_{3B}, q_{3C}, q_{3E}, q_{3G}\}.$$

Using C as our basepoint, we obtain 25 generators of $\pi_1(H)$, one for each edge of $\tilde{K}$ not included in our tree. Note that $\tilde{K}$ has 48 2-cells. We may reduce the number of generators of $\pi_1(H)$ using the 48 relations arising from these 2-cells. We give an example to show how this works. The edge $q_{1A} \notin T$. Since $(A)q_1 = A$, the generator corresponding to q_{1A} is $u_{1A} = \alpha_A q_{1A} \alpha_A^{-1}$ where $\alpha_A = q_{1C}q_{2B}$ is the unique simple path in T from C to A. Now consider the lift of $q_1q_3q_1^{-1}q_3^{-1}$ starting at C. It is $q_{1C}q_{3B}q_{1K}^{-1}q_{3C}^{-1}$ and corresponds to the attaching map of a 2-cell in $\tilde{K}$. Since q_{1C}, q_{3B} and q_{3C} are in T, this 2-cell induces the relation $u_{1K} = 1$ in $\pi_1(H)$, enabling us to throw out this generator. Using 22 of the 48 relations in this manner, we are able to reduce the number of generators from 25 to 3. (In most cases, the relations don't kill generators directly, but express one in terms of some remaining ones, enabling us to eliminate one.) For the reader's convenience, the following table lists the 22 2-cells we used and, for each, the generator it eliminated.

2-cell boundary	generator eliminated
$q_{1C}q_{3B}q_{1K}^{-1}q_{3C}^{-1}$	u_{1K}
$q_{1G}q_{3E}q_{1C}^{-1}q_{3G}^{-1}$	u_{1G}
$q_{1C}q_{2B}q_{1A}q_{2B}^{-1}q_{1C}^{-1}q_{2C}^{-1}$	u_{1A}
$q_{1D}q_{2D}q_{1B}q_{2H}^{-1}q_{1B}^{-1}q_{2D}^{-1}$	u_{1D}
$q_{2D}q_{3B}q_{2I}q_{3B}^{-1}q_{2D}^{-1}q_{3D}^{-1}$	u_{2I}
$q_{1E}q_{3E}q_{2B}q_{3A}^{-1}q_{2B}^{-1}q_{3E}^{-1}$	u_{2E}
$q_{1B}q_{3H}q_{1I}^{-1}q_{3B}^{-1}$	u_{3H}
$q_{1E}q_{3K}q_{1B}^{-1}q_{3E}^{-1}$	u_{3K}
$q_{1F}q_{3C}q_{1E}^{-1}q_{3F}^{-1}$	u_{3F}
$q_{1I}q_{3G}q_{1F}^{-1}q_{3I}^{-1}$	u_{1I}
$q_{1F}q_{2C}q_{1C}q_{2D}^{-1}q_{1D}^{-1}q_{2F}^{-1}$	u_{2C}
$q_{2C}q_{3C}q_{2K}q_{3J}^{-1}q_{2K}^{-1}q_{3C}^{-1}$	u_{3J}
$q_{2G}q_{3L}q_{2L}q_{3C}^{-1}q_{2C}^{-1}q_{3G}^{-1}$	u_{2G}
$q_{1B}q_{2H}q_{3H}q_{3G}q_{2C}q_{1C}$	u_{3I}
$q_{1C}q_{2B}q_{3A}q_{3A}q_{2A}q_{1F}$	u_{2A}
$q_{1J}q_{2J}q_{3G}q_{3C}q_{2K}q_{1J}$	u_{2J}
$q_{1B}q_{2H}q_{1H}q_{2A}^{-1}q_{1A}^{-1}q_{2B}^{-1}$	u_{1H}
$q_{1L}q_{2L}q_{1K}q_{2I}^{-1}q_{1K}^{-1}q_{2L}^{-1}$	u_{1L}
$q_{1E}q_{2K}q_{1J}q_{2K}^{-1}q_{1E}^{-1}q_{2E}^{-1}$	u_{1J}
$q_{2B}q_{3A}q_{2A}q_{3I}^{-1}q_{2I}^{-1}q_{3B}^{-1}$	u_{3D}
$q_{2F}q_{3D}q_{2D}q_{3E}^{-1}q_{2E}^{-1}q_{3F}^{-1}$	u_{1E}
$q_{1G}q_{2E}q_{1E}q_{2L}^{-1}q_{1L}^{-1}q_{2G}^{-1}$	u_{3L}

The three remaining generators of $\pi_1(H)$ are

$$u_{2F} = \alpha_F q_{2F} \alpha_D^{-1} = q_{1F}q_{2F}q_{2D}q_{1C}^{-1}$$

$$u_{2H} = \alpha_H q_{2H} \alpha_H^{-1} = q_{1C}q_{1B}q_{2H}q_{1B}^{-1}q_{1C}^{-1}$$

$$u_{3A} = \alpha_A q_{3A} \alpha_A^{-1} = q_{1C}q_{2B}q_{3A}q_{2B}^{-1}q_{1C}^{-1}$$

Expressing these in terms of the original Q_i's (and inverting them) produces the following three generators for the image of $\pi_1(H,C)$ in $B_4(S^2)$:

$$Q_1Q_2^{-2}Q_1,\ Q_1^2Q_2Q_1^{-2},\ \text{and}\ Q_1Q_2Q_3Q_2^{-1}Q_1^{-1}.$$

Our next task is to calculate a basis for $H_1(\Sigma_C)$ (using the algorithm of section 2), and then to calculate the action of the above generators of $\pi_1(H)$ on $H_1(\Sigma_C)$ in terms of this basis. The signature of Σ_C is $((132),(12),(23),(132))$. As a maximal tree in $\tilde{\Gamma}$, we select $\{w_{11}, w_{12}\}$. For each w_{ij} not in the maximal tree, we obtain a generator u_{ij} for $H_1(\Sigma_C)$. The remainder of the algorithm yields the fact that $\{u_{22}, u_{32}\}$ is a basis for $H_1(\Sigma_C)$. To be precise, the curves in Σ_C corresponding to the basis elements are $u_{22} = w_{12}^{-1}w_{22}$ and $u_{32} = w_{12}^{-1}w_{32}w_{11}^{-1}$. The images of these elements in $\pi_1(S^2 - \{x_1,x_2,x_3,x_4\})$ are simply $u_{22} = w_1^{-1}w_2$

and $u_{32} = w_1^{-1} w_3 w_1^{-1}$. In addition, the algorithm expresses the other 5 generators in terms of this basis as follows:

$$u_{13} = 0$$

$$u_{21} = -u_{22}$$

$$u_{23} = 0$$

$$u_{31} = 0$$

$$u_{33} = -u_{32}$$

To calculate the action of one of our generators Q of $\pi_1(H)$ on $H_1(\Sigma_C)$ we proceed as follows. First, calculate the map $F_* : H_1(\Sigma_C) \to H_1(\Sigma'_C)$ defined by the diagram

$$\begin{array}{ccc} \Sigma_C & \stackrel{F}{\to} & \Sigma'_C \\ \downarrow & & \downarrow \\ S^2 & \stackrel{Q}{\to} & S^2 \end{array}$$

where we are thinking of Q as an automorphism of S^2 which fixes the set $\{x_1, x_2, x_3, x_4\}$, and the map induced by Q on $\pi_1(S^2 - \{x_1, x_2, x_3, x_4\})$ is defined in terms of the generators Q_i by

$$Q_i(w_i) = w_{i+1}$$

$$Q_i(w_{i+1}) = w_{i+1}^{-1} w_i w_{i+1}$$

$$Q_i(w_j) = w_j \text{ for } j \neq i, i+1.$$

The signature of the Riemann surface Σ'_C is calculated using the action of $B_4(S^2)$ on the set of signatures described on page 11 above.

Next, since Q stabilizes the cover $\Sigma_C \to S^2$ up to equivalence, the signatures of Σ_C and Σ'_C are conjugate by an element γ of the symmetric group S_3. This permutation γ gives us an identification of these two branched covers, and hence a map $H_1(\Sigma'_C) \to H_1(\Sigma_C)$. Composing this map with F_* gives the action of Q.

We now carry this out for the generator $Q = Q_1 Q_2^{-2} Q_1$. Applying Q to the signature C gives the signature $((132), (23), (31), (132))$ for Σ'_C. Since Q is in the stabilizer of C, these two signatures must be conjugate by an element of S_3. The conjugating permutation is $\gamma = (123)$, which induces an equivalence between Σ_C and Σ'_C. It is natural to use this equivalence to transport the computations we have made on Σ_C directly to Σ'_C. Applying γ to the second subscript of each symbol and adding bars everywhere yields, for Σ'_C, the maximal tree $\{\bar{w}_{12}, \bar{w}_{13}\}$, the basis $\bar{u}_{23} = \bar{w}_{13}^{-1} \bar{w}_{23}$ and $\bar{u}_{33} = \bar{w}_{13}^{-1} \bar{w}_{33} \bar{w}_{12}^{-1}$ for $H_1(\Sigma'_C)$, and the expressions

$$\bar{u}_{11} = 0$$

$$\bar{u}_{22} = -\bar{u}_{23}$$

$$\bar{u}_{21} = 0$$

$$\bar{u}_{32} = 0$$

$$\bar{u}_{31} = -\bar{u}_{33}$$

for the other natural generators. Using the formulae given just above, we calculate the effect of Q on $\pi_1(S^2 - \{x_1, x_2, x_3, x_4\})$ to be:

$$Q(w_1) = Q_1 Q_2^{-2} Q_1(w_1) = Q_1 Q_2^{-2}(w_2) = Q_1 Q_2^{-1}(w_2 w_3 w_2^{-1}) =$$

$$Q_1(w_2 w_3 w_2 w_3^{-1} w_2^{-1}) = w_2^{-1} w_1 w_2 w_3 w_2^{-1} w_1 w_2 w_3^{-1} w_2^{-1} w_1^{-1} w_2.$$

Similarly, we obtain

$$Q(w_2) = w_2^{-1} w_1 w_2 w_3 w_2^{-1} w_1^{-1} w_2 w_3^{-1} w_2^{-1} w_1^{-1} w_2 w_1 w_2 w_3 w_2^{-1} w_1 w_2 w_3^{-1} w_2^{-1} w_1^{-1} w_2$$

and

$$Q(w_3) = w_2^{-1} w_1 w_2 w_3 w_2^{-1} w_1^{-1} w_2.$$

We calculate the effect of Q on u_{22} and u_{32}, the basis elements of $H_1(\Sigma)$. Using the images of these in $\pi_1(S^2 - \{x_1, x - 2, x_3, x_4\}$ we compute:

$$Q(u_{22}) = Q(w_1^{-1} w_2) = w_2^{-1} w_1 w_2 w_3 w_2^{-1} w_1^{-1} w_2 w_3^{-1} w_2^{-1} w_1^{-1} w_2 \cdot$$

$$w_2^{-1} w_1 w_2 w_3 w_2^{-1} w_1^{-1} w_2 w_3^{-1} w_2^{-1} w_1^{-1} w_2 w_1 w_2 w_3 w_2^{-1} w_1 w_2 w_3^{-1} w_2^{-1} w_1^{-1} w_2$$

$$= w_2^{-1} w_1 w_2 w_3 w_2^{-1} w_1^{-2} w_2 w_3^{-1} w_2^{-1} w_1^{-1} w_2 w_1 w_2 w_3 w_2^{-1} w_1 w_2 w_3^{-1} w_2^{-1} w_1^{-1} w_2.$$

To compute $F_*(u_{22})$, we need to lift this element up to Σ'_C, which involves inserting a second subscript into each generator using the signature obtained by applying Q to C. The resulting lift is:

$$\bar{w}_{21}^{-1} \bar{w}_{11} \bar{w}_{23} \bar{w}_{32} \bar{w}_{23}^{-1} \bar{w}_{11}^{-1} \bar{w}_{12}^{-1} \bar{w}_{22} \bar{w}_{31}^{-1} \bar{w}_{21}^{-1} \bar{w}_{12}^{-1} \bar{w}_{22} \bar{w}_{13} \bar{w}_{22} \bar{w}_{33} \bar{w}_{21}^{-1} \bar{w}_{11} \bar{w}_{23} \bar{w}_{32}^{-1} \bar{w}_{23}^{-1} \bar{w}_{11}^{-1} \bar{w}_{21}.$$

To express this in terms of the generators $\bar{u}_{23}$ and $\bar{u}_{33}$, we first drop the edges $\bar{w}_{12}$ and $\bar{w}_{13}$ which lie in the maximal tree, then replace each $\bar{w}_{ij}$ by the corresponding generator $\bar{u}_{ij}$, and finally apply the relations expressing each of these in terms of the two basis elements. The result is that $F_*(u_{22}) = \bar{u}_{23}^{-3} \bar{u}_{33}^2$, where the order of the factors doesn't matter since H_1 is abelian. We identify this element with an element of $H_1(\Sigma_C)$ (using the equivalence between the branched covers Σ_C and Σ'_C) by replacing each second subscript using the inverse of the permutation γ. The result of this computation is that under our representation, Q takes u_{22} to $u_{22}^{-3} u_{32}^2$. We make similar computations to calculate the action of Q on u_{32}, and then to calculate the action of Q, for Q equal to the other two generators of $\pi_1(H)$, on the two basis elements of $H_1(\Sigma_C)$. The results are as follows:

$$Q_1 Q_2^{-2} Q_1 \mapsto \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}$$

$$Q_1^2 Q_2 Q_1^{-2} \mapsto \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$

$$Q_1Q_2Q_3Q_2^{-1}Q_1^{-1} \mapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

Note that the first of these is hyperbolic (hence of infinite order), the second is parabolic (hence of infinite order) and the third is elliptic, of order 6. The fact that these elements generate an infinite subgroup of $SL(2,Z)$ is of interest for the following reason. Let $\mathcal{M}_1$ denote the moduli space of Riemann surfaces of genus 1. Following Fried [F2], there is a map $\Psi : H \to \mathcal{M}_g$ defined by taking the branched cover $(X \to S^2)$ to the point $[X]$. Fried points out that this is an algebraic map, and considers the problem of computing the dimension of its image, which is a subvariety of $\mathcal{M}_g$. Part of Theorem 3.6 of [F2] states that, for $g = 1$ or 2, the image of the monodromy representation of $X \to S^2$ is a finite subgroup of $\mathrm{Aut}(H_1(X))$ if and only if the map Ψ is constant. In the case we have been considering, this theorem tells us that Ψ is not constant, and, since $\dim(\mathcal{M}_g) = 1$, we conclude that the image of Ψ is a Zariski open subset.

Note: The current authors are preparing a joint paper with Mike Fried in which similar techniques will be used to show that a generic genus 1 Riemann surface admits an analytic map to S^2 with monodromy group A_n, for any $n \geq 4$. This will answer a question of Guralnick and Neubauer in [GN], p. 328.

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