Finite Singularities and Hypergeometric Solutions of Linear Recurrence Equations

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Abstract

In this paper the notion of finite singularities of difference operators is introduced. The study of singularities has many applications. One application will be given, a new method for computing hypergeometric solutions of difference equations.

1 Introduction

The goal of this paper is to introduce the notion of finite singularities of difference equations (recurrence equations) and to give some of their properties. So far only the singularity at infinity has been studied because that is the only point in $P^1(\mathbb{C}) = \mathbb{C} \bigcup \{\infty\}$ which is invariant under the shift operator τ, where $\tau(x) = x + 1$. Since $x \mapsto x + 1$ does not leave the elements of $\mathbb{C}$ invariant, the finite singularities should be elements of $\mathbb{C}/\mathbb{Z}$ instead of $\mathbb{C}$.

In the theory of differential operators, a singularity p is studied by considering differential operators over $\mathbb{C}((x-p))$, or $\mathbb{C}((1/x))$ if $p = \infty$. The shift operator $\tau : \mathbb{C}(x) \to \mathbb{C}(x)$ can be extended to $\mathbb{C}((1/x))$ but not to $\mathbb{C}((x-p))$ for $p \in \mathbb{C}$, and so for finite singularities we can not use a construction similar to the one in the differential case. For a finite singularity $p = q+\mathbb{Z} \in \mathbb{C}/\mathbb{Z}$ where $q \in \mathbb{C}$ we will consider left solutions $u : \{q - M, q - (M + 1), q - (M + 2), \ldots\} \to \mathbb{C}$ and right solutions $u : \{q + M, q + M + 1, q + M + 2, \ldots\} \to \mathbb{C}$ of the difference equation, where M is a sufficiently large integer. We will show how one can obtain right solutions from left solutions by deforming the difference equation with $x \mapsto x + \epsilon$. After the deformation the left solutions are defined over the field of constants $\mathbb{C}(\epsilon)$, and a 1-1 map from these left solutions to the corresponding right solutions can be given. Then one can study the difference in ϵ-valuation between these left and right solutions. That yields integers which give useful information about the singularity, information that only depends on the *type* of the equation. Furthermore we can also use this construction to define a map from the left solutions to the right solutions of the original (not deformed) equation, and a map from the right to the left solutions. These maps can also be used to give more characterizations of the singularity. We also give two relations between infinite and finite singularities in the case of an equation of order 1, similar to Fuchs' relation in the differential case.

2 Preliminaries and definitions

Let $\mathcal{W}$ be the set of all functions to $\mathbb{C}$ that are defined on some subset of $\mathbb{C}$. Let $k = \mathbb{C}(x)$ and τ the $\mathbb{C}$-automorphism of k defined by $\tau(x) = x + 1$. A *difference operator* is an operator

$$L = a_n\tau^n + \cdots a_0\tau^0 \tag{1}$$

that acts in the following way on $f \in \mathcal{W}$

$$(L(f))(x) = a_n(x)f(x + n) + \cdots + a_0(x)f(x).$$

In this paper the coefficients a_i will be elements of k. The function $L(f) \in \mathcal{W}$ is defined for those $x \in \mathbb{C}$ for which $f(x), \ldots, f(x+n)$ and $a_0(x), \ldots, a_n(x)$ are defined. If $a_n \neq 0$ then n is called the *order* of L.

The set of all difference operators is

$$k[\tau] = \{a_n\tau^n + \cdots a_0\tau^0 | n \in \mathbb{N}, a_0, \ldots, a_n \in k\}.$$

The following

$$\tau \cdot a = \tau(a)\tau \tag{2}$$

defines the multiplication on $k[\tau]$. This turns $k[\tau]$ into a non-commutative ring. Using this relation (2) any product of difference operators can be written in a unique way in the form (1), i.e. the coefficients in k appear on the left of the τ^i. Multiplication in $k[\tau]$ corresponds to composition of difference operators; if $L = L_1 \cdot L_2$ then $L(f) = L_1(L_2(f))$ (on the subset of $\mathbb{C}$ where both are defined).

A sequence $u(0), u(1), u(2), \ldots$ of complex numbers is a function

$$u : \mathbb{N} \to \mathbb{C}.$$

So a sequence is an element of $\mathcal{W}$. Now u is called a solution of L if $L(u) = 0$, i.e.

$$L(u)(x) = a_n u(x+n) + \cdots + a_0 u(x) = 0$$

for all x for which $L(u)(x)$ is defined. So a difference operator $L = a_n\tau^n + \cdots a_0\tau^0$ of order n corresponds to a recurrence relation of order n. For example the sequence $u : \mathbb{N} \to \mathbb{C}$, $u(m) = m!$ is a solution of $L = \tau - (x+1)$.

We would also like to consider sequences that are solutions of an operator like $L = \tau - 1/(x-3)$, however, the coefficients of this L are not defined for all $x \in \mathbb{N}$. To avoid such problems with domains of definition we will consider sets $V_{p,r}$ and $V_{p,l}$ consisting of germs of sequences.

Denote $q + \mathbb{N} = \{q, q+1, q+2, \ldots\} \subset q + \mathbb{Z}$ and $q - \mathbb{N} = \{q, q-1, q-2, \ldots\} \subset q + \mathbb{Z}$. For a, b in $q + \mathbb{Z}$ define an ordering by $a > b$ iff $a - b > 0$.

Define $V_{q,l}$ as the set of germs of functions $u : q - \mathbb{N} \to \mathbb{C}$. So $V_{q,l}$ is the ring of all functions from $q - \mathbb{N}$ to $\mathbb{C}$ modulo the ideal of functions that are non-zero at only finitely many points x in $q - \mathbb{N}$. Note that if $q_1 - q_2 \in \mathbb{Z}$ then $V_{q_1,l}$ and $V_{q_2,l}$ can be identified, and so $V_{p,l}$ can be defined for $p \in \mathbb{C}/\mathbb{Z}$ as well. Similarly define $V_{q,r}$ as the set of germs of functions $u : q + \mathbb{N} \to \mathbb{C}$.

Multiplication and addition in $V_{p,r}$ where $p = q + \mathbb{Z}$ corresponds to multiplication and addition of functions $u : q + \mathbb{N} \to \mathbb{C}$. The invertible elements of $V_{p,r}$ correspond to those u for which $u(x) = 0$ for only finitely many $x \in q + \mathbb{N}$. If u is zero for infinitely many $x \in q + \mathbb{N}$ then $1/u$ is not defined. In this case u is either 0 in $V_{p,r}$ (if the number of x for which $u(x) \neq 0$ is finite) or a zero-divisor in $V_{p,r}$ otherwise.

Each element $f \in k$ is a function from $\mathbb{C}$ to $\mathbb{C}$ that is defined on all but a finite number of elements of $\mathbb{C}$, so f is defined on the set $\{q + M, q + M + 1, \ldots\} \subset \mathbb{C}$ whenever M is sufficiently large. This way k can be embedded in the ring $V_{p,r}$, and also in $V_{p,l}$. Now $V_{p,r}$ and $V_{p,l}$ are k-algebras and $k[\tau]$-modules. The set $V_{0,r}$ is called $\mathcal{S}$ in [11], where it is shown that the Picard-Vessiot ring of L over k can be embedded in $\mathcal{S}$. This Picard-Vessiot ring can be embedded in each of the $V_{p,r}$ and $V_{p,l}$.

Definition 1 *Let* $L = a_n\tau^n + \cdots + a_0\tau^0 \in k[\tau]$ *and* $L \neq 0$. *Denote*

$$\mathrm{ord}(L) = \max\{i | a_i \neq 0\} \; - \; \min\{i | a_i \neq 0\}$$

$$\mathrm{order}(L) = \max\{i | a_i \neq 0\}.$$

L is called normal *if* $a_0 \neq 0$, *i.e. if* $\mathrm{ord}(L) = \mathrm{order}(L)$. *Let*

$$V_{p,r}(L) = \{u \in V_{p,r} | L(u) = 0\}$$

$$V_{p,l}(L) = \{u \in V_{p,l} | L(u) = 0\}$$

be the kernels of L on $V_{p,r}$ *and on* $V_{p,l}$. *These will be called the solution spaces of L in* $V_{p,r}$ *and in* $V_{p,l}$.

The *leading* coefficient of L is the highest non-zero coefficient of L (which is a_n if $a_n \neq 0$). The *trailing* coefficient is the lowest non-zero coefficient of L (which is a_0 if L is normal). L is called *monic* if the leading coefficient is 1.

Proposition 1 *The dimension (as a $\mathbb{C}$-vector space) of the solution spaces is*

$$\dim(V_{p,r}(L)) = \dim(V_{p,l}(L)) = \mathrm{ord}(L).$$

The problems with the domain of definition and problems with vanishing of the leading or trailing coefficients of L have been eliminated by taking germs of sequences (i.e. sequences modulo the given equivalence relation). This makes the proposition easy to prove, it corresponds to the well-known statement that recurrence relations of order n, for which the leading and trailing coefficients do not vanish, have an n-dimensional solution space.

Definition 2 *A k-algebra V is called a* universal extension *of k if*

1. *$\tau : V \to V$ is an automorphism that extends $\tau : k \to k$,*
2. *for every $L \in k[\tau]$ the kernel of $L : V \to V$ is an $\mathrm{ord}(L)$-dimensional $\mathbb{C}$-vector space, and*
3. *for every $u \in V$ there exists a non-zero $L \in k[\tau]$ such that $L(u) = 0$.*

A universal extension exists because one can verify that $\{u \in V_{p,r} | \exists_L L(u) = 0\}$ is a $k[\tau]$-module and k-algebra (to show closure under addition and multiplication use LCLM and symmetric products of operators, a definition follows later). See section 6.2 in [11] for another existence proof, where a universal extension is constructed using a difference ring extension of $\mathbb{C}((1/x))$ (so a local construction at the point $p = \infty$, we used a finite point $p \in \mathbb{C}/\mathbb{Z}$).

Let V be a universal extension of k. V is unique (see [11]) up to difference isomorphisms (i.e. isomorphisms that commute with τ). V can be embedded in each of the $V_{p,r}$ and $V_{p,l}$ for any $p \in \mathbb{C}/\mathbb{Z}$ but not in a unique way because V has many difference automorphisms. We denote the solution space of L in V by $V(L)$. We use this notation $V(L)$ instead of $V_{p,r}(L)$ or $V_{p,l}$ when the $p \in \mathbb{C}/\mathbb{Z}$ and the choice l or r (left or right) are not relevant.

The ring of difference operators $k[\tau]$ is a non-commutative Euclidean ring; one can compute $\mathrm{GCRD}(L_1, L_2)$, the *greatest common right divisor* of two operators L_1 and L_2, c.f. [9, 4]. One can also compute an operator $\mathrm{LCLM}(L_1, L_2)$, the *least common left multiple*. Requiring that the LCLM and GCRD are monic makes them uniquely defined.

Lemma 1 *For every non-zero $M \in k[\tau]$ there exists a unique monic normal $L \in k[\tau]$ for which $V(L) = V(M)$. If L_1 and L_2 are normal operators then*

1. *L_1 is a right-hand factor of L_2 if and only if $V(L_1) \subset V(L_2)$.*
2. *$L_3 = \mathrm{GCRD}(L_1, L_2)$ is normal and $V(L_3) = V(L_1) \bigcap V(L_2)$.*
3. *$L_4 = \mathrm{LCLM}(L_1, L_2)$ is normal and $V(L_4) = V(L_1) + V(L_2)$.*

Later in the paper we will often only consider operators that are normal, or normal and monic. According to the lemma this is no real restriction. The proof of this lemma is based on the fact that every finite dimensional G-invariant subspace of V (such as $V(L_1) \bigcap V(L_2)$ and $V(L_1) + V(L_2)$) is the solution space of some element of $k[\tau]$, where G is the difference Galois group. This result is proven in [12], where it is also shown that the dimension of the solution space is always $\leq \mathrm{ord}(L)$ in any Picard-Vessiot extension of k. In this paper we only use difference rings that are contained in $V_{p,r}$ or $V_{p,l}$ in which case this is easy to prove, but in general this is not trivial.

Definition 3 *An element u of $V_{p,r}$ or $V_{p,l}$ or V is called* rational *if it is an element of k, and it is called* hypergeometric *if it is a solution of an operator of order 1, so $\tau(u) = ru$ for some $r \in k$. The set of all hypergeometric elements of V is denoted by H.*

For example the sequence $u(n) = n!$ is hypergeometric, it corresponds to a hypergeometric element of V. Rational sequences (u is a rational function in n) are also hypergeometric. $H^* = H \setminus \{0\}$ is a group under multiplication; hence for example $u(n) = (n!)^3/(n+3)$ is hypergeometric. If $u \in H^*$ and $L \in k[\tau]$ then $L(u)/u \in k$ and hence $L(u) \in H$. H is not closed under addition.

Lemma 2 *If $a, b \in H^*$ then $a + b \in H \iff a/b \in k^*$.*

Proof: Let $u = a/b \in H^*$. If $u \in k^* = k \setminus \{0\}$ then it is easy to see that $a + b \in H$. If $a + b = 0 \in H$ then clearly $u \in k^*$. Now assume $a, b, c \in H^*$ where $c = a + b$, and $u \notin k$. Then by definition of hypergeometric, $\tau(a) = ra$, $\tau(b) = sb$ and $\tau(c) = tc$ for some $r, s, t \in k^*$. Then $ta + tb = tc = \tau(c) = \tau(a+b) = \tau(a) + \tau(b) = ra + sb$ so $(r-t)a + (s-t)b = 0$. If $r \neq t$ then $u = a/b = -(s-t)/(r-t) \in k$, and if $r = t$ then $0 + (s-t)b = 0$ so $s = t = r$ and hence $\tau(a/b) = (ra)/(sb) = a/b$ which implies $a/b \in \mathbb{C} \subset k$.

□

In a similar way it can be shown more generally that for $u_1, \ldots, u_n \in H^*$, if $u_1 + \cdots + u_n = 0$ then $u_i/u_j \in k^*$ for some $i \neq j$. Similar to the differential case one can define

Definition 4 *Let $L_1, L_2 \in k[\tau]$ be normal and of order ≥ 1. Then the* symmetric product *$L = L_1 \circledS L_2$ is defined as the normal monic operator of minimal order such that $y_1 y_2 \in V(L)$ for all $y_1 \in V(L_1)$ and $y_2 \in V(L_2)$.*

Using the relation $L_1(y_1) = 0$ we can write any $\tau^i(y_1)$ as a k-linear combination of $\tau^0(y_1), \ldots, \tau^{n-1}(y_1)$ where $n = \text{order}(L_1)$, and similarly for the $\tau^i(y_2)$. Then L can be obtained by computing a k-linear relation between $y_1 y_2$, $\tau(y_1)\tau(y_2)$, $\tau^2(y_1)\tau^2(y_2)$, $\ldots$. In [12] it is shown that every finite dimensional G-invariant subspace of V is the solution space of an element of $k[\tau]$. As a consequence $V(L_1 \circledS L_2)$ is spanned by $\{y_1 y_2 | y_1 \in V(L_1),\ y_2 \in V(L_2)\}$.

Lemma 3 *If $L_1 = \tau - r$ with $r \in k^*$, $u \in H^*$ with $L_1(u) = 0$, and L_2 is monic and normal then*

$$su \cdot L_2 \cdot \frac{1}{u} = L_1 \circledS L_2 \in k[\tau] \quad \text{where} \quad s = \tau^{\text{order}(L_2)}(u)/u \in k^*.$$

To prove this note that $su \cdot L_2 \cdot \frac{1}{u}$ is monic and has the same solution space as $L_1 \circledS L_2$. Furthermore $L_2 \cdot v \in v \cdot k[\tau]$ for every $v \in H^*$ so $su \cdot L_2 \cdot \frac{1}{u} \in su \cdot 1/u \cdot k[\tau] = k[\tau]$.

3 The singularity at infinity

It is well-known that one can study the singularity of L at the point $p = \infty$, and that this leads to several algorithms for difference operators. However, the finite singularities of L have not been studied until now, and these are useful as well. In section 5 we will show how the finite singularities can be used to give a more efficient algorithm for computing hypergeometric solutions of difference equations. In this section we will list the things we need to use from the singularity $p = \infty$. The finite singularities $p \in \mathbb{C}/\mathbb{Z}$ will be introduced in section 4.

Let $L = a \cdot (\tau - r)$ for some a and r in k^*. Define the commutative group

$$\mathcal{H}_\infty = \mathbb{C}^* \times \mathbb{Z} \times (\mathbb{C}/\mathbb{Z}).$$

We will use the additive notation for this group, $(c_1, n_1, d_1) + (c_2, n_2, d_2) = (c_1 c_2, n_1 + n_2, d_1 + d_2) \in \mathcal{H}_\infty$. Let $t = 1/x$. Then r can be written as $r = c \cdot t^n \cdot (1 + dt + e)$ for some $c, d \in \mathbb{C}$ and $e \in t^2\mathbb{C}[[t]]$. Now define the *growth* of L at ∞ as

$$g_\infty(L) = (c, n, d + \mathbb{Z}) \in \mathcal{H}_\infty.$$

Define the following groups with multiplication $\circledS$

$$k^*_{\circledS} = \{\tau - r | r \in k^*\} \quad \text{and} \quad k^*_R = \{\tau - \tau(r)/r | r \in k^*\} \subset k^*_{\circledS}$$

We have $(\tau - r_1) \circledS (\tau - r_2) = \tau - r_1 r_2$ so $k^*_{\circledS}$ is isomorphic with the multiplicative group k^*. It is easy to verify that g_∞ is a surjective group homomorphism from $k^*_{\circledS}$ to $\mathcal{H}_\infty$. Furthermore $g_\infty(\tau - r) = g_\infty(\tau - \tau(r))$ and hence k^*_R is a subgroup of the kernel of g_∞. So $g_\infty(L)$ only depends on the type (see definition below) of L.

Definition 5 *Let $L = a(\tau - r)$ where $a, r \in k^*$. Then the* type *of L is defined as the image of $\tau - r$ in the group (with $\circledS$ as multiplication) $k^*_{\circledS}/k^*_R$.*

The type is trivial, i.e. type(L) is the identity in $k^*_{\circledS}/k^*_R$, if and only if the solutions of L are rational. The type can be defined for operators of higher order as well.

Definition 6 *Two operators L_1 and L_2 have the same* type *if and only if* $\mathrm{ord}(L_1) = \mathrm{ord}(L_2)$ *and there exists an operator $r \in k[\tau]$ such that $r(V(L_1)) = V(L_2)$.*

This is equivalent with the existence of two operators $r, r' \in k[\tau]$ for which $r(V(L_1)) = V(L_2)$ and $r'(V(L_2)) = V(L_1)$. This r' can be computed from r by the extended Euclidean algorithm for $k[\tau]$. With this algorithm one can find $r', s \in k[\tau]$ for which $r'r + sL_1 = 1$. For order 1 the two definitions of the type coincide. See [13] for several characterizations of the notion of type for the differential case.

The field $\mathbb{C}((t))$ where $t = 1/x$ is a difference field extension of k, where $\tau(t) = \tau(1/x) = 1/(x+1) = t/(1+t) \in \mathbb{C}((t))$. The definition of $g_\infty(L)$ for a normal operator L with order$(L) = 1$ applies for $\mathbb{C}((t))[\tau]$ as well.

Definition 7 *Let $L \in \mathbb{C}((t))[\tau]$ be normal. Then $\overline{g}_\infty(L)$ is defined as the set of $g_\infty(M)$ for all right-hand factors $M \in \mathbb{C}((t))[\tau]$ of L of order 1.*

If $L \in k[\tau]$ then the set of $g_\infty(M)$ for all first order right-hand factors $M \in k[\tau]$ can be a proper subset of $\overline{g}_\infty(L)$, because L may have more right-hand factors in $\mathbb{C}((t))[\tau]$ than in $k[\tau]$.

Lemma 4 *The number of elements of $\overline{g}_\infty(L)$ is at most* order(L).

This follows from factorization properties in $\mathbb{C}((t))[\tau]$, and also from the structure of the formal solutions at the point ∞. We give a brief sketch, for more details see [3, 5, 14]. The elements $(c, n, d) \in \mathbb{C} \times \mathbb{Z} \times (\mathbb{C}/\mathbb{Z})$ of $\overline{g}_\infty(L)$ can be computed as follows: For the ring $\mathbb{C}((t))[\tau]$ one can define a Newton polygon and Newton polynomial (called the τ-polygon and characteristic equation in [3]). The slopes of that polygon give the possible values for n, and c must be a root of the corresponding Newton polynomial. To find d we take the symmetric product of the operator with $\tau - 1/(ct^n)$, then we need to search for a factor $\tau - (1 + dt + e)$ for some $e \in t^2\mathbb{C}[[t]]$. Let $\delta = \tau - 1$. By writing the operator as a polynomial in δ instead of τ one can define a different Newton polygon (called the δ-polygon in [3]). Then d can be found as a root of the Newton polynomial for slope 1. This way the (c, n, d), and hence $\overline{g}_\infty(L)$, can be found with little computational effort.

4 Finite singularities of difference equations

The definitions in this section are the main results in this paper.

Definition 8 *Let $L = a_n\tau^n + \cdots a_0\tau^0 \in k[\tau]$. Then $p \in \mathbb{C}/\mathbb{Z}$ is called a* singularity *of L if there is an a_i with a pole in $p + \mathbb{Z}$, or if the leading or trailing coefficient (a_n and a_0 if $a_n \neq 0$ and $a_0 \neq 0$) has a root in $p + \mathbb{Z}$. The point ∞ is considered a singularity as well.*

The set of singularities of L is a finite subset of $\mathbb{C}/\mathbb{Z} \bigcup \{\infty\}$. One can multiply L on the left by an element of $\mathbb{C}[x]$ in order to obtain an element of $\mathbb{C}[x, \tau]$ with the same solution space. So without loss of generality we may assume that $L \in \mathbb{C}[x, \tau]$. Then the finite singularities are the roots modulo $\mathbb{Z}$ of the leading and trailing coefficients of L.

Let $q \in \mathbb{C}$ and $r \in k^*$. The *valuation* $v_q(r)$ is the largest integer n such that $r/(x - q)^n \in \mathbb{C}[[x - q]]$. The valuation of 0 is ∞.

Definition 9 *Let $p \in \mathbb{C}/\mathbb{Z}$. Define the group homomorphism $v_p : k^* \to \mathbb{Z}$ as*

$$v_p(r) = \sum_{q \in p+\mathbb{Z}} v_q(r).$$

Let $L = a \cdot (\tau - r)$ for some $a, r \in k^*$. Then the valuation growth *or* growth *of L at p is defined as*

$$g_p(L) = v_p(r) \in \mathbb{Z}.$$

Define $\mathcal{H}$ as the product of the additive group of all functions $\mathbb{C}/\mathbb{Z} \to \mathbb{Z}$ with finite support and the group $\mathcal{H}_\infty$. Then define

$$g(L) = (p \mapsto g_p(u), g_\infty(u)) \in \mathcal{H}.$$

For example $u(n) = 2^n(n!)$. Then $g_p(u)$ is 1 if $p = \mathbb{Z}$, it is $(2, -1, 0)$ if $p = \infty$ and 0 otherwise.

The map g_p where $p \in \mathbb{C}/\mathbb{Z}$ is a group homomorphism from $k^*_{Ⓢ}$ to $\mathbb{Z}$. Furthermore $v_p(\tau(r)) = v_p(r)$ and so k^*_R is contained in the kernel of g_p for finite p as well. Hence $g(L)$ only depends on $\text{type}(L) \in k^*_{Ⓢ}/k^*_R$.

Definition 10 *Let $\mathcal{H}_F$ be the set of all $(f, (c, n, d + \mathbb{Z})) \in \mathcal{H}$ where $f : \mathbb{C}/\mathbb{Z} \to \mathbb{Z}$ and $c, d \in \mathbb{C}$ and $n \in \mathbb{Z}$ for which:*

$$n + \sum_{p \in \mathbb{C}/\mathbb{Z}} f(p) = 0 \quad \text{and} \quad d + \sum_{p \in \mathbb{C}/\mathbb{Z}} f(p)p \equiv 0 \mod \mathbb{Z}. \tag{3}$$

The relations (3) will be called *Fuchs' relations*, because in the differential case a similar relation with the same name (lemma 9.2 in [7]) exists between the *exponential parts*, which are the differential equivalent of the $g_p(L)$. So $\mathcal{H}_F$ is the set of all elements of $\mathcal{H}$ that satisfy the Fuchs' relations. It is a subgroup of $\mathcal{H}$. Note that the two sums are defined because f has a finite support; $f(p) \neq 0$ for only finitely many $p \in \mathbb{C}/\mathbb{Z}$.

Theorem 1 *The map $g : k^*_{Ⓢ} \to \mathcal{H}$ is a group homomorphism. The kernel is k^*_R. The image is $\mathcal{H}_F$. So g induces an isomorphism between the group of types of normal operators of order 1 and the group $\mathcal{H}_F$.*

Proof: To show that the image is contained in $\mathcal{H}_F$ we only need to verify that this is true for the generators $\tau - c$ and $\tau - (x - q)$ of the group $k^*_{Ⓢ}$, where $c \in \mathbb{C}^*$ and $q \in \mathbb{C}$. The map is surjective because if $F = (f, (c, n, d + \mathbb{Z})) \in \mathcal{H}_F$ then $F = g(F')$ where $F' = \tau - c(x - q_1)^{f(q_1)} \cdots (x - q_m)^{f(q_m)}$. Here $q_1, \ldots, q_m$ are representants in $\mathbb{C}$ for the $p_1, \ldots, p_m \in \mathbb{C}/\mathbb{Z}$ for which $f(p_i) \neq 0$, and $p_i \neq p_j$ if $i \neq j$. It is easy to see that such F' can only be in k^*_R if $c = 1$ and $f = 0$, i.e. if $F = (0, (1, 0, \mathbb{Z}))$ which is the identity in $\mathcal{H}_F$. So the kernel is contained in k^*_R, and hence equal to k^*_R.

□

The map $u \mapsto \tau - \tau(u)/u$ (note that u is a solution of $\tau - \tau(u)/u$) induces group isomorphisms $H^*/\mathbb{C}^* \to k^*_{Ⓢ}$ and $H^*/k^* \to k^*_{Ⓢ}/k^*_R$. This way the growths $g(u)$ and the growth $g_p(u)$ of u at $p \in \mathbb{C}/\mathbb{Z} \bigcup \{\infty\}$ can be defined for $u \in H^*$ or $u \in H^*/k^*$.

Example. Let $u \in H^*$ be the gamma function $u(n) = \Gamma(n) = (n-1)!$. This is a function on $\mathbb{C}$ that has poles of order 1 (interpret this as valuation -1) at the non-positive integers. It has no poles and no roots (i.e. valuation 0) at the positive integers. So, going trough the integers from the far left to the far right, the valuation increases by 1. So $g_p(u) = 1$ if $p = \mathbb{Z} \in \mathbb{C}/\mathbb{Z}$. If $p \in \mathbb{C}/\mathbb{Z}$ is not equal to $\mathbb{Z}$ then u has no poles and no roots (i.e. valuation 0) on $p + \mathbb{Z}$, so going trough $p + \mathbb{Z}$ from the left to the right, the valuation remains the same; $g_p(u) = 0$. This u is a solution of $L = \tau - x$, and indeed $g_p(L)$ is 1 if $p = \mathbb{Z}$ and 0 for all other $p \in \mathbb{C}/\mathbb{Z}$. Note that if we consider the gamma function not as $u : \mathbb{C} \to \mathbb{C}$ but as $u : \mathbb{Z} \to \mathbb{C}$ then it becomes unclear what it means that u has valuation -1 at negative integers, it is not clear what for example $u(-3)$ means. For this reason we will extend the field of constants to $\mathbb{C}(\epsilon)$ where ϵ is transcendental over $\mathbb{C}$, so ϵ is a new indeterminate. Then we can consider solutions $\tilde{u} : \mathbb{Z} \to \mathbb{C}(\epsilon)$ of an operator $L_\epsilon = \tau - (x+\epsilon) \in \mathbb{C}(\epsilon, x)[\tau]$. We can take a solution $\tilde{u} \in V_0(L_\epsilon)$ of L_ϵ for which $\tilde{u}(1) = 1$. Then $\tilde{u}(n) = \prod_{i=1}^{n-1}(i+\epsilon)$ for $n \geq 1$, and $\tilde{u}(n) = 1/\prod_{i=n}^{0}(i+\epsilon)$ for $n \leq 0$. Now $\tilde{u}(n)$ has value $u(n)$ at $\epsilon = 0$ for every integer $n \geq 1$. For $n \leq 0$, $\tilde{u}(n)$ has a pole at $\epsilon = 0$ of order 1, i.e. the ϵ-valuation $v_\epsilon(\tilde{u}(n))$ is -1 for $n \leq 0$, a definition follows later. Now the valuation growth is

$$\liminf_{n\to\infty} v_\epsilon(\tilde{u}(n)) \quad - \quad \liminf_{n\to-\infty} v_\epsilon(\tilde{u}(n))$$

which is 1 in this example. Note that substituting $\epsilon = 0$ in $\tilde{u}$ results in an non-zero right-solution of L, i.e. a non-zero element of $V_{0,r}(L)$. Evaluating $\epsilon \cdot \tilde{u}$ at $\epsilon = 0$ results in a non-zero left-solution of L. So apparently there is a canonical way to define maps $V_{0,l}(L) \to V_{0,r}(L)$ and $V_{0,r}(L) \to V_{0,l}(L)$ in this example.

The type of $u \in H^*$ (i.e. the type of $L = \tau - \tau(u)/u$) is determined by the growths $g_p(u)$ at all $p \in \mathbb{C}/\mathbb{Z} \bigcup \{\infty\}$. To determine all hypergeometric solutions $u \in H^*$ of an operator L of order > 1 we first want to compute, for each singularity p, a finite set $\overline{g}_p(L)$ such that $g_p(u) \in \overline{g}_p(L)$ for every hypergeometric solution u of L. Then, by trying all combinations that satisfy Fuchs' relations, we obtain a finite set of possibilities for the type of u. For each type, we divide all solutions of L by a $u' \in H^*$ with the same type, so $u/u' \in k^*$, and then to find u the problem is reduced to finding rational solutions. The main purpose of the rest of this section is how to compute such a set $\overline{g}_p(L)$ for finite singularities p.

Definition 11 *Let $L \in k[\tau]$ and suppose $p \in \mathbb{C}/\mathbb{Z}$ is not a singularity. Then define $V_p(L)$ as the set of functions $u : p + \mathbb{Z} \to \mathbb{C}$ that are solutions of L. The dimension of $V_p(L)$ equals* $\mathrm{ord}(L)$. *If p is not a singularity then $V_{p,l}(L)$ and $V_{p,r}(L)$ can both be identified with $V_p(L)$.*

This identification of $V_{p,l}(L)$ with $V_{p,r}(L)$ for non-singular p yields identification maps $E_{p,l} : V_{p,r}(L) \to V_{p,l}(L)$ and $E_{p,r} : V_{p,l}(L) \to V_{p,r}(L)$. Later we will generalize these maps to singular p as well, in which case we will see that these maps are $1-1$ if and only if p is a *semi-apparent* singularity, definitions follow later.

Definition 12 *Let ϵ be a new indeterminate; ϵ is transcendental over $\mathbb{C}$. Define the action of τ on $\mathbb{C}(\epsilon, x)$ as $\tau(\epsilon) = \epsilon$ and $\tau(x) = x + 1$. This turns $\mathbb{C}(\epsilon, x)$ into a difference field with $\mathbb{C}(\epsilon)$ as the field of constants. For $a \in \mathbb{C}(\epsilon)$ the ϵ-valuation is*

$$v_\epsilon(a) = \sup\{m \in \mathbb{Z} | a \in \epsilon^m \mathbb{C}[[\epsilon]]\} \in \mathbb{Z} \bigcup \{\infty\}.$$

Let $L \in k[\tau]$. Define $L_\epsilon \in \mathbb{C}(\epsilon, x)[\tau]$ as the operator one obtains from L by replacing x by $x + \epsilon$. We call L_ϵ the deformation *of L.*

The map $L \mapsto L_\epsilon$ defines an embedding (as non-commutative rings) of $k[\tau]$ in $\mathbb{C}(\epsilon, x)[\tau]$; if $L = MN$ then $L_\epsilon = M_\epsilon N_\epsilon$.

Suppose $\tilde{u} : p + \mathbb{N} \to \mathbb{C}(\epsilon)$ with $p \in \mathbb{C}$, and $\tilde{L} \in \mathbb{C}(\epsilon, x)[\tau]$. Let $\tilde{u}_{\epsilon=0} : p + \mathbb{N} \to \mathbb{C}$ and $\tilde{L}_{\epsilon=0} \in k[\tau]$ be the result of substituting $\epsilon = 0$ in $\tilde{u}$ and $\tilde{L}$, and assume that this substitution can be done, i.e. that there are no poles at $\epsilon = 0$. Then $(\tilde{L}(\tilde{u}))_{\epsilon=0} = \tilde{L}_{\epsilon=0}(\tilde{u}_{\epsilon=0})$, i.e. substituting $\epsilon = 0$ commutes with applying an operator. In particular, if $\tilde{u} : p + \mathbb{N} \to \mathbb{C}(\epsilon)$ is a solution of L_ϵ, then $\tilde{u}_{\epsilon=0}$ (if the substitution $\epsilon = 0$ can be done) is a solution of L.

If $L \in k[\tau]$ and $p \in \mathbb{C}/\mathbb{Z} \subset \mathbb{C}(\epsilon)/\mathbb{Z}$ then p is not a singularity of L_ϵ. So the $\mathrm{ord}(L)$-dimensional $\mathbb{C}(\epsilon)$-vector spaces $V_{p,l}(L_\epsilon)$ and $V_{p,r}(L_\epsilon)$ can be identified with $V_p(L_\epsilon) = \{\tilde{u} : p + \mathbb{Z} \to \mathbb{C}(\epsilon) | L_\epsilon(\tilde{u}) = 0\}$.

Let $L = a_n\tau^0 + \cdots + a_0\tau^0 \in k[\tau]$ with $a_n \neq 0$ and $a_0 \neq 0$, and let $p \in \mathbb{C}/\mathbb{Z}$. We choose a basis for $V_{p,l}(L)$ as follows. Let q_l (resp. q_r) in $p + \mathbb{Z}$ be the minimum (resp. maximum) of the *problem points at* p, i.e. the elements of the set $p+\mathbb{Z}$ that occur as a root of $a_n a_0$ or as a pole in $\{a_0,\ldots,a_n\}$. If p is not a singularity, so if there are no problem points then we choose arbitrary elements q_l, q_r of $p+\mathbb{Z}$.

Definition 13 *With notations as above, for non-zero $\tilde{u} \in V_p(L_\epsilon)$ define the* left valuation

$$v_{\epsilon,l}(\tilde{u}) = \min\{v_\epsilon(\tilde{u}(m))|m \in q_l - 1 - \mathbb{N}\}$$

the right valuation

$$v_{\epsilon,r}(\tilde{u}) = \min\{v_\epsilon(\tilde{u}(m))|m \in q_r + 1 + \mathbb{N}\}$$

and the valuation growth

$$g_{p,\epsilon}(\tilde{u}) = v_{\epsilon,r}(\tilde{u}) - v_{\epsilon,l}(\tilde{u}) \in \mathbb{Z}.$$

Define the set of valuation growths of L at p

$$\overline{g}_p(L) = \{g_{p,\epsilon}(\tilde{u})|\tilde{u} \in V_p(L_\epsilon),\ \tilde{u} \neq 0\} \subset \mathbb{Z}.$$

If $v_{\epsilon,l}(\tilde{u}) \geq 0$ then the left projection *$\rho_l(\tilde{u}) \in V_{p,l}(L)$ is defined by substituting $\epsilon = 0$ in $\tilde{u}$. Similarly if $v_{\epsilon,r}(\tilde{u}) \geq 0$ then the* right projection *$\rho_r(\tilde{u}) \in V_{p,r}(L)$ is defined.*

Note that for all $m \in q_l - 1 - \mathbb{N}$ the leading and trailing coefficient of L_ϵ have ϵ-valuation 0 and all other coefficients of L_ϵ have ϵ-valuation ≥ 0. Since the $\tilde{u}(m)$ for $m \in q_l - 1 - \mathbb{N}$ are determined by L_ϵ and the $\tilde{u}(m)$ for $m \in \{q_l - 1, q_l - 2, \ldots, q_l - n\}$, it follows that

$$v_{\epsilon,l}(\tilde{u}) = \min\{v_\epsilon(\tilde{u}(m))|m \in \{q_l - 1, q_l - 2, \ldots, q_l - n\}\}$$

and similarly

$$v_{\epsilon,r}(\tilde{u}) = \min\{v_\epsilon(\tilde{u}(m))|m \in \{q_r + 1, q_r + 2, \ldots, q_r + n\}\}.$$

In fact, if S is any set of n consecutive elements of $q_l - 1 - \mathbb{N}$ then one can apply the same argument and show that $v_{\epsilon,l}(\tilde{u}) = \min\{v_\epsilon(\tilde{u}(m))|m \in S\}$. Hence, for $\tilde{u} \in V_p(L_\epsilon)$, $\tilde{u} \neq 0$

$$v_{\epsilon,l}(\tilde{u}) = \liminf_{m \to -\infty} v_\epsilon(\tilde{u}(m))$$

and

$$v_{\epsilon,r}(\tilde{u}) = \liminf_{m \to \infty} v_\epsilon(\tilde{u}(m))$$

where the two limits are taken over $m \in p+\mathbb{Z}$ using the ordering $<$ that was defined on $p+\mathbb{Z}$. It is easy to compute the valuation growth of $\tilde{u} \in V_p(L_\epsilon)$ if L has order $n=1$, and one obtains:

Lemma 5 *If L is normal and* order$(L) = 1$ *then $\overline{g}_p(L) = \{g_p(L)\}$.*

Lemma 6 *Let $L, N, M \in k[\tau]$ be normal, $L = NM$. Then $\overline{g}_p(M) \subset \overline{g}_p(L)$.*

Proof: M is a right-hand factor of L, so M_ϵ is a right-hand factor of L_ϵ, so $V_p(M_\epsilon) \subset V_p(L_\epsilon)$ and hence the lemma follows for $p \in \mathbb{C}/Z$. For $p = \infty$ the statement follows directly from the definitions.

□

Without proof we mention that $\overline{g}_p(N) \subset \overline{g}_p(L)$ as well, and that $\overline{g}_\infty(L) = \overline{g}_\infty(M) \bigcup \overline{g}_\infty(N)$. However, if p is finite then examples show that in general the set $\overline{g}_p(L)$ is not determined by $\overline{g}_p(M)$ and $\overline{g}_p(N)$. The two lemmas show that

$$\{g_p(u)|u \in V(L),\ u \in H^*\} \subset \overline{g}_p(L)$$

(recall that $g_p(u)$ for $u \in H^*$ has been defined as $g_p(\tau - r)$ where $r = \tau(u)/u \in k^*$). Examples show that this inclusion is not always an equality.

With $L = a_n\tau^n + \cdots + a_0\tau^0$, $a_na_0 \neq 0$, q_l and q_r as before, define $u_i : q_l - 1 - \mathbb{N} \to \mathbb{C}$ by: for $1 \leq j \leq n$ let $u_i(q_l - j)$ be 1 if $i = j$ and 0 otherwise. The other values of u_i are determined by the recurrence equation $L(u_i) = 0$. Now $u_1, \ldots, u_n$ is a basis of $V_{p,l}(L)$. Similarly we can define a basis $v_1, \ldots, v_n$ of $V_{p,r}(L)$ where $v_i : q_r + 1 + \mathbb{N} \to \mathbb{C}$. For each basis element u_i of $V_{p,l}(L)$ we can choose $\tilde{u}_i \in V_p(L_\epsilon)$ with left valuation ≥ 0 and $\rho_l(\tilde{u}_i) = u_i$. These $\tilde{u}_i$ are not unique. Similarly we can choose $\tilde{v}_i \in V_p(L_\epsilon)$ with right valuation ≥ 0 and $\rho_r(\tilde{v}_i) = v_i$.

Definition 14 *With notations as above, define the* minimum valuation growth *of L at p as*

$$g_{p,r}(L) = \min\{g_{p,\epsilon}(\tilde{u}_i) | 1 \leq i \leq n\} \in \mathbb{Z}.$$

Define the maximum valuation growth *as*

$$g_{p,l}(L) = \max\{g_{p,\epsilon}(\tilde{v}_i) | 1 \leq i \leq n\} \in \mathbb{Z}.$$

Define the following $\mathbb{C}$-linear maps by their action on the basis elements u_i and v_i

$$\begin{array}{ll} E_{p,r} : V_{p,l}(L) \to V_{p,r}(L) & \quad E_{p,r}(u_i) = \rho_r(\tilde{u}_i / \epsilon^{g_{p,r}(L)}) \\ E_{p,l} : V_{p,r}(L) \to V_{p,l}(L) & \quad E_{p,l}(v_i) = \rho_l(\tilde{v}_i \epsilon^{g_{p,l}(L)}). \end{array}$$

The $\tilde{u}_i$ and $\tilde{v}_i$ were chosen in such a way that every non-zero $\tilde{u} \in V_p(L_\epsilon)$ can be written as

$$\tilde{u} = \epsilon^{v_{\epsilon,l}(\tilde{u})} \sum_i a_i \tilde{u}_i = \epsilon^{v_{\epsilon,r}(\tilde{u})} \sum_i b_i \tilde{v}_i$$

for some $a_i, b_i \in \mathbb{C}(\epsilon)$ with $v_\epsilon(a_i) \geq 0$ and $v_\epsilon(b_i) \geq 0$. As a consequence

$$g_{p,r}(L) = \min(\overline{g}_p(L)) \quad \text{and} \quad g_{p,l}(L) = \max(\overline{g}_p(L)).$$

If $p \in \mathbb{C}/\mathbb{Z}$ is not singular then $\overline{g}_p(L) = \{0\}$. If $p \in \mathbb{C}/\mathbb{Z}$ is singular and $\overline{g}_p(L) = \{0\}$ then p is called an *apparent* singularity. If $\overline{g}_p(L)$ has only 1 element then p is called a *semi-apparent* singularity. If $\overline{g}_p(L)$ has more than 1 element then p is called an *essential* singularity. Note that the definition of $E_{p,r}$ and $E_{p,l}$ depends on L because $g_{p,r}(L)$ and $g_{p,l}(L)$ depend on L. So, if ambiguity could occur, these maps should be denoted as $E_{p,r,L}$ and $E_{p,l,L}$. If $g_{p,\epsilon}(\tilde{u}_i) < g_{p,\epsilon}(\tilde{v}_j)$ then one can verify that

$$\{g_{p,\epsilon}(\tilde{u}_i + c\epsilon^m \tilde{v}_j) | c \in \mathbb{C},\ m \in \mathbb{Z}\} = \{l \in \mathbb{Z} | g_{p,\epsilon}(\tilde{u}_i) \leq l \leq g_{p,\epsilon}(\tilde{v}_j)\}$$

and hence $\overline{g}_p(L)$ is given by the following formula

$$\overline{g}_p(L) = \{l \in \mathbb{Z} | g_{p,r}(L) \leq l \leq g_{p,l}(L)\} \subset \mathbb{Z}.$$

The valuation growth g_p has been defined for (solutions of) operators L of order 1 (c.f. definition 9) and corresponds to the valuation growth $g_{p,\epsilon}$ of solutions of L_ϵ. For higher order L, if $u \in V(L)$ is hypergeometric, the growth rate $g_p(u)$ has been defined as the growth rate of the corresponding right-hand factor $\tau - \tau(u)/u \in k[\tau]$ of L. However, in general $g_p(u)$ can not be defined for every $u \in V(L)$. Because if the growth of $u \in V(L)$ at p can be defined, and p is an essential singularity, then the growth of u is either not minimal, or not maximal (or both), so (use any embedding $V \to V_{p,r}$ and $V \to V_{p,l}$) we have $u \in \mathrm{Ker}(E_{p,r})$, or $u \in \mathrm{Ker}(E_{p,l})$ (or both). This can not hold for all $u \in V(L)$ because a vector space $V(L)$ can not be the union of two vector spaces $\mathrm{Ker}(E_{p,r})$ and $\mathrm{Ker}(E_{p,l})$ of smaller dimension. Hence one can not define $g_p(u)$ for all $u \in V(L)$ in a way that corresponds to the definition of $g_{p,\epsilon}$.

If L_1 and L_2 are normal operators of order 1, then we have seen that $g(L_1 \circledS L_2) = g(L_1) + g(L_2)$. It is not difficult to prove the following generalization

Lemma 7 *Let $L_1, L_2 \in k[\tau]$ be normal, and let* $\mathrm{order}(L_2) = 1$. *Then $\overline{g}_p(L_1 \circledS L_2) = \overline{g}_p(L_1) + g_p(L_2)$ for all $p \in \mathbb{C}/\mathbb{Z} \bigcup \{\infty\}$.*

Lemma 8 *If $g_{p,r}(L) = g_{p,l}(L)$ (i.e. p is a semi-apparent singularity) then $E_{p,r}$ and $E_{p,l}$ are each other's inverses. If $g_{p,r}(L) \neq g_{p,l}(L)$ (i.e. p is an essential singularity) then $E_{p,r} \circ E_{p,l} = 0$ and $E_{p,l} \circ E_{p,r} = 0$, in other words*

$$\mathrm{Im}(E_{p,r}) \subset \mathrm{Ker}(E_{p,l}) \quad \text{and} \quad \mathrm{Im}(E_{p,l}) \subset \mathrm{Ker}(E_{p,r}).$$

Proof: $E_{p,r}(E_{p,l}(v_i)) = E_{p,r}(\rho_l(\tilde{v}_i \epsilon^{g_{p,l}(L)})) = \rho_r(\tilde{v}_i \epsilon^{g_{p,l}(L)}/\epsilon^{g_{p,r}(L)}) = \rho_r(\tilde{v}_i \epsilon^s)$ where $s = g_{p,l}(L) - g_{p,r}(L)$. Now $v_{\epsilon,r}(\tilde{v}_i \epsilon^s) = v_{\epsilon,r}(\tilde{v}_i) + s = s$, which is > 0 if p is an essential singularity and 0 otherwise. In the former case $\rho_r(\tilde{v}_i \epsilon^s) = 0$, and in the latter case $\rho_r(\tilde{v}_i \epsilon^s) = v_i$. The linear map $E_{p,l} \circ E_{p,r}$ can be computed in the same way.

□

If $p \in \mathbb{C}/\mathbb{Z}$ is not a singularity then $E_{p,l}$ and $E_{p,r}$ are isomorphisms of G-modules where G is the *difference Galois group* (for a definition see [11]) of L. However, at essential singularities the linear maps $E_{p,r}$ and $E_{p,l}$ are in general not G-homomorphisms, because at essential singularities the kernels and images are non-trivial even when L is irreducible.

5 Computing hypergeometric solutions

Computing hypergeometric solutions is equivalent to computing first order right-hand factors; if $u \in H^*$ is a non-zero hypergeometric solution then $\tau - \tau(u)/u \in k[\tau]$ is a right-hand factor of L. The type of u is determined by $g(u)$, so when $g(u)$ is known, an operator $\tau - r$ with the same type can be constructed. By taking the symmetric product with $\tau - 1/r$ we get a right-hand factor $\tau - (\tau(u)/u)/r$ which has the trivial type and hence a rational solution. So u can be found by trying all possible values of $g(u)$ and computing rational solutions for each case. This leads to the following algorithm.

Algorithm hypergeometric solutions
Input: $L = a_n \tau^n + \cdots + a_0 \tau^0$ with $a_n \neq 0$, $a_0 \neq 0$ and $n \geq 2$.
Output: All hypergeometric solutions, parametrized by constants c_i.

1. Let $S = \{p_1, \ldots, p_m\} \subset \mathbb{C}/\mathbb{Z} \bigcup \{\infty\}$ be the set of singularities.
2. result := {}
3. For all $h = (h_1, \ldots, h_m)$ with $h_i \in \overline{g}_{p_i}(L)$ that satisfy Fuchs' relations (3) do
   (a) Construct $r \in k^*$ such that all singularities of $\tau - r$ are in S, and $g_{p_i}(\tau - r) = h_i$.
   (b) Let $b_1, \ldots, b_q \in k$ be a basis of rational solutions of $L \circledS (\tau - 1/r)$.
   (c) If $q > 0$ then result := result $\bigcup \{u \cdot (c_1 b_1 + \cdots + c_q b_q)\}$ where c_i are parameters and $u \in H^*$ is a solution of $\tau - r$.
4. Output: result.

Note that we could remove all apparent finite singularities from S, that would not change the computation. Furthermore we can compute $r \in k^*$ such that $\{g_p(\tau - r)\} = \overline{g}_p(L)$ for all semi-apparent singularities $p \in \mathbb{C}/\mathbb{Z}$ and such that $\tau - r$ does not have other finite singularities. Then $L \circledS (\tau - 1/r)$ has an apparent singularity at all these p, and this way these p can be discarded as well.

For $L \in k[\tau]$ the *field of definition* is the smallest subfield of $\mathbb{C}$ over which all coefficients are defined. This is always a finite extension of $\mathbb{Q}$ and hence not an algebraically closed field. One of the main benefits of our algorithm over the algorithm in [10] is that instead of having to compute in the splitting field of all singularities, our algorithm "only" needs to compute in the splitting field of the essential singularities, which is likely to be much smaller. Furthermore the number of cases (the number of loops in step 3) in our algorithm is usually much smaller than (but at most equal to) the number of cases in [10].

6 Improvements

A small improvement is the following: when during the algorithm a hypergeometric solution u is found (equivalently: when a first order right hand factor $\tau - r$ is found where $r = \tau(u)/u \in k^*$), we can write $L = L_2 \cdot (\tau - r)$. Let $M = L \circledS (\tau - 1/r) = L_3 \cdot (\tau - 1)$ for some L_3 with $\mathrm{order}(L_3) = \mathrm{order}(L) - 1$. The non-constant hypergeometric solutions of M can be obtained by computing the hypergeometric solutions of L_3 and applying Gospers algorithm (c.f. [6] or [1]) on those. Multiplying the hypergeometric solutions of M by u gives the hypergeometric solutions of L. This process is called *reduction of order*. It may speed up the algorithm in the previous section if a hypergeometric solution is found early in the computation.

If all singularities are semi-apparent then the algorithm in the previous section is fast because there are few cases to check in step 3. If $p \in \mathbb{C}/\mathbb{Z}$ is not semi-apparent then in some cases we can also use the images and kernels of $E_{p,r}$ and $E_{p,l}$ to find factors or hypergeometric solutions of L. For example if L is normal and has order 2 then by lemma 8 we have $\mathrm{Im}(E_{p,r}) = \mathrm{Ker}(E_{p,l})$ and $\mathrm{Im}(E_{p,l}) = \mathrm{Ker}(E_{p,r})$. If $u \in H^*$ is a hypergeometric solution of L, then either $g_p(u) = g_{p,r}(L)$ or $u \in \mathrm{Ker}(E_{p,r})$. And either $g_p(u) = g_{p,l}(L)$ or $u \in \mathrm{Ker}(E_{p,l})$. So $u \in \mathrm{Ker}(E_{p,r})$ or $u \in \mathrm{Ker}(E_{p,l})$ (or both). Let $r = \tau(u)/u \in k^*$. If we had bounds on the degrees of the numerator and denominator of r then r can be computed by checking two cases $u \in \mathrm{Ker}(E_{p,r})$ or $u \in \mathrm{Ker}(E_{p,l})$. So there are at most two possible r. Such bounds can be obtained by a careful study of the algorithm in the previous section. Let $K_p \subset \mathbb{C}$ be the smallest field over which p and L are defined. Then all right-hand factors $\tau - r$ that we find are defined over K_p, because $\mathrm{Ker}(E_{p,r})$ and $\mathrm{Ker}(E_{p,l})$ are 1-dimensional and are defined over K_p. So then r must be an element of $K_p(x)$ for every essential singularity p, and we can conclude

Theorem 2 *Let $L \in K(x)[\tau]$ be an operator of order 2, where $K \subset \mathbb{C}$ and let $K_p = K(q)$ if $p = q + \mathbb{Z}$ for some $q \in \mathbb{C}$. Let $S \subset \mathbb{C}/\mathbb{Z}$ be the set of essential singularities. If $S \neq \emptyset$ then the number of first order monic right-hand factors $\tau - r$ is at most two, and they are elements of $\tilde{K}(x)[\tau]$ where $\tilde{K} = \bigcap_{p \in S} K_p$.*

If $S = \emptyset$ then computing the first order right-hand factors is easy. In this case the only algebraic extension that can occur is the $c \in \mathbb{C}^*$ in (c, n, d), which can be at most an extension of degree $\mathrm{order}(L) = 2$. If $S \neq \emptyset$ then we can skip all cases in the algorithm that would lead to an extension larger than $\tilde{K}$. This way computations with exponentially large algebraic extensions are avoided in the algorithm in the case of order 2.

A similar result can be obtained for order 3. To compute the hypergeometric solutions efficiently, we first try only those r in step 3a of the algorithm that involve no algebraic extensions. Splitting fields over K are not needed for this computation. If no hypergeometric solutions are found this way then we use the lemma and theorem below in order to find all hypergeometric solutions (including the ones that involve algebraic extensions) without having to compute with splitting fields.

Lemma 9 *Let $L \in K(x)[\tau]$ a monic normal operator of order 3. If L has a first order right-hand factor $\tau - r \in k[\tau]$ and no first order right-hand factor $\tau - r \in K(x)[\tau]$ then L can be written as $L = \mathrm{LCLM}(\tau - r_1, \tau - r_2, \tau - r_3)$ for some $r_i \in \overline{K}(x)$ or as $L = L_1 L_2$ for some $L_i \in K(x)[\partial]$ with* $\mathrm{order}(L_2) = 2$.

Proof: If L has a first order right-hand factor in $k[\tau]$ then using the algorithm we can find a first order right-hand factor $\tau - r \in \overline{K}(x)[\tau]$. Then take L_2 as the LCLM of $\tau - r$ and its conjugates over K. One sees that L_2 is invariant under the Galois group of $\overline{K}/K$ and hence $L_2 \in K(x)[\partial]$. If $\mathrm{order}(L_2) = 3$ then the first case holds, otherwise the second case.

□

In the second case in the lemma we can find L_1 by computing those hypergeometric solutions of the adjoint of L (see [1] for a definition) that involve no algebraic extensions of K. Then to compute the hypergeometric solutions of L efficiently, we apply theorem 2 on L_2. The remaining case is that L is an LCLM of three first-order operators in $\overline{K}(x)[\tau]$.

Theorem 3 *Let $L \in K(x)[\tau]$ be a monic normal operator of order 3 and assume $L = \mathrm{LCLM}(\tau - r_1, \tau - r_2, \tau - r_3)$ for some $r_i \in \overline{K}(x)$. Let $p \in \mathbb{C}/\mathbb{Z}$ be an essential singularity. Then L is reducible in $K_p(x)[\tau]$.*

Proof: Let $u_i \in H^*$ be a solution of $\tau - r_i$, for $i \in \{1,2,3\}$. The solution space of $\mathrm{LCLM}(\tau - r_1, \tau - r_2, \tau - r_3)$ is spanned by the u_i, and hence the u_i are linearly independent. For each of the u_i, the valuation growth is not minimal (then $u_i \in \mathrm{Ker}(E_{p,r})$) or not maximal (then $u_i \in \mathrm{Ker}(E_{p,l})$). The dimension of these kernels is at most 2 (the order minus 1). At least one of the following two cases must hold: at least two u_i do not have minimal valuation growth, or at least two u_i do not have maximal valuation growth. Hence at least one of the the two kernels is spanned by two of the u_i. Assume that $\mathrm{Ker}(E_{p,r})$ is spanned by u_1, u_2. Let $M = \mathrm{LCLM}(\tau - r_1, \tau - r_2)$. Then $V_{p,l}(M) = \mathrm{Ker}(E_{p,r})$. This solution space is defined over the field K_p, or equivalently, the Galois group of $\overline{K}/K_p$ maps $V_{p,l}(M)$ to itself. Since the monic normal operator M is uniquely determined by its solution space, it follows that M is invariant under this Galois group as well, and hence L has a right-hand factor $M \in K_p(x)[\tau]$ of order 2. Now $L = NM$ for some $N \in K_p(x)[\tau]$ of order 1. To find a right-hand factor of L of order 1, we can compute those hypergeometric solutions of L that have the same type as N.

□

Note that if there are no essential singularities, then the only algebraic extension that can occur in the r of step 3a in the algorithm is the c of the (c,n,d) at the point infinity. This is an algebraic extension of degree ≤ 3.

The consequence of theorems 2 and 3 is that computing hypergeometric solutions of operators of order ≤ 3 can be done very efficiently, because either a singularity p is semi-apparent (and can be discarded after a transformation of L), or p greatly reduces the algebraic extension of K over which solutions have to be searched. For order ≤ 3 the algebraic extensions we have to compute with will be small because of this.

The following remarks are topics for a subsequent paper. The algorithm given in the previous section allows several other improvements, for example intermediate results in the computations of the $\overline{g}_p(L)$ can also be used to find rational solutions in step 3b in an efficient way. Furthermore $\overline{g}_p$ can also be defined and computed for systems of equations $\tau(Y) = AY$ where $A \in GL_n(k)$, without having to use cyclic vectors. See also [8] where these ideas are used to compute rational solutions of systems. The methods in this paper can be applied for q-difference equations as well.

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