CONVOLUTION WITH MEASURES ON POLYNOMIAL CURVES

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This paper is concerned with convolution estimates for certain measures on degenerate curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. (These questions are not yet completely understood for the nicest curves in $\mathbb{R}^n$ if $n \geq 4$, though there has been some recent progress—see [O3] and [O4].) Here is some of the history of this problem. Ideas going back to [L] show, for example, that if μ is the measure given by dt on the circle $(\cos(t), \sin(t))$ or on the parabola (t, t^2), then

$$\mu * L^{\frac{3}{2}}(\mathbb{R}^2) \subseteq L^3(\mathbb{R}^2). \tag{1}$$

And it is easy to see that these estimates are optimal—see [O1] for more on this. The feature, common to these two curves, which in retrospect gives rise to (1) is the fact that on both of them the measure dt is a multiple of the measure $\kappa^{\frac{1}{3}}(s)ds$ where ds is arclength and κ is curvature. Drury [D] was the first to notice the importance of the measures μ given by $d\mu = \kappa^{\frac{1}{3}}(s)ds$ in the context of (1). In particular, it was Drury's idea to obtain (1) for the measure $d\mu = \kappa^{\frac{1}{3}}(s)ds$ on degenerate curves. His result (Theorem 1 in [D]) applies to curves of the form $(t, p(t))$, so that $d\mu = |p''(t)|^{\frac{1}{3}}dt$, where the convex function p satisfies certain regularity conditions. The paper [O2] contains a similar result, valid for any real-valued polynomial p. And that estimate is uniform for polynomials of a fixed degree. Theorem 1 below generalizes this: the estimate (1) holds for curves $(p_1(t), p_2(t))$ with $d\mu = \kappa^{\frac{1}{3}}(s)ds$ if p_1 and p_2 are real-valued polynomials, and the convolution bounds are uniform in p_1 and p_2 if the degree of these polynomials is fixed.

Part of the motivation for the above-mentioned work of Drury stems from the fact that convolution estimates for curves in $\mathbb{R}^2$ can be used to obtain convolution estimates for curves in $\mathbb{R}^3$ - see [O1]. The main result in [O1] is the following: suppose that $p_1(t)$ and $p_2(t)$ are polynomials and that the two vectors $(p_1^{(j)}(t), p_2^{(j)}(t)), j = 1, 2$, are linearly independent for every $t \in [a, b]$. Then the measure μ given by $\chi_{[a,b]}dt$ on the curve $(t, p_1(t), p_2(t))$ satisfies

$$\mu * L^{\frac{3}{2}}(\mathbb{R}^3) \subseteq L^2(\mathbb{R}^3). \tag{2}$$

This result, and its proof, were generalized in several papers, e.g., [P1], [P2], [D], where the main emphasis was the study of the curves

$$(t, t^\alpha, t^\beta) \tag{3}$$

with the measures $t^{(1+\alpha+\beta)/6-1}dt$. (The method of [O1] is not the only one applicable to the curves (3)—see [O3] and, in particular, [Se] where the definitive result is obtained by modifying a homogeneity argument of Christ [Chr].)

If $\gamma(t)$ is a curve in $\mathbb{R}^3$, we will write $D(t)$ for the absolute value of the determinant of the matrix

$$\begin{pmatrix} \gamma'(t) \\ \gamma''(t) \\ \gamma^{(3)}(t) \end{pmatrix}.$$

When $\gamma(t)$ is given by (3), a computation shows that, up to a constant, $D(t) = t^{1+\alpha+\beta-6}$. The convolution results for these curves lead to the conjecture that, under mild additional hypotheses, the measure μ given by $D^{1/6}(t)\,dt$ on the curve $\gamma(t)$ will satisfy (2). Theorem 2 below shows that this conjecture is true for curves $\gamma(t) = (t, p_1(t), p_2(t))$ when p_1 and p_2 are real-valued polynomials.

The recent papers [Cho1] and [Cho2] contain, among other interesting results, special cases of our Theorems 1 and 2 obtained by specializing to compact or homogenenous curves.

The remainder of this paper, then, is devoted to the proofs of the following results:

Theorem 1. *Fix a positive integer N. There is a positive constant $C(N)$ such that if $p_1(t)$ and $p_2(t)$ are real-valued polynomials of degree not exceeding N and if μ is the measure on the curve $(p_1(t), p_2(t))$, $-\infty < t < \infty$, given by*

$$|\,p_1'(t)\,p_2''(t) - p_1''(t)\,p_2'(t)\,|^{\frac{1}{3}}\,dt,$$

then

$$\|\mu * f\|_{L^3(\mathbb{R}^2)} \le C(N)\,\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}$$

for functions f on $\mathbb{R}^2$.

Theorem 2. *Suppose $p_1(t)$ and $p_2(t)$ are real-valued polynomials. Let μ be the measure on the curve $(t, p_1(t), p_2(t))$, $-\infty < t < \infty$, given by*

$$|p_1''(t)\,p_2^{(3)}(t) - p_1^{(3)}(t)\,p_2''(t)|^{\frac{1}{6}}\,dt.$$

Then there is a positive constant C such that

$$\|\mu * f\|_{L^2(\mathbb{R}^3)} \le C\,\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}$$

for functions f on $\mathbb{R}^3$.

It seems likely that the convolution bound in Theorem 2 is, as in Theorem 1, a function only of the degrees of p_1 and p_2. A uniform version of Lemma 4 would give this, but our current proof of that lemma does not seem to yield such an estimate.

The following lemma furnishes a Fourier transform estimate used in the proof of Theorem 1. It is an extension of the case $n = 2$ of Theorem 2 in [O2] and we postpone its proof until after the proofs of our main results.

Lemma 3. *Given $N = 2, 3, ...$ and $\lambda \in \mathbb{R}$ there is a constant $C(N, \lambda)$ such that if $s \in \mathbb{R}$ and if p and q are real-valued polynomials of degree not exceeding N, then we have*

$$\left|\int_a^b e^{ip(t)}|p''(t)|^{\frac{1}{2}+is}|q(t)|^{i\lambda s}dt\right| \leq C(N, \lambda)(1+|s|)^{\frac{1}{2}}$$

independently of $a, b \in \mathbb{R}$.

Proof of Theorem 1. Let (a, b) be any interval on which both $p_1^{'}p_2^{''} - p_1^{''}p_2^{'}$ and $p_1^{''}p_2^{(3)} - p_1^{(3)}p_2^{''}$ are of constant sign. Write $\kappa(t)$ for $|(p_1^{'}p_2^{''} - p_1^{''}p_2^{'})(t)|$ and define

$$Tf(x_1, x_2) = \int_a^b f\big(x_1 - p_1(t), x_2 - p_2(t)\big)\, \kappa^{\frac{1}{3}}(t)\, dt.$$

It is enough to show that

$$\|Tf\|_3 \leq C(N)\, \|f\|_{\frac{3}{2}}.$$

We will treat the case where the signs of $p_1^{'}p_2^{''} - p_1^{''}p_2^{'}$ and $p_1^{''}p_2^{(3)} - p_1^{(3)}p_2^{''}$ are the same. The other case is similar. Define an analytic family of operators by

$$T_z f(x_1, x_2) =$$

$$\frac{1}{\Gamma(\frac{z+1}{2})}\int_0^\infty \int_a^b f\big(x_1 - p_1(t) - up_1^{''}(t), x_2 - p_2(t) - up_2^{''}(t)\big)\,(\kappa(t))^{1+\frac{2z}{3}}dt\,|u|^z\,du.$$

Since T_{-1} is a multiple of T, it will suffice, by analytic interpolation, to observe that

$$\|T_{is}f\|_\infty \leq C(N)\, \|f\|_1 \tag{4}$$

and

$$\|T_{-\frac{3}{2}+is}f\|_2 \leq C(N)\,(1+|s|)^{\frac{1}{2}}\, \|f\|_2. \tag{5}$$

To see (4), just observe that the absolute value of the Jacobian of the map

$$(t, u) \rightarrow (p_1(t), p_2(t)) + u(p_1^{''}(t), p_2^{''}(t))$$

is

$$|(p_1^{'}p_2^{''} - p_1^{''}p_2^{'}) + u(p_1^{''}p_2^{(3)} - p_1^{(3)}p_2^{''})|$$

which, by our assumption on the signs of $p_1^{'}p_2^{''} - p_1^{''}p_2^{'}$ and $p_1^{''}p_2^{(3)} - p_1^{(3)}p_2^{''}$, exceeds κ. For (5) we must estimate the Fourier transform of $T_{-3/2+is}$ at $\xi \in \mathbb{R}^2$. If we write $p(t) = \xi \cdot (p_1(t), p_2(t))$ and $q(t) = (p_1^{'}p_2^{''} - p_1^{''}p_2^{'})(t)$, then a well-known calculation shows that this Fourier transform is a multiple of

$$\int_a^b e^{ip(t)}|p''(t)|^{\frac{1}{2}+is}|q(t)|^{\frac{2is}{3}}\,dt.$$

This integral is controlled by Lemma 3, and so the proof of Theorem 1 is complete.

The proof of Theorem 2 is an adaptation of the proof in [O1]. It depends on Theorem 1 and on Lemma 4 below. The proof of Lemma 4 is elementary but tedious, and we postpone it until the end of the paper.

Lemma 4. *Suppose f and g are real-valued polynomials on $\mathbb{R}$. Define*

$$G(a,b)=(f'g''-f''g')(a)(f'g''-f''g')(b),$$

$$F(a,b)=\frac{(f(b)-f(a))(g'(b)-g'(a))-(f'(b)-f'(a))(g(b)-g(a))}{(b-a)^2}$$

if $a,b\in\mathbb{R}$, $a\neq b$, and

$$F(a,a)=(f'g''-f''g')(a).$$

Then there are a finite partition of $\mathbb{R}$ into a union of intervals I_j and a positive constant M such that

$$|G(a,b)|^{\frac{1}{2}}\leq M|F(a,b)|$$

whenever a and b are both in the same I_j.

Proof of Theorem 2. Fix polynomials p_1 and p_2, take $f=p_1^{'}$ and $g=p_2^{'}$, and let I_j be as in Lemma 4. If $I_j=[a,b]$, $\gamma(t)=(t,p_1(t),p_2(t))$, and $D(t)=|p_1^{''}(t)p_2^{(3)}(t)-p_1^{(3)}(t)p_2^{''}(t)|$, define

$$Tf(x)=\int_a^b f(x-\gamma(t))D^{\frac{1}{6}}(t)dt$$

for $x\in\mathbb{R}^3$ and functions f on $\mathbb{R}^3$. It is enough to prove that T maps $L^{3/2}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$. We will do this by applying Theorem 1 in conjunction with the method of [O1]. By the "method of T^*T ", it is enough to show that, if S is the operator given by

$$Sf(x)=\int_a^b\int_a^b f(x-\gamma(t)+\gamma(s))D^{\frac{1}{6}}(t)D^{\frac{1}{6}}(s)dt\,ds,$$

then S maps $L^{3/2}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$. Writing $x=(x_1,x')$ for $x\in\mathbb{R}^3=\mathbb{R}\times\mathbb{R}^2$ and $\phi(t)=(p_1(t),p_2(t))$ and then changing variables leads to

$$Sf(x_1,x')=\int_{a-b}^{b-a}\int_{I_u} f(x_1-u,x'-\phi(s+u)+\phi(s))D^{\frac{1}{6}}(s+u)D^{\frac{1}{6}}(s)\,ds\,du,$$

where I_u is the appropriate subinterval of $[a-b,b-a]$. Writing

$$p_{1,u}(s)=p_1(s+u)-p_1(s)$$

and similarly for $p_{2,u}$, the conclusion of Lemma 4 shows that $|Sf(x)|$ is majorized by

$$Pf(x_1,x')=$$

$$\int_{a-b}^{b-a}\int_{I_u}|f|\big(x_1-u,x'-\big(p_{1,u}(s),p_{2,u}(s)\big)\big)\ |(p_{1,u}^{'}p_{2,u}^{''}-p_{1,u}^{''}p_{2,u}^{'})(s)|^{\frac{1}{3}}ds\ |u|^{-\frac{2}{3}}du.$$

For fixed x_1 and u, Theorem 1 shows that

$$\Big\|\int_{I_u}|f|\big(x_1-u,x'-\big(p_{1,u}(s),p_{2,u}(s)\big)\big)\ |(p_{1,u}^{'}p_{2,u}^{''}-p_{1,u}^{''}p_{2,u}^{'})(s)|^{\frac{1}{3}}\ ds\Big\|_{3,x'}$$

is bounded by a constant times $\|f(x_1-u,\cdot)\|_{3/2}$, and so

$$\|Pf\|_3 \le C \,\Big\| \int_{a-b}^{b-a} \|f(x_1-u,\cdot)\|_{3/2}\, |u|^{-\frac{2}{3}} du \,\Big\|_{3,x_1}.$$

The boundedness of the one-dimensional Riesz potential of order $\frac{1}{3}$ as a mapping of $L^{3/2}(\mathbb{R})$ into $L^3(\mathbb{R})$ now completes the proof of Theorem 2.

The two lemmas which follow are used in the proof of Lemma 3. The first is Lemma 3 in [O2].

Lemma 5. *Fix a positive integer N. There are positive constants $K = K(N)$ and $L = L(N)$ such that if*

$$r(t) = \prod_{j=1}^{J_1}(t-a_j) \prod_{j=J_1+1}^{J_2} [(t-a_j)^2 + {b_j}^2]$$

is a monic polynomial of degree not exceeding N with the a_j's distinct and each b_j real, then there exists a collection $\{I_l\}_{l=1}^{L_1}$, with $L_1 \le L$, of pairwise disjoint subintervals of $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}\sim\cup I_l} \left|\frac{r'}{r}\right| \le K$$

and such that for each l there are $C = C(l) \in (0,\infty)$, $j = j(l) \in \{1,2,...,J_2\}$, and a nonnegative integer $n = n(l)$ with

$$\frac{C}{K}|t-a_j|^n \le |r(t)| \le CK|t-a_j|^n,\ t \in I_l,$$

and

$$\frac{1}{K|t-a_j|} \le \left|\frac{r'}{r}\right| \le \frac{K}{|t-a_j|},\ t \in I_l.$$

Lemma 6. *Given a positive integer N, there is a positive constant $C = C(N)$ such that if $p(t)$ is a real-valued polynomial of degree not exceeding N, then, for any $\rho > 1, K > 0$,*

$$\int_{\{K \le |tp(t)| \le \rho K\}} \frac{dt}{|t|} \le C \cdot (\log(\rho) + 1).$$

Proof of Lemma 6. Without loss of generality we can write

$$tp(t) = t^{l_1} \prod (t-a_j) \prod ((t-b_j)^2 + c_j^2) \prod (t^2 + d_j^2) \doteq t^{l_1} \prod p_j(t)$$

where the number of factors p_j does not exceed N. Let C be a constant depending only on N, but which may not be the same at each occurrence. For nonnegative numbers A and B, we will write $A \sim B$ if $B \backslash C \leq A \leq CB$. We begin by observing that for each p_j there is a partition

$$\mathbb{R} = I_j \cup (\bigcup_l I_j^l)$$

of $\mathbb{R}$ into at most ten intervals such that

$$\int_{I_j} \frac{dt}{|t|} \leq C$$

and such that on each I_l^j either $|p_j| \sim c_l^j$ for some positive constant c_l^j or $|p_j(t)| \sim |t|$ or $|p_j(t)| \sim t^2$. (For example, if $p_j(t) = t - a_j$ with $a_j > 0$, then

$$|p_j(t)| \sim |t| \text{ if } t \leq \frac{-a_j}{2},$$

$$|p_j(t)| \sim |a_j| \text{ if } \frac{-a_j}{2} \leq t \leq \frac{a_j}{2},$$

$$\int_{\frac{a_j}{2}}^{\frac{3a_j}{2}} \frac{dt}{|t|} \leq \log(3),$$

and

$$|p_j(t)| \sim |t| \text{ if } \frac{3a_j}{2} \leq t.)$$

It is a consequence of this observation that the complement of $\cup I_j$ can be represented as a union of at most C disjoint intervals J_l on each of which $|tp(t)| \sim c_l |t|^{n_l}$ for some positive c_l and some nonnegative integer n_l. Then

$$\int_{\{K \leq |tp(t)| \leq \rho K\} \cap J_l} \frac{dt}{|t|} \leq \int_{\{\frac{K}{(Cc_l)} \leq |t|^{n_l} \leq \frac{\rho K C}{c_l}\}} \frac{dt}{|t|} \leq C(\log(\rho) + 1). \quad \square$$

Proof of Lemma 3. This is similar to, but more complicated than, the proof of Theorem 2 in [O2]. We begin with some reductions: replacing q by a power of q shows that we can assume $0 < \lambda \leq 1$. It is clear that we may asssume that $q(t)$ is monic, and a scaling argument shows that we may assume $p'(t)$ to be monic. Then an approximation argument shows that it is enough to prove Proposition 3 under the additional hypothesis that both $r(t) \doteq p'(t)$ and $r(t) \doteq q(t)$ meet the other requirements of Lemma 5. Finally, it will suffice to show that the conclusion of Proposition 3 holds if p', p'', and

$$|\frac{p''}{(p')^2}| - \frac{1}{10(1 + |s|)}$$

are of constant sign on $(a, b) \doteq I$.

Case 1: $\frac{1}{10(1+|s|)} \leq |\frac{p''}{(p')^2}|$ **on** I. The argument here is identical to that for Case II in the proof of Theorem 2 in [O2].

Case 2: $|\frac{p''}{(p')^2}| \leq \frac{1}{10(1+|s|)}$ **on** I. After making the change of variables $u = p(t)$, we need to estimate an integral of the form

$$\int_J e^{i(u+2s\log|p'(p^{-1}(u))|+\lambda s\log|q(p^{-1}(u))|)} \left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}+is} du, \tag{6}$$

where $J = p(I)$. The derivative of the phase function is

$$1 + 2s\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2} + \lambda s \frac{q'(p^{-1}(u))}{q(p^{-1}(u))p'(p^{-1}(u))}. \tag{7}$$

For any subinterval J' of J we have

$$\int_{J'} \left|\frac{d}{du}\left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}+is}\right| du =$$

$$\left|\frac{1}{2}+is\right| \int_{J'} \left|\frac{d}{du}\left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}}\right| du \leq$$

$$C(N)\left|\frac{1}{2}+is\right| \sup\left\{\left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}} : u \in J'\right\} \leq C(N)(1+|s|)^{\frac{1}{2}}.$$

Here the first inequality follows from the fact that, since p is a polynomial of degree not exceeding N,

$$\frac{d}{du}\left|\frac{p''(p^{-1}(u))}{p'(p^{-1}(u))^2}\right|^{\frac{1}{2}}$$

will have at most $C(N)$ sign changes on J'. The second inequality is a consequence of the Case 2 assumption. It follows from a variant of van der Corput's lemma ([St], p.334), that if J' is a subinterval of J on which the absolute value of (7) exceeds, say, $\frac{1}{10}$, then the part of (6) corresponding to J' is bounded by $C(N)(1+|s|)^{\frac{1}{2}}$. Since J is a union of at most $C(N)$ intervals on each of which either $|(7)| > \frac{1}{10}$ or $|(7)| \leq \frac{1}{10}$, it suffices to estimate (6) with J replaced by some J' on which $|(7)| \leq \frac{1}{10}$. From the Case 2 assumption it follows that then

$$\frac{7}{10|s|\lambda} \leq \left|\frac{q'(p^{-1}(u))}{q(p^{-1}(u))p'(p^{-1}(u))}\right| \leq \frac{13}{10|s|\lambda} \tag{8}$$

on J' and again that

$$\left|\frac{p''(p^{-1}(u))}{(p'(p^{-1}(u)))^2}\right| \leq \frac{1}{10(1+|s|)} \leq \frac{7}{10|s|\lambda} \leq \left|\frac{q'(p^{-1}(u))}{q(p^{-1}(u))p'(p^{-1}(u))}\right| \tag{9}$$

on J'. Now take $r = q$ in Lemma 5 and let the intervals ${I'}_l$ be such that

$$\int_{\mathbb{R}\sim\cup {I'}_l} \Big|\frac{q'}{q}\Big| \leq C. \tag{10}$$

Let $I' = p^{-1}(J')$ so that on I' we have the inequalities

$$\frac{7}{10|s|\lambda} \leq \Big|\frac{q'(t)}{q(t)p'(t)}\Big| \leq \frac{13}{10|s|\lambda} \tag{8'}$$

and

$$\Big|\frac{p''(t)}{(p'(t))^2}\Big| \leq \Big|\frac{q'(t)}{q(t)p'(t)}\Big| \tag*{(9').}$$

From (9′) and (8′) it follows that on I' we have

$$|p''| \leq C|p'|\Big|\frac{q'}{q}\Big|,$$

$$|p'| \leq C|s|\Big|\frac{q'}{q}\Big|,$$

and so

$$|p''|^{\frac{1}{2}} \leq C|s|^{\frac{1}{2}}\Big|\frac{q'}{q}\Big|. \tag{11}$$

Now (10) and (11) give

$$\int_{I'\sim\cup {I'}_l} |p''|^{\frac{1}{2}} \leq C|s|^{\frac{1}{2}}.$$

On the other hand, on an ${I'}_l$ we have, by Lemma 5,

$$\Big|\frac{q'(t)}{q(t)}\Big| \sim \frac{1}{|t-c|} \tag{12}$$

for some c. With (8′) this gives the inequalities

$$\frac{1}{C|s|\lambda} \leq \frac{1}{|p'(t)||t-c|} \leq \frac{C}{|s|\lambda}$$

on $I' \cap {I'}_l$. And with (11) and (12) this gives

$$\int_{I'\cap {I'}_l} |p''|^{\frac{1}{2}} \leq C|s|^{\frac{1}{2}} \int_{I'\cap I'_l} \Big|\frac{q'}{q}\Big| \leq C|s|^{\frac{1}{2}} \int_{\{\frac{1}{C|s|\lambda} \leq \frac{1}{|(t-c)p'(t)|} \leq \frac{C}{|s|\lambda}\}} \frac{dt}{|t-c|}.$$

Thus Lemma 6 completes the proof of Lemma 3.

Proof of Lemma 4. This is a consequence of the following two facts:

Sublemma A. *If $x_0 \in \mathbb{R}$ then there are $\delta > 0$ and $M < \infty$ such that the inequality*

$$|G(a,b)|^{\frac{1}{2}} \leq M\,|F(a,b)| \tag{13}$$

holds if $a, b \in (x_0 - \delta, x_0 + \delta)$.

Sublemma B. *There are positive constants P and M such that (13) holds if $a, b \geq P$ or $a, b \leq -P$.*

Proof of Sublemmas A and B. Without loss of generality we will take $x_0 = 0$. Let n be the maximum of the degrees of f and g. Write

$$f(x) = \sum_{j=0}^{n} c_j x^j, \;\; g(x) = \sum_{j=0}^{n} d_j x^j.$$

Letting T_k stand for the sum

$$\sum_{l=0}^{k} a^{k-l} b^l,$$

we see that

$$\frac{f(b) - f(a)}{b - a} = \sum_{j=1}^{n} c_j T_{j-1}$$

and

$$\frac{g'(b) - g'(a)}{b - a} = \sum_{j=2}^{n} j d_j T_{j-2}.$$

With similar expressions for

$$\frac{f'(b) - f'(a)}{b - a} \text{ and } \frac{g(b) - g(a)}{b - a}$$

this leads to

$$F(a,b) = \sum_{j_1=1, j_2=2}^{n,n} (c_{j_1} d_{j_2} - d_{j_1} c_{j_2}) j_2 T_{j_1 - 1} T_{j_2 - 2}. \tag{14}$$

Let $n(j_1, j_2, l)$ stand for the cardinality of the set

$$\{(l_1, l_2) : 0 \leq l_1 \leq j_1 - 1, \, 0 \leq l_2 \leq j_2 - 2, \, l_1 + l_2 = l\}.$$

Then the coefficient of $a^j b^l$ in (14) is

$$\sum_{\substack{j_1=1, j_2=2 \\ j_1+j_2-3=j+l}}^{n,n} (c_{j_1} d_{j_2} - d_{j_1} c_{j_2}) j_2 n(j_1, j_2, l) \doteq \sum_{\substack{j_1, j_2=1 \\ j_1<j_2 \\ j_1+j_2-3=j+l}}^{n} (c_{j_1} d_{j_2} - d_{j_1} c_{j_2}) m(j_1, j_2, l).$$

Thus

$$F(a,b) = \sum_{J \geq 0} \sum_{j+l=J} a^j b^l \sum_{\substack{j_1, j_2=1 \\ j_1<j_2 \\ j_1+j_2-3=J}}^{n} (c_{j_1} d_{j_2} - d_{j_1} c_{j_2}) m(j_1, j_2, l). \tag{15}$$

We will need to know that the term $m(j_1, j_2, l)$ is positive if $0 \leq l \leq j_1 + j_2 - 3$. Since $m(j_1, j_2, l) = j_2 n(j_1, j_2, l) - j_1 n(j_2, j_1, l)$ and $j_2 > j_1$, it is enough to check that $n(j_1, j_2, l) \geq n(j_2, j_1, l) > 0$. But, by definition,

$$n(j_1, j_2, l) = |\{(l_1, l_2) : l_1 + l_2 = l,\ 0 \leq l_1 \leq j_1 - 1,\ 0 \leq l_2 \leq j_2 - 2\}|$$

and so

$$n(j_2, j_1, l) = |\{(l_1, l_2) : l_1 + l_2 = l,\ 0 \leq l_1 \leq j_1 - 2,\ 0 \leq l_2 \leq j_2 - 1\}|.$$

A picture in the $l_1 l_2$-plane now shows that $n(j_1, j_2, l) \geq n(j_2, j_1, l) > 0$ as desired.

In addition to (15) we will use

$$(f'g'' - f''g')(x) = \sum_{J \geq 0} x^J \sum_{\substack{j_1, j_2 = 1 \\ j_1 < j_2 \\ j_1 + j_2 - 3 = J}}^{n} j_1 j_2 (j_2 - j_1)(c_{j_1} d_{j_2} - d_{j_1} c_{j_2}). \tag{16}$$

We will also need the following fact:

Sublemma C. *For some fixed J suppose that either*

$$c_{j_1} d_{j_2} - d_{j_1} c_{j_2} = 0 \text{ whenever } j_1 + j_2 - 3 < J, \tag{17}$$

or

$$c_{j_1} d_{j_2} - d_{j_1} c_{j_2} = 0 \text{ whenever } j_1 + j_2 - 3 > J. \tag{18}$$

Then there is at most one pair (j_1, j_2) with $1 \leq j_1 < j_2 \leq n$ and $j_1 + j_2 - 3 = J$ such that $c_{j_1} d_{j_2} - d_{j_1} c_{j_2} \neq 0$.

Proof of Sublemma C. Suppose that (17) holds (the proof under the hypothesis (18) is similar) and that $c_1 d_{J+2} - c_{J+2} d_1 \neq 0$. Then either $c_1 \neq 0$ or $d_1 \neq 0$. Without loss of generality, assume $c_1 \neq 0$. Suppose also that $1 < j_1 < j_2 \leq n$ and that $j_1 + j_2 - 3 = J$. We will start by observing that $c_{j_1} d_{j_2} - d_{j_1} c_{j_2} = 0$. Since $1 + j_1 - 3 < 1 + j_2 - 3 < J$, we have $c_1 d_{j_2} - c_{j_2} d_1 = 0$ and $c_1 d_{j_1} - c_{j_1} d_1 = 0$ by assumption. Multiplying the first of these by c_{j_1} and the second by c_{j_2} and subtracting leads to $c_{j_1} d_{j_2} - d_{j_1} c_{j_2} = 0$ as desired. Thus our conclusion holds if $c_1 d_{J+2} - c_{J+2} d_1 \neq 0$. The next case, $c_2 d_{J+1} - c_{J+1} d_2 \neq 0$, and all subsequent cases, are handled similarly. $\square$

Conclusion of proof of Sublemma A. Let J_1 be the first J such there are j_1 and j_2 with $j_1 + j_2 - 3 = J$ and $c_{j_1} d_{j_2} - d_{j_1} c_{j_2} \neq 0$. Then, by (15) and Sublemma C, there are j_1 and j_2 with $j_1 + j_2 - 3 = J_1$ and

$$F(a, b) = (c_{j_1} d_{j_2} - d_{j_1} c_{j_2}) \sum_{j + l = J_1} a^j b^l m(j_1, j_2, l) + O\Big(\sum_{j + l = J_1 + 1} a^j b^l\Big).$$

It follows from (16) that

$$(f'g'' - f''g')(x) = O(|x|^{J_1}).$$

Thus Sublemma A follows from the fact that the $m(j_1, j_2, l)$'s are positive along with the inequality

$$|ab|^{\frac{J_1}{2}} \leq \sum_{j+l=J_1} a^j b^l .$$

The proof of Sublemma B is similar, starting with the choice of J_1 as the greatest J such that there are j_1 and j_2 with $j_1 + j_2 - 3 = J$ and $c_{j_1} d_{j_2} - d_{j_1} c_{j_2} \neq 0$.

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