PRIME IDEALS OF FINITE HEIGHT IN POLYNOMIAL RINGS

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Abstract. We investigate the structure of prime ideals of finite height in polynomial extension rings of a commutative unitary ring R. We consider the question of finite generation of such prime ideals. The valuative dimension of prime ideals of R plays an important role in our considerations. If X is an infinite set of indeterminates over R, we prove that every prime ideal of $R[X]$ of finite height is finitely generated if and only if each $P \in \operatorname{Spec}(R)$ of finite valuative dimension is finitely generated and for each such P every finitely generated extension domain of R/P is finitely presented. We prove that an integrally closed domain D with the property that every prime ideal of finite height of $D[X]$ is finitely generated is a Prüfer v-multiplication domain, and that if D also satisfies d.c.c. on prime ideals, then D is a Krull domain in which each height-one prime ideal is finitely generated.

1. Introduction.

All rings considered in this paper are assumed to be commutative and to contain a unity element. Suppose $X = \{x_i\}_{i=1}^{\infty}$ is a countably infinite set of indeterminates over a Noetherian ring R and T is a localization of $R[X]$ with respect to a multiplicatively closed set of $R[X]$. (In particular, we are including the case where $T = R[X]$.) It is readily seen that a prime ideal of T is finitely generated if and only if it is of finite height (cf. [GH2, Theorem 4, page 2]). In relation to this result, it is shown in [GH3, Theorem 3.3] that an ideal $\mathbf{c}$ of T is finitely generated if and only if $\mathbf{c}$ has only finitely many associated prime ideals and each of the associated prime ideals of $\mathbf{c}$ is finitely generated. Moreover, if this occurs, then $\mathbf{c}$ has a finite primary decomposition.

Motivation for our work in the present paper comes from the following specific questions concerning a converse to the finite generation result.

Question 1.1. Suppose $X = \{x_i\}_{i=1}^{\infty}$ is a countably infinite set of indeterminates over a ring R.

(1) If every prime ideal of $R[X]$ of finite height is finitely generated, does it follow that every prime ideal of R of finite height is finitely generated?

(2) Assume that each prime ideal of R has finite height. If each prime ideal of $R[X]$ of finite height is finitely generated, does it follow that R is Noetherian?

We do not know the answer, in general, to either part of Question 1.1. For ease of reference in considering (1.1), we use the following terminology; here FH stands for finite height.

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Definition. Suppose $X = \{x_i\}_{i=1}^{\infty}$ is a countably infinite set of indeterminates over a ring R. We say that R is an FH-*ring* if every prime ideal of $R[X]$ of finite height is finitely generated.

The concept of valuative dimension is important in the consideration of Question 1.1. We recall that if D is an integral domain with quotient field K, then the **valuative dimension of** D, denoted $\dim_v D$, is the positive integer h if there exists a valuation overring[1] of D of rank h and no valuation overring of D of rank greater than h. If there exist valuation overrings of D of rank greater than h for every positive integer h, then D is said to have valuative dimension ∞. The **valuative dimension** of a commutative ring R is defined to be the supremum of the valuative dimensions of domain homomorphic images of R [J, page 56]. For $P \in \operatorname{Spec}(R)$, the **valuative dimension of** P is $\dim_v R_P$.

In general, for D an integral domain and $P \in \operatorname{Spec}(D)$, $\dim_v D/P$ is at most $\dim_v D - \dim D_P$ [J, Prop. 2, page 57]. Since one also has $\dim D \leq \dim_v D$ [J, Théorème 1, page 56], $\dim_v D/P$ is at most $\dim_v D - \operatorname{ht} P$. A summary of some basic properties of valuative dimension is given in [G1, page 36]. An important property for us is:

Remark 1.2. If $P \in \operatorname{Spec}(R)$ has finite valuative dimension h, where h is also the height of P (so $\dim R_P = \dim_v R_P$), then for X a set of indeterminates over R, the height of $PR[X]$ in $R[X]$ is also h (cf. [J, Théorème 3, page 62]).

Discussion 1.3.

(1) In view of Cohen's theorem that a ring is Noetherian if every prime ideal of the ring is finitely generated [N, (3.4)], an affirmative answer to part (1) of (1.1) implies that the answer to part (2) of (1.1) is also affirmative.

(2) Suppose P is a prime ideal of R and Y is a set of indeterminates over R. Then $Q = PR[Y]$ is a prime ideal of $S = R[Y]$. Since S is a free R-module, it is readily seen that Q is finitely generated in S if and only if P is finitely generated in R. Moreover, if $Y = \{y_1, \ldots, y_n\}$ is a finite set and P has finite height, then Q also has finite height. Indeed, if P has height h, then the height of $PR[y_1]$ is at least h and at most $2h$ (cf. [G2, (30.2)]). Therefore if the set Y is finite, then $Q = PR[Y]$ has finite height if P has finite height and the question analogous to (1.1) for a finite set of indeterminates has an affirmative answer.

(3) In the setting of (1.1), it is possible that there exists in R a prime ideal P having finite height such that $Q = PR[X]$ has infinite height in $R[X]$. Indeed, if R is an integral domain, then $Q = PR[X]$ has infinite height precisely if the domain R_P has infinite valuative dimension (cf. [G2, page 360], [J, page 63]).

[1]By an **overring** of an integral domain D with quotient field K we mean a subdomain of K that contains D.

Suppose R is an FH-ring and Y is a set of indeterminates over R. Is every prime ideal of $R[Y]$ of finite height also finitely generated? We show in (1.4) below that this question has an affirmative answer if Y is infinite. On the other hand, if Y is finite, we show in (1.5) that an affirmative answer to this question is equivalent to an affirmative answer to Question 1.1.

Proposition 1.4. Suppose R is an FH-ring and Y is an arbitrary infinite set of indeterminates over R. Then each prime ideal of $R[Y]$ of finite height is finitely generated.

Proof. Let P be a prime ideal of $R[Y]$ of finite height h and let $P_0 < P_1 < \cdots < P_h = P$ be a chain of prime ideals of $R[Y]$ of length h with terminal element P. Choose a polynomial $f_i \in P_i - P_{i-1}$ for $i = 1, 2, \ldots, h$. There exists a finite subset $\{y_i\}_{i=1}^n$ of Y such that each $f_j \in R[y_1, \ldots, y_n]$. It follows that $P \cap R[y_1, \ldots, y_n]$ has height at least h. Extend $\{y_i\}_1^n$ to a countably infinite subset Y' of Y. Then $P \cap R[Y']$ has height at least h, $P^* = (P \cap R[Y'])R[Y] \subseteq P$ has height at least h, and hence $P = (P \cap R[Y'])R[Y]$. It follows that $P \cap R[Y']$ has height h. Since R is an FH-ring, $P \cap R[Y']$ is finitely generated. Consequently, P is finitely generated. $\square$

Remark 1.5. Suppose x is an indeterminate over a ring R. As noted in part (2) of (1.3), a prime ideal P of R is finitely generated if and only if $Q = PR[x]$ is finitely generated in $R[x]$, and Q has finite height if P has finite height. Thus if Y is a finite set of indeterminates over R, and if every prime ideal of $R[Y]$ of finite height is finitely generated, then R also has this property. The converse, however, is not true. There exists an integral domain R having the property that there exists in R no nonzero prime ideal of finite height and which also has the property that there exists in $R[x]$ a prime ideal Q of height one that is not finitely generated. To obtain such a domain R one can begin with a valuation domain V of infinite rank having no nonzero prime ideal of finite height and having the form $V = F(t) + M$, where M is the maximal ideal of V, F is a field and $F(t)$ is a simple transcendental extension field of F. Let $R = F + M$ and let Q be the kernel of the canonical R-algebra homomorphism $R[x] \to R[t]$ of the polynomial ring $R[x]$ mapping x to t. Then Q is a prime ideal of $R[x]$ of height one, for if K denotes the quotient field of R, then $R[x]_Q$ is a localization of the polynomial ring $K[x]$ and hence is a DVR. Moreover, Q is not finitely generated, for the content ideal of Q in R is M and M as an ideal of R is not finitely generated.

In this example, the prime ideal Q of R has valuative dimension one. Hence if $x = x_1$, and $X = \{x_i\}_{i=1}^{\infty}$, then $QR[X]$ is a non-finitely generated prime ideal of $R[X]$, and by (1.2), $QR[X]$ has height one. Therefore the converse of part (1) of (1.1) is not true; that is, there exists a ring R in which each prime ideal of finite height is finitely generated such that $R[X]$ fails to have this property.

Question 1.6. Suppose R is an FH-ring and $\mathbf{c}$ is an ideal of $R[X]$ having finitely many associated primes, each of which is finitely generated.

(1) Does it follow that $\mathbf{c}$ is finitely generated?

(2) Does it follow that $\mathbf{c}$ has a finite primary decomposition?

Remark 1.7.

(1) If R is an FH-ring, then every height-zero prime of R is finitely generated. For if P is a height-zero prime of R, then $PR[X]$ is a height-zero prime of $R[X]$. Thus $PR[X]$ is finitely generated and so P is finitely generated. It follows that R has only finitely many height-zero primes [GH3, Theorem 1.6].

(2) In view of (1.4) and [GH2, Theorem 4], every Noetherian ring, or polynomial ring over a Noetherian ring, is an FH-ring. As we note in (2.1) below, it is also true in general that a localization of an FH-ring is again an FH-ring.

(3) The case of (1.1) where R is an integral domain is already quite interesting. We consider this case in §3.

2. Stability properties of FH-rings and valuative dimension.

Proposition 2.1. Suppose R is an FH-ring.

(1) If U is a multiplicatively closed subset of R, then the localization $U^{-1}R = R_U$ is again an FH-ring.

(2) If Y is a set of indeterminates over R, then the polynomial ring $R[Y]$ is an FH-ring.

Proof. Since $R[X]_U$ is canonically isomorphic to $R_U[X]$ and since a prime ideal Q of $R[X]_U$ has finite height if and only if $Q \cap R[X]$ has finite height in $R[X]$, the first assertion is clear. For (2), suppose X is a countably infinite set of indeterminates over $R[Y]$. By (1.4), every prime ideal of $R[Y][X]$ of finite height is finitely generated. Therefore $R[Y]$ is an FH-ring. $\square$

Notation 2.2. We use $R^{(n)}$ to denote the polynomial ring in n indeterminates over a ring R.

Proposition 2.3. Suppose X is an infinite set of indeterminates over a ring R and $P \in \mathrm{Spec}(R)$. Then the following are equivalent.

(1) $P[X]$ has finite height in $R[X]$.

(2) $PR_P[X]$ has finite height in $R_P[X]$.

(3) R_P has finite valuative dimension.

Consequently, if R is an FH-ring having finite valuative dimension, then R is Noetherian.

Proof. The equivalence of (1) and (2) is clear. If R_P has finite valuative dimension h, then for n sufficiently large, the height of $P(R_P)^{(n)}$ is the height of $PR_P[X]$, which is h (cf. [J, Théorème 3, page 62]). Thus (3) implies (2). On the other hand, if R_P has infinite valuative dimension, then the sequence $\{\text{ht } P(R_P)^{(n)}\}_{n=1}^{\infty}$ is unbounded (cf. [J, Théorème 4, page 63]). Hence $PR_P[X]$ has infinite height and (2) implies (3). $\square$

Proposition 2.4. Suppose R is a ring and $P \in \text{Spec}(R)$ contains only finitely many height-zero primes $P_1, \ldots, P_k$ of R. Let X be an infinite set of indeterminates over R. The following are equivalent:

(1) $PR[X]$ has finite height.

(2) $PR[X]/P_iR[X]$ has finite height for each i, $1 \leq i \leq k$.

(3) The domain R_P/P_iR_P has finite valuative dimension for each i, $1 \leq i \leq k$.

Proof. The equivalence of (1) and (2) follows from the fact that $\{P_i[X]\}_1^k$ is the set of height-zero primes of $R[X]$ contained in $P[X]$. In view of the fact that $P[X]/P_i[X] \cong (P/P_i)[X]$ and $(R/P_i)_{P/P_i} \cong R_P/P_iR_P$, the equivalence of (2) and (3) follows from Proposition 2.3. $\square$

Theorem 2.5. A ring R is an FH-ring if and only if for each positive integer n, each prime ideal of $R^{(n)}$ of finite valuative dimension is finitely generated.

Proof. Suppose R is an FH-ring and $Q \in \text{Spec}(R^{(n)})$ is of finite valuative dimension. By (2.1), $R^{(n)}$ is an FH-ring and by (1.7), $R^{(n)}$ has only finitely many height-zero primes. Hence (2.4) implies that $QR^{(n)}[X]$ has finite height, where X is an infinite set of indeterminates over $R^{(n)}$. Therefore $QR^{(n)}[X]$, and hence Q, is finitely generated.

Conversely, assume that each prime of $R^{(n)}$ of finite valuative dimension is finitely generated. It follows that every height-zero prime of R is finitely generated. Hence by [GH3, Theorem 1.6]. R has only finitely many height-zero primes. Let P be a prime ideal of $R[X]$ of finite height h. There is a finite subset Y of X such that $P \cap R[Y]$ has height at least h. We necessarily have $(P \cap R[Y])R[X] = P$, since the prime ideal $(P \cap R[Y])R[X]$ is contained in P and has height at least h. By (2.4), it follows that $P \cap R[Y]$ has finite valuative dimension. By hypothesis, this means that $P \cap R[Y]$ is finitely generated, so that $P = (P \cap R[Y])R[X]$ is also finitely generated. Consequently, R is an FH-ring. $\square$

Proposition 2.6. Suppose R is a ring, n is a positive integer, $Q \in \text{Spec}(R^{(n)})$, and $P = Q \cap R$. Then Q has finite valuative dimension if and only if P has finite valuative dimension.

Proof. By passing from R to R_P, we may assume that R is quasilocal with maximal ideal P. If P has finite valuative dimension h, then $R^{(n)}$ has valuative dimension $h + n$ [J, Théorème 2, page 60]. Since $Q \in \text{Spec}(R^{(n)})$, it follows that Q has finite valuative dimension. On the other hand, if P

has infinite valuative dimension, then $PR^{(n)}$ has infinite valuative dimension. Since $R^{(n)}_{PR^{(n)}}$ is a localization of $R^{(n)}_Q$, it follows that Q has infinite valuative dimension. $\square$

Remark 2.7. Suppose $S = R[\zeta_1, \dots, \zeta_n]$ is a finitely generated extension ring of R. If $Q' \in \mathrm{Spec}(S)$ has infinite valuative dimension, then $P = Q' \cap R$ also has infinite valuative dimension. For S is an R-algebra homomorphic image of $R^{(n)}$ and the preimage Q of Q' in $R^{(n)}$ has infinite valuative dimension and $Q \cap R = Q' \cap R = P$. Hence by (2.6), P has infinite valuative dimension. However, as we observe in Remark 3.7 below, it can happen that there exists a prime ideal $Q' \in \mathrm{Spec}(S)$ of finite valuative dimension such that $Q' \cap R = P$ has infinite valuative dimension.

Discussion 2.8.

(1) Since every ring is a homomorphic image of a polynomial ring over $\mathbb{Z}$ and since, as noted in part (2) of (1.7), a polynomial ring over a Noetherian ring is an FH-ring, the property of being an FH-ring is not in general preserved under homomorphic image.

(2) It is unclear whether for P a height-zero prime of an FH-ring R it follows that R/P is again an FH-ring. A problem here is that for $Q \in \mathrm{Spec}(R)$ with $P < Q$ it may happen that $QR[X]$ has infinite height, but $QR[X]/PR[X]$ has finite height.

(3) It would be interesting to know if a finitely generated extension ring of an FH-ring is again an FH-ring.

3. FH-domains and condition (ρ).

Discussion 3.1. Let D be an integral domain with quotient field K and let $x_1, \dots, x_n$ be indeterminates over K. Then $K[x_1, \dots, x_n] = K^{(n)}$ is a localization of $D[x_1, \dots, x_n] = D^{(n)}$. Hence for $P \in \mathrm{Spec}(K^{(n)})$ we have $(K^{(n)})_P = (D^{(n)})_{P \cap D^{(n)}}$. Therefore $P \cap D^{(n)}$ is of finite valuative dimension. In view of Theorem 2.5, for each positive integer n, an FH-domain D satisfies the following condition which we denote by (ρ_n).

(ρ_n) For each $P \in \mathrm{Spec}(K^{(n)})$, the contraction $P \cap D^{(n)}$ is finitely generated.

We say the integral domain D satisfies condition (ρ) if D satisfies (ρ_n) for each positive integer n.

Remark 3.2. An equivalent form of condition (ρ) on an integral domain D is that every finitely generated extension domain of D is finitely presented. It was proved by Nagata in [N2] that a valuation domain has this property, and a result of Raynaud and Gruson in [RG, (3.4.7), page 26] implies that a Prüfer domain also has this property.

Condition (ρ) modulo prime ideals of finite valuative dimension of a ring R relates nicely to R being an FH-ring as we observe in Theorem 3.3.

Theorem 3.3. A ring R is an FH-ring if and only if each $P \in \mathrm{Spec}(R)$ of finite valuative dimension is finitely generated and for each such P the integral domain R/P satisfies condition (ρ).

Proof. Assume that R is an FH-ring. By Theorem 2.5, each $P \in \mathrm{Spec}(R)$ of finite valuative dimension is finitely generated. To show R/P satisfies condition (ρ) it suffices to show that if Q' is a prime ideal of the polynomial ring $(R/P)^{(n)}$ such that $Q' \cap (R/P) = (0)$, then Q' is finitely generated. Let Q denote the preimage of Q' in $R^{(n)}$. Then $Q \cap R = P$. By (2.6), Q has finite valuative dimension. Since R is an FH-ring, Q is finitely generated by (2.5). Therefore Q' is finitely generated.

Assume conversely that each $P \in \mathrm{Spec}(R)$ of finite valuative dimension is finitely generated and R/P satisfies condition (ρ). To show R is an FH-ring, by Theorem 2.5, it suffices to show for each positive integer n that each prime Q of $R^{(n)}$ of finite valuative dimension is finitely generated. Proposition 2.6 implies that $P = Q \cap R$ is of finite valuative dimenion in R. Therefore P is finitely generated. Since R/P satisfies condition (ρ), the image of Q in $(R/P)^{(n)}$ is finitely generated. Therefore Q is finitely generated. □

A test case for (1.1)(b) asks whether a one-dimensional quasilocal FH-domain D is Noetherian. By (2.3), the answer is affirmative if $\dim_v D$ is finite. On the other hand, Theorem 3.3 implies that a one-dimensional quasilocal domain having infinite valuative dimension and satisfying condition (ρ) is an FH-domain: hence the existence of such a domain would provide a negative answer to (1.1)(b).

Let D be an integral domain with quotient field K. We recall that D is said to be **quasi-coherent** if $I^{-1} = D :_K I = \{a \in K : aI \subseteq D\}$ is finitely generated for each nonzero finitely generated ideal I of D [GaH].

Proposition 3.4. If D satisfies condition (ρ), then D is quasi-coherent.

Proof. Suppose $I = (a_1, \ldots, a_n)D$ is a nonzero finitely generated ideal. Let $x_1, \ldots, x_n$ be indeterminates over K and let $f = a_1x_1 + \cdots + a_nx_n$. Then $fK[x_1, \ldots, x_n]$ is a height-one prime ideal of $K[x_1, \ldots, x_n] = K^{(n)}$. Let $P = fK^{(n)} \cap D^{(n)}$. Since D satisfies condition (ρ), P is a finitely generated homogeneous ideal, where $D^{(n)}$ is regarded as a graded ring with D of degree zero and each x_i of degree one. The degree one piece of P is $I^{-1}f$, and P finitely generated as an ideal of $D^{(n)}$ implies that $I^{-1}f$ is finitely generated as a D-module. Therefore I^{-1} is finitely generated as a fractional ideal of D. □

From (3.3) and (3.4), we have the following corollary.

Corollary 3.5. If R is an FH-ring, then for each $P \in \mathrm{Spec}(R)$ of finite valuative dimension, the domain R/P is quasi-coherent. In particular, since the ideal (0) of an integral domain is a prime ideal of finite valuative dimension, if D is an FH-domain, then D is quasi-coherent.

Question 3.6. Suppose $E = D[\zeta]$ is a simple integral extension of domains. Is there an implication in either (or both) directions between the condition that D is an FH-domain and the condition that E is an FH-domain?

Remark 3.7. In relation to Question 3.6, we remark that there can exist in E a maximal ideal M_2 of finite valuative dimension such that $M_2 \cap D = M$ has infinite valuative dimension. This is illustrated by [GH1, Example 5.8, page 161], where A is the field of algebraic numbers, $A((x))$ is the quotient field of the formal power series ring $A[[x]]$, V_1 is a valuation domain of infinite rank on $A((x))$ of the form $V_1 = A + M_1$, and $V_2 = A[[x]] = A + M_2$, where $M_2 = xA[[x]]$. Then with $M = M_1 \cap M_2$, and $\zeta \in M_1$ such that ζ is a unit in V_2, we define $D = A + M$ and $E = D[\zeta]$.

It is easy to see that condition (ρ) lifts from D to E. More generally we have:

Proposition 3.8. If $n \geq 2$, and if an integral domain D satisfies condition (ρ_n), then a simple extension domain $E = D[\zeta]$ of D satisfies condition (ρ_{n-1}). Thus if D satisfies condition (ρ), then every finitely generated extension domain of D also satisfies condition (ρ).

Proof. Suppose $P' \in \mathrm{Spec}(E^{(n-1)})$ is such that $P' \cap E = (0)$. Under the canonical D-algebra homomorphism of $D^{(n)}$ onto $D[\zeta]^{(n-1)}$ mapping $x_n \to \zeta$, the preimage of P' is a prime ideal $P \in \mathrm{Spec}(D^{(n)})$ such that $P \cap D = (0)$. Since D satisfies condition (ρ_n), P is finitely generated. Therefore P' is finitely generated and $E = D[\zeta]$ satisfies condition (ρ_{n-1}). The second statement of (3.8) follows from the first statement. □

Corollary 3.9. Suppose R is an FH-ring and $P \in \mathrm{Spec}(R)$ is of finite valuative dimension. Then every finitely generated extension domain of R/P is quasi-coherent. In particular, if D is an FH-domain, then every finitely generated extension domain of D is quasi-coherent.

Proof. Apply (3.5) and (3.8). □

4. Integrally closed FH-domains.

We recall that an integral domain D is a **Prüfer v-multiplication ring**,[2] abbreviated PVMD, if the divisorial ideals of D of finite type form a group [Kr, page 667], [G2, page 427], [MZ]. It is well known that an integrally closed quasi-coherent domain is a PVMD. A simple direct proof for this is to observe that if I is a nonzero finitely generated ideal of a quasi-coherent domain D, then $J = II^{-1}$ is a finitely generated integral ideal of D with the property that $J^{-1} = J : J$. Since J

[2]The term v-multipliation ring is used in [Gr], while Bourbaki [B, page 96] calls such domains pseudo-Prüfer.

is finitely generated, the elements of $J:J$ are integral over D. If D is also integrally closed, then $J^{-1} = J:J = D$, and it follows that D is a PVMD.

Proposition 4.1. Suppose R is an FH-ring and $P \in \mathrm{Spec}(R)$ is of finite valuative dimension. Then every finitely generated integrally closed extension domain of R/P is a PVMD. In particular, if D is an integrally closed FH-domain, then D is a PVMD.

Proof. This is immediate from (3.9) and the fact that an integrally closed quasi-coherent domain is a PVMD. □

Corollary 4.2. Suppose D is a one-dimensional FH-domain such that the integral closure D' of D is a finitely generated D-module. Then D is Noetherian. In particular, a one-dimensional integrally closed FH-domain is a Dedekind domain.

Proof. By (4.1), D' is a PVMD. Since a one-dimensional PVMD is Prüfer, it follows that D', and hence D, has valuative dimension one. Therefore, by (2.5), each prime ideal of D is finitely generated, and D is Noetherian. □

In preparation for showing that certain integrally closed FH-domains are Krull domains, we note the following.

Proposition 4.3. A nontrivial valuation domain V is an FH-domain if and only if V is either a rank-one discrete valuation domain (DVR), or $\mathrm{Spec}(V)$ contains no prime ideal of finite positive height.[3]

Proof. If V contains a prime ideal of finite positive height and V is not a DVR, then V contains a non-finitely generated prime ideal P of finite height. Then $PV[X]$ is of finite height in $V[X]$ and is not finitely generated. On the other hand, it is clear that if V is a DVR, then V is an FH-domain. If $\mathrm{Spec}(V)$ contains no prime ideal of finite positive height, then Theorem 3.3 implies that V is an FH-domain, for as noted in (3.2), V satisfies condition (ρ). □

Theorem 4.4. Suppose D is an integrally closed FH-domain that satisfies the descending chain condition (d.c.c.) on prime ideals. Then D is a Krull domain, and each prime ideal of D of height one is finitely generated.

Proof. By Proposition 4.1, D is a PVMD. Hence there exists a set $\{P_a\}_{a\in A}$ of prime ideals of D such that $D = \cap_a D_{P_a}$, where each D_{P_a} is a valuation domain. By (2.1), each D_{P_a} is an FH-domain. Since D and therefore D_{P_a} satisfies d.c.c. on prime ideals, either $P_a = (0)$, or D_{P_a}

[3] A nontrivial valuation domain V has no prime ideal of finite positive height if and only if the nonzero prime ideals of V intersect in (0).

is a DVR. Therefore P_a has finite valuative dimension, so by (2.5) each P_a is finitely generated. Suppose $d \in D$ is a nonzero non-unit, and let P be a minimal prime of (d). Then D_P is a PVMD whose maximal ideal PD_P is the radical of a principal ideal. It follows that D_P is a valuation domain, thus a DVR, and P is finitely generated. Therefore each minimal prime of (d) is finitely generated. Hence by [GH3, Theorem 1.6] (d) has only finitely many minimal primes. It follows that the representation $D = \cap_a D_{P_a}$ is locally finite, and D is a Krull domain in which each height-one prime ideal is finitely generated. $\square$

Question 4.5. Suppose $(R, \mathbf{m})$ is a 2-dimensional quasilocal integrally closed FH-domain. Must R be Noetherian?

With notation as in (4.5), we note that if P is a height-one prime of R, then P is finitely generated and has finite valuative dimension. Therefore R/P is a one-dimensional quasilocal domain that satisfies condition (ρ) and hence is quasi-coherent. If R/P is Noetherian, then $\mathbf{m}$ is finitely generated and R is Noetherian.

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