REAL STRUCTURES OF TEICHMÜLLER SPACES, DEHN TWISTS, AND MODULI SPACES OF REAL CURVES

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Abstract. An orientation reversing involution σ of a topological compact genus g, $g > 2$, surface Σ induces an antiholomorphic involution $\sigma^* : T^g \to T^g$ of the Teichmüller space of genus g Riemann surfaces. Two such involutions σ^* and τ^* are conjugate in the mapping class group if and only if the corresponding orientation reversing involutions σ and τ of Σ are conjugate in the automorphism group of Σ. This is equivalent to saying that the quotient surfaces $\Sigma/\langle\sigma\rangle$ and $\Sigma/\langle\tau\rangle$ are homeomorphic. Hence the Teichmüller space T^g has $m_g = \lfloor \frac{3g+4}{2} \rfloor$ distinct antiholomorphic involutions, which are also called real structures of T^g ([5]).

This result is a simple fact that follows from Royden's theorem ([3]) stating that the the mapping class group is the full group of holomorphic automorphisms of the Teichmüller space($g > 2$).

Let $\sigma^* : T^g \to T^g$ and $\tau^* : T^g \to T^g$ be two real structures that are not conjugate in the mapping class group. In this paper we construct a real analytic diffeomorphism $d : T^g \to T^g$ such that

$$\sigma^* = d^{-1} \circ \tau^* \circ d. \tag{1}$$

This mapping d is a product of full and half Dehn–twists around certain simple closed curves on the surface Σ.

This has applications to the moduli spaces of real algebraic curves. A compact Riemann surface (Σ, X) admitting an antiholomorphic involution $\sigma : (\Sigma, X) \to (\Sigma, X)$ is a real algebraic curve of the topological type $\Sigma/\langle\sigma\rangle$.

All fixed–points of the real structure σ^* of the Teichmüller space T^g, are real curves of the above topological type and every real curve of that topological type is represented by an element of the fixed–point set of σ^*. The fixed–point set $T^g_{\sigma^*}$ is the Teichmüller space of real algebraic curves of the corresponding topological type. Given two different real structures σ^* and τ^*, let d the the real analytic mapping satisfying (1). It follows that d maps $T^g_{\sigma^*}$ onto $T^g_{\tau^*}$ and is an explicit real analytic diffeomorphism between these Teichmüller spaces.

1. Preliminaries

1.1. **Orientation reversing involutions of compact surfaces.** We start with reviewing known facts and hereby introducing the notations used in this paper.

Let Σ be a topological compact and oriented surface. An orientation reversing involution $\sigma : \Sigma \to \Sigma$ is called a *symmetry* of the surface Σ and the pair (Σ, σ) is a *symmetric surface.*

Lemma 1 (Weichhold). *A symmetric surface (Σ, σ) is determined topologically by the following invariants:*

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1. *The genus* g *of* Σ.
2. *The number* $n = n(\sigma)$ *of connected components of the fixed–point set* Σ_σ *of the mapping* σ.
3. *The index of orientability,* $k = k(\sigma)$*, which is defined setting* $k = 2-$ *the number of connected components of* $\Sigma \setminus \Sigma_\sigma$.

These invariants satisfy:

1. $0 \leq n \leq g+1$.
2. *For* $k = 0$, $n > 0$ *and* $n \equiv g+1 (\mathrm{mod}\ 2)$.
3. *For* $k = 1$, $0 \leq n \leq g$.

Observation 2. *Observe that if* $n = g+1$ *or* $n = 0$*, the index* k *has only one choice; it is 0 if* $n = g+1$ *and 1 otherwise. Hence, in these cases, the topological type is fully determined by the index* n *only.*

Observe also that these are the only restrictions for topological types of involutions of a genus g *surface* Σ. *Hence there are*

$$m_g = \left\lfloor \frac{3g+4}{2} \right\rfloor \tag{2}$$

different topological types of orientation reversing involutions of a compact genus g *surface.*

All this is well known ([6]). In the above, two involutions σ and τ are considered to be topologically the same if there is an automorphism $f : \Sigma \to \Sigma$ such that $\tau = f^{-1} \circ \sigma \circ f$. This is equivalent to saying that the quotient surfaces $\Sigma/\langle\sigma\rangle$ and $\Sigma/\langle\tau\rangle$ are homeomorphic. The quotient surface $S = \Sigma/\langle\sigma\rangle$ is generally a non–orientable surface with boundary. The boundary components of S correspond to the components of Σ_σ. The surface S is orientable if and only if $k(\sigma) = 0$.

A surface $S = \Sigma/\langle\sigma\rangle$ is called *non–standard.* The *genus* of a non–standard surface $S = \Sigma/\langle\sigma\rangle$ equals that of Σ by the definition. Non–standard surfaces are either non–orientable or they have a non–empty boundary (or both).

Let now (Σ, σ) be a fixed symmetric genus g surface and let $\mathcal{M}(\Sigma)$ be the set of those complex structures on Σ which agree with the orientation of Σ. The group $\mathrm{Homeo}(\Sigma)$ of automorphisms of Σ acts of $\mathcal{M}(\Sigma)$ by pull–back, i.e., for $f \in \mathrm{Homeo}(\Sigma)$ we define the mapping $f^* : \mathcal{M}(\Sigma) \to \mathcal{M}(\Sigma)$ by saying that for an $X \in \mathcal{M}(\Sigma)$, $f^*(X)$ is that complex structure of Σ for which the mapping

$$f : (\Sigma, f^*(X)) \to (\Sigma, X)$$

is either holomorphic or antiholomorphic depending on whether f is orientation preserving or not. This is the usual definition. Let $\mathrm{Homeo}_0(\Sigma)$ denote the subgroup consisting of mappings homotopic to the identity. The *Teichmüller space* of Σ is then

$$T(\Sigma) = \mathcal{M}(\Sigma)/\mathrm{Homeo}_0(\Sigma) \tag{3}$$

The Teichmüller space of a compact oriented surface Σ of genus g is a complex manifold homeomorphic to $\mathbb{C}^{3g-3}$ ([2]).

Let $\mathrm{Homeo}^+(\Sigma)$ denote the subgroup of orientation preserving automorphisms of Σ. Then $\mathrm{Homeo}_0 \subset \mathrm{Homeo}^+$ and For $g > 2$, the *mapping class group*

$$\Gamma = \Gamma^g = \mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$$

acts on the Teichmüller space of Σ as the full group of holomorphic automorphisms of $T(\Sigma)$ ([3]). A symmetry $\sigma : \Sigma \to \Sigma$ induces an antiholomorphic involution

$\sigma^* : T(\Sigma) \to T(\Sigma)$, $[X] \mapsto [\sigma^*(X)]$. Such a mapping is called a *real structure* of the Teichmüller space ([1]).

The symmetries σ and τ are topologically equivalent if and only if the corresponding real structures are conjugate in the mapping class group.

We say that two real structures of the Teichmüller space are *distinct* if they cannot be conjugated to each other in the mapping class group. It follows that the Teichmüller space of an oriented compact genus g surface Σ has $m_g = \lfloor \frac{3g+4}{2} \rfloor$ distinct real structures. For $g > 2$, one can show further that these are all the real structures of $T(\Sigma)$ ([5]).

1.2. **Fractional twists.** Fractional Dehn twists provide a way to deform complex structures of topological surfaces. Let us recall the usual way of picturing them. This usual definition is slightly imprecise; in the sequel we will give also a precise definition for the fractional Dehn twist.

Let $\alpha \subset \Sigma$ be an oriented simple closed curve. Assume that $g \geq 2$, g being the genus of Σ. Then every complex structure of Σ determines a hyperbolic metric of (Σ, X). Two complex structures X and Y are biholomorphically equivalent if and only if the corresponding metrics are directly isometric.

Standard arguments show that, in the definition of the Teichmüller space of Σ, we may consider only such complex structures X for which the given simple closed curve α is a geodesic curve. Let us do this.

Since α was assumed to be oriented, we can distinguish between the left– and the right–hand sides of α. Let $\ell_\alpha(X)$ denotes the length of α on X. For $t, -1 \leq t \leq 1$, we deform a complex structure X of Σ in the following way. If $t \geq 0$, we cut (Σ, X) open along the geodesic curve α, turn the left hand side of α in the positive direction of α the distance $t\ell_\alpha(X)$. Then we glue the different sides of α back together. For negative t we turn in the negative direction of α. Since α was assumed to be a geodesic curve, the resulting structure is again a complex structure of Σ. Let us denote it by $X_{t\alpha}$.

Properly speaking, the cutting and pasting yields a new surface which maybe identified with Σ in a natural way. A more precise description of $(\Sigma, X_{t\alpha})$ is given in Section 3.

For any t, the mapping

$$d = d_{t\alpha} = T(\Sigma) \to T(\Sigma), [X] \mapsto [X_{t\alpha}] \tag{4}$$

is well defined. It is a *fractional Dehn twist.* Fractional Dehn twists are real analytic diffeomorphisms of the Teichmüller space of Σ. For $t = \pm 1$, $d_{t\alpha}$ is actually a holomorphic automorphism ([7]).

2. Deforming involutions of surfaces

For our applications it is necessary to get a concrete picture of the different topological involutions of a surface. Consider first involutions σ with $k(\sigma) = 0$. Let n be an integer with $g - (n-1) = g + 1 - n$ even.

Take first a Riemann surface of genus $(g+1-n)/2$. Delete n open disks from it. Assume that the disks are chosen in such a manner that their closures are disjoint. Then one gets a Riemann surface Y of genus $(g+1-n)/2$ with n boundary components.

Let $\overline{Y}$ denote the Riemann surface obtained from Y by replacing the complex structure of Y with its conjugate structure, i.e. by replacing all local variables

z with their complex conjugates $\overline{z}$. $\overline{Y}$ is simply the mirror image of Y. Glue the Riemann surfaces Y and $\overline{Y}$ together identifying the boundary points. In that way one gets a compact Riemann surface $X = (\Sigma, X)$ of genus g. The identity mapping $Y \to \overline{Y}$ induces an antiholomorphic involution $\sigma : X \to X$ such that the curves of X corresponding to the boundary curves of Y remain point–wise fixed. Therefore $n(\sigma) = n$ and $k(\sigma) = 0$ for this involution. This is how one can construct topologically all symmetries σ of a genus g Riemann surface X satisfying $k(\sigma) = 0$ and $n(\sigma) \equiv g + 1 (\mathrm{mod}\ 2)$.

Let α be a closed curve left point–wise fixed under the above involution σ. Let A be a tubular neighborhood of α. Then the universal covering of A is

$$\pi : \tilde{A} = \{z \in \mathbf{C} | \ -1 < \operatorname{Im} z < 1\} \to A$$

Furthermore we may suppose that σ maps A onto itself and that the complex conjugation is a lifting of $\sigma : A \to A$ onto $\tilde{A}$. Then the real axis covers the curve α.

Everything here is only topological. Hence we may assume, without loss of generality,

- that the covering group of $\tilde{A} \to A$ is generated by $z \mapsto z + 2$
- $\pi : [0, 2] \to A$ is a *constant speed* parametrization of α

In particular, the lengths (on X) of the arcs $\pi([0, 1])$ and $\pi([1, 2])$ are the same.

Define the function $H : \tilde{A} \to \tilde{A}$ by setting $H(x + iy) = (x + 1 - y + iy)$. Then the complex conjugation $\tau(x+iy) = x - iy$ and $H \circ \tau$ are both self–mappings of $\tilde{A}$. Both of them map the real axis onto itself but only the complex conjugation keeps it point–wise fixed.

Let $f_\alpha : X \to X$ be defined by setting $f_\alpha(p) = p$ for $p \in X \setminus A$. In A define f_α as the mapping induced by $H : \tilde{A} \to \tilde{A}$. Then $f_\alpha : X \to X$ is continuous and $f_\alpha \circ \sigma$ is also an involution of X. For this involution we have

$$k(f_\alpha \circ \sigma) = 1 \text{ and } n(f_\alpha \circ \sigma) = n(\sigma) - 1 \tag{5}$$

Figure 1 illustrates how the involution $f_\alpha \circ \sigma$ maps a curve that intersects the curve α.

This is how one can construct topological models for symmetric Riemann surfaces.

An alternative way of constructing the deformed symmetry $f_\alpha \circ \sigma$ is provided by half twists. Define the mapping

$$d_{\alpha/2} : \Sigma \setminus \{\alpha\} \to \Sigma \setminus \{\alpha\}$$

in the following way.

1. In $\Sigma \setminus A$, let $d_{\alpha/2} = \mathrm{Id}$.
2. On that component of $A \setminus \{\alpha\}$ which lies on the right hand side of α, define $d_{\alpha/2}$ also as the identity.
3. On the remaining component of $A \setminus \{\alpha\}$ define $d_{\alpha/2}$ as the mapping induced by the above mapping $H : \tilde{A} \to \tilde{A}$.

The mapping $d_{\alpha/2}$ cannot be extended to α. In spite of that, the mapping

$$\tau = d_{\alpha/2} \circ \sigma \circ d_{\alpha/2}^{-1} \tag{6}$$

of $\Sigma \setminus \{\alpha\}$ onto itself extends to α. It is immediate by the definitions that

$$d_{\alpha/2} \circ \sigma \circ d_{\alpha/2}^{-1} = f_\alpha \circ \sigma. \tag{7}$$

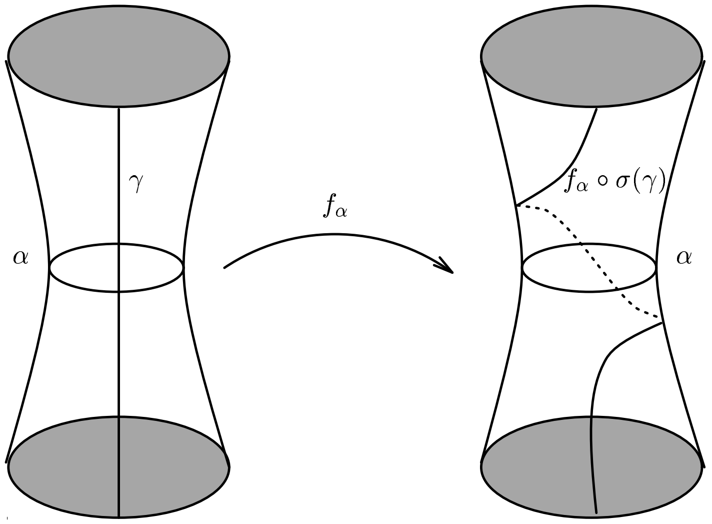

FIGURE 1. The twisted involution $f_\alpha \circ \sigma$ does not keep the curve α pointwise fixed. Curve γ, intersecting α, will be first reflected in α and then twisted as shown in this figure.

3. Real structures of Teichmüller spaces

If X is a complex structure of Σ such that the curve α is a geodesic curve, then we can define a complex structure $d^*_{\alpha/2}(X)$ of Σ by requiring the mapping

$$d_{\alpha/2} : (\Sigma, d^*_{\alpha/2}(X)) \setminus \{\alpha\} \to (\Sigma, X) \setminus \{\alpha\}$$

be holomorphic. The above condition first defines the complex structure $d^*_{\alpha/2}(X)$ on $\Sigma \setminus \{\alpha\}$ only. Since α is a geodesic on (Σ, X), this complex structure extends to all of the surface Σ.

The mapping $d^*_{\alpha/2} : T(\Sigma) \to T(\Sigma)$ is a particular fractional Dehn twist, it is the *half Dehn twist* along the oriented (geodesic) curve α. Hence $d^*_{\alpha/2}$ is a real analytic diffeomorphism of $T(\Sigma)$ onto itself (since all fractional Dehn twists are such).

Theorem 3. *Let σ^* and τ^* be two real structures of the Teichmüller space, T^g, of genus g, $g > 2$, Riemann surfaces. There exists a real analytic mapping $d : T^g \to T^g$ such that we have*

$$\tau^* = d^* \circ \sigma^* \circ (d^*)^{-1} \tag{8}$$

This real analytic mapping d is a product of a number of half Dehn twists and full Dehn twists.

Proof. The real structures σ^* and τ^* are induced by orientation reversing involutions σ and τ of the surface Σ. Let $\alpha_1, \ldots, \alpha_k$ be the components of the fixed–point set of σ.

By (7) and (5) the involution

$$\sigma_0 = d_{\alpha_1/2} \circ \cdots \circ d_{\alpha_k/2} \circ \sigma \circ d^{-1}_{\alpha_1/2} \circ \cdots \circ d^{-1}_{\alpha_k/2} \tag{9}$$

does not have fixed–points.

In the same way we can deform τ to get a fixed–point free involution

$$\tau_0 = d_{\beta_1/2} \circ \cdots \circ d_{\beta_l/2} \circ \tau \circ d^{-1}_{\beta_1/2} \circ \cdots \circ d^{-1}_{\beta_l/2}. \tag{10}$$

Since σ_0 and τ_0 are both orientation reversing involutions without fixed–points, there is, by Observation 2, a homeomorphisms $f : \Sigma \to \Sigma$ such that

$$\tau_0 = f \circ \sigma_0 \circ f^{-1}. \tag{11}$$

Setting now

$$d = d^{-1}_{\beta_1/2} \circ \ldots d^{-1}_{\beta_l/2} \circ f \circ d_{\alpha_1/2} \circ d_{\alpha_k/2} \tag{12}$$

equations (9),(10), and (11) imply that

$$\tau = d \circ \sigma \circ d^{-1}. \tag{13}$$

This, in turn, implies that $d^* : T(\Sigma) \to T(\Sigma)$ is a real analytic self–mapping of $T(\Sigma)$ that conjugates the real structure σ^* to τ^*. Since f itself is a product of Dehn twists, the proof is complete. □

4. Teichmüller spaces of real algebraic curves

Let (Σ, σ) be a compact symmetric surface and X a complex structure on Σ for which the mapping $\sigma : (\Sigma, X) \to (\Sigma, X)$ is an orientation reversing involution. Then (Σ, X) is a a real algebraic curve.

To simplify matters, let us assume that the genus g of Σ satisfies $g > 1$. The cases of genus 0 or 1 curves are elementary and can be treated by explicit methods

Let

$$S = \Sigma / \langle \sigma \rangle \tag{14}$$

be the quotient surface. As mentioned earlier, S is, in general, a non–orientable surface with boundary components. Recall the indices which determine the topological type of an involution (Lemma 1): $n(\sigma)$ is the number of boundary components of S, and S is orientable if and only if if $k(\sigma) = 0$.

The Teichmüller space of S is defined in the usual way ([4]). In this case the structures to be considered on S are *dianalytic* or general conformal in that they are defined by atlases whose co–ordinate transition functions are either analytic or antianalytic, i.e., directly or indirectly conformal. All non–orientable surfaces can be equipped with such atlases. For more details, see [4] and the references given there.

Let now

$$\mathcal{M}(S) = \{\text{dianalytic structures of } S\}.$$

Again the group Homeo(S) of homeomorphic automorphisms of S acts on $\mathcal{M}(S)$ by pull–back. Let Homeo$_0(S)$ denote the subgroup consisting of mappings homotopic to the identity.

As before, the *Teichmüller space* of real curves of topological type S is

$$T(S) = \mathcal{M}(S)/\mathrm{Homeo}_0(S) \tag{15}$$

and the *moduli space* of such curves is

$$M(S) = \mathcal{M}(S)/\mathrm{Homeo}(S) = T(S)/\Gamma(S), \tag{16}$$

where $\Gamma(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$ is the *mapping class group* of the surface S.

All these definitions are, word by word, the same as in the case of orientation preserving surfaces with the exception that, in the definition of the moduli space, we did not consider orientation preserving mappings only. In this case our surfaces are not oriented, and we cannot hence speak of orientation preserving mappings.

This does not, however, cause any difficulties. Any dianalytic structure X on S lifts to a unique conformal structure X^d of Σ that agrees with the orientation of Σ. It is immediate, that any X has two liftings to a conformal structure of Σ. These two liftings are mirror images of each other. The additional requirement regarding the orientation selects one of these two liftings.

We have ([4]):

Theorem 4. *The mapping*

$$\pi^* : T(S) \to T(\Sigma), [X] \mapsto [X^d] \tag{17}$$

is a well defined injection for which

$$\pi^*(T(S)) = T(\Sigma)_{\sigma^*},$$

where $T(\Sigma)_{\sigma^*}$ is the set of fixed–points of the action of σ on the Teichmüller space.

This result is not completely obvious. One has to prove that if f and g are homotopic self–mappings of S, then they have liftings to mappings of Σ that are also homotopic. Usual results about liftings of homotopies cannot be applied, since the covering $\pi : \Sigma \to S = \Sigma/\langle\sigma\rangle$ is branched. In fact, this property is not true for genus 1 surfaces. There we have to use other techniques.

Theorem 4 allows us to identify the Teichmüller space of real curves of the topological type S with $T(\Sigma)_{\sigma^*}$. Let us do that now.

Explicit considerations of deformations of Klein surfaces by Beltrami differentials give an easy proof for the fact that $T(\Sigma)_{\sigma^*}$ is connected. In fact, $T(S) = T(\Sigma)_{\sigma^*}$ is real analytically diffeomorphic to $\mathbb{R}^{3g-3}$ ([4]). Hence, the Teichmüller spaces of two different topological types of real curves are real analytically isomorphic as long as the genera of the curves are the same. Details can be again found in [4]. There the real analytic homeomorphism of $T(S)$ onto $\mathbb{R}^{3g-3}$ was constructed by means of the Bers' embedding of Teichmüller spaces into spaces of quadratic differentials. This embedding is quite transcendental in nature and depends on several choices.

We can use considerations of Section 3 to construct explicit real analytic diffeomorphisms between the Teichmüller spaces $T(S)$ for various surfaces S.

Let σ and τ be two orientation reversing involutions of a surface Σ as in Section 3. Let d be the mapping defined by (12) which conjugates σ to τ. Then the induced mapping $d^* : T(\Sigma) \to T(\Sigma)$ is real analytic and

$$\tau^* = (d^*)^{-1} \circ \sigma^* \circ d^*. \tag{18}$$

.

This implies that $T(\Sigma)_{\sigma^*} = d^*(T(\Sigma)_{\tau^*})$. Hence d^* is a real analytic diffeomorphism between the Teichmüller spaces of the $\Sigma/\langle\sigma\rangle$ and $\Sigma/\langle\tau\rangle$.

5. Moduli spaces

Let X^d be the unique lifting of a dianalytic structure X of S to a conformal structure of Σ such that $\pi : (\Sigma, X^d) \to (S, X)$ is dianalytic. Consider the mapping (17), $\pi^* : T(S) \to T(\Sigma)$, $X \mapsto [X^d]$. Via this mapping we identify $T(S)$ with the set of fixed–points $T(\Sigma)_{\sigma^*}$ of the action of σ on $T(\Sigma)$.

The moduli spaces are

$$M(S) = T(S)/\Gamma(S) \text{ and } M(\Sigma) = T(\Sigma)/\Gamma(\Sigma).$$

Observe first that the mapping (17) does *not* induce a one–to–one mapping between the moduli spaces. The reason is that elements of $\Gamma(S)$ lift to elements of $\Gamma(\Sigma)$ but they form a proper subgroup of $\Gamma(\Sigma)$.

Hence the mapping between moduli spaces induced by the lifting π^* is *not* one–to–one.

Lemma 5. *Lifting of* $\Gamma(S)$ *to a subgroup of* $\Gamma(\Sigma)$ *is the normalizer of* σ^* *in* $\Gamma(\Sigma)$.

Proof. The projection $\pi : \Sigma \to S = \Sigma/\langle\sigma\rangle$ is a ramified covering. Any homeomorphism $f : S \to S$ has exactly two liftings to homeomorphisms $\tilde{f} : \Sigma \to \Sigma$ such that the following diagram commutes:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\tilde{f}} & \Sigma \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ S & \xrightarrow{f} & S \end{array} \tag{19}$$

One of these liftings is orientation preserving and the other is orientation reversing. Let $\tilde{f}$ be the orientation preserving lifting. Then $\sigma \circ \tilde{f}$ is the orientation reversing lifting and

$$\tilde{f} \circ \sigma = \sigma \circ \tilde{f}. \tag{20}$$

Equation (20) implies that, on the level of Teichmüller spaces we have

$$\tilde{f}^* \circ \sigma^* = \sigma^* \circ \tilde{f}^*.$$

Hence $\tilde{f} \in \Gamma(\Sigma, \sigma^*)$, the normalizer of σ^* in $\Gamma(\Sigma)$.

The converse is less obvious. Let $h^* \in \Gamma(\Sigma, \sigma^*)$ be an element in the normalizer of σ^*. We have to show that h^* is induced by a homeomorphism $H : S \to S$.

To this end we will use parametrizations of Teichmüller spaces by Beltrami differentials. Let $[X] \in T(\Sigma)_{\sigma^*}$. Without loss of generality we can suppose that X is such a complex structure of Σ for which $\sigma : (\Sigma, X) \to (\Sigma, X)$ is an antiholomorphic involution.

Next observe that, since $h^* \circ \sigma^* = \sigma^* \circ h^*$, also $h^*([X]) \in T(\Sigma)_{\sigma^*}$.

Let now μ be a Beltrami differential on X. Denote by X^μ that complex structure of Σ for which the identity mapping $(\Sigma, X) \to (\Sigma, X^\mu)$ is μ–quasiconformal. (For details about Beltrami differentials on orientable and on non–orientable surfaces see [4].) We choose μ such that $[X^\mu] = [h^*(X)] \in T(\Sigma)$ and such that μ is σ–invariant. This latter statement simply means that also the mapping

$$\sigma : (\Sigma, X^\mu) \to (\Sigma, X^\mu)$$

is an antiholomorphic involution. The fact that we can choose μ in such a way follows from Lemma 1.2 (page 16) of [4].

The mapping $h^* : T(\Sigma) \to T(\Sigma)$ is induced by some orientation preserving homeomorphism $h : \Sigma \to \Sigma$. Since $[h^*(X)] = [X^\mu]$ in the Teichmüller space, there is a homeomorphism $\alpha \in \mathrm{Homeo}_0(\Sigma)$ such that

$$\alpha : (\Sigma, X^\mu) \to (\Sigma, h^*(X))$$

is holomorphic. Consider the diagram

$$\begin{array}{ccccc} (\Sigma, X^\mu) & \xrightarrow{\alpha} & (\Sigma, h^*(X)) & \xrightarrow{h} & (\Sigma, X) \\ \big\downarrow{\scriptstyle\sigma} & & \big\downarrow{\scriptstyle\alpha\sigma\alpha^{-1}} & & \big\downarrow{\scriptstyle\sigma} \\ (\Sigma, X^\mu) & \xrightarrow{\alpha} & (\Sigma, h^*(X)) & \xrightarrow{h} & (\Sigma, X) \end{array} \tag{21}$$

The left–hand side of diagram (21) clearly commutes. We would like to show that the whole diagram commutes. To that end observe that the horizontal mappings are all holomorphic, and the vertical mappings are antiholomorphic.

All mappings are mappings of the topological surface Σ onto itself. Since h^* lies in the normalizer of σ^*, $h^* \circ \sigma^* = \sigma^* \circ h^*$. This implies that $(\sigma \circ h)^* = (h \circ \sigma)^*$, i.e., that the mappings $\sigma \circ h$ and $h \circ \sigma$ are homotopic as mappings of Σ.

The mapping $\alpha : \Sigma \to \Sigma$ is homotopic to the identity. Hence also $\sigma \circ \ h \circ \alpha$ and $h \circ \alpha \circ \sigma$ are homotopic self–mappings of Σ. Since they are both anticonformal mappings of (Σ, X^μ) onto (Σ, X), they must agree. This follows since a homotopy class of mappings between compact Riemann surfaces contains at most one holomorphic or antiholomorphic homeomorphism.

This implies that the diagram (21) commutes.

The mapping $\tilde{h} = h \circ \alpha$ is homotopic to h and hence induces the same element of the mapping class group as h itself. Furthermore,

$$\tilde{h} \circ \sigma = \sigma \circ \tilde{h}.$$

Hence $\tilde{h}$ is a lifting of some homeomorphism $H : S \to S$. This proves that every element of the normalizer $\Gamma(\Sigma, \sigma)$ of σ^* in $\Gamma(\Sigma)$ is induced by a homeomorphism $S \to S$. This concludes the proof. □

Lemma 5 yields immediately:

Theorem 6. *Let $S = \Sigma/\langle\sigma\rangle$ be a possibly non–orientable compact surface with boundary. Let $\Gamma(\Sigma, \sigma)$ denote the normalizer of $\sigma^* : T(\Sigma) \to T(\Sigma)$ in the mapping class group $\Gamma(\Sigma)$ of Σ. Then*

$$M(S) = T(\Sigma)_{\sigma^*}/\Gamma(\Sigma, \sigma). \tag{22}$$

Let now σ and τ be two orientation reversing involutions of the surface Σ, and let $S_1 = \Sigma/\langle\sigma\rangle$ and $S_2 = \Sigma/\langle\tau\rangle$ be the corresponding possibly non–orientable quotients. Let $d : \Sigma \to \Sigma$ be the orientation preserving mapping that conjugates σ to τ. Then $d^* \circ \tau^* = \sigma^* \circ d^*$ (equation (18)). As observed in Section 4, the mapping d^* defines a real analytic diffeomorphism between the Teichmüller spaces $T(S) = T(\Sigma)_{\sigma^*}$ and $T(T) = T(\Sigma)_{\tau^*}$. However, usually

$$d^*\Gamma(\Sigma, \tau)(d^*)^{-1} \not\subset \Gamma(\Sigma, \sigma)$$

which means that d^* usually does not induce any mapping between the corresponding moduli spaces.

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