The arithmetic and geometry of Salem numbers

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Abstract

A Salem number is a real algebraic integer, greater than 1, with the property that all of its conjugates lie on or within the unit circle, and at least one conjugate lies on the unit circle. In this paper we survey some of the more recent appearances that Salem numbers have made in parts of geometry and arithmetic, and we discuss in particular, the possible implications for the so called 'minimization problem'. This is an old question in number theory which asks whether the set of Salem numbers is bounded away from 1.

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1 Introduction

A *Salem number* is a real algebraic integer, greater than 1, with the property that all its conjugates lie on or within the unit circle, and at least one conjugate lies on the unit circle. This paper deals with the following basic question:

Problem 1 (Minimization problem) *Is the set of Salem numbers bounded away from 1?*

The minimization problem is still unsolved. In fact, it is closely related to the following intriguing question asked by Lehmer [22] in 1933:

Problem 2 (Lehmer's question) *Is there a $\delta > 0$ such that the Mahler measure of every irreducible, non-cyclotomic, monic polynomial with integer coefficients is larger than $1 + \delta$?*

Recall that the *Mahler measure*, $M(P)$, of a monic polynomial $P(x) \in \mathbb{Z}[x]$, is the product of the absolute values of the roots of $P(x)$ which lie outside the unit circle:

$$M(P) = \prod_{P(\theta)=0,\ |\theta|\geq 1} |\theta|.$$

Thus $M(P) \geq 1$ and it is a fact that $M(P) = 1 \iff P(x)$ is a cyclotomic polynomial.

Of special interest to Lehmer were *palindromic* polynomials (also sometimes called *reciprocal* or *symmetric* polynomials): these are integer polynomials $P(x)$ that satisfy

$$P(x) = x^m P(1/x),$$

where m is the degree of $P(x)$. Equivalently, palindromic polynomials are polynomials with integer coefficients that read the same whether read from left to right or vice versa. For an irreducible palindromic polynomial of degree $m > 1$, m must be even, since odd palindromic polynomials always have -1 as a root. In [22], Lehmer found the monic palindromic polynomials of degrees 2, 4, 6, and 8 with smallest Mahler measure. For degree 10 and higher, the best polynomial Lehmer could find was

$$L(x) \;=\; x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1, \tag{1}$$

now referred to as the *Lehmer polynomial*. This polynomial still stands today as the monic polynomial, palindromic or not,[1] with smallest Mahler measure:

$$M(L) = 1.17628\ldots$$

Let us call the minimal polynomial of a Salem number, ϵ, a *Salem polynomial*. Its Mahler measure is clearly just ϵ. It is not too difficult to check that Salem polynomials are always palindromic.

The Lehmer polynomial $L(x)$ is a Salem polynomial: it is the minimal polynomial of the Salem number $\epsilon_L = 1.17628\cdots = M(L)$. Thus ϵ_L is both the smallest known Salem number, as well as the smallest known Mahler measure.

There is already a vast literature on Salem numbers. These numbers arose in the study of Fourier analysis and uniform distributions [29]. The reader can find a convenient survey of some of the rather remarkable properties of Salem numbers, as well as those of the closely related Pisot-Vijayraghavan numbers, in [5] and [2]. Here too the reader will find a more detailed discussion of certain aspects of the minimization problem.

The purpose of this note is to expose some of the *more recent* appearances that Salem numbers, and more generally, Mahler measures, have made in various settings in arithmetic and

[1] Smyth [34] has meanwhile shown that $1.32471\ldots$, or the unique real root of $S(x) = x^3 - x - 1$, is a lower bound for the set of Mahler measures of *non-palindromic* polynomials. Note that $1.32471\ldots > 1.17628\ldots$.

geometry.[2] We have found that the topics we have chosen to describe are not as well known as they ought to be, and we have thought it important to collect here, in one place, some of this work that spreads across the authors' disciplines. We expect that different parts of this article will benefit our readers in different ways. Indeed, we hope that the inter-disciplinary viewpoint we have adopted will shed a fresh perspective on the minimization problem.

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2 Overview of the paper

Here is a brief overview of the contents of this paper.

Sections 3.1 to 3.5 describe a relation between the minimization problem for Salem numbers and a minimization problem for the lengths of closed geodesics on arithmetic hyperbolic surfaces, i.e. hyperbolic surfaces attached to arithmetic Fuchsian groups. This material was worked out by the authors, but it is certainly not new (see for instance Neumann and Reid [24]). An essentially equivalent formulation of these ideas in terms of lattices in semisimple real Lie groups was first worked out by Sury in [39]. We recall this briefly in Section 3.6.

Section 4 is more speculative. Motivated by the Selberg trace formula, and recent work of Katz-Sarnak [20] and Rubenstein [28] on the distribution of zeroes of certain zeta functions, we raise some questions, whose answers may be relevant in solving the minimization problem.

Interestingly enough, Salem numbers also arise naturally from non-arithmetic Fuchsian groups. In fact, Floyd and Plotnick [15] showed that some Salem numbers, including ϵ_L above, are equal to the asymptotic growths (see the body of the paper for definitions!) of certain hyperbolic planar reflection groups. The smallest Salem number that arises in this way is again ϵ_L, as shown by the second author in [18]. These matters are described in Section 5.1.

Sometimes the hyperbolic reflection groups of Section 5.1 are isomorphic to arithmetic Fuchsian groups, and so yield Salem numbers, as described in Section 3. In Section 5.2 we give an example to show that these Salem numbers do not seem to be connected in any obvious way to the Salem numbers of Section 5.1.

The curious, though mostly unexplored, role that Alexander polynomials may play in the study of Salem numbers and the minimization problem, is discussed in Section 5.3. The second author has shown [18] that the Salem polynomials, arising, as in Section 5.1, from polygonal reflection groups, are the Alexander polynomials of certain pretzel knots. A corollary is that the minimization problem has been solved for this family of polynomials.

In Section 6 we explain how Salem numbers and Mahler measures show up in the special values of L-functions. We start, in Section 6.1, with an observation of Chinburg [9], who noticed that Salem numbers turn out to be related to Stark units, and so are connected to the values of certain (abelian) Artin L-functions at $s = 0$. Then, in Section 6.2, we describe work and conjectures of Costa, Friedman and Skoruppa [16] on the ratios of regulators of extensions of number fields, and its relation to the minimization problem.

Pursing this theme further, we discuss, in Section 6.3, some recent striking results of Smyth, Ray, Boyd, Deninger, Bornhorn and Rodriguez Villegas, that show that the special values of more general L-functions are connected to the logarithms of Mahler measures of polynomials in many variables. The possible implications for the minimization problem are discussed following Boyd in [6].

For completeness we conclude with Section 7, where we review some of the sharpest results (known to us) in the direction of solving the minimization problem. These results have been

[2]For connections with dynamical sytems, we refer the reader to the recent monographs [31] and [14], as well as [11]. The jumping of point here is that the entropy of certain dynamical systems attached to polynomials equals the Mahler measure of these polynomials.

established through elementary, though ingenious, number theoretic reasoning. If these methods do eventually yield a solution to the minimization problem, then, reversing our perspective, we see that they will also enrich some of the diverse areas of mathematics described above.

3 Geodesics on arithmetic hyperbolic surfaces

In this section we show that the minimization problem for Salem numbers is equivalent to the minimization problem for the lengths of closed geodesics on arithmetic hyperbolic surfaces.

3.1 Salem extensions

We start by describing what the number field generated by a Salem number looks like.

Recall that a *number field* L is a field extension of $\mathbb{Q}$, of finite degree over $\mathbb{Q}$. We may write $L = \mathbb{Q}(\theta)$ for an element $\theta \in L$ which generates L over $\mathbb{Q}$. Consequently we may identify L with $\mathbb{Q}[x]/P(x)$, where $P(x) \in \mathbb{Z}[x]$ is the minimal polynomial of θ. Then the *degree* of L is defined to be the degree of the polynomial $P(x)$, and it is independent of the choice of the generator θ above.

If ω is any root of $P(x)$ (ω is called a *conjugate* of θ), then there is an isomorphism from L to $\mathbb{Q}(\omega)$ given by sending x to ω. Thus, each root ω of $P(x)$ defines a $\mathbb{Q}$-algebra monomorphism

$$\sigma_\omega : L \xrightarrow{\simeq} \mathbb{Q}(\omega) \subset \mathbb{C}.$$

Composition by complex conjugation yields another (possibly identical) embedding

$$\overline{\sigma_\omega} \quad (= \sigma_{\overline{\omega}})\,.$$

A pair of such embeddings $\{\sigma, \overline{\sigma}\}$ is called an *infinite place* of L. An infinite place is *real* if $\sigma = \overline{\sigma}$, otherwise it is *complex.* In the former case we set $L_\sigma = \mathbb{R}$, and in the latter case we set $L_\sigma = \mathbb{C}$. If all the infinite places of L are real, then we say L is a *totally real* number field.

Since $\mathbb{C}$ has a natural topology, each of the infinite places of L induce (distinct) topologies on L. However there are other topologies on L, the so called archimedean topologies. The natural embeddings of L into the $\mathfrak{q}$-adic completions of L, as $\mathfrak{q}$ varies through the prime ideals of L,

$$\sigma_{\mathfrak{q}} : L \to L_{\mathfrak{q}},$$

induce such topologies on L. For this reason, we will refer to the primes $\mathfrak{q}$, and sometimes even to the corresponding embeddings $\sigma_{\mathfrak{q}}$, as the *finite places* of L. It is a fact that the finite and infinite places together account for all the mutually distinct topologies that one can put on L.

Now let ϵ be a Salem number, as defined in the Introduction. Let $K = \mathbb{Q}(\epsilon)$ be the number field generated by ϵ. We identify K with $\mathbb{Q}[x]/P(x)$ where $P(x)$ is the minimal polynomial of ϵ. As mentioned already in the Introduction, the degree of K must be even, since $P(x)$ is irreducible and palindromic. Say this degree is $2n$.

Note that all the conjugates of ϵ, except for $\frac{1}{\epsilon}$, which is also real, lie *on* the unit circle. This is because if any complex root z of $P(x)$ were to lie strictly within the unit circle, then the root $\frac{1}{z}$ would lie outside it, contradicting the definition of ϵ. Thus a list of the conjugates of ϵ can be written as

$$\epsilon, \frac{1}{\epsilon}, z_1, \overline{z_1}, \dots, z_{n-1}, \overline{z_{n-1}},$$

where $|z_i| = 1$, for $i = 1, \dots, n-1$. A plot of the set of conjugates of a typical Salem number ϵ is given in Figure 3.1.

By the general discussion above we see that K has $n-1$ complex places, say $\tau_1, \dots, \tau_{n-1}$, corresponding to the roots $z_1, \dots, z_{n-1}$, and two real places, say τ and τ', corresponding to ϵ and $1/\epsilon$ respectively.

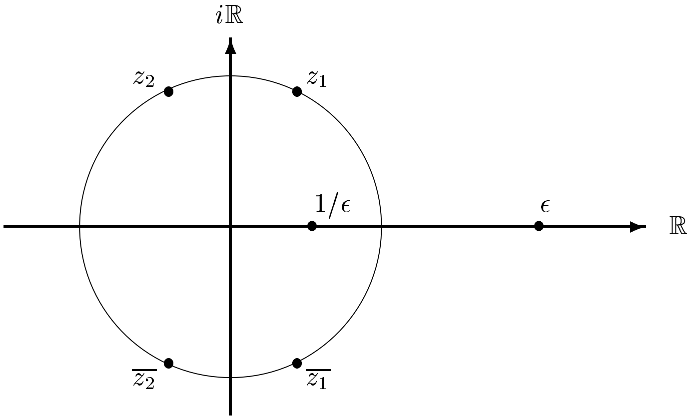

Figure 1: A (degree six) Salem number ϵ and its conjugates

Since $P(x)$ is palindromic, we may further write

$$P(x) = x^n Q(x + \frac{1}{x}),$$

for some monic polynomial $Q(x) \in \mathbb{Z}[x]$ of degree $n \geq 2$ (recall that a Salem number has at least one conjugate on the unit circle). We see that $Q(x)$ has n real roots, namely $\alpha = \epsilon + \frac{1}{\epsilon}$, and $\alpha_i = z_i + \overline{z_i}$, for $i = 1, \ldots, n-1$. Set $k = \mathbb{Q}(\alpha)$, where $\alpha = \epsilon + \frac{1}{\epsilon}$ is the unique (real) root of $Q(x)$ larger than 2. Then k is a number field of degree n.

Note that k is a totally real field. Let σ denote the real place of k corresponding to α, and let $\sigma_1, \ldots \sigma_{n-1}$, denote the other real places of k, corresponding to $\alpha_1, \ldots, \alpha_{n-1}$.

The places of K and k are related. Namely τ and τ' *lie over* the place σ of k. This means that $\sigma(k) \subset \tau(K)$ and $\sigma(k) \subset \tau'(K)$. Similarly the places $\tau_1, \ldots, \tau_{n-1}$ of K lie over the places $\sigma_1, \ldots, \sigma_{n-1}$ of k.

Note that ϵ is a root of the polynomial

$$f(x) = x^2 - \alpha x + 1.$$

Conversely, for any real algebraic integer $a > 2$ (of degree larger than 2) whose conjugates are all real and lie in (-2,2), the root e larger than 1 of the polynomial

$$f(x) = x^2 - ax + 1,$$

is a Salem number.

We see that $K = k(\epsilon)$ is a quadratic extension of k. In the sequel we will refer to the quadratic extension of number fields K/k as a *Salem extension.*

Now let L be an *arbitrary* totally real field of degree $d \geq 2$. We say that a Salem number ϵ is a *Salem number over* L if the totally real field $k = \mathbb{Q}(\alpha)$ corresponding to ϵ equals L. Let $\mathcal{S}(L)$ denote the set of all Salem numbers over L.

Let ι be a real place of L. If, in addition, the embedding σ of k coincides with the embedding ι of L, then we say ϵ is a Salem number over L *of type* ι. Let $\mathcal{S}(L, \iota)$ denote the set of all Salem number over L of type ι. Clearly we have

$$\mathcal{S}(L) \quad = \quad \bigcup_{\iota} \mathcal{S}(L, \iota),$$

as ι varies through all the infinite places of L.

Lemma 3 *The set $\mathcal{S}(L)$ has a smallest element.*

Proof. It suffice to show that each $\mathcal{S}(L,\iota)$ has a smallest element. But, after the discussion above, we see that there is a one to one correspondence between the elements of $\mathcal{S}(L,\iota)$, and the set of $\alpha \in L$ which satisfy

- α generates L (that is α has degree d),
- $\iota(\alpha) > 2$, and,
- the conjugates of α satisfy $|\iota'(\alpha)| < 2$, for the remaining infinite places $\iota' \neq \iota$ of L.

Thus one must hunt for the smallest $\alpha \in L$ satisfying the above conditions. Since $\mathcal{O}_L$, the ring of integers of L, embeds in $\mathbb{R}^d$ in the usual manner as a lattice, we are guaranteed the existence of such an α. Then we simply take

$$\epsilon = \frac{\alpha + \sqrt{\alpha^2 - 4}}{2}.$$

This ϵ is the smallest element of the set $\mathcal{S}(L,\iota)$.

■

An alternative proof of this easy fact, which uses hyperbolic geometry, will be given later when the degree d of L is odd (see Corollary 10 below).

3.2 Arithmetic Fuchsian groups

A group Γ is a *Fuchsian group of the first kind* if Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ and the quotient $\Gamma\backslash H$ of the upper half plane H by Γ has finite volume.

An *arithmetic Fuchsian group* is a Fuchsian group of the first kind which can be constructed out of a totally real number field in a manner that we shall describe in this section. More detailed treatments can be found in [33] and [44].

Let L be any number field. A *central simple quaternion algebra over* L is an associative L-algebra D with unit, such that

- the center of D is L,
- the only two sided ideals of D are the trivial ideal and D itself, and,
- the dimension of D over L is 4.

It is a consequence of a general theorem of Wedderburn [19], that any such algebra is either a *division algebra* (that is D has no zero divisors), or that D is isomorphic to $\mathrm{M}_2(L)$, the algebra of 2 by 2 matrices over L.

D comes equipped with the *reduced trace* and *reduced norm* maps:

$$\mathrm{red\,Tr} : D \to L, \quad \text{and,} \quad \mathrm{red\,Nm} : D^\times \to L^\times,$$

defined as follows. Fix an extension F of L that *splits* D. This means there is an isomorphism

$$h : D \otimes_L F \quad \xrightarrow{\sim} \quad \mathrm{M}_2(F).$$

It is a fact that such an F always exists. Now define

$$\mathrm{red\,Tr}(x) = \mathrm{trace}\, h(x) \;\text{ and }\; \mathrm{red\,Nm}(x) = \det h(x).$$

One checks that, so defined, red Tr and red Nm take values in $L \subset F$, and that their definition is independent of the choice of F and h.

We say that D is *unramified* or *split* at a place

$$\sigma : L \hookrightarrow L_\sigma$$

of L, which may be either finite or infinite, if the tensor product,

$$D \otimes_{L,\sigma} L_\sigma,$$

viewed as a central simple quaternion algebra over L_σ, is isomorphic to $\mathrm{M}_2(L_\sigma)$. Otherwise (Wedderburn again!) $D \otimes_{L,\sigma} L_\sigma$ is a division algebra, and in this case, we say that D is *ramified* at σ.

If σ is a complex place, then D is necessarily split at σ. If σ is a real place, then D may ramify at σ, in which case

$$D \otimes_{L,\sigma} \mathbb{R} \simeq \mathbb{H}$$

is the usual Hamilton algebra of quaternions over $\mathbb{R}$.

An important property of the quaternion algebras D in this context is that every such D is ramified at an *even* number of non-complex places. Conversely, for any finite collection Σ of non-complex places of L, of even cardinality, there is a *unique* (up to isomorphism) central simple quaternion algebra D over L which is ramified at exactly the places in Σ. In particular, if D is unramified everywhere ($\Sigma = \emptyset$), then D must be isomorphic to $\mathrm{M}_2(L)$.

Remark. Much of the above discussion can be described very elegantly by the following exact sequence from class field theory (see for example [33], Chapter 10 or [1]):

$$0 \to \mathrm{Br}(L) \to \bigoplus_\sigma \mathrm{Br}(L_\sigma) \to \mathbb{Q}/\mathbb{Z} \to 0, \tag{2}$$

where the sum runs over all places σ, both finite and infinite, of L. Here $\mathrm{Br}(L)$, respectively $\mathrm{Br}(L_\sigma)$, denotes the Brauer group of L, respectively of L_σ. We make this remark simply to orient the more knowledgeable reader. In essence all the information contained in (2) that we shall need in the sequel, has already been discussed above in words.

Now suppose that L is totally real, with infinite places $\iota_1, \ldots, \iota_d$. Fix a place ι ($= \iota_j$ for some $j = 1, \ldots, d$) of L. We would like to impose the following condition on the 'ramification of D at ∞'. We assume that D is ramified at all the infinite places except ι. That is, we assume that

$$D \otimes \mathbb{R} \quad = \quad \textstyle\bigoplus_{r=1}^d D \otimes_{L,\iota_r} \mathbb{R} \quad = \quad \mathbb{H} \oplus \cdots \oplus \mathbb{H} \oplus \mathrm{M}_2(\mathbb{R}) \oplus \mathbb{H} \oplus \cdots \oplus \mathbb{H}, \tag{3}$$

with the single summand $\mathrm{M}_2(\mathbb{R})$ corresponding to the place ι, and $d-1$ copies of the Hamilton algebra $\mathbb{H}$ corresponding to the other places.

An *order* $\mathcal{O}$ in D is a subring of D, such that

- $\mathcal{O}$ contains the ring $\mathcal{O}_L$ of integers of L, and,
- the $\mathcal{O} \otimes_{\mathcal{O}_L} L = D$.

A *maximal order* is an order which is not properly contained in any other. Let $\mathcal{O}$ be a maximal order in D and let U be the set of unit elements of $\mathcal{O}$ of reduced norm one.

When L is totally real, and D has ramification at ∞ given by (3), we let $\Gamma(D, \mathcal{O}, \iota) \subset \mathrm{SL}_2(\mathbb{R})$ be the image of U under the embedding ι fixed above. It turns out that $\Gamma(D, \mathcal{O}, \iota)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, and that the volume of the quotient surface $\Gamma(D, \mathcal{O}, \iota)\backslash H$ is finite. Thus $\Gamma(D, \mathcal{O}, \iota)$ is a Fuchsian group of the first kind. If the defining field L is not equal to $\mathbb{Q}$, then the quotient surface is also compact.

We call $\Gamma(D, \mathcal{O}, \iota)$ an *arithmetic Fuchsian group.*[3] The corresponding quotient surface $\Gamma(D, \mathcal{O}, \iota)\backslash H$ is called an *arithmetic hyperbolic surface.*

[3]More generally, any Fuchsian group of the first kind *commensurable* with $\Gamma(D, \mathcal{O}, \iota)$, that is one which admits a homomorphism to $\Gamma(D, \mathcal{O}, \iota)$ with finite kernel and co-kernel, is also called an *arithmetic Fuchsian group*, but we will not be using this more general form of the definition until later in Section 5.2.

We note that up to conjugation, $\Gamma(D, \mathcal{O}, \iota)$ does not depend on the choice of $\mathcal{O}$. Indeed although there is more than one maximal order in D, it is a fact that, since D has ramification, all its maximal orders are conjugate to each other, and hence the corresponding Fuchsian group is determined up to inner automorphism of $\mathrm{SL}_2(\mathbb{R})$.

In view of the remarks at the beginning of this section, if the degree d of L is odd, there is a unique D (up to isomorphism) which is unramified at all the finite places, and is ramified at all infinite places except at ι. We will denote this D by D_ϕ. On the other hand, if d is even, then a choice of a finite place p of L will again give a canonical choice of D (up to isomorphism), which is unramified at all the finite places except p and ramified at all the infinite places except ι. We shall denote this D by D_p, to show that it depends on the prime p of L. The corresponding arithmetic Fuchsian groups are denoted by $\Gamma(D_\phi, \mathcal{O}, \iota)$ and $\Gamma(D_p, \mathcal{O}, \iota)$ respectively.

3.3 Salem numbers and hyperbolic matrices

In this section we show how Salem numbers are closely related to the eigenvalues of hyperbolic elements in arithmetic Fuchsian groups.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be an arbitrary Fuchsian group of the first kind. An element $\gamma \in \Gamma$ is said to be *hyperbolic* if its eigenvalues are distinct and real. Let $f_\gamma(x)$ be the characteristic polynomial of γ. Then since $\det(\gamma) = 1$, we have

$$f_\gamma(x) = x^2 - \mathrm{tr}(\gamma)x + 1.$$

Thus, γ is hyperbolic if and only if $|\mathrm{tr}(\gamma)| > 2$. Moreover, in this case the roots of $f_\gamma(x)$ form a pair of positive real reciprocal numbers. Let $\epsilon(\gamma)$ be the root which is larger than 1, and let

$$N(\gamma) = \epsilon(\gamma)^2$$

be the *norm* of $\gamma \in \Gamma$.

Now suppose that L is a totally real field of degree $d \geq 2$, and ι is an infinite place of L. Let $\Gamma = \Gamma(D, \mathcal{O}, \iota)$ be an arithmetic Fuchsian group, for some central simple quaternion division algebra D over L, and some maximal order $\mathcal{O}$ of D, as constructed in Section 3.2. We have the following proposition:

Proposition 4 *If $\gamma = \iota(u) \in \Gamma(D, \mathcal{O}, \iota)$ is not equal to the identity or its negative, then the conjugates of* $\mathrm{red\,Tr}(u) \in L$ *lie in the interval* $(-2, 2)$.

Proof. By construction (see Section 3.2) the conjugates $\iota'(u)$ of u (for $\iota' \neq \iota$) lie in the group of norm one units of the Hamilton quaternion algebra over $\mathbb{R}$. Thus, they are of the form

$$\iota'(u) = a + bi + cj + dk,$$

with a, b, c and $d \in \mathbb{R}$, and with

$$\mathrm{norm}(\iota'(u)) = a^2 + b^2 + c^2 + d^2 = 1.$$

Since a is real, we see that $|a| \leq 1$, and hence that $|\iota'(\mathrm{red\,Tr}(u))| = |\mathrm{trace}(\iota'(u))| = |2a| \leq 2$. If $|a| = 1$, we have $\iota'(u) = \pm 1$, and hence u, and therefore γ, is either the identity or its negative. The proposition follows.

■

Proposition 5 *Let $\Gamma = \Gamma(D, \mathcal{O}, \iota)$ be an arithmetic Fuchsian group as above, and let $\gamma \in \Gamma$ be a hyperbolic element. Then $\epsilon(\gamma)$ is a Salem number.*[4]

[4]By this we mean that $\epsilon(\gamma)$ is the image of a Salem number ϵ under the unique real infinite place of $\mathbb{Q}(\epsilon)$ which realizes ϵ as a real number larger than 1.

Proof. Assume that $\gamma = \iota(u)$. By Proposition 4 above, $\alpha = \text{red}\,\text{Tr}(u)$ is an integer in L, with $\iota(\alpha) > 2$, and with all other conjugates lying in $(-2, 2)$. Thus, since $\epsilon(\gamma)$ is the larger root of

$$f_\gamma(x) = x^2 + \iota(\alpha)x + 2,$$

$\epsilon(\gamma)$ is a Salem number.

■

Let us define $\mathcal{S}(\Gamma(D, \mathcal{O}, \iota))$ to be the set of those Salem numbers ϵ satisfying:

- ϵ is the larger eigenvalues of a hyperbolic element in $\Gamma(D, \mathcal{O}, \iota)$,[5] and,
- the degree of ϵ is $2d$, that is, $\alpha = \epsilon + 1/\epsilon$ generates L.

Note that the set $\mathcal{S}(\Gamma(D, \mathcal{O}, \iota))$ does not depend on the choice of the maximal order $\mathcal{O}$.

In Section 3.1 we had defined $\mathcal{S}(L, \iota)$ as the set of all Salem numbers over L of type ι. Proposition 5 shows that

$$\mathcal{S}(\Gamma(D, \mathcal{O}, \iota)) \subset \mathcal{S}(L, \iota).$$

We now ask whether it is possible to capture the entire set $\mathcal{S}(L, \iota)$ by *one* division algebra. When the degree d of L is odd, this is indeed possible:

Proposition 6 *Assume that d is odd, and that D_ϕ is defined as in the end of Section 3.2. Then*

$$\mathcal{S}(L, \iota) = \mathcal{S}(\Gamma(D_\phi, \mathcal{O}, \iota)).$$

Proof. One containment has been shown. To show the other, let $\epsilon \in \mathcal{S}(L, \iota)$. Let K/k be the corresponding Salem extension (see Section 3.1 for the definition of a Salem extension, and for notation that we shall use below). We have $k = L$, $n = d$, and $\sigma = \iota$. We claim that K is isomorphic to a maximal subfield of D_ϕ. A general theorem about division algebras show that it suffices to check that K splits D_ϕ. By the remarks in Section 3.2 applied to K, we see that it is enough to check that each completion $K_\wp$ of K (finite or infinite) splits $D_\phi \otimes_k K$. But

$$D_\phi \otimes_k K \otimes_K K_\wp = D_\phi \otimes_k k_p \otimes_{k_p} K_\wp,$$

where $p = \wp \cap \mathcal{O}_k$ is the prime of k lying under the prime $\wp$ of K. By hypothesis, D_ϕ is already split by k_p for finite p, so there is nothing to check at the finite places. At the infinite places, a similar phenomena occurs at σ. As for the other infinite places σ_i $(i = 1, \ldots, n-1)$, we just note that the Hamiltonian algebra is split by the complex numbers.

Thus, K splits D_ϕ, and so the Salem extension K/k sits inside D. Since the Salem number ϵ generates K, it gives an element $\gamma_\epsilon \in \Gamma(D_\phi, \mathcal{O}, \iota)$. Note that γ_ϵ is hyperbolic, since its trace is just $\sigma(\text{tr}_{K/k}(\epsilon)) = \sigma(\epsilon + \frac{1}{\epsilon}) = \sigma(\alpha) > 2$. Clearly the larger eigenvalue $\epsilon(\gamma_\epsilon)$ of γ_ϵ is just $\tau(\epsilon)$. Finally the degree of ϵ is $2d$. This shows the other inclusion.

■

When the degree d of L is even, $\mathcal{S}(L, \iota)$ may not be equal to $\mathcal{S}(\Gamma(D, \mathcal{O}, \iota))$ for one particular D. Before we see this, let us first remind the reader of some terminology. Let K/k be an arbitrary quadratic extension of number fields. A prime ideal p of k is said to be *inert* in K if the ideal $\wp$ generated by p in $\mathcal{O}_K$ is a prime ideal of K. Also, p is said to *split* in K if the ideal generated by p in $\mathcal{O}_K$ factors as a product of two distinct prime ideals of K.

Let us now assume that $\epsilon \in \mathcal{S}(L, \iota)$. Let K/k denote the Salem extension corresponding to ϵ. Fix a division algebra D over L with ramification as in (3). Let $\Sigma_f(D)$ denote the set of finite places of D where D is ramified. Recall that $\Sigma_f(D)$ is a finite set of odd cardinality. The following proposition gives a criterion to check whether ϵ is the (larger) eigenvalue of a hyperbolic matrix in $\Gamma(D, \mathcal{O}, \iota)$.

[5] Here, and in the sequel, we will sometimes identify Salem numbers with their images under the embeddings which realize them as real numbers larger than 1. See the previous footnote.

Proposition 7 *Assume that d is even.*

1. *If for each $p \in \Sigma_f(D)$, p is inert in K, then K sits inside D.*
2. *If K sits inside D, then for all $p \in \Sigma_f(D)$, p does not split in K.*

Proof. The proof of the first statement of Proposition 7 is exactly the same as the proof of Proposition 6, except that, additionally, for each $p \in \Sigma_f$, we have to check the splitting of $D_p \otimes_k K$ at $K_\wp$, where $\wp$ is the prime of K generated by p. But since

$$D_p \otimes_k K \otimes_K K_\wp = D_p \otimes_k k_p \otimes_{k_p} K_\wp,$$

it suffices to show that the extension $K_\wp/k_p$ splits $D_p \otimes_k k_p$. But this follows from the general fact that over a local field, a quaternion algebra is always split by the unique unramified extension of degree 2 (see [33], Chapter 12).

For the second statement, fix $p \in \Sigma_f$. If $p = \wp\wp'$ splits in K and K sits in D, then $D \otimes_k k_p$ would contain the subalgebra $K \otimes_k k_p = K_\wp \times K_{\wp'}$ which has zero divisors. This is a contradiction!

■

Proposition 7 shows that, when the degree d of L is even, we may not be able to find a single, or even finitely many, division algebras D which account for all Salem numbers over L (of type ι). We must be content with the following result:

Corollary 8 *Assume d is even, and let D_p be the division algebra over L defined as in the end of Section 3.2. Then*

$$\mathcal{S}(L,\iota) = \bigcup_p \mathcal{S}(\Gamma(D_p, \mathcal{O}, \iota)) = \bigcup_D \mathcal{S}(\Gamma(D, \mathcal{O}, \iota)),$$

where p varies through all finite places of L, and D varies over all division algebras over L, which is ramified at exactly one infinite place.

Proof. Let $\epsilon \in \mathcal{S}(L,\iota)$, and let K/k denote the corresponding Salem extension. Choose a prime p of k such that p is inert in K. Then, by Proposition 7, $\epsilon \in \mathcal{S}(\Gamma(D_p, \mathcal{O}, \iota))$. All the other containments have already been shown or are obvious.

■

3.4 Closed geodesics on hyperbolic surfaces

In this section we set up a bijection between hyperbolic elements in cocompact Fuchsian groups Γ, and closed geodesics on the quotient surface $\Gamma\backslash H$. In the next section we will apply this to the cocompact arithmetic Fuchsian groups $\Gamma(D, \mathcal{O}, \iota)$ constructed from totally real fields $L \neq \mathbb{Q}$ as in Section 3.2.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be an arbitrary Fuchsian group of the first kind. Recall that we had called a matrix $\gamma \in \Gamma$ hyperbolic $\iff |\mathrm{tr}(\gamma)| > 2$. We say that a matrix $\gamma \in \Gamma$ is *elliptic*, respectively *parabolic*, if $|\mathrm{tr}(\gamma)| < 2$, respectively $|\mathrm{tr}(\gamma)| = 2$. It is an easily checked fact (cf. [44], page 115) that, as a transformation of $\mathbb{C} \cup \infty$,

$$\begin{aligned}
\gamma \text{ is hyperbolic} \quad &\iff \quad \gamma \text{ has two distinct fixed points in } \mathbb{R} \cup \infty, \\
\gamma \text{ is elliptic} \quad &\iff \quad \gamma \text{ has two distinct fixed points of the form } z, \overline{z}, \text{ for } z \in \mathbb{C}, \text{ and}, \\
\gamma \text{ is parabolic} \quad &\iff \quad \gamma \text{ has a unique fixed point in } \mathbb{R} \cup \infty.
\end{aligned}$$

Let us now assume that Γ is *cocompact*, that is, the quotient surface $\Gamma\backslash H$ is compact. This is eqivalent to assuming that Γ does not have any parabolic elements.

The elliptic elements in Γ are precisely the elements of finite order in Γ. Thus, if in addition Γ does not have any such elements, then $\Gamma\backslash H$ is a smooth surface, that is, it is a compact *Riemann surface*. Otherwise it is simply a compact *orbifold*, whose singularities are in bijection with the set of elliptic elements.

Proposition 9 *Let Γ be a cocompact Fuchsian group. Then there is a bijection between the hyperbolic elements $\gamma \in \Gamma$ and closed geodesics $g(\gamma)$ on $\Gamma\backslash H$. In this correspondence, the length of $g(\gamma)$ is $\log N(\gamma)$.*

Proof. Say $\gamma \in \Gamma$ is hyperbolic. Then γ has two distinct fixed points on $\mathbb{R} \cup \infty$. Let G be the geodesic in H which joins them. Then, it is a fact that for each point P on G, the point $\gamma(P)$ also lies on G, and, some simple integration shows that the distance between P and $\gamma(P)$ is $\log N(\gamma)$ (see [44], Proposition 2.8, for more details). Let us call $G(\gamma, P)$ the (open) geodesic joining P and $\gamma(P)$ in H. This projects down to a closed geodesic $g(\gamma)$ on $\Gamma\backslash H$ of length $\log N(\gamma)$, which does not depend on the choice of P.

Conversely, let g be a closed geodesic on $\Gamma\backslash H$, and let G be its preimage in H. Then G is a geodesic on H and its stabilizer under the action of Γ must contain a hyperbolic element γ (see [43]). For this γ, we have $g = g(\gamma)$.

■

3.5 The minimization problem

In this section, we finally show how the minimization problem for Salem numbers is equivalent to the problem of finding a geodesic of minimal length amongst the set of all geodesics on all arithmetic hyperbolic surfaces.

Let us first give another proof of Lemma 3, in Section 3.1 when the degree d of L is odd, using hyperbolic geometry.

Corollary 10 *When $d > 1$ is odd, $\mathcal{S}(L)$ has a minimal element.*

Proof. As usual, it suffices to show that each $\mathcal{S}(L, \iota)$ has a minimal element. By Proposition 6, there is a correspondence between Salem numbers ϵ over L, of type ι, and the larger eigenvalues of hyperbolic elements $\gamma \in \Gamma = \Gamma(D_\phi, \mathcal{O}, \iota)$ having trace $\iota(\epsilon + 1/\epsilon) > 2$, with $\epsilon + 1/\epsilon$ generating L.

Since $L \neq \mathbb{Q}$, the arithmetic Fuchsian group Γ is cocompact. By Proposition 9, each $\gamma \in \Gamma$ as above yields a closed geodesic of length $\log N(\gamma) = 2\log\epsilon$ on the compact surface $\Gamma\backslash H$. It is known that, amongst such geodesics, there is one of *minimal length*, and it corresponds to the minimal Salem number over L, of type ι.

■

When the degree of L is even we may not immediately deduce anything about the existence of a minimal Salem number over L using hyperbolic geometry. However, since Lemma 3 holds unconditionally, we may invert our viewpoint to deduce:

Corollary 11 *Assume that the degree d of L is even. Then there is a geodesic of minimal length amongst all closed geodesics on all arithmetic hyperbolic manifolds coming from division algebras over L.*

Proof. This follows as above from Corollary 8 and Proposition 9.

■

Now suppose that we allow L to vary through ALL totally real fields (of degree > 1). The discussion above shows that the problem of the existence of a minimal Salem number (cf. Problem 1) is actually equivalent to the following conjecture:

Conjecture 12 (Minimization problem for geodesics) *There is a geodesic of minimal length amongst all closed geodesics on* ALL *arithmetic hyperbolic surfaces.*

It seems that settling Conjecture 12 either way, is just as difficult a problem as Problem 1, even though the corresponding statement for *volumes* of arithmetic hyperbolic surfaces is known. However, as we shall see in Section 7, this conjecture is true if the set of possible L under consideration is restricted to the set of totally real number fields of a *fixed degree*.

3.6 Cocompact lattices

In this section we present a simple reformulation of the previous sections in terms of lattices in semisimple Lie groups.

Theorem 13 (Sury [38], [39]) *The set of Salem numbers is bounded away from 1 if and only if there is a neighborhood of the identity element $U \subset \mathrm{SL}_2(\mathbb{R})$, such that for all arithmetic Fuchsian groups Γ, the intersection $\Gamma \cap U$ consists only of elements of finite order.*

Proof. We have already seen that the problem of minimizing Salem numbers ϵ is equivalent to the problem of minimizing the traces $\epsilon + 1/\epsilon$ of hyperbolic elements in arithmetic Fuchsian groups. Since hyperbolic matrices have infinite order, the existence of the neighborhood U is equivalent to the statement that the set of these traces is bounded away from 1.

■

It has been pointed out to us[6] that a positive answer to (the more general) Problem 2 is implied by the truth of the following conjecture:

Conjecture 14 (Margulis [23]) *Let G be a connected semisimple group over $\mathbb{R}$. Suppose that $rank(G) \geq 2$. Then there is a neighborhood $U \subset G(\mathbb{R})$ of the identity such that for any irreducible cocompact lattice $\Gamma \subset G(\mathbb{R})$, the intersection $\Gamma \cap U$ consists only of elements of finite order.*

We refer the reader to [23] (especially page 322) for the definitions of terms used in this conjecture, and for its connection with Problem 2.

4 Selberg trace formula and zeroes of zeta functions

In Section 3 we saw that the minimization problem for Salem numbers is equivalent to the minimization problem for geodesics on arithmetic hyperbolic surfaces, or equivalently, to the minimization problem for the traces of hyperbolic matrices arising from arithmetic Fuchsian groups in $\mathrm{SL}_2(\mathbb{R})$. Yet, it must be admitted, that these re-formulations of Problem 1 have not brought us much closer to solving it.

In this speculative section, we propose that studying the distribution properties of Salem numbers may lead to some information about Problem 1. In formulating the questions we ask below, we were motivated by two considerations. The first is the well known Selberg trace formula, which relates the 'length spectrum' of an arithmetic hyperbolic surface to the 'eigenvalue spectrum' of the Laplacian. The second is some recent work of Rubenstein and Katz-Sarnak on the distribution properties of zeroes of certain zeta functions.

Let us describe the Selberg trace formula fist. We follow Hejhal [17]. Fix an arithmetic Fuchsian group Γ as in Section 3.2. Assume, for simplicity, that the corresponding arithmetic

[6] We thank T. N. Venkataramana for informing us about the conjecture in [23].

hyperbolic surface $\Gamma\backslash H$ is smooth. That is, assume Γ has no elements of finite order, so that $\Gamma\backslash H$ is a (compact) Riemann surface (of genus ≥ 2).

Let $d\mu(z) = \frac{dxdy}{y^2}$ denote the volume form on $\Gamma\backslash H$, and let $L_2(\Gamma\backslash H)$ denote the Hilbert space of smooth function on $\Gamma\backslash H$, with respect to the inner product

$$(f,g) \quad = \quad \int_{\Gamma\backslash H} f(z)\overline{g(z)}d\mu(z).$$

The Laplacian $\nabla = y^2(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2})$ acts on $L_2(\Gamma\backslash H)$, and we denote its eigenvalues by:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots .$$

This allows us to decompose $L_2(\Gamma\backslash H)$ discretely as

$$L_2(\Gamma\backslash H) = \bigoplus_{n=0}^{\infty} < \phi_n >,$$

where $\nabla\phi_n = \lambda_n\phi_n$.

Now define r_n by

$$\lambda_n = \frac{1}{4} + r_n^2,$$

where we assume that $\mathrm{Arg}(r_n) = 0$ or $-\pi/2$.

Let $h(r)$ denote any analytic function on the strip in the complex plane defined by

$$\{|\mathrm{Im}(r)| \leq \frac{1}{2} + \delta\},$$

where $\delta > 0$ is small. Assume that

- $h(-r) = h(r)$, and,
- $|h(r)| \leq \frac{A}{(1+r)^{2+\delta}}$, for some $A > 0$.

Also let $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-irt}dr$ denote the Fourier transform of $h(r)$.

Then we have (cf. Hejhal [17], Theorem 5.4):

Theorem 15 (Selberg Trace Formula)

$$\sum_{n=1}^{\infty} h(r_n) = \frac{\mu(\Gamma\backslash H)}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\,dr \; + \sum_{\gamma} \frac{\log N(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}}\, g(\log N(\gamma))$$

Here the sum on the right is over a set of representatives γ of the hyerbolic conjugacy classes in Γ. Also, $\gamma = \gamma_0^k$, for some $k \in \mathbb{Z}$, where γ_0 is a *primitive* element, that is $Z_\Gamma(\gamma_0) =< \gamma_0 >$.

Theorem 15 shows us that there is a deep connection between the 'eigenvalue spectrum' and the 'length spectrum' of an arithmetic Fuchsian group Γ. So, to obtain information about the latter, it may profit to study the former. In this context let us recall the definition of the Selberg Zeta function

$$Z(s) = \prod_{\gamma_0}\prod_{k=0}^{\infty}\left(1 - \log N(\gamma_0)^{-(s+k)}\right),$$

for $\mathrm{Re}(s) > 1$. It turns out that $Z(s)$ has the following remarkable property (cf. [17]):

Proposition 16 *The non-trivial zeros of $Z(s)$ are located at $\frac{1}{2} \pm r_n$, for $n = 1, 2, \ldots$.*

Recently Rubenstein [28] and Katz-Sarnak [20] have studied the distribution properties of the zeroes of certain zeta-functions, such as those attached to Dirichlet characters, modular forms and elliptic curves. They also study, and this is more relevant to our purposes, the distribution properties of the lowest lying non-trivial zeroes of families of such zeta-functions. Motivated by these discoveries, the following questions seem natural:[7]

Problem 17 *Do the mean spacings of the zeroes of the Selberg zeta function, follow any distribution laws?*

Problem 18 *Do the lowest lying zeroes of the family of all Selberg zeta functions follow any distribution laws?*

The above questions lead us naturally to similar ones about the distribution properties of the length spectrum. Let us set up some notation.

For each integer d, let ϵ_d denote the smallest Salem number of degree d (as mentioned earlier, we will see in Section 7 that such a Salem number exists). Now set

$$X_d = \{\epsilon_{d'} \mid d' \leq d\}$$

Fix an interval $[a,b] \subset (1,\infty)$, and set $X_d[a,b] = X_d \cap [a,b]$. Then we may ask

Problem 19 *Is there a measure ν on $[a,b]$ such that the sets $X_d[a,b]$ are equidistributed with respect to ν?*

In other words, is there a measure ν such that for each continuous function $g : [a,b] \to \mathbb{R}$,

$$\lim_{d\to\infty} \frac{1}{d} \sum_{x\in X_d[a,b]} g(x) = \int_a^b g(x)\nu(x)?$$

Problem 20 *What would the existence of such a ν bring to bear on the minimization problem?*

5 Non-arithmetic Fuchsian groups and pretzel knots

In this section we show how Salem numbers arise from certain possibly non-arithmetic Fuchsian groups in an entirely different way from that described in Section 3. We also make some connections with knot theory.

5.1 Growth series of Coxeter groups

Let G be any group and S a collection of generators for G. The *growth series for G with respect to S* is the formal power series

$$f(x) = \sum_{n=1}^{\infty} N_S(n)x^n,$$

where $N_S(n)$ is the number of words in G of length n with respect to the set of generators S. The quantity

$$\lim_{n\to\infty} N_S(n)^{1/n},$$

which is the reciprocal of the radius of convergence of $f(x)$, is called the *asymptotic growth* of the group G with respect to the generators S.

A group G is a *Coxeter group*, if it is a group of isometries of the (spherical), hyperbolic or Euclidean plane generated by a finite set of reflections, through (great circles or) hyperplanes. The set S of generating reflections is called the set of *standard generators* of the Coxeter group.

[7] See [30] and the references therein for some numerical evidence pertaining to (at least the first of) these questions.

We will restrict our discussion to the case when the quotient space has finite volume (in the case when the fundamental domain is a polygon). Let $\chi(G)$ be the orbifold Euler characteristic of the quotient space. If $\chi(G) < 0$, G acts on the hyperbolic plane and is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, hence it is a Fuchsian group of the first kind as defined in Section 3.2.

Steinberg [37] showed that if G is a Coxeter group, and S is a standard set of generators then the corresponding growth series $f(x)$ is a rational function of x (cf. [4]). Floyd and Plotnick ([15], page 503), expanding on Cannon's work [7], show that for any group G of Euclidean or hyperbolic planar isometries, the growth series with respect to suitable 'geometric' generators is a reciprocal or anti-reciprocal function:

$$f(x) = \pm f(\frac{1}{x}),$$

up to a factor of $(1-x)$ in some mild exceptional cases. In particular they show that for a Coxeter group (G, S), if $f(x)$ is written as a quotient of relatively prime polynomials, then the denominator $\Delta(x)$ of $f(x)$ is a palindromic polynomial.

Specifically, consider the Coxeter reflection group $G_{p_1,\ldots,p_d}$ generated by reflections through the sides of a spherical, hyperbolic, or planar polygon whose angles are

$$\frac{\pi}{p_1}, \ldots, \frac{\pi}{p_d},$$

where $p_1, \ldots, p_d$ are any positive integers. Then it is not hard to verify that $G_{p_1,\ldots,p_d}$ has the presentation

$$G_{p_1,\ldots,p_d} = \langle\ g_1, \ldots, g_d \mid (g_i)^2 = 1,\ (g_i g_{i+1})^{p_i} = 1\ \rangle.$$

The second set of relations is because $g_i g_{i+1}$ is rotation around a vertex of the polygon by angle

$$\frac{2\pi}{p_i}.$$

Thus, its order is p_i.

The orbifold Euler characteristic of the quotient surface is given by

$$\chi(G_{p_1,\ldots,p_d}) = \frac{1}{p_1} + \cdots + \frac{1}{p_d} - d + 2.$$

When $(p_1, p_2, p_3) = (p, q, r)$, with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, we will call $G_{p,q,r}$ a (p, q, r)-*hyperbolic triangle group*. A picture of the fundamental domain for a (p, q, r)-hyperbolic triangle group is given in Figure 2.

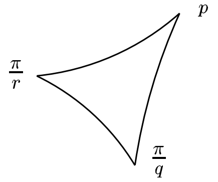

Figure 2. Fundamental domain of a (p, q, r)-hyperbolic triangle group

Let $\Delta_{p_1,\ldots,p_d}(x)$ be the denominator of the growth series $f(x)$ of $G_{p_1,\ldots,p_d}$. For any positive integer n, let

$$[n] = 1 + x + \cdots + x^{n-1}.$$

Theorem 21 (Cannon-Wagreich [7], Floyd-Plotnick [15], Parry [25]) *The polynomial $\Delta_{p_1,\ldots,p_d}(x)$ is given by*

$$\Delta_{p_1,\ldots,p_d}(x) = [p_1]\ldots[p_d](x-d+1) + \sum_{i=1}^{d}[p_1]\ldots\widehat{[p_i]}\ldots[p_d].$$

Furthermore, $\Delta_{p_1,\ldots,p_d}(x)$ is a product of cyclotomic polynomials and at most one Salem polynomial. The Salem polynomial occurs if and only if $G_{p_1,\ldots,p_d}$ is hyperbolic, that is,

$$\frac{1}{p_1} + \cdots + \frac{1}{p_d} < d-2.$$

It follows that the asymptotic growth of a hyperbolic Coxeter reflection group with respect to the standard generators is a Salem number!

The polynomials with smallest Mahler measure of degrees 2, 4, 6 and 8 (found by Lehmer in [22]) all arise as factors of $\Delta_{p_1,\ldots,p_d}$ for some positive integers $p_1,\ldots,p_d$ (see [18], Section 3). Also the Lehmer polynomial $L(x) = \Delta_{2,3,7}(x)$, the denominator of the growth series of the $(2,3,7)$-hyperbolic triangle group.

As the next result shows, the minimization problem has been solved for the family of polynomials $\Delta_{p_1,\ldots,p_d}(x)$.

Theorem 22 (Hironaka [18]) *Lehmer's Salem number ϵ_L is the smallest Salem number arising as the root of $\Delta_{p_1,\ldots,p_d}(x)$, where $p_1,\ldots,p_d$ are any positive integers.*

A suggestive coincidence is that the $(2,3,7)$-hyperbolic triangle has the smallest volume among hyperbolic polygons. This leads to the question:

Problem 23 *Is there a direct connection between the volumes of hyperbolic polygons and the asymptotic growths of the underlying Coxeter reflection groups?*

5.2 Arithmeticity of Coxeter groups

As mentioned in Section 3.2, the definition of an arithmetic Fuchsian group Γ is often generalized to include those Γ which are *commensurable* with Fuchsian groups of the form $\Gamma(D,\mathcal{O},\iota)$, which we introduced in Section 3.2. Recall that the word commensurable here means that there is a homomorphism from Γ to $\Gamma(D,\mathcal{O},\iota)$ with finite kernel and co-kernel.

Some of the Coxeter reflection groups $\Gamma = G_{p_1,\ldots,p_d}$ act by isometries on the hyperbolic plane, and it is natural to ask which of these are arithmetic (in the more general sense of the definition of arithmeticity given above). In [41], Takeuchi answers this question completely. His classification is based on the following useful result.

Theorem 24 (Takeuchi [40], Theorem 1, Propositions 4, 5) *Let Γ be a Fuchsian group of the first kind, and let $\Gamma^{(2)}$ denote the set of squares of elements in Γ. Then*

$$\Gamma \text{ is arithmetic} \iff \Gamma^{(2)} \subset \Gamma(D,\mathcal{O},\iota),$$

where $\Gamma(D,\mathcal{O},\iota)$ is the arithmetic Fuchsian group constructed out of a division quaternion algebra D over a totally real field L, a maximal order $\mathcal{O}$ of D, and a place ι of L, as in Section 3.2.

It turns out (cf. [41]) that there are exactly 85 hyperbolic triangle groups which are arithmetic.[8] Included among these is the triangle group $G_{2,3,7}$.

Recall that the asymptotic growth of $G_{2,3,7}$ is equal to ϵ_L, Lehmer's Salem number. Naively, one might hope that the arithmeticity of $G_{2,3,7}$ might be connected in some way to the arithmetic of ϵ_L. There does not however seem to be any obvious connection! Indeed, in [41], Takeuchi shows that the $(2,3,7)$-hyperbolic triangle group is commensurable to an arithmetic Fuchsian

[8]However, there are only 19 totally real fields which are responsible for the arithmeticity of these triangle groups.

group associated to the totally real field $k_{2,3,7} := \mathbb{Q}(\cos(\frac{\pi}{7}))$. The degree of $k_{2,3,7}$ over $\mathbb{Q}$ is 6, since it is a quadratic extension of the cubic field $\mathbb{Q}(\cos(\frac{2\pi}{7}))$. (This latter field is cubic since it has degree equal to half the degree of the cyclotomic field generated by a primitive 7th root of unity). On the other hand, the totally real field $k_L := \mathbb{Q}(\epsilon_L + 1/\epsilon_L)$ associated to Lehmer's Salem number ϵ_L has degree 5 (half the degree of the Lehmer polynomial).

Thus, comparing degrees shows that ϵ_L is not quadratic over $k_{2,3,7}$, and in particular, ϵ_L can not appear, as described in Section 3.3, as the larger eigenvalue of a hyperbolic matrix in $G_{2,3,7}$. Since ϵ_L *does* appear as the larger eigenvalue of a hyperbolic matrix in $\Gamma = \Gamma(D_\phi, \mathcal{O}, \iota)$, the arithmetic Fuchsian group attached to the division algebra D_ϕ over the totally real field k_L (see Section 3 for an explanation of the notation), we see that $G_{2,3,7}$ and Γ are not isomorphic subgroups of $\mathrm{SL}_2(\mathbb{R})$.

5.3 Alexander polynomials of pretzel links

The Alexander polynomial is a standard polynomial invariant of a knot or link L embedded in the three sphere S^3. Here is one definition. Let $M = S^3 \setminus L$ be the complement of L in S^3. Then the abelianization of the fundamental group $\pi_1(M)$ of M, is a free abelian group of rank r, where r is the number of components of the link. Choose an orientation on the connected components of L. Consider the map

$$\phi : \pi_1(M) \to \mathbb{Z},$$

given by sending all positively oriented meridian loops around the components of L (which are well-defined up to conjugacy class) to the positive generator of $\mathbb{Z}$.

The map ϕ canonically determines an infinite cyclic abelian unbranched covering U of M on which $\mathbb{Z}$ acts as covering automorphisms. In particular, $\mathbb{Z}$ acts on the first homology group of U relative to a fiber b of the covering map $U \to M$. Since M is connected, different choices of base point for $\pi_1(M)$, and similarly different choices of fiber b, don't affect the isomorphism class of the fundamental group and relative homology.

It turns out that $\mathcal{H} = \mathrm{H}_1(U, b; \mathbb{Z})$ is a finitely presented module over the group ring Λ of $\mathbb{Z}$, which is canonically isomorphic to the ring of Laurent polynomials in one variable. Given a finite presentation of $\mathcal{H}$ as a Λ-module, the relations can be made into a matrix A with entries in Λ, which is called the *Alexander matrix* of the presentation. We have the exact sequence:

$$0 \to \Lambda^n \xrightarrow{A} \Lambda^m \to \mathcal{H} \to 0.$$

There are various ways to compute Alexander matrices for an oriented knot or link explicitly (see, for example, [27]).

The *fitting ideal* I_s of a matrix A is the ideal of $m - s \times m - s$ minors of A. Since Λ is a principle ideal domain, each fitting ideal is generated by a single polynomial. By the definition, one can see that there is an increasing sequence of ideals:

$$I_0 \subset I_1 \subset \cdots \subset I_m = \Lambda.$$

The polynomial generating the smallest non-trivial fitting ideal in this sequence is called the *Alexander polynomial* of the oriented link L. One can check that this definition does not depend on the presentation of $\mathcal{H}$ (it does depend on the choice of orientation above).

Alexander polynomials form a rich testing ground for the minimization problem, because of a theorem of Seifert [32] that any monic palindromic polynomial $P(x) \in \mathbb{Z}[x]$ satisfying

$$P(1) = \pm 1$$

is the Alexander polynomial of some knot K.

Remark. If there is a palindromic polynomial $P(x)$ with minimal Mahler measure it seems likely that it might lie among polynomials with $P(1) = \pm 1$. This is based on the following heuristic argument. Firstly, one might expect that the minimal Mahler measure, if it exists, is a Salem number. This is because the more roots $P(x)$ has outside the unit circle, the larger $M(P)$, the Mahler measure of $P(x)$, tends to become.

Secondly, the real graph of a Salem polynomial $P(x)$ has a single real zero $1/\epsilon$ in the open interval $(0,1)$, and another real zero $\epsilon \in (1,\infty)$. Since $P(x)$ is reciprocal, ϵ is close to 1 if and only if $1/\epsilon$ is close to 1. Heuristically, the easiest way for this to happen is for the non-zero integer $P(1)$ to be as close to 0 as possible, since otherwise the slope of $P(x)$ around 1 would have to be steep to compensate for the height which $P(x)$ would have to climb or fall between its two real zeros and $(1, P(1))$. For instance, since $P'(1) = n \cdot P(1)$, where $2n$ is the degree of $P(x)$, it seems that $P(1) = \pm 1$ is optimal.

Figure 3 is a graph of the Lehmer polynomial $L(x)$. Although it is a Salem polynomial and satisfies $L(1) = -1$, there is so far no concrete proof which shows that to solve Problem 2, one may restrict one's attention to Salem polynomials $P(x)$ with $P(1) = \pm 1$.

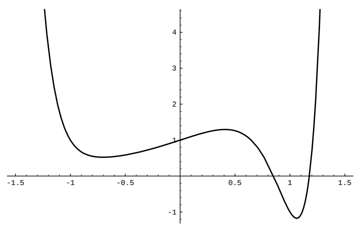

Figure 3. Real graph of Lehmer polynomial

Let us now restrict our attention to a particular class of knots, the so called *pretzel knots*. We start with an example. The (-2, 3, 7)-pretzel knot is obtained as follows. One takes 3 pairs of strings and twists them -2, 3, and 7 times respectively. The negative sign with the 2 means a negative twist. Then one joins the top and bottom strands as in Figure 4.

By replacing (-2, 3, 7) by any list of nonzero integers $(p_1, \ldots, p_d)$, we get a corresponding knot or link $L_{p_1,\ldots,p_d}$, called the $(p_1, \ldots, p_d)$-pretzel knot or link. If $p_1, \ldots, p_d$ are all odd, or at most one is even, then $L_{p_1,\ldots,p_d}$ is a knot, otherwise it is a link with number of components equal to one less than the number of even integers among $p_1, \ldots, p_d$. By choosing orientations on the components of the link, one obtains a one-variable Alexander polynomial for the link. This polynomial may be different for different choices of orientation.

As Kirby points out in his problem list ([21], page 340), the Lehmer polynomial $L(x)$ is the Alexander polynomial of the (-2, 3, 7)-pretzel knot, drawn in Figure 4.

In Section 5.1, we had pointed out that the Lehmer polynomial $L(x)$ is also the Salem polynomial $\Delta_{2,3,7}(x)$ which is the denominator of the growth series of the $(2,3,7)$-hyperbolic triangle group $G_{2,3,7}$. This leads naturally to the question: is $\Delta_{p_1,\ldots,p_d}(x)$ also the Alexander polynomial of a pretzel knot or link? This question is answered by the following result.

Theorem 25 (Hironaka [18], Theorem 1.1) *Let d be odd, and let $p_1, \ldots, p_d$ be positive integers. Then the Alexander polynomial of $L_{p_1,\ldots,p_d,-1}$, with respect to a suitable orientation of*

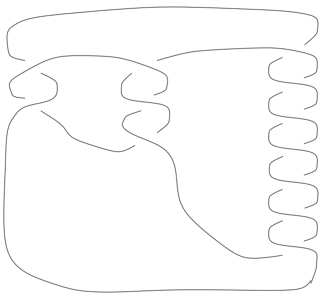

Figure 4. (-2,3,7)-pretzel knot

its components, is $\Delta_{p_1,\ldots,p_d}(x)$.

This theorem applies to the (-2, 3, 7)-pretzel knot, which is equivalent to the (2, 3, 7, -1)-pretzel knot. The Alexander polynomials for the ($p_1, \ldots, p_d$, -1)-pretzel links are among the polynomials covered in Theorem 22, and hence we have the following corollary.

Corollary 26 *Among Alexander polynomials of* $(p_1, \ldots, p_d, -1)$*-pretzel links, where* d *is odd and* $p_1, \ldots, p_k$ *are positive, the Mahler measure is minimized by the Lehmer polynomial.*

So far, questions about Salem numbers and Mahler measures arising from Alexander polynomials of more arbitrary links have not been fully addressed.

Problem 27 *Can the Mahler measure be minimized for monic non-cyclotomic Alexander polynomials?*

Finding a concrete relation between the geometry of oriented knot and link complements, and the Mahler measures of Alexander polynomials could lead to new insights into this problem, and to Lehmer's question itself.

6 Special values of L-functions

We now continue our survey of the apparent ubiquity of Salem numbers in mathematics, by describing how they show up in the special values of L-functions. It has to be said that not too much is known here: one is still trying to understand a handful of some very beautiful examples. Here we hope we can give the reader a glimpse of what is turning out to be a promising and exciting new area for research.

6.1 Stark units and Salem numbers

Given an arbitrary Galois extension K/k of number fields with group G, and a finite dimensional complex representation $\rho : G \to GL(V)$ one has the Artin L-function

$$L(s, K/k, \rho) = \prod_p \det(1 - Fr_p\big|_{V^{I_p}} Np^{-s})^{-1},$$

where the product runs over all (archimedean) primes p of k, V^{I_p} is the I_p (inertia at p) fixed points of V, and Fr_p is a Frobenius element at p.

Stark has a conjectural description of the leading term in the Taylor expansion of this L-function at $s = 0$, which states that it is essentially a certain $r \times r$ determinant, where r is the order of vanishing of $L(s, K/k, \rho)$ at $s = 0$. A more precise statement can be found in Tate's beautiful book [42]. When $r = 1$, Stark's conjecture is known to be true (cf. [36]), and the above mentioned determinant is essentially just the logarithm of a special unit of K, now called a *Stark unit.*

We are interested in the case when K/k is a Salem extension, and ρ is the non-trivial quadratic character of K/k. In this case we have

$$L(s, K/k, \rho) = \frac{\zeta_K(s)}{\zeta_k(s)},$$

where $\zeta_F(s)$ is the Dedekind zeta function of the number field F. Now, a famous formula of Dedekind says that the leading term in the Taylor expansion of $\zeta_F(s)$ at $s = 0$ is given by

$$\zeta_F(s) = \frac{h_F R_F}{w_F} s^{r_1+r_2-1} + \text{ higher order terms} + \cdots ,$$

where

- h_F is the class number of F,
- R_F is the regulator of F,
- w_F is the number of roots of 1 in F, and,
- s_F (respectively t_F) is the number of real (respectively complex) places of F.

Using this, we see that when K/k is a Salem extension, $r = 1$, and

$$L'(0, K/k, \rho) = \frac{h_K R_K}{h_k R_k}. \tag{4}$$

The formula (4) is already very close to what is predicted by Stark's conjecture. In fact, Stark showed:

Proposition 28 (Stark [36]) *Let K/k be a Salem extension. Then*

$$L'(0, K/k, \rho) = \frac{h_K \; 2^{n-2} \log(e)}{h_k \; u} \tag{5}$$

where $u = 2$ when K is generated over k by a square root of a unit of k, and $u = 1$ otherwise, and e is a unit of K, which together with the units of k generates a subgroup of index $2u$ in $\mathcal{O}_K^\times$.

The relation between Stark's conjecture and our subject is seen by the following result:[9]

Proposition 29 (Chinburg [9]) *The unit e of K is a Salem number. Conversely every Salem number ϵ in K is of the form $\epsilon = e^{m/2}$ for some positive integer m.*

6.2 Lower bounds for relative regulators

The formulas (4) and (5) of the previous section show that one might make some progress towards the minimization problem if one could establish a good absolute lower bound for the quotient of regulators R_K/R_k corresponding to a Salem extension K/k. This observation was made by Costa, Friedman and Skoruppa, who in fact made the following general conjecture (cf. [16]):

[9] We also refer the reader to [8], where Chinburg derives some partial results for the minimization problem. He also uses Proposition 29 to deduce some information on the relative class number h_K/h_k of a Salem extension.

Conjecture 30 *Let L/F denote an arbitrary extension of number fields, and let*

$$\mathrm{Reg}(L/F) := \frac{1}{[\mathcal{O}_F^\times : \mu_F N_{L/F}(\mathcal{O}_L^\times)]} \frac{R_L}{R_F},$$

where μ_F is the groups of roots of 1 in F. Then there are absolute constants f_0 and f_1 such that:

$$\mathrm{Reg}(L/F) \geq f_0 f_1^{r_{L/F}},$$

where $r_{L/F} \geq 0$ denotes the difference in the unit ranks of L and F.

In [16], the authors point out that the minimization problem would have an affirmative answer if one knew the truth of Conjecture 30 in the case when $r_{L/F} = 1$. They also prove Conjecture 30 when $[L : F]$ is 'large', and point out that it also holds in the case when L (and so F) is totally real (see [10]).

6.3 Mahler measures and L-values

The results of Section 6.1 point towards a connection between the (logarithms of) Mahler measures of polynomials and the special values of L-functions. In fact the first examples of this phenomena were discovered by Smyth for polynomials in more than one variables. Let $P(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ be such a polynomial. Define its Mahler measure $M(P)$ by

$$\log M(P) = \int \cdots \int_{S^1 \times \cdots \times S^1} |\log P(x_1, \ldots, x_n)| \, \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

It is easy to check that when $n = 1$ we recover the definition made in the Introduction. In [35] Smyth showed that

$$\log M(1 + x + y) = L'(\chi_{-3}, -1),$$
$$\log M(1 + x + y + z) = 14\, \zeta'(-2),$$

where χ_{-3} is the Dirichlet character attached to the quadratic extension $\mathbb{Q}(\sqrt{-3})$, and $\zeta(s)$ is just the Riemann zeta function. Other striking examples of this kind were given by Ray [26].

The first 'non-abelian' example is due to Deninger [12] and Boyd [6]. They showed

$$\log M(y^2 + (x^2 + x + 1)y + x^2) = L'(E_{15}, 0), \tag{6}$$

where E_{15} is the elliptic curve of conductor 15 defined by the polynomial appearing in (6) above. Actually this formula has only been checked to 50 decimals places of accuracy, though in [12] Deninger has been able to give a heuristic reason for its validity. Moreover in some cases Deninger (see also [3]) has been able to interpret Mahler measures as Deligne periods of mixed motives.

Many other formulae similar to (6) have been numerically identified by Boyd [6], and via different methods, by Rodriguez Villegas [45]. Interestingly, proofs, and not just numerical coincidence, of these formulae have so far been forthcoming mostly for curves E with complex multiplication, in which case both the L-value and the Mahler measure reduce to the value of a Eisenstein-Kronecker series evaluated at the corresponding point of CM. We refer the reader to the papers mentioned above for more details.

In closing this section we would like to point out an interesting curiosity, discovered by Boyd [6]. Boyd noticed that there is a formula similar to (6) above for $L'(0, E_{14})$ where E_{14} is the elliptic curve over $\mathbb{Q}$ with conductor 14:

$$\log M\big((x + 1)y^2 + (x^2 + x + 1)y + (x^2 + x)\big) = 0.2274812230.... = L'(0, E_{14}).$$

This formula has again only been checked numerically. Thanks to the general identity

$$\lim_{m\to\infty} M(P(x,x^m)) = M(P(x,y)),$$

we see that $\exp(L'(0,E_{14})) = 1.25543...$ is a (potential) limit point of the set of Mahler measures of polynomials in one variable. As it turns out, it is the smallest known limit point.

Now apart from E_{14} there is exactly one more elliptic curves over $\mathbb{Q}$ with conductor smaller than 15, namely the curve E_{11} of conductor 11. Yet no such analogous formula has so far been discovered for E_{11}. Since $\exp(L'(E_{11},0)) = 1.16433... < \epsilon_L = 1.17628...$, the existence of such a formula for E_{11} would imply that one would be able to find *infinitely many polynomials* in one variable, with Mahler measure *smaller* than that of Lehmer's polynomial $L(x)$!

7 Best results

This section contains the best results (known to us) concerning lower bounds on Salem numbers and Mahler measures of irreducible monic polynomials $P(x) \in \mathbb{Z}[x]$ in *one* variable.

In 1979, Dobrowolski [13] obtained a lower bound $B(d)$ for the Mahler measure of polynomials of degree d. Since then, his methods have been refined by various mathematicians. As Boyd points out ([5]) the existence of $B(d)$ is not in question since there are only finitely many possible values of the Mahler measure, less than a fixed constant, for polynomials of a fixed degree d. In any case, a sample of such a result is the

Theorem 31 (Voutier, [46]) *If $P(X)$ is not a cyclotomic polynomial, and has degree $d > 1$, then*

$$\log M(P) > \frac{1}{4}\Big(\frac{\log(\log(d))}{\log(d)}\Big)^3.$$

Note that such theorems have strong implications for Conjecture 12. In fact we now see that it is true if we restrict ourselves to the family of arithmetic surfaces coming from totally real fields of degree k of a fixed degree d. On the other hand, we are still far from the truth of Conjecture 12 in general. The problem of course is that the bound above tends to 0 as $d \to \infty$.

One also has bounds that depend on the number of real roots of $P(X)$. For instance Lehmer had already conjectured that a lower bound for $M(P)$ for those polynomials $P(X)$ which have at least one non-real root outside the unit circle is 1.2013... This is the Mahler measure of the polynomial

$$x^{18} + x^{17} + x^{16} - x^{15} - x^{11} - x^9 - x^7 - x^5 + x^2 + x^1 + 1,$$

which has two complex-conjugate roots outside the unit circle. In this vein we have the

Theorem 32 (Laurent, cf. [23]) *Let P(X) be a non-cyclotomic polynomial, with r real roots, of degree d. Then*

$$\log M(P) \geq c\,\frac{r^2}{d\log(1 + d/r)},$$

where $c > 0$ is an absolute constant.

Note that for $r = 2$, which is the minimization problem for Salem numbers, this is a better bound then the one above, yet it still tends to 0 as $d \to \infty$.

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