A BESSEL FUNCTION MULTIPLIER

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Abstract. We obtain nearly sharp estimates for the $L^p(\mathbb{R}^2)$ norms of certain convolution operators.

For $n \geq 1$ let λ_n be the measure on $\mathbb{R}^2$ obtained by multiplying normalized arclength measure on $\{|x| = 1\}$ by the oscillating factor $e^{in\arg(x)}$. For $1 \leq p \leq \infty$, let $C(p, n)$ denoted the norm of the operator $T_n f \doteq \lambda_n * f$ on $L^p(\mathbb{R}^2)$. The purpose of this note is to estimate the rate of decay of $C(p, n)$ as $n \to \infty$. By duality, it is enough to consider $p \geq 2$. Examples below will show that

$$C(p, n) \geq C(p) n^{-\frac{1}{6} - \frac{1}{3p}} \quad \text{if} \quad 2 \leq p \leq 4, \tag{1}$$

and

$$C(p, n) \geq C(p) n^{-\frac{1}{p}} \quad \text{if} \quad 4 \leq p \leq \infty. \tag{2}$$

On the other hand, we will observe that

$$\begin{aligned} C(2, n) &\leq C n^{-\frac{1}{3}} \\ C(\infty, n) &\leq C \end{aligned} \tag{3}$$

and then prove the following result.

Theorem. *There is a positive number a such that*

$$C(4, n) \leq C n^{-\frac{1}{4}} (\log(n))^a. \tag{4}$$

Interpolating (3) and (4) gives upper bounds for $C(p, n)$ which differ only by a power of $\log(n)$ from the lower bounds of (1) and (2), thus providing nearly sharp estimates for $C(p, n)$.

The above question naturally arises when considering the $L^p(\mathbb{R}^3)$ mapping properties of the operator T given by convolution with respect to a compact piece of arclength measure on the helix

$$t \to (\cos t, \sin t, t).$$

T is an example of a folding Fourier integral operator in dimension 3, whose singular set is of dimension 1. The sharp $L^p \to L^2$ mapping properties of T were established by the first author in

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[O]. The operator T_n arises when considering the L^p smoothing properties of T; that is, for which values of α_p is $|D|^{\alpha_p} T$ bounded on $L^p(\mathbb{R}^3)$. Since

$$T\big(e^{inx_3} f(x_1,x_2)\big) = e^{inx_3}(T_n f)(x_1,x_2)\,,$$

the exponents in (1) and (2) give upper bounds on α_p. In particular, the smoothing exponent for T is less than that of averaging in $\mathbb{R}^2$ over the cubic $t \rightarrow (t,t^3)$, where the corresponding value of α_p is

$$\alpha_p = \begin{cases} \frac{1}{3} & \text{if} \quad 2 \le p < 3\,, \\ \\ \frac{1}{p} & \text{if} \quad 3 < p < \infty\,. \end{cases}$$

See, for example, [SW] or [SS].

To see (2), apply the operator $T_n f \doteq \lambda_n * f$ to $f(x) = e^{-in\arg(x)}\chi_A(x)$ where A is the annulus $\{1 \le |x| \le 1 + 1/n\}$. One observes that there is a constant C such that $|T_n f(x)| \ge C$ if $|x| \le C/n$ and (2) follows (for all p, but (1) is better for $p \le 4$).

The example for (1) is a little more complicated: for fixed n, and $1 \le j \le n^{1/3}$, let $\theta_j = jn^{-1/3}$, $\omega_j = (\cos(\theta_j), \sin(\theta_j))$, and $\omega_j^{'} = (-\sin(\theta_j), \cos(\theta_j))$. Let B_j be the disk $\{|x-\omega_j| \le \varepsilon n^{-\frac{1}{3}}\}$ where ε is a positive number independent of n and small enough to insure that, for any n, the disks B_j are pairwise disjoint. Let

$$f_j(x) = e^{in(x\cdot\omega_j^{'})}\chi_{B_j}(x).$$

One can check that

$$|T_n f_j(x)| \ge cn^{-\frac{1}{3}} \quad \text{if} \quad |x| \le cn^{-\frac{1}{3}} \tag{5}$$

for some small positive c independent of n and j. Let r_j be the jth Rademacher function on $[0,1]$ and put

$$f(t,x) = \sum_{j=1}^{n^{\frac{1}{3}}} r_j(t) f_j(x).$$

Then

$$||f(t,\cdot)||_p \le Cn^{-\frac{1}{3p}}. \tag{6}$$

Also

$$\int_0^1 ||T_n f(t,\cdot)||_p^p\,dt \ge \int_{|x|\le cn^{-1/3}} \Big(\sum_j |T_n f_j(x)|^2\Big)^{p/2} dx \ge c^{2+p} n^{-\frac{2}{3}-\frac{p}{6}}\,,$$

where the third inequality uses (5). With (6) this yields (1).

A computation shows that $\widehat{T_n}(\xi) = e^{in\arg(\xi)} J_n(|\xi|)$ (whence the name of this note). Thus (3) follows from the estimate, uniform in n,

$$|J_n(r)| \le Cr^{-\frac{1}{3}} \quad \text{if} \quad r \ge 1$$

(see p.357 in [S]) combined with the observation

$$|J_n(r)| \le \frac{C}{n} \quad \text{if} \quad 0 \le r \le \frac{3n}{4}. \tag{7}$$

To begin the proof of (4), let ρ be a smooth cutoff function which is equal to 1 on the annulus $\{\frac{3}{4} \le |\xi| \le \frac{5}{4}\}$ and is supported in the annulus $\{\frac{1}{2} \le |\xi| \le \frac{3}{2}\}$. Let S_n be the operator defined by $\widehat{S}_n(\xi) = \widehat{T}_n(\xi)\rho(|n^{-1}\xi|)$. The easy estimate

$$|J_n(r)| \le Cn^{-\frac{1}{2}} \quad \text{if} \quad r \ge \frac{5n}{4}$$

combines with (7) to show that the $L^2(\mathbb{R}^2)$ operator norm $||T_n - S_n||_{2,2}$ is $O(n^{-\frac{1}{2}})$. Interpolating this with $||T_n - S_n||_{\infty,\infty} = O(1)$ yields $||T_n - S_n||_{4,4} = O(n^{-\frac{1}{4}})$. Thus (4) will follow from

$$||S_n||_{4,4} \le Cn^{-\frac{1}{4}}(\log(n))^a, \tag{8}$$

which is our principal result. The Fourier transform $\widehat{S}_n(\xi)$ is supported in the annulus $A_n = \{\frac{n}{2} \le |\xi| \le \frac{3n}{2}\}$. Having fixed n, we will decompose S_n by decomposing A_n into a union of annuli A_n^j as follows:

$$\text{for } j \ge 1, \text{ set } A_n^j = \{n + 2^j n^{\frac{1}{3}} \le |\xi| \le n + 2^{j+1} n^{\frac{1}{3}}\}\,;$$

$$\text{set } A_n^0 = \{n - 2n^{\frac{1}{3}} \le |\xi| \le n + 2n^{\frac{1}{3}}\}\,;$$

$$\text{for } j \le -1, \text{ set } A_n^j = \{n - 2^{|j|+1} n^{\frac{1}{3}} \le |\xi| \le n - 2^{|j|} n^{\frac{1}{3}}\}\,.$$

Introducing a suitable partition of unity on the Fourier transform side leads to the decomposition

$$S_n = \sum_j S_n^j\,.$$

For fixed n, the number of terms S_n^j is $O(\log(n))$. Thus (8) will follow from

$$||S_n^j||_{4,4} \le Cn^{-\frac{1}{4}}(\log(n))^b \tag{9}$$

for all j and n and some $b > 0$. At this point we make a further decomposition of A_n^j into sectors A_n^{jl} of opening angle $\delta \doteq 2^{|j|/2} n^{-\frac{1}{3}}$. This leads to a decomposition

$$S_n^j = \sum_{l=1}^{\delta^{-1}} S_n^{jl}\,.$$

The function $\widehat{S}_n^{jl}$ is supported in a set R^{jl} obtained from the intersection of the annulus $n + \frac{1}{2}n\delta^2 \le |\xi| \le n + 3n\delta^2$ with a sector of angle δ; thus, R^{jl} is essentially a rectangle of dimensions $n\delta$ by $n\delta^2$, with major dimension $n\delta$ normal to the vector through the center of R^{jl}.

Lemma.

$$||S_n^{jl}||_{4,4} \le Cn^{-\frac{1}{4}}\delta^{\frac{1}{4}}\,.$$

Proof. We will obtain the lemma by interpolating the following estimates

$$\begin{aligned} ||S_n^{jl}||_{2,2} &\le C\,(n\delta)^{-\frac{1}{2}}\,, \\ ||S_n^{jl}||_{\infty,\infty} &\le C\,\delta\,. \end{aligned} \tag{10}$$

The first estimate in (10) is a bound on $J_n(r)$ over the annulus A_n^j. The desired estimates are well known, but we provide the simple argument here for completeness.

For $j=0$, the desired bounds follow from the uniform bound $|J_n(r)| \le C\, n^{-\frac13}$. For $j \neq 0$, it suffices to show that

$$\Bigl| \int_0^\pi e^{int - in(1\pm\delta^2)\sin t}\, dt \Bigr| \le C\,(n\delta)^{-\frac12}\,,$$

where C is uniform over $n \in \mathbb{Z}$ and $\delta^2 \le 1/2$.

We let $\phi(t) = t - (1\pm\delta^2)\sin t$. On the interval $0 \le t \le \delta$, we have $|\phi'(t)| \ge c\delta^2$, and $\phi'(t)$ is monotonic, so Proposition 2 of [S], page 332, implies that

$$\Bigl| \int_0^\delta e^{int - in(1\pm\delta^2)\sin t}\, dt \Bigr| \le C\,(n\delta^2)^{-1} \le C\,(n\delta)^{-\frac12}\,.$$

On the interval $\delta \le t \le \pi-\delta$, it follows that $|\phi''(t)| \ge c\delta$, and the same proposition implies that

$$\Bigl| \int_\delta^{\pi-\delta} e^{int - in(1\pm\delta^2)\sin t}\, dt \Bigr| \le C\,(n\delta)^{-\frac12}\,.$$

On the interval $\pi-\delta \le t \le \pi$, $|\phi'(t)| \ge 1$, and the integral is bounded by n^{-1}.

For the second estimate of (10), it suffices to consider the term $S_n^{j\,0}$, associated to the rectangle $R_n^{j\,0}$ with center on the positive ξ_2 axis. The partition of unity element associated to this rectangle is of the form $\widehat{\psi}\bigl((n\delta)^{-1}\xi_1, (n\delta^2)^{-1}(\xi_2-n)\bigr)$, where ψ is a Schwartz function, whose seminorms are bounded by constants independent of n, j, l. Thus, the convolution kernel associated to $S_n^{j\,0}$ is of the form

$$K_n^{j\,0}(x) = n^2\delta^3 \int_{-\pi}^{\pi} e^{in(x_2 - \sin t) + int}\, \psi\bigl(n\delta(x_1-\cos t), n\delta^2(x_2-\sin t)\bigr)\, dt\,.$$

We need to show that

$$\int \bigl| K_n^{j\,0}(x) \bigr|\, dx \le C\,\delta\,. \tag{11}$$

The contribution from the integral over $|t| \le \delta$ trivially satisfies (11), so it suffices to consider the following term

$$\widetilde{K}(x) = n^2\delta^3 \int e^{in(t-\sin t)}\, \chi(\delta^{-1}t)\, \psi\bigl(n\delta(x_1-\cos t), n\delta^2(x_2-\sin t)\bigr)\, dt\,,$$

where $\chi(s)=1$ for $|s|\ge 2$, and $\chi(s)=0$ for $|s|\le 1$. Integration by parts yields

$$\widetilde{K}(x) = in\delta^3 \int e^{in(t-\sin t)}\, \frac{\partial}{\partial t}\left[\frac{\chi(\delta^{-1}t)}{1-\cos t}\, \psi\bigl(n\delta(x_1-\cos t), n\delta^2(x_2-\sin t)\bigr) \right] dt\,.$$

The term where the derivative falls on the term in front of ψ satisfies (11), since

$$\int \left| \frac{\partial}{\partial t}\left(\frac{\chi(\delta^{-1}t)}{1-\cos t} \right) \right| dt \le C\delta^{-2} \le Cn\delta\,.$$

The term where the derivative falls on the x_2 place of ψ also satisfies (11), since

$$\int \left| \frac{\chi(\delta^{-1}t)\, \cos t}{1-\cos t} \right| dt \le C\delta^{-1}\,.$$

The term where the derivative falls on the x_1 place of ψ would appear to lead to bounds comparable to $\delta\log(\delta^{-1})$; however, one further integration by parts shows that this term too satisfies (11). $\square$

We now prove (9) by noting that the angle δ was chosen so that the sets $R^{j\,l}+R^{j\,l'}$ have bounded overlap for $R^{j\,l}$ and $R^{j\,l'}$ in the same quadrant, i.e., so that the orthogonality argument of [F] applies. This argument yields

$$\Big\| \sum_l S_n^{j\,l} f \Big\|_4 \le C \Big\| \Big(\sum_l |S_n^{j\,l} f|^2 \Big)^{\frac{1}{2}} \Big\|_4 .$$

The number of indices l is $O(\delta^{-1})$, so

$$\sum_l \big|S_n^{j\,l} f(x)\big|^2 \le C\delta^{-\frac{1}{2}} \Big(\sum_l \big|S_n^{j\,l} f(x)\big|^4 \Big)^{\frac{1}{2}} .$$

With $\widehat{f}_{j\,l}$ representing the localisation of $\widehat{f}$ to an appropriate sector, we thus have

$$\begin{aligned} \Big\| \sum_l S_n^{j\,l} f \Big\|_4 &\le C\delta^{-\frac{1}{4}} \Big\| \Big(\sum_l |S_n^{j\,l} f|^4 \Big)^{\frac{1}{4}} \Big\|_4 \\ &\le Cn^{-\frac{1}{4}} \Big\| \Big(\sum_l |f_{j\,l}|^4 \Big)^{\frac{1}{4}} \Big\|_4 \\ &\le Cn^{-\frac{1}{4}} \Big\| \Big(\sum_l |f_{j\,l}|^2 \Big)^{\frac{1}{2}} \Big\|_4 . \end{aligned}$$

A result of Córdoba [C] gives

$$\Big\| \Big(\sum_l |f_{j\,l}|^2 \Big)^{\frac{1}{2}} \Big\|_4 \le C(\log(n))^b \|f\|_4$$

for some positive b, which completes the proof of (9).

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