HOMOMORPHIC IMAGES OF AN INFINITE PRODUCT OF ZERO-DIMENSIONAL RINGS

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Abstract. Let $R = \prod_{a \in A} R_a$ be an infinite product of zero-dimensional chained rings. It is known that R is either zero-dimensional or infinite-dimensional. We prove that a finite-dimensional homomorphic image of R is of dimension at most one. If each R_a is a PIR and if R is infinite-dimensional, then R admits one-dimensional homomorphic images. However, without the PIR hypothesis on the rings R_a, we present examples to show that R may be infinite-dimensional while each finite-dimensional homomorphic image of R is zero-dimensional. We prove that a prime ideal of R of positive height is of infinite height, and we give conditions for an infinite product of zero-dimensional local rings to admit a one-dimensional local domain as a homomorphic image.

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All rings considered in this paper are assumed to be commutative and unitary. If R is a subring of a ring S, we assume that the unity of S is contained in R, and hence is the unity of R. All allusions to the dimension, $\dim(R)$, of a ring R refer to its Krull dimension; thus $\dim(R)$ is the nonnegative integer n if there exists a chain $P_0 < P_1 < \cdots < P_n$ of prime ideals of $\mathrm{Spec}(R)$ and no chain of longer length. If there is no upper bound on the length of finite chains of primes of $\mathrm{Spec}(R)$, we say that R is *infinite-dimensional.* In stating that a ring R has positive dimension or that a prime ideal P has positive height, we include the possibility that R is infinite-dimensional or that P has infinite height. For n a positive integer, we say that an ideal I of a ring R is *n-generated* if there exist elements $r_1, \ldots, r_n \in I$ such that $I = (r_1, \ldots, r_n)R$. We say that R is a *chained ring* if the ideals of R are linearly ordered with respect to inclusion.

Let $\{R_a\}_{a\in A}$ be a family of zero-dimensional quasilocal rings, let $R = \prod_{a\in A} R_a$, and let $p_a : R \to R_a$ denote the canonical projection homomorphism. We are interested in the structure of homomorphic images of R. We begin with several general remarks.

Remark 0.1. (i) For each maximal ideal M of R, the localization R_M of R is a homomorphic image of R [GH2, Proposition 2.4]. Thus, each homomorphic image of R_M is also a homomorphic image of R.

(ii) If I is a finitely generated ideal of R and if, for each $a \in A$, we let $I_a = p_a(I)$, then $I = \prod_{a\in A} I_a$ (cf. [GH2, Proposition 2.1]). Therefore $R/I \cong \prod_a (R_a/I_a)$. Since a product of zero-dimensional rings is either zero-dimensional or infinite-dimensional [GH1, Theorem 3.4], it follows for a finitely generated ideal I of R that either $\dim(R/I) = 0$ or $\dim(R/I) = \infty$.

It is shown in [GH2, Remark 3.9] that certain products of zero-dimensional local PIR's admit a discrete rank-one valuation domain (DVR) as a homomorphic image. In Section 1 we consider in a more general context the existence of one-dimensional homomorphic images of R. We relate this to the existence of maximal ideals M of R of positive height such that MR_M is finitely generated. If $R = \prod R_a$, where each R_a is quasilocal, we give in (1.3) necessary and sufficient conditions for MR_M to be finitely generated for each maximal ideal M of R.

In Sections 2 and 3 we consider a family $\{R_a\}_{a\in A}$ of zero-dimensional chained rings and $R = \prod_{a\in A} R_a$. In Theorem 2.3 we prove that if $P \in \mathrm{Spec}(R)$ is a prime ideal of positive height, then there exists an infinite strictly descending chain $P > P_1 > P_2 > \dots$ of prime ideals of R. In particular, each such prime ideal P has infinite height. In Theorem 3.3 and Corollary 3.4 we prove that if J is an ideal of R such that $\dim(R/J) > 1$, then there exists an infinite strictly ascending chain of prime ideals of R/J. Therefore each homomorphic image of R that satisfies the ascending chain condition on prime ideals is of dimension at most one. We observe in Remark 3.5 that if R is an infinite-dimensional product of zero-dimensional PIR's, then R admits a one-dimensional homomorphic image. In Theorem 3.7 we establish existence of infinite-dimensional products of zero-dimensional chained rings with the property that every finite-dimensional homomorphic image is zero-dimensional.

1. Finitely generated maximal ideals and finite-dimensional homomorphic images.

In considering homomorphic images of a product $R = \prod_{a\in A} R_a$, it is convenient to regard R as the set of functions f from A to the union of the rings R_a with the property that $f(a) \in R_a$ for each $a \in A$. From this perspective, we use the following notation.

Notation 1.1. (i) For $f \in R$, let $\mathcal{Y}(f) = \{a \in A : f(a)$ is a nonunit of $R_a\}$. Note that f is a unit of R if and only if $\mathcal{Y}(f) = \emptyset$.

(ii) For an ideal J of R, let $\mathcal{Y}(J) = \{\mathcal{Y}(f) : f \in J\}$.

(iii) For $X \subseteq A$, let $e_X \in R$ denote the characteristic function of the set X; thus, $e_X(a) = 1 \in R_a$ if $a \in X$ and $e_X(a) = 0$ if $a \notin X$. For $a \in A$, we write e_a in place of $e_{\{a\}}$; the set $\{e_a : a \in A\}$ generates the direct sum ideal of R—that is, the ideal of R consisting of all functions f that are finitely nonzero. Note that if $f \in R$ and $X = \mathcal{Y}(f)$, then $(1 - e_X)R \subseteq fR$.

It is also useful in considering the ideal structure of $R = \prod_{a\in A} R_a$ to employ the terminology of filters and ultrafilters on A (cf. [B, Chapter I, Section 6], [GJ, Chapter 2] and [LLS]). We recall that a *filter* on A is a nonempty collection $\mathcal{F}$ of subsets of A such that $\emptyset \notin \mathcal{F}$ and such that $\mathcal{F}$ is closed under finite intersection and taking supersets in A. An *ultrafilter* on A is a maximal filter. Each filter on A is contained in an ultrafilter on A and a filter $\mathcal{F}$ on A is an ultrafilter if and only if for each subset X of A, either X or its complement $X' = A - X$ is in $\mathcal{F}$. We say that a filter $\mathcal{F}$ is *fixed* if there exists $a \in A$ such that each $X \in \mathcal{F}$ contains a. If there does not exist such an a, then we call $\mathcal{F}$ a *free filter*.[1] The set A admits a free ultrafilter if and only if A is infinite; if $\mathcal{F}$ is a free ultrafilter on A, then $\mathcal{F}$ contains the free filter consisting of the cofinite subsets of A. In the case where each R_a is quasilocal, Proposition 2.3 of [GH2] shows that there is a natural bijection between the set of maximal ideals of R and the set of ultrafilters on A. On the set of maximal ideals of R, this bijection is the map $M \to \mathcal{Y}(M)$; the inverse map is given by $U \to \{f \in R : \mathcal{Y}(f) \in U\}$. In particular, $M = \{f \in R : \mathcal{Y}(f) \in \mathcal{Y}(M)\}$ for each maximal ideal M of R if each R_a is quasilocal. We say that M is a *free maximal ideal* if $\mathcal{Y}(M)$ is a free ultrafilter.

In Proposition 1.2 we give necessary and sufficient conditions on a maximal ideal M of a product R of quasilocal rings for MR_M to be finitely generated.

Proposition 1.2. Let $R = \prod_{a \in A} R_a$, where each R_a is a quasilocal ring. Let M be a maximal ideal of R and let $\mathcal{F} = \mathcal{Y}(M)$ be the ultrafilter on A associated with M. For each positive integer n, let $X_n = \{a \in A :$ the maximal ideal M_a of R_a is n-generated $\}$. Then MR_M is n-generated if and only if $X_n \in \mathcal{F}$. Therefore MR_M is finitely generated if and only if $X_n \in \mathcal{F}$ for some positive integer n.

Proof. Suppose $X_n \in \mathcal{F}$. Define $f_1, \dots, f_n \in R$ so that $(f_1(a), \dots, f_n(a))R_a = M_a$ for each $a \in X_n$, while $f_i(a) = 1$ for each i and for each $a \in A - X_n$. Then f_i is in M since $\mathcal{Y}(f_i) = X_n \in \mathcal{Y}(M)$. Moreover, for $g \in M$ we have $g(a) \in (f_1(a), \dots, f_n(a))R_a$ for each $a \in \mathcal{Y}(g) \cap X_n$. Since $\mathcal{Y}(g) \cap X_n \in \mathcal{F}$, it follows from part (i) of [GH2, Proposition 2.4] that $gR_M \subseteq (f_1, \dots, f_n)R_M$. Therefore $MR_M = (f_1, \dots, f_n)R_M$.

[1] The referee has indicated that the terms "principal" and "nonprincipal" are more standard in reference to ultrafilters [BS, page 15]. We prefer the terms "fixed" and "free" as in [GJ] because they seem to fit better with ideal-theoretic considerations.

Conversely, assume that $f_1, \ldots, f_n \in M$ are such that $(f_1, \ldots, f_n)R_M = MR_M$, and suppose that $X_n \notin \mathcal{F}$. Then $(A - X_n) \cap \mathcal{Y}(f_1) \cap \cdots \cap \mathcal{Y}(f_n) = Y \in \mathcal{F}$ and for each $a \in Y$ we have $(f_1(a), \ldots, f_n(a))R_a < M_a$. We define $g \in R$ so that $g(a) \in M_a - (f_1(a), \ldots, f_n(a))R_a$ for each $a \in Y$ while $g(a) = 1$ for $a \in A - Y$. Then $g \in M$ since $\mathcal{Y}(g) = Y \in \mathcal{F}$, but part (i) of [GH2, Proposition 2.4] shows that $gR_M \nsubseteq (f_1, \ldots, f_n)R_M$. This contradicts the equality $MR_M = (f_1, \ldots, f_n)R_M$. Hence $X_n \in \mathcal{F}$. $\square$[2]

The following corollary of (1.2) concerns finite generation of all the localized maximal ideals of a product of quasilocal rings.

Corollary 1.3. Let $R = \prod_{a \in A} R_a$, where each R_a is a quasilocal ring.

(i) If there exists a positive integer n such that the maximal ideal M_a of R_a is n-generated for each $a \in A$, then MR_M is n-generated as an ideal of R_M for each maximal ideal M of R.

(ii) If the condition in (i) fails — that is, if there is no positive integer n such that each M_a is n-generated — then MR_M is not finitely generated for some maximal ideal M of R.

Proof. (i) is immediate from Proposition 1.2. In (ii), assume that $X_n = \{a \in A : M_a \text{ is } n\text{-generated}\} < A$ for each positive integer n. Then $\{A - X_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty subsets of A. The set $\mathcal{G}$ of all subsets of A that contain one of the sets $A - X_n$ is a filter on A. Hence, there exists an ultrafilter $\mathcal{F}$ on A such that $A - X_n \in \mathcal{F}$ for each positive integer n. Let M be the maximal ideal of R associated with $\mathcal{F}$. By (1.2), MR_M fails to be n-generated for each positive integer n — that is, MR_M is not finitely generated. $\square$

It can happen that each R_a is zero-dimensional local and yet there exists a maximal ideal M of $R = \prod_{a \in A} R_a$ such that MR_M is not finitely generated. Indeed, in the following Example 1.4, each free maximal ideal of R fails to be locally finitely generated.

[2] The referee has indicated that an alternate proof of (1.2) can be obtained by using ultraproducts and Łoš's theorem [BS, Theorem 2.1 of Ch. 5]. Note that Łoš's theorem is applicable because, for a fixed positive integer n, the property "R is a quasilocal ring and the maximal ideal of R is generated by n elements" is a first order property of commutative rings.

Example 1.4. Let $\{x_i\}_{i=1}^{\infty}$ be a set of indeterminates over a field k and for each positive integer n, let $R_n = k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^2$. Let $R = \prod_{n=1}^{\infty} R_n$. If M is a free maximal ideal of R, then by Proposition 1.2, MR_M is not finitely generated as an ideal of R_M.

If the maximal ideal M of a valuation domain R, or more generally of a chained ring R, is the union of prime ideals properly contained in M, then it is readily seen that the only homomorphic images of R of finite dimension are R/M and the zero ring. Thus R admits no homomorphic image of finite positive dimension. In the case of a quasilocal ring (R, M) of positive dimension, we note in (1.5) a sufficient condition for R to admit a one-dimensional homomorphic image.

Proposition 1.5. Let (R, M) be a quasilocal ring of positive dimension.

(i) If M is finitely generated, then R admits a one-dimensional local domain as a homomorphic image.

(ii) If M is the radical of a finitely generated ideal, then R admits a one-dimensional quasilocal domain as a homomorphic image.

Proof. If M is the radical of a finitely generated ideal, then the set of prime ideals of R property contained in M is inductive, and hence by Zorn's lemma has a maximal element. Thus there exists a prime ideal $P < M$ such that $\dim(R/P) = 1$. If M is finitely generated, then by a theorem of Cohen [M, page 17], R/P is Noetherian. $\square$

Corollary 1.6. Let $R = \prod_{a \in A} R_a$ be a product of zero-dimensional local rings and assume that R is of positive dimension. If there exists a positive integer n such that the maximal ideal of R_a is n-generated for each $a \in A$, then for each maximal ideal M of R of positive height, R_M (and hence R) admits a one-dimensional local domain as a homomorphic image.

Proof. Apply Propositions 1.5 and 1.2. $\square$

2. Prime ideals of infinite height.

In order to prove that a prime ideal of positive height in a product of zero-dimensional chained rings is of infinite height, we note in (2.1) that a certain divisibility condition implies the existence of an infinite descending chain of prime ideals.

Proposition 2.1. Let (S, N) be a quasilocal ring of positive dimension having a unique minimal prime ideal. Assume that for each nonnilpotent element $f \in N$, there exists a nonnilpotent element $g \in \bigcap_{n=1}^{\infty} f^n S$. Then each prime ideal P of S of positive height contains an infinite strictly decending chain of prime ideals. Hence S, and each localization of S of positive dimension, is of infinite dimension.

Proof. Let $f \in P$ be a nonnilpotent element and let g be a nonnilpotent element with $g \in \bigcap_{n=1}^{\infty} f^n S$. Since P contains g, P contains a minimal prime P' of gS. We wish to prove that $P' < P$. Suppose that $P' = P$. Then P is a minimal prime of gS and hence also a minimal prime of fS. Let $Q < P$ be a prime ideal of S maximal with respect to not containing f. Let f' and g' denote the images of f and g in S/Q. By the choice of Q, the prime ideal P/Q of the integral domain S/Q is of height one. Since $g \in f^n S$ for each positive integer n, we have $g' \in (f')^n(S/Q)$ for each positive integer n. But this implies that $g' = 0$ and hence that $g \in Q$. This contradicts our assumption that $P' = P$. Hence $P' < P$. Using the nonnilpotent element g, we iterate the above argument to obtain a prime ideal $P'' < P'$ and by induction an infinite strictly decreasing chain of prime ideals below P. $\square$

The proof of Theorem 2.3 uses the following terminology. If x is a nilpotent element of a ring, the *index of nilpotency of* x is the smallest positive integer n such that $x^n = 0$. We write $\eta(x)$ for the index of nilpotency of x. In the proofs of Theorem 2.3 and Theorem 3.3 we use the following remark.

Remark 2.2. If $\mathcal{F}$ is an ultrafilter on the set A and if $X \in \mathcal{F}$, then for any subset Y of X, either Y or $X - Y$ is in $\mathcal{F}$; equivalently, $\{Y \in \mathcal{F} : Y \subseteq X\}$ is an ultrafilter on X.

Theorem 2.3. Let $\{R_a\}_{a \in A}$ be a family of zero-dimensional chained rings and let $R = \prod_{a \in A} R_a$. If $P \in \operatorname{Spec}(R)$ is a prime ideal of positive height, then there exists an infinite strictly descending chain $P > P_1 > P_2 > \dots$ of prime ideals of R. In particular, P has infinite height.

Proof. Let M be a maximal ideal of R containing P, let $S = R_M$, and let

$N = MR_M$. Then (S, N) is a chained ring [GH2, Proposition 2.6] of positive dimension, and to prove Theorem 2.3 it suffices, in view of (2.1), to prove that the hypothesis of that result is satisfied in (S, N). Thus, suppose $f \in M$ is such that fS is a nonnilpotent ideal of S. Since M is a prime ideal of positive height, $\mathcal{Y}(M) = \mathcal{F}$ is a free ultrafilter on A ([GH2, Proposition 2.3]). Since $f \in M$, the set $\mathcal{Y}(f) = X \in \mathcal{F}$. In the argument below we use the fact that for any $g \in R$ and $Y \in \mathcal{F}$, the ideal gS is completely determined by $\{g(a) : a \in Y\}$ in the sense that if $h \in R$ is such that $g(a)R_a = h(a)R_a$ for each $a \in Y$, then $gS = hS$ ([GH2, Proposition 2.4]).
CLAIM. There exists $g \in M$ such that gS is not nilpotent and $gS \in f^n S$ for each positive integer n.

To establish the claim[3] we define $g \in R$ as follows:

$$g(a) = \begin{cases} 1 & \text{if } a \in A - X \\ f(a)^{\lfloor \sqrt{n} \rfloor} & \text{if } a \in X \text{ and } \eta(f(a)) = n. \end{cases}$$

We verify that $gS \subseteq f^n S$ for each $n \in \mathbb{Z}^+$ and that gS is not nilpotent. Thus, let $E_n = \{a \in A : f(a)^{n^2} = 0\}$. Since fS is not nilpotent, $E_n \notin \mathcal{F}$, so the set $X_n = X - E_n$ is in $\mathcal{F}$. By definition of g, we have $g(a) \in f(a)^n R_a$ for each $a \in X_n$, and hence $gS \subseteq f^n S$. Moreover, $\{a \in A : g(a)^n = 0\}$ is a subset of $A - X_n$, and hence is not an element of $\mathcal{F}$. Consequently, $g^n S \neq (0)$ and gS is not nilpotent. This establishes the claim, and by Proposition 2.1, completes the proof of Theorem 2.3. □

3. Infinite-dimensional homomorphic images.

We record in (3.1) a basic divisibility property of principal ideals in a quasilocal ring. In the statement of (3.1), we use the notation $\operatorname{rad}(J)$ to denote the radical of an ideal J of a ring S. Thus $\operatorname{rad}(J)$ consists of all elements $s \in S$, some power of which belongs to J; equivalently, $\operatorname{rad}(J)$ is the intersection of the set of prime ideals of S that contain J. We say that J is a *radical ideal* if $J = \operatorname{rad}(J)$.

Proposition 3.1. Let (S, N) be a quasilocal ring. If $g, h \in N$ are such that $g \neq 0$ and $g \in \bigcap_{n=1}^{\infty} h^n S$, then $\operatorname{rad}(gS) < \operatorname{rad}(hS)$.

[3] The proof given for this claim is due to the referee. It streamlines significantly our previous presentation.

Proof. We clearly have $\mathrm{rad}(gS) \subseteq \mathrm{rad}(hS)$. Since $g \neq 0$, it follows that h^n is a nonzero nonunit of S for each positive integer n. Because S is quasilocal, it then follows that $gS \subseteq h^{2n}S < h^nS$ for each n. Hence $h^n \notin gS$, and $\mathrm{rad}(gS) < \mathrm{rad}(hS)$.

In (3.2) we observe that a certain divisiblility condition on elements of a ring S implies failure of the ascending chain condition (a.c.c.) on radical ideals of S.

Proposition 3.2. Let (S, N) be a quasilocal ring. Assume that $P \in \mathrm{Spec}(S)$ has the property that for each nonzero $g \in P$ such that $\mathrm{rad}(gS) < P$, there exists an element $h \in P$, h depending on g, such that $\mathrm{rad}(hS) < P$ and $g \in \bigcap_{n=1}^{\infty} h^nS$. Then for each nonzero $g \in P$ with the property that $\mathrm{rad}(gS) < P$, the ring S/gS contains an infinite strictly ascending chain of radical ideals contained in P/gS. Thus, if P has positive height and if the prime ideals of S contained in P are linearly ordered with respect to inclusion (so in particular, if S is a chained ring), then $\dim(S/gS) = \infty$ for each $g \in P$ such that $\mathrm{rad}(gS) < P$.

Proof. Let $g = g_1 \in P$ be a nonzero element such that $\mathrm{rad}(g_1S) < P$, and let $h_1 \in P$ be such that $\mathrm{rad}(h_1S) < P$ and $g_1 \in \bigcap_{n=1}^{\infty} h_1^nS$. Then by (3.1), $\mathrm{rad}(g_1S) < \mathrm{rad}(h_1S)$. Let $g_2 = h_1$ and choose $h_2 \in P$ such that $\mathrm{rad}(h_2S) < P$ and $g_2 \in \bigcap_{n=1}^{\infty} h_2^nS$. Then $\mathrm{rad}(g_2S) < \mathrm{rad}(h_2S)$. Continuing in this way, we obtain by induction an infinite ascending chain of radical ideals of S contained in P and having initial term $\mathrm{rad}(gS)$. □

Theorem 3.3. Let $\{R_a\}_{a\in A}$ be a family of zero-dimensional chained rings and let $R = \prod_{a\in A} R_a$. Suppose $P \in \mathrm{Spec}(R)$ is a prime ideal of positive height. Let M be a maximal ideal of R containing P and let $S = R_M$. Then there exists $g \in P$ such that $gS \neq (0)$ and $\mathrm{rad}(gS) < PS$. For each such g, S/gS contains an infinite strictly ascending chain of prime ideals contained in PS/gS. In particular, $\dim(S/gS) = \infty$.

Proof. By part (vi) of (2.4) of [GH2], MS consists of zero divisors of S. In particular, S is not an integral domain. Since S is a chained ring and since P has positive height, there exists $g \in P$ such that $gS \neq (0)$ and $\mathrm{rad}(gS) < PS$. Consider any such element g. Since S is chained, $\mathrm{rad}(gS) = QS$, where

$Q < P$ is a prime ideal of R. If $f \in P - Q$, then $gS \subseteq f^n S$ for each positive integer n. Let $\mathcal{Y}(M) = \mathcal{F}$. Since M has positive height, $\mathcal{F}$ is a free ultrafilter on A. Let

$$X = \mathcal{Y}(f) \cap \{a \in A \ : 0 \neq g(a) \in f(a)R_a \ \}.$$

Since $gS \neq (0)$, the set $\{a \in A \ : \ g(a) \neq 0\}$ is in $\mathcal{F}$. It follows that $X \in \mathcal{F}$. For each positive integer n, let $X_n = \{a \in X \ : \ g(a) \in f(a)^{n^2} R_a \ \}$ and let $Y_n = X_n - X_{n+1}$. Since each R_a is zero-dimensional and quasilocal, $X = \bigcup_{n=1}^{\infty} Y_n$. Moreover, since $gS \subseteq f^{n^2} S$, we have $X_n \in \mathcal{F}$ and, by (2.2), $Y_n \notin \mathcal{F}$ for each positive integer n. We define $h \in R$ by setting $h(a) = f(a)^n$ for all $a \in Y_n$ and $h(a) = 0$ for $a \in A - X$. Since $X_n \in \mathcal{F}$, we have $hS \subseteq f^n S$ and $gS \subseteq h^n S$ for each positive integer n. It follows that $hS \subseteq \bigcap_{n=1}^{\infty} f^n S$ and $gS \subseteq \bigcap_{n=1}^{\infty} h^n S$. By Proposition 3.1, $\operatorname{rad}(gS) < \operatorname{rad}(hS) < \operatorname{rad}(fS) \subseteq PS$. Then Proposition 3.2 shows that there exists an infinite strictly ascending chain of prime ideals of S/gS contained in PS/gS. □

Corollary 3.4. Let $\{R_a\}_{a \in A}$ be a family of zero-dimensional chained rings and let $R = \prod_{a \in A} R_a$. If J is an ideal of R such that $\dim(R/J) > 1$, then there exists in R/J an infinite strictly ascending chain of prime ideals. In particular, $\dim(R/J) = \infty$. Therefore each homomorphic image of R that satisfies a.c.c. on prime ideals is of dimension at most one.

Proof. If $\dim(R/J) > 1$, then there exists a prime ideal Q of R such that $J \subseteq Q$ and a chain $Q < P_1 < M$ of prime ideals of R with M a maximal ideal. Let $S = R_M$ and let $g \in P_1 - Q$. Then $gS \neq (0)$ and $\operatorname{rad}(gS) < MS$. Hence by Theorem 3.3, S/gS contains an infinite strictly ascending chain of prime ideals. Since S is a chained ring, we have $JS < gS$. Therefore S/JS contains an infinite strictly ascending chain of prime ideals. Since S/JS is a localization of R/J, it follows that R/J also contains such a chain. The remaining assertions of (3.4) are transparent. □

Remark 3.5. In relation to (3.4) we observe that if $R = \prod_{a \in A} R_a$ is an infinite-dimensional product of zero-dimensional rings such that all but finitely many of the rings R_a are principal ideal rings (PIR's), then R admits one-dimensional homomorphic images. Indeed, each free maximal ideal M of R of positive height contains a prime ideal P such that R/P is a DVR. In proving this there is no loss of generality in assuming that each R_a which is a PIR is local, for a zero-dimensional PIR is a finite product of local PIR's. For the proof, let $f \in R$ be such that $f(a) \in R_a$ generates the maximal ideal of R_a for each $a \in A$ such that R_a is a PIR. By part (i) of (2.4) of [GH2], the maximal ideal MR_M of R_M is principal, generated by the image of f. Also by (2.4) of [GH2], R_M is a homomorphic image of R. Since M is of positive height, the ideals $f^n R_M, n = 1, 2, \ldots$ strictly descend. It follows that $\bigcap_{n=1}^{\infty} f^n R_M = PR_M$ is a prime ideal of R_M (cf. [G, Exercise 16, page 477]) and $R_M/PR_M \cong R/P$ is a DVR that is a homomorphic image of R.[4]

We show in Theorem 3.7 that an infinite-dimensional product of zero-dimensional chained rings need not admit a homomorphic image of finite positive dimension. The proof of Theorem 3.7 uses the following lemma. The *index of nilpotency* of a ring S, denoted $\eta(S)$, is defined as

$$\eta(S) = \sup\{\eta(x) : x \text{ is a nilpotent element of } S\}.$$

Lemma 3.6. Suppose (S, M) is a zero-dimensional chained ring such that $M = M^2$. Then $M = \bigcup\{x^n S \ : \ x \in M\}$ for each $n \in \mathbb{Z}^+$. Hence $\eta(S) = \infty$ if $M \neq (0)$.[5]

Proof. If $y \in M = M^n$, then $y \in I^n$ for some finitely generated ideal I contained in M. Since S is chained, I is principal, so $y \in \bigcup\{x^n S \ : \ x \in M\}$ and $M \subseteq \bigcup\{x^n S \ : \ x \in M\}$; the reverse inclusion is clear. If $M \neq (0)$ and if y is a nonzero element of M, then for $n \in \mathbb{Z}^+$, $y \in x^n S$ for some $x \in M$. Hence $\eta(x) \geq n$ and $\eta(S) = \infty$. $\square$

Theorem 3.7. Let $\{(R_a, M_a)\}_{a \in A}$ be an infinite family of zero-dimensional chained rings such that $M_a = M_a^2 \neq (0)$ for each $a \in A$. Let M be a free maximal ideal of $R = \prod_{a \in A} R_a$, let $S = R_M$, and assume that $\mathcal{U} = \mathcal{Y}(M)$ has the property that there exists a sequence $\{X_i\}_{i=1}^{\infty}$ of elements of $\mathcal{U}$ such that $\bigcap_{i=1}^{\infty} X_i = \emptyset$. Then for each $g \in M$ there exists $h \in M$ such that

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[4] The argument just given proves that a quasilocal ring of positive dimension with principal maximal ideal admits a DVR homomorphic image.

[5] The hypothesis here that R is a chained ring is needed. There exists a zero-dimensional quasilocal ring (S, M) such that $M = M^2 \neq (0)$ with the property that $x^2 = 0$ for each $x \in M$, so $\eta(S) = 2$.

$gS \subseteq \bigcap_{n=1}^{\infty} h^n S$. For each ideal $I < MS$ of S, the ring S/I is infinite-dimensional.

Proof. Let $Y = \mathcal{Y}(g) \in \mathcal{U}$. By replacing $\{X_i\}_{i=1}^{\infty}$ by the sequence $\{\bigcap_{j=1}^{i} X_j\}_{i=1}^{\infty}$ in $\mathcal{U}$ we may assume without loss of generality that $X_1 > X_2 > \ldots$. (We may assume that the inclusions are proper because $\bigcap_{i=1}^{\infty} X_i = \emptyset$.) Define $h \in R$ as follows:

$$h(a) = \begin{cases} 1 & \text{if } a \in A - (X_1 \cap Y) \\ x_a & \text{if } a \in (Y \cap X_i) - (Y \cap X_{i+1}) \text{ for } i \geq 1, \\ & \text{where } x_a \in M_a \text{ is such that } g(a) \in x_a^i R_a; \\ & \text{existence of } x_a \text{ follows from Lemma 3.5.} \end{cases}$$

Since $\bigcap_{i=1}^{\infty} X_i = \emptyset$, h is defined on A. Now $\mathcal{Y}(h) = Y \cap X_1 \in \mathcal{U}$, so $h \in M$. For $i \geq 1$, $Y \cap X_i \subseteq \{a \in A \ : \ g(a) \in h^i(a) R_a\}$, and since $Y \cap X_i \in \mathcal{U}$, $gS \subseteq h^i S$. Thus $gS \subseteq \bigcap_{i=1}^{\infty} h^i S$. It then follows from Propositions 3.1 and 3.2 that S/gS is infinite-dimensional, and hence S/I is infinite-dimensional for each ideal $I < MS$ of S $\square$

If the index set A in the statement of Theorem 3.7 is countable, it is easy to see that each free ultrafilter $\mathcal{F}$ on A satisfies the condition of that result. For example, $\mathcal{F}$ contains the sequence $\{A - \{a\} \ : \ a \in A\}$, which has empty intersection. Thus, if the set A in the statement of (3.7) is countable, then R admits no homomorphic image of finite dimension (see Corollary 3.8 below). P. Eklof pointed out to us that the condition on $\mathcal{Y}(M)$ described in the statement of Theorem 3.7 is equivalent to the condition that $\mathcal{Y}(M)$ is not closed under countable intersection; he also enlightened us in regard to the connection between this condition and the concept of an ω-measurable cardinal. Definitions are as follows. Let ω denote the cardinality of $\mathbb{Z}$. An ultrafilter $\mathcal{U}$ is said to be *ω-complete* [BS, p. 111] if $\mathcal{U}$ is closed under countable intersection; otherwise $\mathcal{U}$ is *ω-incomplete*. An infinite cardinal α is said to be *ω-measurable* [BS, p. 112] if each set of cardinality α admits a free ω-complete ultrafilter. The statement that ω-measurable cardinals do not exist is consistent with ZFC [BS, pp. 112-3]. Besides ω, the first uncountable cardinal fails to be ω-measurable [BS, pp. 112, 292, 296]. In the terminology just described, the condition on $\mathcal{Y}(M)$ in the statement of (3.7) is that $\mathcal{Y}(M)$ is ω-incomplete, and this condition is satisfied for each free ultrafilter on A if $|A|$ is not ω-measurable. Thus we obtain the following corollary.

Corollary 3.8. Let $\{(R_a, M_a)\}_{a\in A}$ be an infinite family of zero-dimensional chained rings such that $M_a = M_a^2 \neq (0)$ for each $a \in A$. If $|A|$ is not ω-measurable (so in particular, if $|A|$ is countably infinite), then $R = \prod_{a\in A} R_a$ admits no homomorphic image of finite positive dimension.

Remark 3.9. Let $\{R_a\}_{a\in A}$ be an infinite family of zero-dimensional rings and let M be a free maximal of $R = \prod_{a\in A} R_a$. Since each $f \in R$ having finite support is in the kernel of the canonical homomorphism of $R \to R_M$, Theorem 3.7 and Corollary 3.8 also hold if all but finitely many of the rings R_a are chained rings with nonzero idempotent maximal ideal.

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