# AN ESTIMATE FOR A RESTRICTED X-RAY TRANSFORM


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Fix $d \geq 2$ and, for $s \in \mathbb{R}$, let $\gamma(s) = (1, s, s^2, \dots, s^{d-1}) \in \mathbb{R}^d$. Let $\Pi(s)$ be the space of vectors in $\mathbb{R}^d$ which are orthogonal to $\gamma(s)$. We consider a restricted x-ray transform $R$ on $\mathbb{R}^d$ defined by

$$Rf(s,p) = \int_{-\infty}^{\infty} f(p + t\gamma(s))\, dt, \;\; s \in \mathbb{R},\; p \in \Pi(s).$$

Our interest is in estimates

$$\left( \int_{\{|s|\leq 1\}} \int_{\{p \in \Pi(s): |p| \leq 1\}} |Rf(s,p)|^q \, dp\, ds \right)^{\frac{1}{q}} \leq C(p,q) \|f\|_p \tag{1}$$

related to the mapping properties of $R$ as an operator on $L^p(\mathbb{R}^d)$. It is observed in [GS] that if (1) holds then $(\frac{1}{p}, \frac{1}{q})$ must belong to the triangle $\mathcal{T} = \text{hull}\ (A, B, C)$, where $A = (0,0)$, $B = (1,1)$, $C = (\frac{d^2-d+2}{d^2+d}, \frac{d^2-d}{d^2+d})$. Let $D = (\frac{d^2-d+2}{2d^2-2d}, \frac{1}{2})$, so that $D$ is on the lower edge $AC$ of $\mathcal{T}$. Proposition 5.2 in [GS] shows that $R$ is bounded if $(\frac{1}{p}, \frac{1}{q})$ belongs to hull $(A, B, D)$. Our purpose is to prove another partial result of this type: let $E = (\frac{d-1}{d}, \frac{d-2}{d-1})$, so that $E$ lies on the upper edge $CB$ of $\mathcal{T}$.

**Theorem.** *$R$ is of restricted weak type $(\frac{d}{d-1}, \frac{d-1}{d-2})$.*

It follows by interpolation that $R$ is bounded if $(\frac{1}{p}, \frac{1}{q}) \neq E$ belongs to hull $(A, B, E)$. The method of proof is geometrical. It is an adaptation of the proof in [O1]. Our motivation for studying the mapping properties of $R$ is that $R$ can be regarded as a model for the operator $Tf(x) = \int_0^1 f(x - \Gamma(s))\, ds$, where $\Gamma(s) = (s, s^2/2, \dots s^d/d)$ — see [O2], [M], [O3], [O4].

*Proof of theorem.* The notation $|\cdot|$ will stand for the Lebesgue measure of a set in $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$ or $\{(s,p) : s \in \mathbb{R}, p \in \Pi(s)\} \sim \mathbb{R}^d$, the choice being clear from the context. Fix $\lambda > 0$ and a measurable set $E \subseteq \mathbb{R}^d$. Let $F = \{|s|, |p| \leq 1, R\chi_E(s,p) > \lambda\}$.


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We need to show that

$$\lambda\,|F|^{\frac{d-2}{d-1}} \leq C|E|^{\frac{d-1}{d}} \tag{2}$$

for some $C$ independent of $E$ and $\lambda$. The trivial estimate

$$\lambda\,|F| \leq 2|E|$$

gives (2) if $\lambda\,|F|^{-1/(d-1)}$ is small, so we consider the other case. Let $k$ be a positive number so small that $k/2-(2k)^d c_1 > 0$ — here $c_1$ is a positive constant that will appear later in the proof. Assuming that $k\,\lambda^{\frac{1}{(d-1)}}\,|F|^{\frac{-1}{(d-2)^2}} \geq 1$, i.e., that $\lambda\,|F|^{\frac{-1}{(d-1)}}$ is not small, choose an integer $N$ such that

$$k\lambda^{\frac{1}{d-1}}|F|^{\frac{-1}{(d-1)^2}} \leq N < k\lambda^{\frac{1}{d-1}}|F|^{\frac{-1}{(d-1)^2}}+1 \leq 2k\lambda^{\frac{1}{d-1}}|F|^{\frac{-1}{(d-1)^2}}. \tag{3}$$

For $|s|\leq 1$, let $\widetilde{E}(s)=\{p\in\Pi(s): |p|\leq 1,\ R\chi_E(s,p)>\lambda\}$. Then $\int_{-1}^{1}|\widetilde{E}(s)|\,ds = |F|$. Assume without loss of generality that $\int_0^1|\widetilde{E}(s)|\,ds \geq |F|/2$. Choose $s_1,\dots,s_N$ with

$$0<s_1<s_2<\ldots<s_N,\ s_{j+1}-s_j=\frac{1}{N},$$

$$\frac{1}{N}\sum_{j=1}^{N}|\widetilde{E}(s_j)| = \int_0^1|\widetilde{E}(s)|\,ds \geq \frac{|F|}{2}.$$

Let $P_j$ denote the orthogonal projection of $\mathbb{R}^d$ onto the plane $\Pi(s_j)$ and set

$$E(s_j)=\{x\in E: P_jx\in\widetilde{E}(s_j)\}.$$

Then

$$|E(s_j)| = \int\limits_{\Pi(s_j)} R\chi_E(s,p)\chi_{\widetilde{E}(s_j)}(p)\,dp \geq \lambda|\widetilde{E}(s_j)|, \tag{4}$$

where the inequality follows from the definition of $\widetilde{E}(s_j)$. To simplify notation we will write $E_j$ for $E(s_j)$ and $\widetilde{E}_j$ for $\widetilde{E}(s_j)$. We will establish (2) by using the following counting lemma to estimate $|E|$ from below.

**Lemma 1.** *The following inequality holds:*

$$\left|\bigcup_{j=1}^{N}E_j\right| \geq \frac{1}{d}\left(\sum_{j=1}^{N}|E_j| - \sum_{1\leq j_1<\ldots<j_d\leq N}|E_{j_1}\cap\ldots\cap E_{j_d}|\right).$$

*Proof of Lemma 1.* Let $G_k=\{x\in E_k : \sum_{1\leq j<k}\chi_{E_j}(x)<d-1\}$. Then

$$\chi_{E_k}\left(1-\sum_{1\leq j_1<\ldots<j_{d-1}<k}\chi_{E_{j_1}}\cdots\chi_{E_{j_{d-1}}}\right)\leq\chi_{G_k},\quad \sum_1^N\chi_{G_k}\leq d\chi_{\bigcup_1^N E_j},$$

and so

$$\sum_1^N\left(\chi_{E_k}-\sum_{1\leq j_1<\ldots<j_{d-1}<k}\chi_{E_{j_1}}\cdots\chi_{E_{j_{d-1}}}\chi_{E_k}\right)\leq d\chi_{\bigcup_1^N E_j}.$$

Integrating this inequality yields the desired result. □

**Lemma 2.** $|E_{j_1} \cap \cdots \cap E_{j_d}| \leq C \prod_{1 \leq k < \ell \leq d} |s_{j_k} - s_{j_\ell}|^{\frac{-1}{d-1}} \prod_{p=1}^{d} \left| \widetilde{E}_{j_p} \right|^{\frac{1}{d-1}}$.

*Proof of Lemma 2.* The change of variables $(u_1, \ldots, u_d) \mapsto x \doteq \sum_{\ell=1}^{d} u_\ell \gamma(s_{j_\ell})$ has Jacobian $J$ given by the Vandermonde determinant

$$\prod_{1 \leq k < \ell \leq d} (s_{j_k} - s_{j_\ell}).$$

Thus

$$\begin{aligned}
|E_{j_1} \cap \cdots \cap E_{j_d}| &= |J| \int \prod_{k=1}^{d} \chi_{E_{j_k}} \big(\sum_{\ell=1}^{d} u_\ell \gamma(s_{j_\ell})\big) du_1 \cdots u_d \\
&\leq |J| \int \prod_{k=1}^{d} \chi_{\widetilde{E}_{j_k}} \Big(P_{j_k}\big(\sum_{\ell=1}^{d} u_\ell \gamma(s_{j_\ell})\big)\Big) du_1 \cdots u_d \\
&= |J| \int \prod_{k=1}^{d} \chi_{\widetilde{E}_{j_k}} \Big(P_{j_k}\big(\sum_{\substack{\ell=1 \\ \ell \neq k}}^{d} u_\ell \gamma(s_{j_\ell})\big)\Big) du_1 \cdots u_d.
\end{aligned}$$

The following result of Blei [B] applies to the last integral: suppose $f_1, \ldots, f_d$ are nonnegative functions of $u_1, \ldots, u_d$ with $f_k$ independent of $u_k$ for $k = 1, \ldots, d$, so that $f_k$ is a function on $\mathbb{R}^{d-1}$. Then

$$\int \prod_{k=1}^{d} f_k(u_1, \ldots, u_d)\, du_1 \cdots u_d \leq \prod_{k=1}^{d} \|f_k\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

This leads to

$$\begin{aligned}
&\big|E_{j_1} \cap \cdots \cap E_{d_j})\big| \\
&\leq |J| \prod_{k=1}^{d} \left( \int \chi_{\widetilde{E}_{j_k}} \Big(P_{j_k}\big(\sum_{\substack{\ell=1 \\ \ell \neq k}}^{d} u_\ell \gamma(s_{j_\ell})\big)\Big)\, du_1 \cdots u_{k-1} u_{k+1} \cdots u_d \right)^{\frac{1}{d-1}}.
\end{aligned}$$

Now

$$\int \chi_{\widetilde{E}_{j_k}} \Big(P_{j_k}\big(\sum_{\substack{\ell=1 \\ \ell \neq k}}^{d} u_\ell \gamma(s_{j_\ell})\big)\Big)\, du_1 \cdots u_{k-1} u_{k+1} \cdots u_d = \frac{|\gamma(s_{j_k})|}{|J|} \left| \widetilde{E}_{j_k} \right|,$$

which one can see, for example, by considering an orthogonal transformation of $\mathbb{R}^d$ that takes $\gamma(s_{j_k})$ onto $(|\gamma(s_{j_k})|, 0, 0, \ldots, 0)$. Lemma 2 follows since $|s| \leq 1$.

Lemma 2 will be applied in conjunction with a multilinear inequality for non-negative sequences:

$$\sum_{j_1,\cdots,j_d>0}\prod_{\ell=1}^{d}a_{j_\ell}\prod_{1\le k<\ell\le d}\frac{1}{1+|j_k-j_\ell|^{\frac{1}{d-1}}}\le C(\sum a_j^2)^{\frac{d}{2}}. \tag{5}$$

This inequality is a discrete version of Proposition 2.2 in [C] and follows from that result or from its proof. Using Lemma 2, (5), and the equality $|s_{j_k}-s_{j_\ell}|=N^{-1}|j_k-j_\ell|$, we get

$$\begin{aligned}
\sum_{1\le j_1<\cdots<j_d\le N}|E_{j_1}\cap\cdots\cap E_{j_d}| &\le c_1N^{\frac{d}{2}}\left(\sum_1^N|\widetilde{E}_j|^{\frac{2}{d-1}}\right)^{\frac{d}{2}}\\
&\le c_1N^{\frac{d}{2}+\frac{(d-3)d}{2(d-1)}}\left(\sum_1^N\widetilde{E}_j\right)^{\frac{d}{d-1}}\\
&= c_1N^{\frac{d}{2}+\frac{(d-3)d}{2(d-1)}+\frac{d}{d-1}}\left(\frac{1}{N}\sum_1^N\widetilde{E}_j\right)^{\frac{d}{d-1}}\\
&= c_1N^d\left(\int_0^1|\widetilde{E}(s)|\,ds\right)^{\frac{d}{d-1}}.
\end{aligned}$$

The last equality here follows from the fact that the $s_j$ were chosen so that

$$\frac{1}{N}\sum_{j=1}^N|\widetilde{E}_j|=\int_0^1|\widetilde{E}(s)|\,ds.$$

Since

$$\frac{|F|}{2}\le\int_0^1|\widetilde{E}(s)|\,ds\le|F|,$$

Lemma 1 and (4) yield

$$d\Big|\bigcup_{j=1}^N E_j\Big|\ge\lambda N\frac{|F|}{2}-c_2N^d|F|^{\frac{d}{d-1}}.$$

With (3) this gives

$$d\Big|\bigcup_{j=1}^N E_j\Big|\ge\lambda^{\frac{d}{d-1}}\,|F|^{\frac{d(d-2)}{(d-1)^2}}\left(\frac{k}{2}-(2k)^dc_1\right).$$

Since $\cup E_j\subseteq E$ and since $k$ was chosen so that $k/2-(2k)^dc_1$ is positive, (2) follows and the proof is complete. $\square$

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