Submodules of the Deficiency Modules and an Extension of Dubreil's Theorem

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In this paper, we consider a basic question in commutative algebra: if I and J are ideals of a commutative ring S, when does $IJ = I \cap J$? More precisely, our setting will be in a polynomial ring $k[x_0, \dots, x_n]$, and the ideals I and J define subschemes of the projective space $\mathbb{P}^n_k$ over k. In this situation, we are able to relate the equality of product and intersection to the behavior of the cohomology modules of the subschemes defined by I and J. By doing this, we are able to prove several general algebraic results about the defining ideals of certain subschemes of projective space.

Our main technique in this paper is a study of the *deficiency modules* of a subscheme V of $\mathbb{P}^n$. These modules are important algebraic invariants of V, and reflect many of the properties of V, both geometric and algebraic. For instance, when V equidimensional and $\dim V \geq 1$, the deficiency modules of V are invariant (up to a shift in grading) along the even liaison class of V ([13], [10], [14], [6]), although they do not in general completely determine the even liaison class, except in the case of curves in $\mathbb{P}^3$, [13]. On the algebraic side, at least for curves in $\mathbb{P}^3$, the deficiency modules have been shown to have connections to the number and degrees of generators of the saturated ideal defining V, [11]. One of our main goals in this paper is to extend these results to subschemes of arbitrary codimension in any projective space $\mathbb{P}^n$.

We now describe the contents of this paper more precisely. In the first section, we set up our notation and give the basic definitions which we will use. Then we prove our main technical result: if I and J define subschemes V and Y, respectively, of $\mathbb{P}^n$, we relate the quotient module $(I \cap J)/IJ$ to the cohomology of V, at least when V and Y meet properly. We are then able to give a different proof of a general statement due to Serre about when there is an equality of intersection and product.

1991 *Mathematics Subject Classification.* 14M06, 13C40

In the second section, we give an extension of Dubreil's Theorem on the number of generators of ideals in a polynomial ring. Specifically, our generalization works for an ideal I defining a locally Cohen–Macaulay, equidimensional subscheme V of any codimension in $\mathbb{P}^n$, and relates the number of generators of the defining ideal to the length of certain Koszul homologies of the cohomology of V. The results in this section depend crucially on the identification done in Section 1 of the intersection modulo the product.

Finally, in Section 3, we give an extension of a surprising result of Amasaki, [1], showing a lower bound for the least degree of a minimal generator of the ideal of a Buchsbaum subscheme. Originally, Amasaki gave a bound in the case of Buchsbaum curves in $\mathbb{P}^3$. Easier proofs were subsequently given by Geramita and Migliore in [5], based on a determination of the free resolution of the ideal from a resolution of its deficiency module. For Buchsbaum codimension 2 subschemes of $\mathbb{P}^n$ whose intermediate cohomology vanishes, we are able to extend these considerations.

1 When does $I \cap J = IJ$?

Let $S = k[x_0, \ldots, x_n]$ be a polynomial ring over the algebraically closed field k. Let I and J be ideals defining subschemes V and Y, respectively, of the projective space $\mathbb{P}^n_k = \mathbb{P}^n$ over k. In particular, both I and J are homogeneous, saturated ideals. In this section, we will derive a relationship between the quotient module $(I \cap J)/IJ$ and the cohomology of V, when V and Y meet in the expected dimension.

In general, if V is a subscheme of $\mathbb{P}^n$, with saturated homogeneous defining ideal $I = I_V$, the cohomology modules of V (or, less precisely, of I) are defined, for $i = 0, \ldots, n-1$, by

$$H^i_*(\mathcal{I}_V) = H^i_*(V) = \bigoplus_j H^i(\mathbb{P}^n, \mathcal{I}_V(j)),$$

where $\mathcal{I}_V = \widetilde{I_V}$ is the ideal sheaf of V. These are all graded S-modules. Moreover, $H^0_*(V) = I_V$ and $H^i_*(V) = 0$ for $i > \dim V + 1$. Usually, when $i = 1, \ldots, \dim V$, we will call $H^i_*(V)$ a deficiency module. This name comes from the fact that the $H^i_*(V)$, $i = 1, \ldots, \dim V$, measure the failure of V to be an arithmetically Cohen–Macaulay subscheme, since they vanish whenever V is aCM.

We will also have need to use the cohomology of modules. If M is a (finitely generated) S-module, let $\widetilde{M}$ be its sheafification. Then, exactly as in the case of ideal sheaves, we define the cohomology module of $\widetilde{M}$ to be

$$H^i_*(\widetilde{M}) = \bigoplus_j H^i(\mathbb{P}^n, \widetilde{M}(j)).$$

These are again graded S-modules. We note here for future reference that the cohomology modules of M are related to the local cohomology modules $H^i_{\mathfrak{m}}(M)$ of M with respect to the homogeneous maximal ideal $\mathfrak{m}$ as follows:

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0_*(\widetilde{M}) \to H^1_{\mathfrak{m}}(M) \to 0 \tag{1}$$

$$H^i_{\mathfrak{m}}(M) \cong H^i_*(\widetilde{M}) \qquad \text{for } i > 1.$$

See [16, Chapter 0] for a good discussion of graded and local cohomology.

In this paper, we will sometimes require subschemes of $\mathbb{P}^n$ to be locally Cohen–Macaulay and equidimensional. This is equivalent to saying that all the cohomology modules have finite length, except of course for the top cohomology $H^{d+1}_*(V)$, $d = \dim V$. By Serre's vanishing theorem, this is again equivalent to having $[H^i_*(V)]_j = 0$ for $j \ll 0$, and $1 \leq i \leq d$, since in any case the cohomology modules vanish in high degrees.

Now, let I and J be as above, let $s = \text{pd } J$ be the projective dimension of J, and write a minimal graded free resolution of J as follows:

$$0 \;\to\; F_s \;\xrightarrow{\phi_s}\; \dots \;\xrightarrow{\phi_2}\; F_1 \;\xrightarrow{\phi_1}\; F_0 \;\to\; J \;\to\; 0. \tag{2}$$

where $F_j = \bigoplus_i S(-a_{ji})$ are free modules. For each $j = 1, \dots, s$, let K_j be the j-th syzygy module, so that there are short exact sequences

$$0 \to K_{j+1} \xrightarrow{\psi_{j+1}} F_j \xrightarrow{\eta_j} K_j \to 0,$$

where the maps ψ_j and η_j are the canonical inclusions and projections, respectively. Note that for $j = s$, we have $K_s = F_s$, $\psi_s = \phi_s$ and $\eta_s = id$.

For S-modules M and N, and a map $f : M \to N$, we denote by $f^i : H^i_*(\widetilde{M \otimes I}) \to H^i_*(\widetilde{N \otimes I})$ the map induced on cohomology by $f \otimes id : M \otimes I \to N \otimes I$.

Our main technical result for this paper is the following Theorem.

Theorem 1.1 *Suppose the ideals I and J as above define disjoint subschemes V and Y, respectively. Then for each $i \geq 1$, there are isomorphisms*

$$\ker \psi^1_i \cong \text{Tor}^S_i(S/I, S/J)$$

and, for each $i, j \geq 1$, a long exact sequence

$$0 \to \mathrm{im}\ \psi_{i+1}^j \to \ker \phi_i^j \to \ker \psi_i^j \to \ker \psi_{i+1}^{j+1} \to \mathrm{coker}\ \phi_i^j \to \mathrm{coker}\ \psi_i^j \to 0. \tag{3}$$

Proof: We remark that for $i > s$, both statements are trivial, since then $\phi_i = \psi_i$ is the zero map. Also, if $j > \dim S/I$, then because $H_*^j(V) = 0$ again the second statement is trivial.

Now, let $\mu : I \otimes J \to IJ$ be the natural surjection, and note that $\ker \mu \cong \mathrm{Tor}_2^S(S/I, S/J)$. This follows, for instance, by tensoring

$$0 \to I \to S \to S/I \to 0$$

with J, comparing the resulting sequence with

$$0 \to IJ \to J \to J/IJ \to 0$$

via the multiplication map, and using that $\mathrm{Tor}_1^S(S/I, J) \cong \mathrm{Tor}_2^S(S/I, S/J)$. Note especially that $\ker \mu$ has finite length since it is annihilated by $I + J$. In particular, by sheafifying and taking cohomology of the short exact sequence

$$0 \to \ker \mu \to I \otimes J \to IJ \to 0,$$

we see that $H_*^i(\widetilde{I \otimes J}) \cong H_*^i(\widetilde{IJ})$ for all $i \geq 0$.

Next, using the functorial map $M \to H_*^0(\widetilde{M})$ for any S-module M, we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H_{\mathfrak{m}}^0(I \otimes J) & \to & H_{\mathfrak{m}}^0(IJ) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Tor}_2^S(S/I, S/J) & \to & I \otimes J & \xrightarrow{\mu} & IJ & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & H_*^0(\widetilde{I \otimes J}) & \to & H_*^0(\widetilde{IJ}) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H_{\mathfrak{m}}^1(I \otimes J) & \to & H_{\mathfrak{m}}^1(IJ) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{4}$$

But IJ is an ideal, so $H_{\mathfrak{m}}^0(IJ) = 0$. Hence the kernel of the map $I \otimes J \to H_*^0(\widetilde{I \otimes J})$ is $\mathrm{Tor}_2^S(S/I, S/J)$.

Now, with these preliminaries out of the way, we prove the isomorphisms by induction on i. For $i = 1$, tensor the exact sequence

$$0 \to K_1 \xrightarrow{\psi_1} F_0 \to J \to 0$$

by I. This yields an exact sequence

$$0 \to \mathrm{Tor}_1^S(I, J) \to K_1 \otimes I \xrightarrow{\psi_1 \otimes 1} F_0 \otimes I \to J \otimes I \to 0.$$

Since $\mathrm{Tor}_1^S(I, J) \cong \mathrm{Tor}_3^S(S/I, S/J)$ is annihilated by $I + J$, in particular it has finite length. Thus, taking cohomology and comparing with the original sequence gives a diagram

$$\begin{array}{ccccccccc} 0 \to \mathrm{Tor}_3^S(S/I, S/J) \to & K_1 \otimes I & \to & F_0 \otimes I & \to & J \otimes I & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to H_*^0(\widetilde{K_1 \otimes I}) & \to & H_*^0(\widetilde{F_0 \otimes I}) & \to & H_*^0(\widetilde{J \otimes I}) & \to & \ker \psi_1^1 \to 0. \end{array} \tag{5}$$

Here, the middle vertical map is an isomorphism, since I is saturated. Thus the snake lemma shows that there are exact sequences

$$0 \to \mathrm{Tor}_3^S(S/I, S/J) \to K_1 \otimes I \to H_*^0(\widetilde{K_1 \otimes I}) \to \mathrm{Tor}_2^S(S/I, S/J) \to 0,$$

and

$$0 \to \mathrm{Tor}_2^S(S/I, S/J) \to J \otimes I \to H_*^0(\widetilde{J \otimes I}) \to \ker \psi_1^1 \to 0.$$

But from the above discussion, the last sequence implies the short exact sequence

$$0 \to IJ \to H_*^0(\widetilde{IJ}) \to \ker \psi_1^1 \to 0.$$

Since I and J define disjoint varieties, we have $H_*^0(\widetilde{IJ}) = I \cap J$. Thus, the above sequence shows that

$$\ker \psi_1^1 \cong \frac{I \cap J}{IJ} \cong \mathrm{Tor}_1^S(S/I, S/J).$$

By induction, we may assume that $\ker \psi_i^1 \cong \mathrm{Tor}_i^S(S/I, S/J)$, and that there is an exact sequence

$$0 \to \mathrm{Tor}_{i+2}^S(S/I, S/J) \to K_i \otimes I \to H_*^0(\widetilde{K_i \otimes I}) \to \mathrm{Tor}_{i+1}^S(S/I, S/J) \to 0.$$

Tensoring the exact sequence

$$0 \to K_{i+1} \xrightarrow{\psi_{i+1}} F_i \to K_i \to 0$$

with I yields

$$0 \to \mathrm{Tor}_1^S(K_i, I) \to K_{i+1} \otimes I \to F_i \otimes I \to K_i \otimes I \to 0.$$

Here, $\mathrm{Tor}_1^S(K_i, I) \cong \mathrm{Tor}_{i+1}^S(I, J) \cong \mathrm{Tor}_{i+3}^S(S/I, S/J)$. In particular, it is finite length. Hence, taking cohomology and comparing yields a diagram

$$\begin{array}{ccccccccc} 0 \to \mathrm{Tor}_{i+3}^S(S/I,S/J) \to & K_{i+1}\otimes I & \to & F_i \otimes I & \to & K_i \otimes I & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to H^0_*(\widetilde{K_{i+1}\otimes I}) & \to & H^0_*(\widetilde{F_i\otimes I}) & \to & H^0_*(\widetilde{K_i\otimes I}) & \to & \ker\psi^1_{i+1} \to 0. \end{array} \tag{6}$$

But by the inductive hypothesis, we know the kernel and cokernel of the right-hand vertical map. Thus the snake lemma implies that

$$\ker\psi^1_{i+1} \cong \mathrm{Tor}_{i+1}^S(S/I, S/J),$$

and that there is a long exact sequence

$$0 \to \mathrm{Tor}_{i+3}^S(S/I,S/J) \to K_{i+1}\otimes I \to H^0_*(\widetilde{K_{i+1}\otimes I}) \to \mathrm{Tor}_{i+2}^S(S/I,S/J) \to 0,$$

which finishes the proof of the isomorphisms.

Next, we show that the long exact sequence exists. Fix an $i \geq 1$. Thus there is an exact sequence

$$0 \to K_{i+1} \xrightarrow{\psi_{i+1}} F_i \xrightarrow{\eta_i} K_i \to 0.$$

Tensor this sequence with I, to obtain

$$0 \to \mathrm{Tor}_1^S(K_i, I) \to K_{i+1}\otimes I \xrightarrow{\psi_{i+1}\otimes 1} F_i \otimes I \xrightarrow{\eta_i \otimes 1} K_i \otimes I \to 0,$$

and note that $\mathrm{Tor}_1^S(K_i, I) = \mathrm{Tor}_{i+3}^S(S/I, S/J)$, has finite length. Thus, after sheafifying and taking cohomology, at the j-th stage this yields isomorphisms

$$\begin{aligned} \ker\eta_i^j &\cong \mathrm{im}\ \psi^j_{i+1} \\ \mathrm{coker}\,\eta_i^j &\cong \ker\psi^{j+1}_{i+1}. \end{aligned}$$

Now, using the functoriality of tensor products and of cohomology, we obtain a commutative square

$$\begin{array}{ccc} H^j_*(\widetilde{F_i\otimes I}) & \xrightarrow{\sim} & H^j_*(\widetilde{F_i\otimes I}) \\ \downarrow \eta_i^j & & \downarrow \phi_i^j \\ H^j_*(\widetilde{K_i\otimes I}) & \xrightarrow{\psi_i^j} & H^j_*(\widetilde{F_{i-1}\otimes I}). \end{array} \tag{7}$$

Applying the snake lemma to the columns, and using the two isomorphisms above shows that there is a sequence

$$0 \to \text{im } \psi_{i+1}^{j} \to \ker \phi_i^j \to \ker \psi_i^j \to \ker \psi_{i+1}^{j+1} \to \text{coker } \phi_i^j \to \text{coker } \psi_i^j \to 0,$$

which is what we claimed. $\square$

We note that this greatly extends the arguments in [11, Section 1]. The situation there was much simpler in that it only considered the case that J was codimension 2 and arithmetically Cohen–Macaulay (so most of the terms in the sequence (3) vanish), and only the case $i = j = 1$ was studied, so it focused on $\ker \psi_1^1 = (I \cap J)/IJ$. Our extension makes no assumptions on the Cohen–Macaulayness of J, nor on its codimension. Of course, our conclusion is much more complicated, reflecting the fact that so much information is encoded in the free resolution of J.

As an application of this technical result, in the next theorem we give a proof of a statement due to Serre on when there is an equality $I \cap J = IJ$.

Theorem 1.2 [15, Corollaire, p. 143] *Suppose the ideals I and J define disjoint subschemes of $\mathbb{P}^n$. Then $IJ = I \cap J$ if and only if* $\dim S/I + \dim S/J = \dim S$ *and both S/I and S/J are Cohen–Macaulay.*

Proof: Suppose first that S/I and S/J are Cohen–Macaulay with $\dim S/I + \dim S/J = \dim S$. Then by the Auslander–Buchsbaum formula, $s = \text{pd } J = \dim S/I - 1$, and moreover $H_*^i(\widetilde{I}) = 0$ for $i = 1, \ldots, s$. In particular, $\ker \phi_i^i = 0 = \text{coker } \phi_i^i$ for $i = 1, \ldots, s$. Since $\psi_s = \phi_s$, by reverse induction the sequence (3) with $j = i$ shows that $\ker \psi_i^i = 0$ for $i = 1, \ldots, s$. Thus $(I \cap J)/IJ = \ker \psi_1^1 = 0$.

Conversely, since the subschemes defined by I and J are disjoint, we have $\dim S/I + \dim S/J \leq \dim S$ Hence

$$\dim S/I \leq \dim S - \dim S/J \leq \dim S - \text{depth } S/J = s + 1 \tag{8}$$

where the last equality is by the Auslander–Buchsbaum formula. Now, if $IJ = I \cap J$, then $\text{Tor}_1^S(S/I, S/J) = 0$, and so by rigidity, $\text{Tor}_i^S(S/I, S/J) = 0$ for $i \geq 1$. Hence the isomorphisms of Theorem 1.1 show that $\ker \phi_s^1 = 0$. But this implies that $H_*^1(I) = 0$. Thus also $\ker \phi_i^1 = 0 = \text{coker } \phi_i^1$ for all $i = 1, \ldots, s$, and since $\ker \psi_1^1 = (I \cap J)/IJ = 0$, the exact sequence (3) with $j = 1$, $i = 1, \ldots, s$ implies that $\ker \psi_i^2 = 0$, for $i = 2, \ldots, s$. Since $\psi_s = \phi_s$, this shows $\ker \psi_s^2 = 0$ and hence also $H_*^2(\widetilde{I}) = 0$. Continuing inductively, we see that $\ker \psi_i^j = 0$ for all i and j with $i \geq j$. In particular, since $\psi_s = \phi_s$ we get that $H_*^j(\widetilde{I}) = 0$ for $j = 1, \ldots, s$.

Let $d = \dim S/I$. We have seen that $d - 1 \leq s$. If this inequality were strict, then in particular $H_*^d(\widetilde{I}) = 0$, which is impossible. Hence we have $d - 1 = s$ and $H_*^j(\widetilde{I}) = 0$ for $j = 1, \ldots, d-1$; that is S/I is Cohen–Macaulay. But furthermore, each of the inequalities in (8) is actually in equality. This shows both that $\dim S/J = \operatorname{depth} S/J$, i.e., S/J is Cohen–Macaulay, and that $\dim S/I + \dim S/J = \dim S$, which finishes the proof. $\square$

2 An Extension of Dubreil's Theorem

In this section, we wish to use the results of Section 1 to extend a theorem of Dubreil on the number of generators of certain ideals. Let $\nu(I)$ denote the minimal number of generators of I, and let $\alpha(I)$ denote the least degree of a minimal generator. In its most basic form, Dubreil's Theorem states:

Theorem 2.1 *Let I be a homogeneous ideal of $k[x, y]$. Then $\nu(I) \leq \alpha(I) + 1$.*

See [3] for a proof of this; note however that it is essentially a consequence of the Hilbert–Burch theorem. Dubreil's theorem is easily extended to the case that I is a codimension 2 arithmetically Cohen–Macaulay ideal in any polynomial ring $k[x_0, \ldots, x_n]$; again, see [3] for the details. On the other hand, when I is not arithmetically Cohen–Macaulay, or when I is not codimension 2, not much is known in this direction. However, in the case of an ideal defining a subscheme of $\mathbb{P}^3$, the following theorem of Migliore shows that the general case will involve the cohomology of the subscheme.

Theorem 2.2 [11, Corollary 3.3] *Suppose I defines a subscheme V of $\mathbb{P}^3$, of codimension at least 2. Let $A = (L_1, L_2)$ be the complete intersection of two general linear forms, and let K_A denote the submodule of $H_*^1(V)$ annihilated by A. Then*

$$\nu(I) \leq \alpha(I) + 1 + \nu(K_A).$$

We note in particular that this formula is valid both for the case that V is codimension 2, not necessarily arithmetically Cohen–Macaulay, and the case that V is codimension 3. In the latter case, even though $H^1_*(V)$ is not finitely generated, we still have that at least K_A is finitely generated (see [11, Theorem 2.1] or our Lemma 2.4), so the theorem still has useful content. Furthermore, in case I defines an arithmetically Buchsbaum curve, so that $H^1_*(V)$ is a k-vector space, $K_A = H^1_*(V)$ and $\nu(K_A) = \dim_k H^1_*(V)$. Thus the Buchsbaum case is particularly easy to calculate in examples. In this special case, the bound can be obtained from [1].

In this section, we will give a generalization of Dubreil's Theorem to ideals defining subschemes of $\mathbb{P}^n$ of arbitrary codimension. As an easy consequence, we recover by our methods the above two theorems, and also part of a result of Chang, [2], on the number of generators of an ideal defining a Buchsbaum codimension 2 subscheme of $\mathbb{P}^n$, which again seems to be the best understood case. Our generalization is a corollary to the technical statement Theorem 1.1 in Section 1, underscoring the usefulness of identifying the difference between intersections and products.

Our generalization will be based on the Koszul homologies of the cohomology modules of an ideal I defining a subscheme of projective space. As such, we will make some general remarks concerning Koszul homology. These comments are basic, and can be found, for instance, in [9]. We first set the notation. If R is a ring, and $y_1, \ldots, y_s$ elements of R, we let $\mathbb{K}((y_1, \ldots, y_s); R)$ denote the Koszul complex with respect to $y_1, \ldots, y_s$. If M is an R-module, put $\mathbb{K}((y_1, \ldots, y_s); M) = \mathbb{K}((y_1, \ldots, y_s); R) \otimes M$, the Koszul complex on M with respect to $y_1, \ldots, y_s$. Set $\mathbb{H}_i((y_1, \ldots, y_s); M)$ to be the i-th homology module of $\mathbb{K}((y_1, \ldots, y_s); M)$; this is the Koszul homology on M with respect to $y_1, \ldots, y_s$. We will need the following facts:

Remark 2.3 (i) If $y_1, \ldots, y_n$ forms a regular sequence on M, then $\mathbb{K}((y_1, \ldots, y_n); M)$ is acyclic.

(ii) Let $J = (y_1, \ldots, y_s)$. Then for each $i = 0, \ldots, s$, $J \subseteq \operatorname{ann} \mathbb{H}_i((y_1, \ldots, y_s); M)$.

(iii) Suppose there is a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Then there is a long exact sequence on Koszul homology

$$\begin{aligned} \cdots \to \mathbb{H}_{i+1}((y_1, \dots, y_n); M_3) \to \mathbb{H}_i((y_1, \dots, y_n); M_1) \\ \to \mathbb{H}_i((y_1, \dots, y_n); M_2) \to \mathbb{H}_i((y_1, \dots, y_n); M_3) \to \cdots . \end{aligned}$$

(iv) For each $i = 0, \dots, s$, there is an isomorphism

$$\mathbb{H}_i((0, y_2, \dots, y_s); M) \cong \mathbb{H}_i((y_2, \dots, y_s); M) \oplus \mathbb{H}_{i-1}((y_2, \dots, y_s); M).$$

Throughout this section, let I be the saturated defining ideal of a locally Cohen–Macaulay, equidimensional subscheme V of $\mathbb{P}^n$; put $d = \dim V$. Let $J = (L_1, \dots, L_{n-1})$ be the complete intersection of $n-1$ general linear forms. In particular, $\mathbb{K}((L_1, \dots, L_{n-1}); R)$ is a free resolution of S/J.

Recall that the highest non-zero cohomology module $H^{d+1}_*(\widetilde{I})$ is never finitely generated, when $d \geq 0$. In the next result, we show that nonetheless, most of its Koszul homologies are finitely generated. As general notation, for a module M and an ideal A, let M_A denote the submodule of M which is annihilated by A; that is, $M_A = (0 :_M A)$.

Proposition 2.4 *Let I and $J = (L_1, \dots, L_{n-1})$ be as above. Then the Koszul homology $\mathbb{H}_i((L_1, \dots, L_{n-1}); H^{d+1}_*(\widetilde{I}))$ is finitely generated for each $i \geq d+2$. In particular, if $d \geq 0$, the Koszul homology has finite length.*

Proof: By changing coordinates if necessary, we may, without loss of generality, assume that $L_i = x_{i-1}$. We will prove the proposition by using induction on d. If $d = -1$, i.e., I defines the empty subscheme, then $H^0_*(\widetilde{I}) = I$ is already finitely generated, so each of its Koszul homologies is also finitely generated.

Suppose $d \geq 0$. The exact sequence

$$0 \to I \xrightarrow{x_0} I \to I/x_0 I \to 0$$

induces the long exact sequence on cohomology

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & A \arrow[r] & H^d_*(\widetilde{I/x_0 I}) \arrow[rr] \arrow[dr] & & H^{d+1}_*(\tilde{I}) \arrow[r, "x_0"] & H^{d+1}_*(\tilde{I}) \arrow[r] & 0 \\ & & & H^{d+1}_*(\tilde{I})_{(x_0)} \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd} \qquad (9)$$

where $A = H^d_*(\widetilde{I})/x_0H^d_*(\widetilde{I})$. Note in particular that A is finitely generated, since it is the quotient of two finitely generated modules.

From the right-hand part of this sequence, we obtain a long exact sequence of Koszul homology (see Remark 2.3)

$$\begin{aligned}\cdots \xrightarrow{x_0} \mathbb{H}_{i+1}((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})) \to \mathbb{H}_i((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)}) \\ \to \mathbb{H}_i((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})) \xrightarrow{x_0} \mathbb{H}_i((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})) \to \cdots.\end{aligned}$$

But for any module M, $\mathbb{H}_i((x_0,\ldots,x_{n-2}); M)$ is annihilated by x_0, so the long exact sequence breaks into short exact sequences

$$\begin{aligned}0 \longrightarrow \mathbb{H}_{i+1}((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})) \longrightarrow \mathbb{H}_i((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)}) \longrightarrow \\ \mathbb{H}_i((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})) \longrightarrow 0.\end{aligned}$$

Thus to show that $\mathbb{H}_i((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I}))$ is finitely generated for $i \geq d+2$, it will suffice to show that $\mathbb{H}_i((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)})$ is finitely generated for $i \geq d+2$. However, x_0 kills $H^{d+1}_*(\widetilde{I})_{(x_0)}$, and so

$$\begin{aligned}\mathbb{H}_i((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)}) &= \mathbb{H}_i((0,x_1,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)}) \\ &= \mathbb{H}_i((x_1,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)}) \oplus \mathbb{H}_{i-1}((x_1,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)}),\end{aligned}$$

and we can calculate this over $R = S/(x_0) = k[x_1,\ldots,x_n]$.

Now, the left-hand part of (9) yields a long exact sequence of Koszul homology

$$\begin{aligned}\cdots \to \mathbb{H}_{j+1}((x_1,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)}) \to \mathbb{H}_j((x_1,\ldots,x_{n-2}); A) \to \\ \mathbb{H}_j((x_1,\ldots,x_{n-2}); H^d_*(\widetilde{I/x_0I})) \to \mathbb{H}_j((x_1,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)}) \to \cdots.\end{aligned} \tag{10}$$

Here, the saturation of $I/x_0I \subseteq R$ defines a subscheme $\overline{V}$ of $\mathbb{P}^{n-1}$, with $\dim \overline{V} = d-1$. Thus, by the induction hypothesis, $\mathbb{H}_j(H^d_*(\widetilde{I/x_0I}))$ is finitely generated for each $j \geq d+1$. In particular, since $i \geq d+2$, this is true for $j = i, i-1$. Also, since A is finitely generated, all of its Koszul homologies are also finitely generated. But then (10) shows that both $\mathbb{H}_i((x_1,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)})$ and $\mathbb{H}_{i-1}((x_1,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I})_{(x_0)})$ are finitely generated. This implies that $\mathbb{H}_i((x_0,\ldots,x_{n-2}); H^{d+1}_*(\widetilde{I}))$ is finitely generated, which finishes the proof of the first statement.

For the second statement, recall that the Serre vanishing theorem says that $H^{d+1}(\widetilde{I}(t))$ vanishes for large t, and hence the Koszul homology $\mathbb{H}_i((L_1,\ldots,L_{n-1}); H^{d+1}_*(\widetilde{I}))$ also vanishes in high degrees. But since it is finitely generated, it must also vanish in low degrees, and we can conclude that it must have finite length. $\square$

Theorem 2.5 *Suppose I defines a locally Cohen–Macaulay, equidimensional subscheme V of dimension d of $\mathbb{P}^n$. Let $J = (L_1, \ldots, L_{n-1})$ be generated by $n-1$ general linear forms, and let $\mathbb{H}_i((L_1, \ldots, L_{n-1}); H^j_*(\widetilde{I}))$ be the Koszul homologies of $H^j_*(\widetilde{I})$ with respect to J. Then*

$$\nu(I) \leq \alpha(I) + 1 + \sum_{i=1}^{n-2} \dim \mathbb{H}_{i+1}((L_1, \ldots, L_{n-1}); H^i_*(V)). \tag{11}$$

Proof: Note that we have an exact sequence

$$0 \to \frac{I \cap J}{IJ} \to \frac{I}{IJ} \to \frac{I+J}{J} \to 0.$$

Hence, $\nu(I) = \nu(I/IJ) \leq \nu(\frac{I+J}{J}) + \nu(\frac{I \cap J}{IJ})$. Now, since $(I+J)/J$ is an ideal in S/J, which is a polynomial ring in two variables, Dubreil's Theorem applies, and says that $\nu((I+J)/J) \leq \alpha((I+J)/J) + 1$. Since J is generated by general linear forms, $\alpha((I+J)/J) = \alpha(I)$, and so $\nu((I+J)/J) \leq \alpha(I) + 1$.

Thus it only remains to estimate $\nu((I \cap J)/IJ)$. Note that I and J define disjoint schemes, since J is generated by general linear forms, and that the Koszul complex $\mathbb{K}((L_1, \ldots, L_{n-1}); S)$ is a free resolution of S/J. In particular, the isomorphisms and exact sequences of Theorem 1.1 hold. For each $i = 1, \ldots, n-1$, let $P_i = \ker \phi^i_i / \operatorname{im} \psi^i_{i+1}$. Note then, that $\mathbb{H}_{i+1}(H^i_*(V))$ naturally maps surjectively onto P_i, since $\operatorname{im} \psi^i_{i+1}$ contains $\operatorname{im} \phi^i_{i+1}$. Thus, in particular, $\dim P_i \leq \dim \mathbb{H}_{i+1}(H^i_*(V))$. Hence it follows from Theorem 1.1 that

$$\begin{aligned}
\nu((I \cap J)/IJ) \leq \dim((I \cap J)/IJ) &\leq \dim \mathbb{H}_2(H^1_*(V)) + \dim \ker \psi^2_2 \\
&\leq \dim \mathbb{H}_2(H^1_*(V)) + \dim \mathbb{H}_3(H^2_*(V)) + \dim \ker \psi^3_3 \\
&\vdots \\
&\leq \sum_{i=1}^{n-2} \dim \mathbb{H}_{i+1}(H^i_*(V)).
\end{aligned}$$

$\square$

Remark 2.6 Since $H^i_*(V) = 0$ whenever $i > \dim V + 1$, many of the terms in the formula (11) vanish. For instance, if V is a curve in $\mathbb{P}^5$, there are only two terms coming from the cohomology of V.

Remark 2.7 In general, for a finite length graded module, $\nu(M)$ is much less than $\dim M$, and so we would like to be able to replace "dim" by "ν" throughout in the above formula. However, counting minimal generators is much more difficult in general than counting vector space dimensions.

One important case in which we can replace "dim" by "ν" is when all but the top cohomology of V is annihilated by the maximal ideal. Recall the definition:

Definition 2.8 A subscheme V of $\mathbb{P}^n$ of dimension d is said to be *arithmetically Buchsbaum* if $H^i_*(V)$ is annihilated by the maximal ideal for each $i = 1, \ldots, d$, and if for each general linear subspace Y of $\mathbb{P}^n$, the cohomology $H^i_*(V \cap Y)$ is annihilated by the maximal ideal for $i = 1, \ldots, \dim V \cap Y$.

Note that the condition on linear subspaces of V is required, for there are examples of subschemes V whose cohomology is annihilated by the maximal ideal which have hypersurface sections whose cohomology is not annihilated by the maximal ideal; see for instance, [12]. In general, we will not require the full strength of this definition, only that the cohomologies are annihilated by the maximal ideal. Such subschemes are called *quasi–Buchsbaum.*

The next corollary was obtained for the case $n = 3$ in [11], and a somewhat better bound is stated in [2] for arithmetically Buchsbaum subschemes of $\mathbb{P}^n$.

Corollary 2.9 *Let I define a codimension 2 subscheme of $\mathbb{P}^n$ which is quasi-Buchsbaum. Then*

$$\nu(I) \leq \alpha(I) + 1 + \sum_{i=1}^{n-2} \binom{n-1}{i+1} \dim_k H^i_*(V).$$

Proof: We only have to note that since $H^i_*(V)$ is annihilated by the maximal ideal for $i = 1, \ldots, n-1$, and since every non-zero entry in a matrix representation for ϕ_i is a linear form, then for each $i = 1, \ldots, n-1$, $\mathbb{H}_{i+1}(H^i_*(V)) = \ker \phi^i_i$ is a direct sum of $\binom{n-1}{i+1}$ copies of (twists of) $H^i_*(V)$. $\square$

More generally, we can apply the same kind of analysis to quasi-Buchsbaum subschemes of arbitrary codimension, except that we now have to consider the top cohomology as well.

Corollary 2.10 *Let V be a d-dimensional quasi-Buchsbaum subscheme of $\mathbb{P}^n$, defined by the saturated ideal I. Let $J = (L_1, \ldots, L_{n-1})$ be generated by $n-1$ general linear forms, and let $\mathbb{H}_i(H_*^j(V))$ be the Koszul homologies of $H_*^j(V)$ with respect to J. Then*

$$\nu(I) \leq \alpha(I) + 1 + \sum_{i=1}^{d} \binom{n-1}{i+1} \dim_k H_*^i(V) + \dim \mathbb{H}_{d+2}(H_*^{d+1}(V)).$$

Proof: Again, since $H_*^i(V)$ is annihilated by every linear form for $i = 1, \ldots, d$, then $\mathbb{H}_{i+1}(H_*^i(V))$ is just a direct sum of $\binom{n-1}{i+1}$ copies of $H_*^i(V)$. □

Remark 2.11 By Lemma 2.4, even though $H_*^{d+1}(V)$ does not have finite length, the Koszul homology $\mathbb{H}_{d+2}(H_*^{d+1}(V))$ does have finite length, and so this corollary really does give a finite bound on the number of generators of I.

Example 2.12 Unfortunately, the bound in Theorem 2.5 does not seem to be very sharp. For example, in $\mathbb{P}^4$, let V be the union of a conic and a line not meeting the plane of the conic. Then V is arithmetically Buchsbaum, and $\dim_k H_*^1(V) = 1$. Also, $\alpha(I_V) = 2$, and $\nu(I_V) = 7$. However, if J is generated by three general linear forms, then $H_*^2(V)_J$ is at least 2, as can be seen, for instance, by a calculation using the inductive procedure in Proposition 2.4. So $\alpha(I_V) + 1 + 3 \dim_k H_*^1(V) + \dim H_*^2(V)_J \geq 8$, but $\nu(I_V) = 7$.

Remark 2.13 During the final preparation of this paper, the authors received the preprint [7] of Hoa, which contains bounds on the number of generators of an ideal based in part on the cohomology of the ideal, but involving different invariants of the ideal than our bounds. Neither Hoa's bounds nor our bounds seem to be particularly sharp in general.

3 On the Least Degree of Surfaces Containing Certain Buchsbaum Subschemes

In this section, we want to use the bound given in Section 2 to extend a result of Amasaki on the minimal degree of the minimal generators of an ideal I defining a codimension 2 Buchsbaum subscheme of $\mathbb{P}^n$. In [1], Amasaki showed that if C is a Buchsbaum curve in $\mathbb{P}^3$, and if $N = \dim_k H_*^1(C)$ is the Buchsbaum invariant of C, then $\alpha(I) \geq 2N$. A different proof was subsequently given in [5] based on combining the upper bound estimate for $\nu(I)$ of

Corollary 2.10 in the case of curves in $\mathbb{P}^3$ together with a lower bound estimate coming from a determination of the free resolution of the ideal I from a free resolution of $H^1_*(C)$. Also, Chang extended Amasaki's bound to a codimension 2 Buchsbaum subscheme of any $\mathbb{P}^n$ in [2], based on a structure theorem for the locally free resolution of the ideal sheaf associated to the subscheme.

Here, we would like to use our methods to give a different proof of Amasaki's bound for certain codimension 2 subschemes of $\mathbb{P}^n$. Specifically, we will give a lower bound for $\alpha(I)$ in terms of $H^1_*(V)$ for a codimension 2 subscheme of V for which $H^1_*(V)$ is annihilated by the maximal ideal, and $H^i_*(V) = 0$ for $i = 2, \ldots, \dim V$. Note that these quasi-Buchsbaum schemes are in fact Buchsbaum, since if H is a general hyperplane defined by a linear form L, the from the standard exact sequence

$$0 \to I \xrightarrow{\times L} I \to I/LI \to 0,$$

it is easy to see that $H^i_*(V \cap H) \cong H^i_*(V)$ for $1 \leq i \leq \dim V - 1$. Our method of proof will be to follow the lines of [5]. That is, we will combine our upper bound estimate of $\nu(I)$ with a lower bound estimate for $\nu(I)$ based on the free resolution of I.

We begin with an extension of a result in [13].

Proposition 3.1 *Suppose V is a codimension 2 subscheme of $\mathbb{P}^n$, such that $H^i_*(V) = 0$ for all $i = 2, \ldots, \dim V$. Let*

$$0 \to L_{n+1} \xrightarrow{\sigma_{n+1}} L_n \xrightarrow{\sigma_n} L_{n-1} \to \ldots \xrightarrow{\sigma_1} L_0 \to H^1_*(V) \to 0$$

be the minimal free resolution of the finite length module $H^1_*(V)$. Then the saturated defining ideal $I = I_V$ of V has a minimal free resolution

$$0 \to L_{n+1} \xrightarrow{\sigma_{n+1}} L_n \xrightarrow{\sigma_n} L_{n-1} \xrightarrow{\sigma_{n-1}} \ldots \xrightarrow{\sigma_4} L_4 \xrightarrow{[\sigma_3\ 0]} L_3 \oplus \bigoplus_{i=1}^{r} S(-b_i) \to \oplus_{j=1}^{p} S(-a_j) \to I \to 0,$$

for some $r \geq 0$, and $p = \nu(I)$.

Proof: Write a minimal free resolution of S/I as follows:

$$0 \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \ldots \xrightarrow{\phi_2} F_1 \to S \to S/I \to 0,$$

and let E_i be the i-th syzygy module. Sheafifying and dualizing the short exact sequence

$$0 \to F_n \xrightarrow{\phi_{n-1}} F_{n-1} \to E_{n-1} \to 0$$

yields the exact sequence

$$0 \to \mathcal{E}_{n-1}^\vee \to \mathcal{F}_{n-1}^\vee \xrightarrow{\phi_{n-1}^\vee} \mathcal{F}_n^\vee \to 0.$$

Taking cohomology then gives a sequence

$$0 \to H^0_*(\mathcal{E}_{n-1}^\vee) \to F_{n-1}^\vee \xrightarrow{\phi_{n-1}^\vee} F_n^\vee \to H^1_*(\mathcal{E}_{n-1}^\vee) \to 0.$$

Note, though, that $H^1_*(\mathcal{E}_{n-1}^\vee) \cong \mathrm{Ext}_S^{n+1}(H^1_*(V), S))$.

Next, for each $i = 2, \ldots, n-2$, consider the sequence

$$0 \to E_{i+1} \to F_i \to E_i \to 0.$$

Sheafifying, dualizing and taking cohomology gives an exact sequence

$$0 \to H^0_*(\mathcal{E}_i^\vee) \to F_i^\vee \to H^0(\mathcal{E}_{i+1}^\vee) \to H^1_*(\mathcal{E}_i^\vee) \to 0.$$

But $H^1_*(\mathcal{E}_i^\vee) = \mathrm{Ext}_S^{n+1}(H^{n-i}_*(V), S) = 0$, by assumption. Hence, we can paste together all these exact sequences to get a long exact sequence

$$F_2^\vee \xrightarrow{\phi_3^\vee} F_3^\vee \xrightarrow{\phi_4^\vee} \ldots \xrightarrow{\phi_n^\vee} \mathrm{Ext}_S^{n+1}(H^1_*(V), S) \to 0.$$

However, a minimal free resolution of $\mathrm{Ext}_S^{n+1}(H^1_*(V), S)$ is given by just dualizing the resolution of $H^1_*(V)$, and so we see that $F_i = L_{i+1}$ and $\phi_i = \sigma_{i+1}$ for $i = 3, \ldots, n$, and $F_2 = L_3 \oplus \bigoplus_{i=1}^r S$ and $\phi_2 = [\sigma_3 \; 0]$, for some $r \geq 0$. This finishes the proof. □

Corollary 3.2 *Let V be as in the previous proposition, and let I be its saturated defining ideal. Then $\nu(I) \geq 1 + \sum_{i=3}^{n+1}(-1)^i \operatorname{rank} L_i$.*

Proof: With the notation as in the statement of the Proposition, we have

$$\nu(I) = p = 1 + \operatorname{rank} L_3 + r - \operatorname{rank} L_4 + \operatorname{rank} L_5 + \ldots \geq 1 + \sum_{i=3}^{n+1}(-1)^i \operatorname{rank} L_i.$$

□

Corollary 3.3 *In addition to the assumptions of the Proposition, suppose that* $H^1_*(V)$ *is annihilated by the maximal ideal. Let* $N = \dim_k H^1_*(V)$*. Then* $\alpha(I) \geq (n-2)N$*.*

Proof: A minimal free resolution of $H^1_*(V)$ is just a direct sum of N copies of the Koszul complex resolving $k = S/\mathfrak{m}$. Thus, rank $L_i = N\binom{n+1}{i}$. By the previous Corollary and Corollary 2.10, we have

$$1 + \sum_{i=3}^{n+1} (-1)^i N \binom{n+1}{i} \leq 1 + \alpha(I) + (n+1)N.$$

A simple arithmetic calculation then reduces this to $(n-2)N \leq \alpha(I)$, as claimed. $\square$

References

[1] M. Amasaki, *On the structure of arithmetically Buchsbaum curves in* $\mathbb{P}^3$, Publ. Res. Inst. Math. Sci. **20** (1984), 793–837.

[2] M.-C. Chang, *Characterization of arithmetically Buchsbaum subschemes of codimension* 2 *in* $\mathbb{P}^n$, J. Diff. Geom. **31** (1991), 323–341.

[3] E. Davis, A. Geramita, and P. Maroscia, *Perfect homogeneous ideals: Dubreil's Theorems revisited*, Bull. Sci. Math. (2) **108** (1984), 143–185.

[4] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, lecture notes.

[5] A. Geramita and J. Migliore, *Generators for the ideal of an arithmetically Buchsbaum curve*, J. Pure Appl. Algebra **58** (1989), 147–167.

[6] R. Hartshorne, *Generalized divisors on Gorenstein schemes*, preprint.

[7] L. T. Hoa, *Bounds for the number of generators of generalized Cohen–Macaulay ideals*, preprint.

[8] M. Martin-Deschamps and D. Perrin, *Sur la classification des courbes gauches*, Astérisque 184–185 (Societé Mathématique de France, Paris, 1990).

[9] H. Matsumura, Commutative Algebra, Benjamin, New York, 1970.

[10] J. Migliore, *Liaison of a union of skew lines in* $\mathbb{P}^4$, Pacific J. Math. **130** (1987), 153–170.

[11] J. Migliore, *Submodules of the deficiency module*, J. London Math. Soc. **48** (1993), 396–414.

[12] C. Miyazaki, *Graded Buchsbaum algebras and Segre products*, Tokyo J. Math **12** (1989), 1–20.

[13] P. Rao, *Liaison among curves in* $\mathbb{P}^3$, Invent. Math. **50** (1979), 205–217.

[14] P. Schenzel, *Notes on liaison and duality*, J. Math. Kyoto Univ. **22** (1982), 485–498.

[15] J.-P. Serre, Algèbre Local–Multiplicités, Lecture Notes in Mathematics **11**, 3rd edition, Springer–Verlag, 1975.

[16] J. Stuckrad and W. Vogel, Buchsbaum Rings and Applications, an interaction between algebra, geometry and topology (VEB Deutscher Verlag der Wissenschaften, Berlin, 1986).

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