# NORMAL IDEALS IN REGULAR RINGS

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## 1. Introduction

This paper studies the depth of graded algebras associated to normal ideals. If $I$ is an ideal in a commutative ring $R$, the integral closure of $I$, denoted $\overline{I}$, is the set of elements $x \in R$ such that $x$ satisfies an equation of the form $x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_j \in I^j$ for $1 \leq j \leq n$. If $x \in \overline{I}$ we say $x$ is integral over $I$. It is not difficult to prove that $\overline{I}$ is an ideal. An ideal $I$ of a commutative Noetherian ring $R$ is said to be normal if all its powers are integrally closed. If $R$ is an integrally closed domain then the normality of $I$ is equivalent to the normality of the Rees algebra of $R$ with respect to $I$, $R[It] = \bigoplus_{n\geq 0} I^n t^n$, thus providing a convenient and useful way to interpret the property.

It often has been noted that when the Rees algebra is normal, it tends to be Cohen-Macaulay. This is especially the case when the base ring is regular or more generally pseudo-rational. While this phenomena has been partially explained by the geometry of the blowup of the ideal, there is still much which is not understood. This paper was originally motivated by trying to prove that the normality of the Rees algebra implied the Cohen-Macaulay property of the Rees algebra for a particular class of ideals in regular local rings. Vasconcelos singled out this class for study, partly because it was the simplest class of ideals in a regular ring where the interplay between normality and Cohen-Macaulayness was not fully understood. Vasconcelos guessed that normality would imply Cohen-Macaulayness in the following special case: prime ideals generated by four elements. However, what we found instead was a type of structure theorem for normal height two ideals (which are generically complete intersections) in a three-dimensional regular local ring. In the case conjectured by Vasconcelos–when the ideal is further prime and four-generated, we can give very explicit structural equations which allow one to predict when the Rees algebra is Cohen-Macaulay (cf.Theorem 2.6). In particular we use these equations to give a negative answer to the question of Vasconcelos (cf. Theorem 2.11). In fact, our arguments do not actually require that $I$ is normal, but instead only the apparently much weaker property that the square of the ideal $I$ satisfies $I^2\mathfrak{m} : \mathfrak{m} = I^2$ (this is strictly weaker than assuming the square is integrally closed).

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The examples constructed by using Theorem 2.6 also allow us to construct an explicitly given $\mathfrak{m}$-primary normal ideal $I$ of a three-dimensional regular local ring such that $H^2(X, O_X) \neq 0$, where $X$ is the blowup of $I$. Recall that if $I$ is an integrally closed $\mathfrak{m}$-primary ideal of a two-dimensional regular local ring $(R, \mathfrak{m})$ (or more generally, the local ring of a rational surface singularity), then $I$ is normal, and has a Cohen-Macaulay Rees algebra. Zariski proved the normality as part of his general theory of "complete ideals" in a two-dimensional regular local ring [ZS, Appendix 5]. One way to prove the Cohen-Macaulay property of their Rees algebras is to apply results of Lipman and Tessier [LT] to obtain that $I$ has reduction number one. In particular, if $X$ is the blowup of such an ideal, then $H^1(X, O_X) = 0$. This is no longer true in higher dimensions. Examples of $\mathfrak{m}$-primary normal ideals $I$ of a three-dimensional regular local ring such that $H^2(X, O_X) \neq 0$, where $X$ is the blowup of $I$, were constructed by Cutkosky [C] using geometric tools. It appears however to be difficult or impossible to give explicit generators for the ideals contructed in [C]. Of course, this was not the point in that paper, but we wished to have examples which can be studied via various computer algebra programs. Our example is given in Theorem 3.12.

A second direction in this paper is motivated by a two-dimensional version, outlined by Huneke in [Hun2], of the Grauert-Riemenschneider vanishing theorem. That version generalized the two-dimensional case of the following formulation of Grauert-Riemenschneider, proved by Sancho de Salas: If $R$ is a reduced Cohen-Macaulay local ring, essentially of finite type over an algebraically closed field of characteristic 0, and $I$ is an ideal of $R$ such that $\mathrm{Proj}(R[It])$ is regular, then there exists an integer $n \gg 0$ such that $G(I^n) = \bigoplus_{k \geq 0} I^{nk}/I^{nk+k}$ is Cohen-Macaulay.

In section 3 we prove a theorem (Theorem 3.1) that implies the two-dimensional version outlined in [Hun2] (see Corollary 3.8), and which also recovers a result of Itoh [I2] concerning the coefficient $e_3(I)$ of the Hilbert-Samuel polynomial of a normal $\mathfrak{m}$-primary ideal $I$ (see Corollary 3.10). Corollary 3.8 assumes only that $(R, \mathfrak{m})$ is Cohen-Macaulay and does not require $\mathrm{Proj}(R[It])$ to be regular. It is known however that the Grauert-Riemenschneider theorem fails in dimension 3 if the assumption on $\mathrm{Proj}(R[It])$ being regular is dropped. Cutkosky's example mentioned above provides a height 3 counterexample in the ring $\mathbb{C}[[x, y, z]]$. We end section 3 with a new counterexample (already discussed above) in the ring $k[x, y, z]$, where $k$ is a field of characteristic not 3.

## 2. Non Cohen-Macaulay Rees algebras

Given a three-dimensional regular local ring $(R, \mathfrak{m})$ and a normal four-generated height two unmixed ideal $I$ of $R$, it is natural to ask about the Cohen-Macaulayness of the Rees algebra $R[It]$. Assume further that $I$ is generically a complete intersection (that is, $I_P$ is a complete intersection for every prime ideal $P$ minimal over $I$). Our main result of this section will characterize, in terms of a presentation matrix of $I$, when $R[It]$ is Cohen-Macaulay for such an ideal. It builds on work of Vasconcelos [Vas1] and Aberbach-Huneke [AH], where techniques for studying Rees algebras via presentation matrices were staged. The assumption that $I$ is normal is not needed for the proof of our theorem however.

Instead we need only the weaker condition that $(\mathfrak{m}I^2 : \mathfrak{m}) = I^2$. If $I^2$ is integrally closed then it satisfies this condition. For if $w \in (\mathfrak{m}I^2 : \mathfrak{m})$ then $w\mathfrak{m} \subset \mathfrak{m}I^2$, thus $w \in \overline{I^2} = I^2$ by the determinant trick.

**Theorem 2.1.** *Let $(R, \mathfrak{m})$ be a $d$-dimensional regular local ring containing a field and $I$ a height $d-1$ unmixed ideal of $R$. Assume that $I$ is generically a complete intersection, $I$ has a $d$-generated reduction, $\mu(I) = d+1$, and $I^2\mathfrak{m} : \mathfrak{m} = I^2$, but $I^2$ is not unmixed. Then there is a generating set $\{x_1, ..., x_d\}$ for $\mathfrak{m}$, positive integers $m \geq n$, and a presentation matrix $\phi$ for $I$, such that $I_1(\phi) = (x_2, ..., x_d, x_1^n)$ and*

$$\phi = \begin{pmatrix} x_2 & \dots & x_d & x_1^m & 0 & \dots & 0 \\ . & . & . & . & . & \dots & . \\ . & . & . & . & . & \dots & . \\ . & . & . & . & . & \dots & . \end{pmatrix}.$$

*Proof.* By our assumption, $\mathfrak{m} \in$Ass$(R/I^2)$. In particular, $(I^2 : \mathfrak{m}) \neq I^2$. Choose an element $e \in (I^2 : \mathfrak{m}) \setminus I^2$. Because $(\mathfrak{m}I^2 : \mathfrak{m}) = I^2$, $e\mathfrak{m} \not\subseteq \mathfrak{m}I^2$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $ex \notin \mathfrak{m}I^2$. Expand $x = x_1$ to a minimal generating set $\{x_1, .., x_d\}$ for $\mathfrak{m}$. Using that $ex \in I^2 \setminus \mathfrak{m}I^2$ we may assume that $I = (p_1, ..., p_{d+1})$ is a minimal generating set for $I$, and that $ex = \alpha p_1^2 + p$ for some $p \in (p_2, ..., p_{d+1})I$ and unit $\alpha$. Set $ex_i = g_i \in I^2$. Consider the homomorphism

$$\psi : R[T_1, ..., T_{d+1}] \to R[It]$$

given by $T_i \to p_i t$. Let $F, G_i \in R[T_1, ..., T_{d+1}]$ be homogeneous of degree 2 (in the $T_i's$) such that $\psi(F) = pt^2$, $\psi(G_i) = g_i t^2$. By our choice of $p$ we can assume that $F \in (T_2, ..., T_{d+1})R[T_1, ..., T_{d+1}]$. Let $Q$ be the kernel of $\psi$, and let $Q_j$ denote the ideal generated by the homogeneous elements of $Q$ having degree at most $j$. The trivial relation $(ex)x_i = (ex_i)x$ forces $x_i(\alpha T_1^2 + P) - xG_i \in Q_2$. But $Q_2 = Q_1$ because $I$ is syzygetic [Hun4], thus there are linear homogeneous polynomials $A_i \in R[T_1, ..., T_{d+1}]$ such that $x_i(\alpha T_1^2 + P) - xG_i = A_1 L_1 + ... + A_s L_s$, where $Q_1 = (L_1, ..., L_s)$. The coefficient of $T_1^2$ must be $x_i - xb_{1i}$ for some $b_{1i} \in R$, and the coefficients of $L_i$ all lie in $\mathfrak{m}$, therefore $Q_1$ contains polynomials of the form $(x_i - xb_{1i})T_1 + b_{2i}T_2 + ... + b_{si}T_s$. The existence of these linear polynomials implies that $x_i - b_{1i}x \in ((p_2, ..., p_{d+1}) : p_1)$. Replacing $x_i$ with $x_i - b_{1i}x$ yields that $x_i \in ((p_2, ..., p_{d+1}) : p_1)$. Therefore invertible row and column operations yield that $\phi$ may be reduced to the form described in (1). In particular, $I_1(\phi) = (x_2, ..., x_d, x^n)$ for some $n \leq m$. $\square$

We are particularly interested in applying (2.1) to curves in 3-space, where several of our assumptions automatically are valid.

**Corollary 2.2.** *Let $(R, \mathfrak{m})$ be a 3-dimensional regular local ring containing a field and $I$ a height 2 unmixed ideal of $R$. Assume that $I$ is generically a complete intersection, $I$ has a 3-generated reduction, $\mu(I) = 4$, and $I^2\mathfrak{m} : \mathfrak{m} = I^2$. There is a generating set $\{x, y, z\}$ for $\mathfrak{m}$, positive integers $m \geq n$, and a presentation matrix $\phi$ for $I$, such that $I_1(\phi) = (y, z, x^n)$*

*and*

$$\phi = \begin{pmatrix} y & z & x^m \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}.$$

*Proof.* This follows at once from Proposition 2.1 as soon as we observe that the assumption that $I^2$ is not unmixed is automatic in this case [HU], and also note that the first syzygies have precisely three generators by the Hilbert-Burch theorem. □

We recall Vasconcelos' construction (see [Vas2, §8.2]). Let $\phi$ be a $4 \times 3$ presentation matrix of $I = (p_1, p_2, p_3, p_4)$ and assume $I_1(\phi)$ (the ideal generated by the entries of $\phi$) is a complete intersection, generated by $\{a, b, c\}$. Consider the homomorphism

$$\psi : R[T_1, T_2, T_3, T_4] \to R[It]$$

given by $T_i \to p_i t$. The symmetric algebra of $I$ is a complete intersection whose defining ideal is generated by 3 elements $L_1, L_2$, and $L_3$. These elements satisfy the matrix equation

$$(\,L_1 \quad L_2 \quad L_3\,) = (\,T_1 \quad T_2 \quad T_3 \quad T_4\,) \cdot \phi.$$

Build an associated matrix $B(\phi)$ via the matrix equation

$$(\,T_1 \quad T_2 \quad T_3 \quad T_4\,) \cdot \phi = (\,L_1 \quad L_2 \quad L_3\,) = (\,a \quad b \quad c\,) \cdot B(\phi). \tag{2.3}$$

We also define $C(\phi)$ to be the image of $B(\phi)$ in $k[T_1, ..., T_4]$, i.e. the matrix $B(\phi)$ reduced modulo the maximal ideal of $R$.

*Remark 2.4.* We will need to perform various changes on the generators of $I$, the choice of presentation $\phi$, and the matrix $B(\phi)$. It is convenient to record exactly what changes we will use and how they affect the rest of the data.

First, consider column operations upon $B(\phi)$. Let $\theta$ be a 3 by 3 invertible matrix with coefficients in $R$. Column operations upon $B(\phi)$ coming from the base ring $R$ are obtained by replacing $B(\phi)$ by $B(\phi)\theta$. To preserve equation (2.3) we must also multiply $\phi$ by $\theta$ and then the corresponding equation (2.3) is valid. This only changes the generators for the syzygies of $I$, and not the chosen generating set. Otherwise stated, $B(\phi\theta) = B(\phi)\theta$.

Row operations on $B(\phi)$ coming from $R$ correspond to multiplying $B(\phi)$ by an invertible 3 by 3 matrix $\theta$ with coefficients in $R$ on the left side of $B(\phi)$. To insure that (2.3) continues to hold, we must then multiply the matrix $(a, b, c)$ by $\theta^{-1}$ on the right, basically changing the choice of generators for the ideal $(a, b, c)$.

Finally, we are free to change the chosen generators of $I$. In this case we replace $\phi$ by $\theta\phi$, with $\theta$ in this case a 4 by 4 invertible matrix with coefficients in $R$. If we change the corresponding $T_i$ by multiplying on the right by $\theta^{-1}$, then (2.3) is still valid without change to the complete intersection $a, b, c$ or the matrix $B(\phi)$.

Vasconcelos proved that if $I$ is a prime ideal such that $\det(C(\phi)) \neq 0$ then $R[It]$ is Cohen-Macaulay. Implicit in the work of Aberbach and Huneke is an improvement of this statement. See [MU] and [PU] for a full generalization of this result.

**Proposition 2.5.** *Let $(R, \mathfrak{m})$ be a three-dimensional regular local ring containing a field of characteristic not 2 and $I$ a four-generated height two unmixed ideal of $R$ which is generically a complete intersection. Assume further that $I$ has a three-generated minimal reduction (automatic if $R/\mathfrak{m}$ is infinite) and $I_1(\phi)$ is a complete intersection. Then $R[It]$ is Cohen-Macaulay if and only if det$(C(\phi)) \neq 0$.*

*Proof.* The proof of the forward direction is contained in the proof of the (1) implies (3) part of the argument given in [AH, Theorem 8.2]. Conversely, if $\det(B(\phi)) \notin \mathfrak{m}R[T_1, T_2, T_3, T_4]$ then we may assume (after changing the generators of $I$ if necessary) that $\det(B(\phi)) = T_1^3 + A$ for some homogeneous (in the $T's$) degree 3 polynomial $A$ contained in $(T_2, T_3, T_4)R[T_1, T_2, T_3, T_4]$. Because the elements $\psi(f)$, $\psi(g)$, and $\psi(h)$ vanish, $\psi(\det(B(\phi)) = 0$. Hence $p_1^3 \in (p_2, p_3, p_4)I^2$ which means $I$ has reduction number two, therefore $R[It]$ is Cohen-Macaulay by [AH, Theorem 8.2]. $\square$

The following theorem is the key theorem of this paper upon which the examples are based.

**Theorem 2.6.** *Let $(R, \mathfrak{m})$ be a 3-dimensional regular local ring containing a field of characteristic not 2 and $I$ a height 2 unmixed ideal of $R$. Assume that $I$ is generically a complete intersection, $I$ has a 3-generated reduction, $\mu(I) = 4$, and $I^2\mathfrak{m} : \mathfrak{m} = I^2$. Further assume that a generating set $\{x, y, z\}$ for $\mathfrak{m}$ and a presentation matrix $\phi$ for $I$ have been chosen as in (2.2). Then $R[It]$ is not C-M if and only if $I_1(\phi) = (y, z, x^n) = (u, v, w)$, and there is a presentation matrix $\theta$ for $I$ of the form*

$$\theta = \begin{pmatrix} v & w & 0 \\ u & 0 & w \\ 0 & u & -v \\ 0 & 0 & 0 \end{pmatrix} \quad mod \quad \mathfrak{m}I_1(\theta).$$

*Proof.* If $\theta$ has the form described in (2.6) then it is easy to see that $\det(C(\theta)) = 0$. Therefore $R[It]$ is not Cohen-Macaulay by Proposition 2.5.

Assume $R[It]$ is not Cohen-Macaulay and that $\phi$ is a presentation matrix for $I$ having the form prescribed in (2.2), that is

$$\phi = \begin{pmatrix} y & z & x^m \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix},$$

where $m \geq n$ and $I_1(\phi) = (y, z, x^n)$. By using invertible row operations we may also assume that $m$ is the least power of $x$ appearing in the last column of $\phi$. We will analyze the matrix $B(\phi)$, which in this case has the form

$$B(\phi) = \begin{pmatrix} T_1 + A & B & C \\ D & T_1 + E & F \\ G & H & x^{m-n}(T_1 + K) \end{pmatrix},$$

where $A, B, ..., K \in R[T_2, T_3, T_4]$. If $m = n$ then $\phi$ satisfies the row condition, leading to the contradiction that $R[It]$ is Cohen-Macaulay [AH, Theorem 8.2]. Therefore we assume $m > n$. Let $C = C(\phi)$ denote the image of $B$ modulo $\mathfrak{m}R[T_1, T_2, T_3, T_4]$, and use lower-case letters to denote images modulo $\mathfrak{m}R[T_1, T_2, T_3, T_4]$. Then

$$C(\phi) = \begin{pmatrix} t_1 + a & b & c \\ d & t_1 + e & f \\ g & h & 0 \end{pmatrix}. \tag{2.7}$$

Here $a, b, c, ..., h \in k[t_2, t_3, t_4]$ are linear forms. In addition, the fact that $I_1(\phi) = (y, z, x^n)$ implies that either $g \neq 0$ or $h \neq 0$ (else the least pure power of $x$ in $I_1(\phi)$ would be greater than $n$).

Since $R[It]$ is not Cohen-Macaulay the determinant of $C(\phi)$ is zero by Proposition 2.5, therefore

$$g(bf - ct_1 - ce) - h(ft_1 + af - cd) = 0. \tag{2.8}$$

We consider two cases; either $g$ and $h$ are relatively prime, or not. First suppose that $g$ and $h$ are relatively prime. Because $a, b, ..., h \in k[t_2, t_3, t_4]$, (2.8) implies that $gc + hf = 0$. Therefore there exists an $\alpha \in k$ such that $c = -\alpha h$ and $f = \alpha g$. Substituting into (2.7) yields

$$C(\phi) = \begin{pmatrix} t_1 + a & b & -\alpha h \\ d & t_1 + e & \alpha g \\ g & h & 0 \end{pmatrix}.$$

Thus

$$0 = \det(C) = \alpha(bg^2 + ghe - gha - dh^2).$$

Using that $g$ and $h$ are relatively prime we obtain that $h$ divides $b$, hence $b = \beta h$ for some $\beta \in k$. Substituting above and factoring out $h$ implies that

$$\beta g^2 + ge - ga - dh = 0.$$

Therefore $g$ divides $d$, hence $d = \gamma g$ for some $\gamma \in k$. Substituting and factoring out $g$ implies that $\beta g + e - a - \gamma h = 0$, therefore

$$e - \gamma h = a - \beta g.$$

Further, after making the substitutions $b = \beta h$ and $d = \gamma g$, $C(\phi)$ takes the form

$$C(\phi) = \begin{pmatrix} t_1 + a & \beta h & -\alpha h \\ \gamma g & t_1 + e & \alpha g \\ g & h & 0 \end{pmatrix}.$$

By using row operations $C(\phi)$ may be reduced to

$$C(\phi) = \begin{pmatrix} t_1 + a - \beta g & 0 & -h \\ 0 & t_1 + e - \gamma h & g \\ g & h & 0 \end{pmatrix}.$$

As in Remark 2.4, these row operations will change the choice of generators of the ideal $(y, z, x^n)$. Let us call the new generators $u, v, w$.

Set $q = a - \beta g = e - \gamma h$. By the above calculation,

$$C(\phi) = \begin{pmatrix} t_1 + q & 0 & -h \\ 0 & t_1 + q & g \\ g & h & 0 \end{pmatrix}.$$

Note that $\{t_1 + q, g, h\}$ are independent linear forms in $k[t_1, t_2, t_3, t_4]$ because $g$ and $h$ are relatively prime linear forms and do not involve $t_1$. Therefore by changing variables we may replace $t_1 + q$, $g$, and $h$ with $t_1$, $t_2$, and $t_3$ (respectively replace $T_1 + Q$, $G$, and $H$ with $T_1$, $T_2$, and $T_3$). As in Remark 2.4, this change also will change our original choice of generators for $I$. Substituting this variable-change yields that

$$C(\phi) = \begin{pmatrix} t_1 & 0 & -t_3 \\ 0 & t_1 & t_2 \\ t_2 & t_3 & 0 \end{pmatrix},$$

and lifting back to $R[T_1, T_2, T_3, T_4]$,

$$B(\phi) = \begin{pmatrix} T_1 & 0 & -T_3 \\ 0 & T_1 & T_2 \\ T_2 & T_3 & 0 \end{pmatrix} \quad \text{mod} \quad mR[T_1, ..., T_4].$$

Finally, to see that the matrix $\phi$ may be chosen to have the form described in the statement of the theorem, use the matrix equation

$$(\,v \quad w \quad u\,)\, B(\phi) = (\,T_1 \quad T_2 \quad T_3 \quad T_4\,)\, \phi,$$

to rebuild $\phi$. This completes the proof of Theorem 2.6 in the case that $g$ and $h$ are relatively prime.

We will finish the proof of Theorem 2.6 by proving that $g$ and $h$ must be relatively prime under our assumption that $I^2\mathfrak{m} : \mathfrak{m} = I^2$.

*Remark 2.9.* Before continuing the proof of Theorem 2.6 it is convenient to introduce some general remarks concerning relations on regular sequences. These remarks easily follow from the existence of the Buchsbaum-Eisenbud multipliers (see [BE] and for a discussion of these multipliers, [Ho, Chapter 7]), but we choose to directly give the details. Suppose that $a, b, c$ form a regular sequence, and let $A$ denote the matrix of Koszul relations:

$$A = \begin{pmatrix} 0 & -c & -b \\ -c & 0 & a \\ b & a & 0 \end{pmatrix}.$$

Let $D = (d_{ij})$ be an arbitrary 3 by 3 matrix and set $E = AD$. Consider the 2 by 2 minors of $E$. If we write $F$ for the transpose of $(a, b, c)$ then the adjoint of $E$ is given by the transpose of $FG$ where

$$G = (\delta_3, \delta_2, \delta_1)$$

and

$$\delta_1 = cd_{12}d_{21} - cd_{11}d_{22} + bd_{12}d_{31} + ad_{22}d_{31} - bd_{11}d_{32} - ad_{21}d_{32}$$

$$\delta_2 = cd_{13}d_{21} - cd_{11}d_{23} + bd_{13}d_{31} + ad_{23}d_{31} - bd_{11}d_{33} - ad_{21}d_{33}$$

$$\delta_3 = cd_{13}d_{22} - cd_{12}d_{23} + bd_{13}d_{32} + ad_{23}d_{32} - bd_{12}d_{33} - ad_{22}d_{33}.$$

In particular this means that the two by two minors of every choice of two columns of $E$ are of the form $(a\delta_i, b\delta_i, c\delta_i)$ for a fixed $\delta_i$, $1 \leq i \leq 3$.

Suppose that $g$ and $h$ are not relatively prime. Then one must be a multiple of the other, and without loss of generality we assume that $g = \alpha h$. By the discussion above $h \neq 0$ in this case. Subtracting $\alpha$ times the second column of $C$ from the first column yields a new $C$ with bottom row $(0, h, 0)$. We make the corresponding change to the matrix $B(\phi)$. Since $\det(C) = 0$ and $h \neq 0$ this implies that the $2 \times 2$ minors of the first and third columns of this changed $C$ are zero.

Consider the generating set $\{L_1, L_2, L_3\}$ of $Q_1$ given by

$$(\, L_1 \quad L_2 \quad L_3 \,) = (\, y \quad z \quad x^n \,)\, B(\phi).$$

By $B'$ denote $B(\phi)$ evaluated at the substitution of $p_i$ for $T_i$. Since the $L_i$ are linear relations on $p_1, ..., p_4$ it follows that

$$0 = (\, y \quad z \quad x^n \,)\, B'.$$

Note that $B'$ has the form,

$$\begin{pmatrix} p_1 + a_1 & b_1 & c_1 \\ (-\alpha p_1 + d_1) & (p_1 + e_1) & f_1 \\ g_1 & h_1 & x^{m-n}(p_1 + k_1) \end{pmatrix},$$

where $a_1, ..., k_1 \in (p_2, p_3, p_4)$. (Recall the slightly different form of $B$ is due to the fact that we changed our original matrix $C$ by subtracting a multiple of the second column from the first column.)

As $y, z, x^n$ form a regular sequence, we obtain that

$$B' = \begin{pmatrix} 0 & -x^n & -z \\ -x^n & 0 & y \\ z & y & 0 \end{pmatrix} D$$

for some $3 \times 3$ matrix $D = (d_{ij})$.

Using Remark 2.9 it follows that the $2 \times 2$ minors of an arbitrary two columns of $B'$ are of the form $(y\delta, z\delta, x^n\delta)$ for some $\delta \in R$.

Recall that the $2 \times 2$ minors of first and third columns of $C$ are zero. Lifting back to $B$, we obtain that the $2 \times 2$ minors of the corresponding two columns of $B$ are in $\mathfrak{m}(T_1, ..., T_4)^2$.

Upon substituting $p_i$ for $T_i$, we find that the $2 \times 2$ minors of the corresponding two columns of $B'$ are in $\mathfrak{m}I^2$. Hence there is an element $\delta$ such that $(y, z, x^n)\delta \subseteq \mathfrak{m}I^2$. This forces $x^{n-1}\delta \in \mathfrak{m}I^2 : \mathfrak{m} = I^2$, the last equality following by assumption. We will reach a contradiction from this. Taking the two by two minor of the first and third rows and columns of $B'$ we see that

$$z\delta = x^{m-n}(p_1 + k_1)(p_1 + a_1) - c_1 g_1.$$

Multiplying through by $x^{n-1}$ yields the equation

$$z(x^{n-1}\delta) = x^{m-1}(p_1 + k_1)(p_1 + a_1) - x^{n-1}c_1 g_1.$$

As $x^{n-1}\delta \in I^2$, we can write $x^{n-1}\delta = rp_1^2 + s$, where $s \in (p_2, p_3, p_4)I$. Combining with the displayed equation above, we see that

$$yr - x^{m-1} \in (p_2, p_3, p_4)I : I^2.$$

The syzygetic property of $I$ implies that $(p_2, p_3, p_4)I : I^2 = (p_2, p_3, p_4) : p_1$. Since we already knew that $(y, z) \subseteq (p_2, p_3, p_4) : I$, we obtain that $(y, z, x^{m-1}) \subseteq (p_2, p_3, p_4) : I$, a contradiction. □

**Corollary 2.11.** *Let $(R, \mathfrak{m})$ be a 3-dimensional regular local ring containing a field of characteristic not 2 and $I$ a height 2 unmixed ideal of $R$. Assume that $I$ is generically a complete intersection, $I$ has a 3-generated reduction, $\mu(I) = 4$, and $I^2$ is integrally closed. Then $R[It]$ is not C-M if and only if there is a generating set $\{x, y, z\}$ for $\mathfrak{m}$ and a presentation matrix $\phi$ of $I$ such that $I_1(\phi) = (x^n, y, z) = (u, v, w)$,*

$$\phi = \begin{pmatrix} v & w & 0 \\ u & 0 & w \\ 0 & u & -v \\ 0 & 0 & 0 \end{pmatrix} \quad mod \quad \mathfrak{m}I_1(\phi).$$

The following corollary implies the validity of [Vas2, Conjecture 8.2.6] for ideals $I$ such that the multiplicity $e(R/I)$ of $R/I$ is relatively small.

**Corollary 2.12.** *Let $(R, \mathfrak{m})$ be a 3-dimensional regular local ring containing a field of characteristic not 2 and $I$ a height 2 unmixed ideal of $R$. If $I$ is generically a complete intersection, $\mu(I) = 4$, $I^2\mathfrak{m} : \mathfrak{m} = I^2$, $I$ has a 3-generated minimal reduction, and the multiplicity of $R/I$ is at most 10, then $R[It]$ is Cohen-Macaulay.*

*Proof.* If $R[It]$ is not Cohen-Macaulay then by using the presentation matrix of Theorem 2.4, $I \subseteq \mathfrak{m}^4$. Let $w$ be a sufficiently general linear combination of the generators of $\mathfrak{m}$. Then

$$e(R/I) = \lambda(R/(I, w)) \geq \lambda(R/(\mathfrak{m}^4, w) = 10.$$

If $e(R/I) = 10$ then $(I, w) = (\mathfrak{m}^4, w)$. This is impossible however, because $\mu(I, w) = 5$ while $\mu(\mathfrak{m}^4, w) = 6$. □

It is easy to use Theorem 2.4 to give interesting examples of ideals whose Rees algebras are not Cohen-Macaulay. The following theorem provides a counterexample to [Vas2, Conjecture 8.2.6]. Searching for a proof of the conjecture motivated us for much of the work in this paper. Several details of the theorem were verified with the help of Macaulay and Maple. The authors also thank Joe Mott and Wolmer Vasconcelos for their help concerning this theorem.

**Theorem 2.13.** *Let $T = \mathbb{Q}[x, y, z]$, $\mathfrak{n} = (x, y, z)$, $R = T_{\mathfrak{n}}$, and $\mathfrak{m} = \mathfrak{n}T_{\mathfrak{n}}$. Set*

$$\phi = \begin{pmatrix} y & z & -x^2 \\ 0 & x & -y \\ x & y^2 & z \\ z^2 & y^5 & 0 \end{pmatrix},$$

*and let $P = I_3(\phi)$. Then $P$ is a four-generated normal height two prime ideal, but $R[Pt]$ is not Cohen-Macaulay.*

*Proof.* By using Macaulay or Maple it is easy to verify that $P$ has a 3-generated reduction. It then follows from Theorem 2.2 that $R[Pt]$ is not Cohen-Macaulay. To see that $P$ is a prime ideal we use the technique of [Vas2, §10.4]. One verifies that $B = \mathbb{Q}[z] \hookrightarrow A = \mathbb{Q}[x, y, z]/P$ is a Noether normalization, that the polynomial

$$f(x, z) = z^{12} + x^8 z^8 + 4x^{10} z^7 + 2x^{12} z^6 + x^{24}$$

is in $P \cap k[x, z]$, and that the degree of $A$ over $B$ is 24. Therefore by [Vas2, Proposition 10.4.19] $P$ will be prime if and only if $f$ is irreducible. Using Maple we verified that $f$ is indeed irreducible over $\mathbb{Z}$.

To verify that $R[Pt]$ is normal requires more work. Set $S = R[T_1, T_2, T_3, T_4]$ and let $I$ be the kernel of the surjection $S \to R[Pt]$ induced by $T_i \to p_i$, where $P = (p_1, p_2, p_3, p_4)$. We need to show that $S/I$ satisfies the conditions $R_1$ (regular in codimension one) and $S_2$ (Serre's condition). Using either Maple or Mathematica one observes that

$$0 \to S \xrightarrow{\phi_4} S^6 \xrightarrow{\phi_3} S^9 \xrightarrow{\phi_2} S^5 \xrightarrow{\phi_1} S \to S/I \to 0 \tag{2.14}$$

is a complex, where (using $T_1 = A, T_2 = B, T_3 = C$, and $T_4 = D$)

$$\begin{gathered} \phi_1 = (z^2A - xB + yC \quad x^2C + zB - yD \quad y^5A - y^2B + zC - xD \\ xy^4zA^2C + y^4AB^2 - xyzABC - zAD^2 - xC^3 - yB^3 \\ xy^3zA^2BC + y^4zA^2C^2 + xy^4A^2CD + y^3AB^3 - xzAB^2C - yzABC^2 - xyABCD - B^4 - C^4 - AD^3), \end{gathered}$$

$$\phi_2 = \begin{pmatrix} y^5A - y^2B + zC - xD & -x^2C - zB + yD & 0 & -y^4AB + yB^2 - CD \\ 0 & z^2A - xB + yC & y^5A - y^2B + zC - xD & y^4zA^2 - yzAB - C^2 \\ -z^2A + xB - yC & 0 & -x^2C - zB + yD & zAD + BC \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} -xC^2-BD & xy^4AC-xyBC-D^2 & -y^3AB^2+B^3 & -C^3 \\ zAD-BC & y^4AB-yB^2-CD & y^3zA^2B+y^4A^2D-zAB^2-yABD & AD^2 \\ xzAC+B^2 & -xC^2+BD & AD^2 & zAC^2+xACD \\ -y & -z & C & B \\ 0 & 0 & -x & -y \end{matrix}$$

$$\left.\begin{matrix} xy^3ABC+y^4AC^2-xB^2C-yBC^2 \\ y^3AB^2-B^3 \\ -C^3 \\ D \\ -z \end{matrix}\right),$$

$$\phi_3 = \begin{pmatrix} -B & xC & -D & 0 & C^2 & 0 \\ -C & -D & y^4A-yB & 0 & 0 & y^3AB-B^2 \\ zA & B & -C & AD & 0 & 0 \\ -y & 0 & -z & B & 0 & D \\ x & z & 0 & -C & -D & 0 \\ 0 & -y & x & 0 & B & -C \\ 0 & 0 & 0 & -y & 0 & -z \\ 0 & 0 & 0 & x & z & 0 \\ 0 & 0 & 0 & 0 & -y & x \end{pmatrix},$$

$$\phi_4 = \begin{pmatrix} -D \\ C \\ B \\ z \\ -x \\ -y \end{pmatrix},$$

and the ideal $I$ is generated by the entries of $\phi_1$. By using the Buchsbaum-Eisenbud criterion for acyclicity it holds that the above complex is acyclic. We sketch some of the details. The criterion requires that the heights of $I_1(\phi_1)$, $I_4(\phi_2)$, $I_5(\phi_3)$, and $I_1(\phi_4)$ are at least 1, 2, 3, and 4 respectively. Clearly $I_1(\phi_1)$ and $I_1(\phi_4)$ satisfy the conditions. Among the $4\times 4$ minors of $\phi_2$ are the polynomials

$$x^2z^4A^2-2x^3z^2AB+2x^2yz^2AC+x^4B^2-2x^3yBC+x^2y^2C^2$$

and

$$\begin{aligned} y^{12}A^2 &- 2y^9AB+2y^7zAC-2xy^7AD \\ &+ y^6B^2-2y^4zBC+y^2z^2C^2+2xy^4BD-2xy^2zCD+x^2y^2D^2. \end{aligned}$$

It is easy to see that this pair generates an ideal of height two, thus $I_4(\phi_2)$ satisfies the required condition. The set of $5\times 5$ minors of $\phi_3$ includes the polynomials

$$\begin{aligned} f &= z^5A-xz^3B+yz^3C \\ g &= x^2z^2AD+x^4C^2+x^2zBC-x^3BD \\ h &= y^7AD-y^4BD+y^2zCD-xy^2D^2. \end{aligned}$$

By using maple one sees that the leading monomials of a Gröbner basis of the ideal $(f, g, h)$ (using rev lex) is the set

$$\{z^5A, x^2zBC, y^7AD, xy^7z^3BD, x^4z^4AC^2, xy^4z^5BCD, x^6z^3AC^3, \\ x^5y^7z^2C^2D, y^4z^7BCD, x^8z^2AC^4, x^7y^7zC^3D, x^9B^2C^4, x^9y^7C^4D\}.$$

The ideal generated by these monomials has height three, therefore $I_5(\phi_3)$ also satisfies the height condition. We conclude that the complex (2.15) is acyclic. A trivial verification gives that $I_1(\phi_1) = J \subseteq I$. To verify that $J = I$ we need only check this at the associated primes of $J$. The resolution of $J$ above shows that all such associated primes have height at most one over $J$. We verify below that $R[T_i]/J$ satisfies Serre's condition $R_1$. This forces $J$ to be a prime and since it is the same height as $I$, it is then equal to $I$.

To verify the $R_1$ condition it suffices to show that the $3 \times 3$ minors of the Jacobian matrix of $J$ generate an ideal of height at least 5. Among the polynomials in a Gröbner basis of this ideal are the following:

$$x^4 + y^4, x^3y^5 + y^6z - x^2y^2z^2 - z^4, B^5 + BC^4 + ABD^3, C^5, D^5.$$

These generate an ideal of height five (for example, by using [E, Proposition 15.15]), therefore $R[T_i]/J$ satisfies $R_1$.

Note that the ideal $I_1(\phi_4)$ has height six, thus if $Q$ is a prime ideal of height five containing $I$ then $S_Q/I_Q$ has projective dimension at most 3, hence by the Auslander-Buchsbaum formula the depth of $S_Q/I_Q$ is at least 2. Further, if $Q$ is a prime ideal of height at least six then (2.15) already yields that the depth of $S_Q/I_Q$ is at least 2. Therefore Serre's $S_2$ condition holds, completing the proof that $R[Pt]$ is normal. □

## 3. Cohen-Macaulay associated graded rings

In this section we will show that normal ideals, while not always producing Cohen-Macaulay Rees algebras, do often yield depth information about associated graded rings. A corollary will provide a 2-dimensional version of Grauert-Riemenschneider vanishing. We will need to use the concept of a superficial element. Recall that if $(R, \mathfrak{m})$ is a local ring and $I$ is an ideal of $R$, then an element $x \in I \setminus I^2$ is said to be *superficial* (of order one) for $I$ if there is a positive integer $c$ such that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. If $R/\mathfrak{m}$ is infinite then superficial elements exist for any ideal of $R$ ([N] or [ZS]). An important property of a superficial element of $I$ is the following; if $x$ is superficial for $I$, and is also an $R$-regular element, then $(I^n : x) = I^{n-1}$ for all $n >> 0$. This follows by an application of the Artin-Rees lemma.

**Theorem 3.1.** *Let $(R, \mathfrak{m})$ be a local ring and let $I$ be an ideal of $R$. Assume $I$ is normal, grade($I$) $\geq 2$, and $I$ is integral over an ideal generated by an $R$-regular sequence. Then there exists $n$ such that depth($G(I^n)$) $\geq 2$.*

*Proof.* By passing to $R(x) = R[x]_{\mathfrak{m}[x]}$ we may assume that $R/\mathfrak{m}$ is infinite. In this case, if we choose a general element of $I$ which has the same value as $I$ on all the Rees valuations of $I$, we obtain that $I^n : x = I^{n-1}$ for all $n$, since $I^n : x$ will be contained in the integral closure of $I^{n-1}$.[1]

Next choose $y \in I \setminus I^2$ such that the image of $y$ in $R/(x)$ is superficial for $I/(x)$, $y$ is an $R/(x)$-regular element, and $\{x, y\}$ form part of a minimal generating set for a minimal reduction of $I$. Note that choosing $y$ to be $R/(x)$-regular is where we require grade$(I) \geq 2$. There exists a positive integer $b$ such that

$$((I^k : y) = (I^{k-1}, x) \text{ for all } k \geq b. \tag{3.3}$$

We are going to show that $\{(x^b)', (y^b)'\}$ form a $G(I^b)$-regular sequence. From (3.2), $(x^b)'$ is a $G(I^b)$-regular element. What remains is to verify that

$$((I^{bn}, x^b) : y^b) = (I^{bn-b}, x^b) \text{ for all } n \geq 2. \tag{3.4}$$

By our choice of $x$ and $y$ we may assume there is a minimal reduction $J$ of $I^b$ having a minimal generating set including $\{x^b, y^b\}$. In other words $J = (x^b, y^b, z_3, ..., z_s)$ is a minimal reduction of $I^b$, and because $I$ is integral over an ideal generated by a regular sequence, $\{x^b, y^b, z_3, ..., z_s\}$ must itself be a regular sequence. To prove (3.4) we will handle the cases $n = 2$ and $n \geq 3$ separately. Suppose first that $n = 2$. By the theorem of Huneke and Itoh ([Hun] or [I, Theorem 1]), $J \cap I^{2b} = JI^b$ (this uses that $I^b$ and $I^{2b}$ are integrally closed). In fact, we claim that

$$(x^b, y^b) \cap I^{2b} = (x^b, y^b)I^b. \tag{3.5}$$

To prove this claim it suffices to show that if $\{z_1, ..., z_s\}$ is an $R$-regular sequence contained in $I^b$ such that

$$(z_1, ..., z_i) \cap I^{2b} = (z_1, ..., z_i)I^b$$

for some $i$, $2 \leq i \leq s$, then

$$(z_1, ..., z_{i-1}) \cap I^{2b} = (z_1, ..., z_{i-1})I^b.$$

Let $u \in (z_1, ..., z_{i-1}) \cap I^{2b}$. Then

$$u \in (z_1, ..., z_i) \cap I^{2b} = (z_1, ..., z_i)I^b,$$

hence we may write

$$u = \sum_{j=1}^{i-1} u_j z_j = \sum_{j=1}^{i} v_j z_j \text{ for some } u_j \in R, v_j \in I^b. \tag{3.6}$$

[1] Alternatively, without assuming $I$ is normal, one begins by choosing a superficial element $x$ for $I$ having the property that $x$ is $R$-regular. Then there exists a positive integer $a$ such that $(I^k : x) = I^{k-1}$ for all $k \geq a$. We claim that $(x^a)'$ is a $G(I^a)$-regular element, or equivalently that $(I^{an} : x^a) = I^{an-a}$ for all $n \geq 1$. Suppose $n \geq 2$ and let $rx^a \in I^{an}$. Then $rx^{a-1} \in (I^{an} : x) = I^{an-1}$, hence $rx^{a-2} \in (I^{an-1} : x) = I^{an-2}$. Continuing we eventually obtain that $r \in I^{an-a}$, proving the claim. One then replaces $I$ by this higher power of $I$.

Therefore $v_i z_i \in (z_1, ..., z_{i-1})$, thus $v_i \in (z_1, ..., z_{i-1})$. Writing $v_i = \sum_{j=1}^{i-1} w_j z_j$ for some $w_j \in R$ and substituting into (3.6) yields

$$\sum_{j=1}^{i-1}(u_j - v_j - w_j z_i) z_j = 0.$$

A consequence, because $\{z_1, ..., z_s\}$ is a regular sequence, is that $u_j - v_j - w_j z_i \in (z_1, ..., z_{i-1})$ for each $j$, $1 \le j \le i-1$. In particular this implies that $u \in (z_1, ..., z_{i-1})I^b$, completing the proof of (3.5).

To finish the proof of (3.4) for the case $n = 2$, let $cy^b \in (I^{2b}, x^b)$. Then $cy^b - dx^b \in I^{2b}$ for some $d \in R$, hence $cy^b - dx^b \in (x^b, y^b)I^b$ by (3.5). Therefore $c \in I^b = (x^b, I^b)$ because $\{x^b, y^b\}$ is a regular sequence.

Now assume $n \ge 3$. We claim that

$$(I^{bn-i}, x^{b-i}) : y^b \subseteq I^{bn-b-i} + x((I^{bn-i-1}, x^{b-i-1}) : y^b) \tag{3.7}$$

for $0 \le i \le b-1$. Let $cy^b \in (I^{bn-i}, x^{b-i})$ and write $cy^b - dx^{b-i} \in I^{bn-i}$ for some $d \in R$. Then $cy^b \in (I^{bn-i}, x)$ so $c \in (I^{bn-b-i}, x)$ by (3.3). Thus we write $c - c_0 x \in I^{bn-b-i}$ for some $c_0 \in R$, which leads to $cy^b - c_0 xy^b \in I^{bn-i}$. Hence $(c_0 y^b - dx^{b-i-1})x \in I^{bn-i}$, therefore $c_0 y^b - dx^{b-i-1} \in I^{bn-i-1}$ by (3.1). In other words, $c_0 y^b \in (I^{bn-i-1}, x^{b-i-1})$. It follows that

$$c \in I^{bn-b-i} + (c_0 x) \subseteq I^{bn-b-i} + x((I^{bn-i-1}, x^{b-i-1}) : y^b),$$

proving (3.7). By applying (3.7) successively to the $i = 0$ and $i = 1$ cases we obtain

$$((I^{bn}, x^b) : y^b) \subseteq I^{bn-2} + x^2((I^{bn-2}, x^{b-2}) : y^b).$$

The pattern is now easily detected and by continuing we eventually obtain

$$((I^{bn}, x^b) : y^b) \subseteq I^{bn-b} + x^{b-1}((I^{bn-b+1}, x) : y^b).$$

But by using (3.3) $b$ successive times, $((I^{bn-b+1}, x) : y^b) = (I^{bn-2b+1}, x)$ (this is where we require that $n \ge 3$). Therefore

$$((I^{bn}, x^b) : y^b) \subseteq (I^{bn-b}, x^b),$$

proving (3.4). The proof of Theorem 3.1 is now complete. $\square$

By looking at some special cases of Theorem 3.1 we are able to provide some interesting corollaries. The first is a 2-dimensional version of Grauert-Riemenshneider vanishing ([Hun2]).

**Corollary 3.8.** *Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring, and let $I$ be a normal $\mathfrak{m}$-primary ideal of $R$. Assume $I$ has a 2-generated minimal reduction (automatic if $R/\mathfrak{m}$ is infinite). Then there exists $n$ such that $G(I^n)$ is Cohen-Macaulay.*

The next corollary gives some information about the coefficient $e_3(I)$ of the Hilbert-Samuel polynomial of a normal $\mathfrak{m}$-primary ideal $I$ of a Cohen-Macaulay local ring. Recall that for any $d$-dimensional local ring $(R, \mathfrak{m})$ and $\mathfrak{m}$-primary ideal $I$ of $R$, the length of $R/I^n$ is given by a polynomial in $n$, of degree $d$, for all large values of $n$. The Hilbert-Samuel polynmial is by definition that polynomial, and it can be expressed in the form

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^{d-1}e_{d-1}(I)n + (-1)^d e_d(I)$$

where the coefficients $e_i(I)$ are integers. Certain bounds on the $e_i(I)$'s are known to hold if $R$ is assumed to be Cohen-Macaulay. In particular it holds that $e_i(I) \geq 0$ for $0 \leq i \leq 2$ (see [No] for $i = 1$ and [Na] for $i = 2$). In [Na] Narita showed that it is possible for $e_3(I)$ to be negative, but Itoh proved that $e_3(I) \geq 0$ if $I$ is assumed to be normal.

In fact Itoh proved a stronger result by considering the filtration $\{\overline{I^n}\}$ of integral closures of the powers of $I$. If $(R, \mathfrak{m})$ is assumed to be analytically unramified then the length of $R/\overline{I^n}$ is a polynomial of degree $d$ for large $n$ and takes the form

$$\overline{P}_I(n) = \overline{e}_0(I)\binom{n+d-1}{d} - \overline{e}_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^{d-1}\overline{e}_{d-1}(I)n + (-1)^d\overline{e}_d(I),$$

where the coefficients $\overline{e}_i(I)$ are integers (the normalized Hilbert coefficients). In [I2] Itoh proved that $\overline{e}_3(I) \geq 0$. Corollaries 3.9 and 3.10 below give new proofs of Itoh's results.

**Corollary 3.9.** *(*[I2, Theorem 3]*)Let $(R, \mathfrak{m})$ be an analytically unramified $d$-dimensional Cohen-Macaulay local ring. If $I$ is an $\mathfrak{m}$-primary ideal of $R$ then $\overline{e}_3(I) \geq 0$.*

*Proof.* By using the usual machinery (spelled out in [I2, Theorem 1]) we may assume $d = 3$. Because $R$ is analytically unramified the ring

$$\overline{R} \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \dots$$

is noetherian, thus there exists a positive integer $k$ such that $\overline{I^{kn}} = (\overline{I^k})^n$ for all $n \geq 1$ (see [B] for example). Furthermore we know that $\overline{e}_3(I) = \overline{e}_3(I^k)$ because $\overline{P}_I(kn) = \overline{P}_{I^k}(n)$. Hence we may replace $I$ with $I^k$ and therefore assume $I$ is normal. By applying Theorem 3.1 we obtain that $\operatorname{depth}(G(I)) \geq 2$. The statement now follows from [M, Corollary 2]. □

**Corollary 3.10.** *(*[I2]*)Let $(R, \mathfrak{m})$ be a $d$-dimensional Cohen-Macaulay local ring. If $I$ is a normal $\mathfrak{m}$-primary ideal of $R$ then $e_3(I) \geq 0$.*

*Proof.* As in (3.9) we may assume that $\dim(R) = 3$. The result [Na, Proposition 2] implies that $e_3(I) = e_3(I^k)$ for all $k \geq 1$. Therefore we may assume that $\operatorname{depth}(G(I)) \geq 2$ by using Theorem 3.1. The statement now follows from [M, Corollary 2]. □

We now give an example showing that Corollary 3.8 does not extend to higher dimensions. The existence of such an example, over $\mathbb{C}$, was proved in [C, §3]. The purpose here is to provide an explicit example, in the sense of giving actual equations. The idea behind this construction is useful: to find an $m$-primary normal ideal having specified properties, one first finds a height two normal prime ideal $p$ with the required properties (often a much easier task) Then one takes the integral closure of $p + m^n$ for large $n$. This ideal 'should' have much the same properties as $p$. However, to make this philosophy work in practice, we depend on the following lemma.

**Lemma 3.11.** *Let $R$ be a graded ring with homogeneous maximal ideal $\mathfrak{m}$ and such that $R_0$ is a field. Let $N$ be a homogeneous ideal of $R$ generated by forms of the same degree $d$. If $N$ is a normal ideal then $N + \mathfrak{m}^{d+1}$ is also a normal ideal.*

*Proof.* Let $f \in \overline{(N + \mathfrak{m}^{d+1})^k}$ be homogeneous of degree $n$. Then there are elements $c_i \in (N + \mathfrak{m}^{d+1})^{ki}$ such that $f^m + c_1 f^{m-1} + ... + c_{m-1} f + c_m = 0$. Note that $n \geq kd$. If $n = kd$ then by considering the homogeneous part of the equation having degree $mkd$ we may assume that $c_i$ is homogeneous of degree $kdi$ for each $i$, $1 \leq i \leq m$. Write $c_i = a_i + b_i$ where $a_i \in N^{ki}$ and $b_i \in \mathfrak{m}^{d+1}(N + \mathfrak{m}^{d+1})^{ki-1}$. Then $\deg(b_i) \geq kdi + 1$, thus $b_i = 0$. It follows that $f \in \overline{N^k}$ so that $f \in N^k$ by normality. In particular $f \in (N + \mathfrak{m}^{d+1})^k$. If $n \geq k(d+1)$ then $f \in \mathfrak{m}^{k(d+1)} \subseteq (N + \mathfrak{m}^{d+1})^k$. Assume that $kd < n < k(d+1)$ and write $n = kd + j$ for some $j$, $0 < j < k$. By decomposing $(N + \mathfrak{m}^{d+1})^{ki}$ into the sum

$$N^{(k-j)i}(N + \mathfrak{m}^{d+1})^{ji} + (\mathfrak{m}^{d+1})^{ji+1}(N + \mathfrak{m}^{d+1})^{ki-ji-1}$$

we may express $c_i$ as $c_i = a_i + b_i$ for some

$$a_i \in N^{ki-ji}(N + \mathfrak{m}^{d+1})^{ji} \text{ and } b_i \in (\mathfrak{m}^{d+1})^{ji+1}(N + \mathfrak{m}^{d+1})^{ki-ji-1}.$$

As above we may assume that $c_i$ is homogeneous, this time having degree $i(kd + j)$. But $\deg(b_i) \geq i(kd + j) + 1$, thus $b_i = 0$ and $c_i = a_i$. In particular $c_i \in N^{(k-j)i}$, therefore $f \in \overline{N^{k-j}}$. By the normality of $N$ again, $f \in N^{k-j}$. This means that $f \in N^{k-j} \cap \mathfrak{m}^{kd+j}$. But by using that $R$ is graded and $\mathfrak{m}$ is its homogeneous maximal ideal it holds that $N^{k-j} \cap \mathfrak{m}^{kd+j} = N^{k-j}\mathfrak{m}^{(d+1)j}$. Therefore $f \in (N + \mathfrak{m}^{d+1})^k$ and the proof of Lemma 3.10 is complete. □

**Theorem 3.12.** *Let $k$ be a field of characteristic not 3. Set $R = k[x, y, z]$. Let*

$$N = (x^4, x(y^3 + z^3), y(y^3 + z^3), z(y^3 + z^3))$$

*and set $I = N + \mathfrak{m}^5$, where $\mathfrak{m} = (x, y, z)R$. Then*

1. *$I$ is a height 3 normal ideal of $R$.*
2. *$G(I^n)$ is not Cohen-Macaulay for any $n \geq 1$.*
3. *If $X$ denotes the blowup of $I$, then $X$ is normal but $H^2(X, O_X) \neq 0$.*

*Proof.* Let $L = (x^4, y^3+z^3)$. We first show that $L$ is normal. The powers of $L$ are unmixed since $L$ is generated by a regular sequence. Further, $L$ is generically normal (i.e. locally normal at its minimal primes). This follows since $y^3+z^3$ will be reduced in characteristic not equal to 3, and then the minimal primes above $L$ are exactly generated by the minimal primes over $y^3+z^3$ together with the element $x$, and locally at each such prime $L$ is generated by $x^4$ together with a regular parameter. But all such ideals are normal. It follows that $L$ is normal.

By induction on $i$ we claim that $L^i \cap m^{4i} = N^i$. It follows immediately from this claim that the powers of $N$ are also integrally closed and so $N$ is normal.

For $i = 1$ the equation is clear. Assume $i > 1$. Clearly $N^i \subseteq L^i \cap m^{4i}$, so we prove the other containment. Let $u \in L^i \cap \mathfrak{m}^{4i}$. Write $u = rx^{4i} + (y^3+z^3)v$ for some $r \in R$ and $v \in L^{i-1}$. Then $(y^3+z^3)v \in \mathfrak{m}^{4i}$ and so $v \in \mathfrak{m}^{4i-3} \subseteq \mathfrak{m}^{4(i-1)}$. Hence $v \in L^{i-1} \cap \mathfrak{m}^{4(i-1)} = N^{i-1}$. Since $v \in \mathfrak{m}^{4i-3}$, we even obtain that $v \in \mathfrak{m}N^{i-1}$. Finally we need only to observe that $(y^3+z^3)v \in (y^3+z^3)\mathfrak{m}N^{i-1} \subseteq N^i$.

Therefore $I$ is a normal ideal by Lemma 3.10. For each $s \geq 1$ we will prove that $G(I^s)$ is not Cohen-Macaulay by showing that $G(I^s_{\mathfrak{m}})$ is not Cohen-Macaulay. For this it suffices to prove that the reduction number of $I^s_{\mathfrak{m}}$ is at least 3 [Vas2, (5.1.18)]. Further, it will be enough to find a single minimal reduction $J_s$ of $I^s_{\mathfrak{m}}$ such that $J_s I^{2s}_{\mathfrak{m}} \neq I^{3s}_{\mathfrak{m}}$. This is because if $JI^{2s}_{\mathfrak{m}} = I^{3s}_{\mathfrak{m}}$ for some minimal reduction $J$ of $I^s_{\mathfrak{m}}$ then $J \cap I^{(j+1)s}_{\mathfrak{m}} = JI^{js}_{\mathfrak{m}}$ for $j \geq 1$, and by the result of Huneke and Itoh ([Hun] or [I1, Theorem 1]) $J \cap I^{2s}_{\mathfrak{m}} = JI^s_{\mathfrak{m}}$, therefore $G(I^s_{\mathfrak{m}})$ is Cohen-Macaulay by [VV, Theorem 2.3]. A consequence of $G(I^s_{\mathfrak{m}})$ being Cohen-Macaulay is that every minimal reduction of $I^s_{\mathfrak{m}}$ has reduction number 2.

We proceed to construct the ideals $J_s$. First observe that $J = (x^4, y(y^3+z^3), z(y^3+z^3))$ is a reduction of $N$ (in fact, $JN^3 = N^4$). For convenience set $a = x^4$, $b = y(y^3+z^3)$, $c = z(y^3+z^3)$, and $d = z^5$. We claim that $J_1 = (a, c, b+d)$ is a minimal reduction of $I$. To see this it suffices to show that $J$ is a reduction of the ideal $K = (a, b, c, d)$, because $K$ is already a reduction of $I$ ($KI^3 = I^4$). Clearly

$$0 = zc^4 - b^3d - c^3d.$$

In particular this implies that $b^3d \in J_1K^3$. But $b^4 = b^3(b+d) - b^3d$, therefore $b^4 \in J_1K^3$. In other words, $K^4 = J_1K^3$ which proves the claim.

Now define $J_s = (a^s, c^s, (b+d)^s)$. Then $J_s$ is a minimal reduction of $I^s$. To show that $J_sI^{2s} \neq I^{3s}$, observe that

$$x^{3s}(y^3+z^3)^{3s} \notin J^sI^{2s}$$

because the multiples of $x$ contained in $J^sI^{2s}$ must be of the form $x^j$ or $x^{4s+j}$ for some $j$, $0 \leq j \leq 2s$. To finish the proof, observe that $J_sI^{2s}_{\mathfrak{m}} \neq I^{3s}_{\mathfrak{m}}$ because $I$ is $\mathfrak{m}$-primary.

To prove the last statement we use Corollary 5.3 of [LT], given below as Lemma 3.13.

**Lemma 3.13.** *(see [LT, Cor. 5.3]) Let $R$ be a normal local domain whose completion is reduced. Let $f_1, ..., f_d \in R$, and for each integer $t > 0$ let $I_t$ be the ideal $(f_1^t, ..., f_d^t)$. Let*

*$Y \to Spec(R)$ be obtained by blowing up $I$ and normalizing. Then, for all sufficiently large $t$ we have (with $I = I_1$): $H^{d-1}(O_Y) = \overline{I_t^d}/I_t(\overline{I_t^{d-1}})$.*

In the notation of this lemma, to prove $H^2(X, \mathcal{O}_X) \neq 0$, it suffices to prove that $\overline{I_t^3}/I_t(\overline{I_t^2}) \neq 0$ for all large $t$. Since $\overline{I_t^3} = I^{3t}$ by above, we need only to prove that $I^{3t} \neq J_t I^{2t}$, which we have done above. $\square$

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