Zero-Dimensional Extension Rings and Subrings*

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Suppose R is a ring. In this paper we consider questions of existence and behavior of zero-dimensional extension rings and subrings of R. The paper has two main sections, representing the dichotomy of its title. In Section 1, on extension rings, our primary emphasis is on the question of existence; in Section 2, on subrings, we focus more on behavior of the family of zero-dimensional subrings of R. In both sections and in discussion of the two problems that constitute Section 3, we consider the special cases of Artinian and von Neumann regular rings.

1. Zero-Dimensional Extension Rings

Suppose R is a ring. Under what conditions is R imbeddable in a zero-dimensional ring? This is the basic question considered by M. Arapovic in [Ar3], one of three papers by the author published consecutively in the 1983 volume of *Glasnik Matematicki*; this is the group of papers that has been a stimulus for much

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* This paper contains material presented in the second 1994 Barrett Lecture, given on April 8. Several of its results come from the papers [GH3], [GH5] and [GH7] (see the list of references for the paper *Background and Preliminaries* ... in this volume), and represent work done jointly with W. Heinzer.

of the recent interest in zero-dimensional rings. Arapovic's main result in regard to the imbedding question appears below as Theorem 1.7; we refer to this as Arapovic's Imbedding Theorem, abbreviated AIT(cf. [Hu2, Thm. 3.5).

It is easy to show that several significant classes of rings consist of rings that are imbeddable in a zero-dimensional ring. In this regard, suppose $\{I_a\}_{a\in A}$ is a family of ideals of the ring R such that $(0) = \cap_{a\in A} I_a$. Then R is imbeddable in $\prod_{a\in A}(R/I_a)$, which is imbeddable in $\prod_{a\in A} q(R/I_a)$, where $q(\bullet)$ denotes the total quotient ring. From this we conclude:

(1.1) If A is finite and each I_a is primary, then R is imbeddable in a zero-dimensional ring, for if $P_a = \mathrm{rad}(I_a)$, then $q(R/I_a) \cong (R/I_a)_{P_a/I_a}$ is zero-dimensional, and hence $\prod_{a\in A} q(R/I_a)$ is also zero-dimensional. Thus each Noetherian ring (or more generally, each Laskerian ring) is imbeddable in a zero-dimensional ring.

(1.2) If each I_a is prime, then $\prod_{a\in A} q(R/I_a)$ is a product of fields, and hence is von Neumann regular. Therefore each reduced ring is imbeddable in a von Neumann regular ring.

Because a subring of a reduced ring is reduced, the converse of the second statement in (1.2) also holds. We state this result as Theorem 1.3.

Theorem 1.3. *The ring R is imbeddable in a von Neumann regular ring if and only if R is reduced.*

Our proof of Theorem 1.7 uses the following lemma.

Lemma 1.4. ([O, Lemma 2], [Gz, Lemma 4.3.9]) *Assume that R is a ring and $x \in R$ is such that the ideal Rx of R is idempotent. Then there exists a unique element $y \in R$ such that $x = x^2y$ and $y = xy^2$.*

Proof. Since Rx is idempotent, there exists an element $z \in R$ such that $x = zx^3$. Considered as a ring, the ideal Rx has identity element zx^2, and $y = zx$ is the inverse of x in the ring Rx. Because xy is the identity element of Rx, we have $x = x^2y$ and $y = xy^2$. If $w \in R$ is such that $x = x^2w$ and $w = xw^2$, then $w \in Rx$ and xw is the identity element of ring Rx. Because x has a unique inverse in Rx, it follows that $y = w$. ∎

Definition 1.5. If x, y and R are as in the statement of Lemma 1.4, we call y the *pointwise inverse* of x in R.

We note that x has a pointwise inverse in R only if the ideal Rx is idempotent. The terminology *pointwise inverse* apparently comes from the fact that if $R = \prod_{a \in A} F_a$ is a product of fields and if $x = \{x_a\}$, then the pointwise inverse of x is the element $\{y_a\}$, where $y_a = 0$ if $x_a = 0$, and $y_a = x_a^{-1}$ if $x_a \neq 0$.

Theorem 1.6. **([GH5, Thm. 3.1])** *Suppose R is a subring of the ring T. Then R is contained in a zero-dimensional subring of T if and only if, for each $x \in R$, some power of the ideal xT is idempotent.*

Proof. If R is a subring of the zero-dimensional subring S of T and if $x \in R$, then $x^m S$ is idempotent for some $m \in \mathbb{Z}^+$ by zero-dimensionality of S. Therefore $x^m T$ is also idempotent.

Conversely, assume that for each $x \in R$ there exists $m(x) \in \mathbb{Z}^+$ such that the ideal $x^{m(x)}T$ is idempotent. Let t_x be the pointwise inverse of $x^{m(x)}$ in T and let $S = R[\{t_x : x \in R\}]$; we show that S is zero-dimensional. Thus, let $P \in \mathrm{Spec}(S)$, let $\overline{R} = (R+P)/P \cong R/(P \cap R)$, and write $\overline{s}$ for the element $s + P$ of S/P. Then $S/P = \overline{R}[\{\overline{t}_x : x \in R\}]$. We show that $S/P = q(\overline{R})$, the quotient field of $\overline{R}$. Thus, if $\overline{r} \in \overline{R} - \{\overline{0}\}$ where $r \in R$, then $r^{m(r)}(1 - r^{m(r)}t_r) \in P$. Since $r \notin P$, it follows that $1 - r^{m(r)}t_r \in P$, and hence $\overline{r}$ is a unit of S/P. Consequently, $q(\overline{R}) \subseteq S/P$, and to prove equality, it suffices to show that each nonzero $\overline{t}_x$ in S/P belongs to $q(\overline{R})$. This follows as above: $\overline{t}_x \neq 0$ implies $\overline{x} \neq 0$, so $\overline{t}_x$ is the inverse of $\overline{x}^{m(x)}$ in S/P. Therefore $\overline{t}_x \in q(\overline{R})$. We have shown that S/P is a field for each $P \in \mathrm{Spec}(S)$, so S is zero-dimensional. ∎

Theorem 1.7. **([Ar3, Thm. 7])** *The ring R is imbeddable in a zero-dimensional ring if and only if there exists a family $\{Q_\lambda\}_{\lambda \in \Lambda}$ of primary ideals of R, where $P_\lambda = \mathrm{rad}(Q_\lambda)$, such that the following conditions (A1) and (A2) are satisfied.*

(A1) $\cap_{\lambda \in \Lambda} Q_\lambda = (0)$.

(A2) For each $a \in R$, there exists $n_a \in \mathbb{Z}^+$ such that $a^{n_a} \notin \cup_{\lambda \in \Lambda}(P_\lambda - Q_\lambda)$.

Proof: Assume first that R is a subring of a zero-dimensional ring S. Let $\mathrm{Spec}(S) = \{M_\lambda\}_{\lambda \in \Lambda}$. Result 4.1 of [BP] shows that $(0) = \cap_{\lambda \in \Lambda} J_\lambda$, where J_λ is M_λ-primary. If $Q_\lambda = J_\lambda \cap R$ and $P_\lambda = M_\lambda \cap R$, then (A1) is satisfied for the family $\{Q_\lambda\}_{\lambda \in \Lambda}$. Moreover, if $a \in R$, then $a^{n_a} S$ is idempotent for some $n_a \in \mathbb{Z}^+$; consequently,

$a^{n_a}(1 - a^{n_a}s) = 0$ for some $s \in S$. We claim that $a^{n_a} \notin \cup_{\lambda \in \Lambda}(P_\lambda - Q_\lambda)$ or, equivalently, $a^{n_a} \notin \cup_\lambda(M_\lambda - J_\lambda)$. Thus, if $\lambda \in \Lambda$ is such that $a^{n_a} \notin J_\lambda$, the equation $a^{n_a}(1 - a^{n_a}s) = 0$ implies that $1 - a^{n_a}s \in M_\lambda$, and hence $a^{n_a} \notin M_\lambda$. Therefore conditions (A1) and (A2) are satisfied.

Conversely, assume that there exists a family $\{Q_\lambda\}$ of primary ideals of R satisfying (A1) and (A2). Then R is imbedded in the ring $S = \prod_{\lambda \in \Lambda} S_\lambda$, where $S_\lambda = (R/Q_\lambda)_{P_\lambda/Q_\lambda}$ is zero-dimensional. One might at this point surmise that (A2) implies that S itself is zero-dimensional. That is not the case, but it is true that R is contained in a zero-dimensional of S. To prove this we apply Theorem 1.6. Hence, if φ denotes the canonical imbedding $a \to \{a+Q_\lambda/1+Q_\lambda\}_{\lambda \in \Lambda}$ of R in S, we show that the ideal $\varphi(a)^{n_a}S = \prod_{\lambda \in \Lambda}(a^{n_a} + Q_\lambda/1 + Q_\lambda)S_\lambda$ is idempotent. For this purpose it suffices to show that the ideal $(a^{n_a} + Q_\lambda/1 + Q_\lambda)S_\lambda$ of S_λ is idempotent for each $\lambda \in \Lambda$. If $a^{n_a} \in Q_\lambda$, this is clear. If $a^{n_a} \notin Q_\lambda$, then condition (A2) implies that $a^{n_a} \notin P_\lambda$, and hence $a^{n_a} + Q_\lambda/1 + Q_\lambda$ is a unit of the ring $(R/Q_\lambda)_{P_\lambda/Q_\lambda} = S_\lambda$. In either case the ideal $(a^{n_a} + Q_\lambda/1 + Q_\lambda)S_\lambda$ is idempotent, and this completes the proof of Theorem 1.7. ∎

Condition (A1) of (1.7) provides a relatively easy means for giving an example of a ring not imbeddable in a zero-dimensional ring. In fact, the next result provides a class of such examples.

Result 1.8. In a chained ring R, the following conditions are equivalent:

(1) R is imbeddable in a zero-dimensional ring.

(2) Condition (A1) of Theorem 1.7 is satisfied in R.

(3) The ideal (0) of R is primary.

Proof. The implications (1)⇒(2) and (3)⇒(1) are clear. To prove that (3) follows from (2) amounts to showing that if $\{Q_\lambda\}_{\lambda \in \Lambda}$ is a family of primary ideals of R, then $I = \cap_\lambda Q_\lambda$ is primary. Toward this end we note that $P = \mathrm{rad}(I)$ is prime since R is chained. If $xy \in I$, where $x \notin P$, we show that $y \in Q_\lambda$ for each λ. Since $y \in P$, this is clear if $P \subseteq Q_\lambda$. And if $P \nsubseteq Q_\lambda$, then $Q_\lambda < P$, so $P = \mathrm{rad}(Q_\lambda)$; in this case $y \in Q_\lambda$ since $xy \in Q_\lambda$ and $x \notin P$. ∎

Thus, if V is a valuation domain and if J is a non-primary ideal of V, then V/J is not imbeddable in a zero-dimensional ring. This observation leads naturally to questions concerning imbeddability in a finite-dimensional ring that Bill Heinzer discussed in his talk at the 1994 Barrett Lecture Conference (see [Hz]).

It is not as easy to produce an example of a ring R such that (A1) is satisfied in R, but R is not imbeddable in a zero-dimensional ring. Such examples can be obtained, however, from Theorem 1.9, a result that is of interest in its own right.

Theorem 1.9. **([GH3, Cor. 8])** *Let $\{R_a\}_{a\in A}$ be a family of zero-dimensional rings and let $R=\prod_{a\in A} R_a$. Then R is imbeddable in a zero-dimensional ring only if R itself is zero-dimensional.*

Proof. Our proof is by contradiction. Thus, we assume that R is imbeddable in a zero-dimensional ring T, but $\dim R>0$. Recall ([GH3, Thm. 3] or [I, Thm. 6]) that the condition $\dim R>0$ implies that, for each $k\in \mathbb{Z}^+$, the set $B_k=\{a\in A : \eta(R_a)>k\}$ is infinite. Thus, there exists an infinite sequence $a_1,a_2,\ldots$ of distinct elements of A such that $\eta(R_{a_i})>i$ for each i. There exists an idempotent e of R such that $Re\cong\prod_{i=1}^{\infty}R_{a_i}$, and Re is imbedded in the zero-dimensional ring Te. Moreover, $\dim(\prod_1^{\infty}R_{a_i})\neq 0$ by choice of the sequence $\{a_i\}_1^{\infty}$. Thus, without loss of generality we assume that $A=\mathbb{Z}^+$ and $\eta(R_i)>i$ for each $i\in\mathbb{Z}^+$. Choose $x\in N(R_i)$ such that $x_i^i\neq 0$. Because the zero ideal of R_i is an intersection of primary ideals of R_i, there exists a primary ideal Q_i of R_i such that $x_i^i\notin Q_i$. Let $P_i=\operatorname{rad}(Q_i)$, let $N_i=R_i-P_i$, and let N be the multiplicative system $\prod_{i=1}^{\infty}N_i$ in R. Then $R_N\cong\prod_{i=1}^{\infty}(R_i)_{N_i}\cong\prod_1^{\infty}(R_i)_{P_i}$ is imbeddable in the zero-dimensional ring T_N, and by choice of Q_i, $\eta((R_i)_{P_i})>1$ for each i. In summary, we have shown that if the statement of Theorem 1.9 is false, then there exists a counterexample where $A=\mathbb{Z}^+$ each R_i is quasilocal, and $\eta(R_i)>i$ for each i. Henceforth we make these additional assumptions on A and on the rings R_i.

Since R is imbeddable in a zero-dimensional ring, there exists a family $\{Q_\lambda\}_{\lambda\in\Lambda}$ of primary ideals of R satisfying conditions (A1) and (A2) of Theorem 1.7. For $i\in\mathbb{Z}^+$, let $e_i\in R$ be the primitive idempotent associated with i and let $\Lambda_i=\{\lambda\in\Lambda : 1-e_i\in Q_\lambda\}$. We have $e_i\in Q_\lambda$ for each $\lambda\in\Lambda-\Lambda_i$, so $e_i[\cap_{\lambda\in\Lambda_i}Q_\lambda]\subseteq\cap_{\lambda\in\Lambda}Q_\lambda=(0)$. Therefore $\cap_{\lambda\in\Lambda_i}Q_\lambda\subseteq\operatorname{Ann}_R e_i=(1-e_i)R$, and because the reverse inclusion is clear, it follows that $(1-e_i)R=\cap_{\lambda\in\Lambda_i}Q_\lambda$. On the other hand, if $\lambda\in\Lambda_i$, then $Q_\lambda=Q_\lambda e_i\oplus(1-e_i)R$, where $Q_\lambda e_i$ is a proper ideal of the ring $Re_i\cong R_i$. Therefore $(1-e_i)R_i=\cap_{\lambda\in\Lambda_i}Q_\lambda=\cap_{\lambda\in\Lambda_i}[Q_\lambda e_i\oplus(1-e_i)R]=(\cap_{\lambda\in\Lambda_i}Q_\lambda e_i)\oplus(1-e_i)R$. We conclude that $\cap_{\lambda\in\Lambda_i}Q_\lambda e_i=(0)$. Choose $y_i\in N(R_i)$ such that $y_i^i\neq 0$, let $y=\{y_i\}_1^{\infty}\in R$, and let s be any positive integer. Then $y^s=\{y_i^s\}_1^{\infty}$ and $y_s^s\neq 0$. Because $\cap_{\lambda\in\Lambda_s}Q_\lambda e_s=(0)$, it follows that there exists $\sigma\in\Lambda_s$ such that $y^s\notin Q_\sigma$. On the other hand, $P_\sigma=\operatorname{rad}(Q_\sigma)=\operatorname{rad}_{Re_s}(Q_\sigma e_s)\oplus(1-e_s)R=N(Re_s)\oplus(1-e_s)R$, so $y\in P_\sigma$ by choice of y_s. Therefore $y^s\in P_\sigma-Q_\sigma$, contrary to our assumption that the family $\{Q_\lambda\}_{\lambda\in\Lambda}$ satisfies (A2). This contradiction completes the proof of Theorem 1.9. ■

If $R_i = \mathbb{Z}/p^i\mathbb{Z}$, where p is prime, then Theorem 1.9 shows that the ring $R = \prod_{i=1}^{\infty} R_i$ is not imbeddable in a zero-dimensional ring. However, $\{(1-e_i)R\}_{i=1}^{\infty}$ is a family of strongly primary ideals[1] of R with $(0) = \cap_1^\infty(1-e_i)R$. The intersection representation $(0) = \cap_1^\infty(1-e_i)R$ is *irredundant* in the sense that $\cap_{j\neq i}(1-e_j)R \not\subseteq (1-e_i)R$ for each i, and $(0) = \cap_1^\infty(1-e_i)R$ is the unique irredundant representation of (0) as an intersection of primary ideals of R. In particular, even strong versions of (A1) need not imply (A2) of Theorem 1.7.

In general, condition (A2) of AIT is not easy to verify. This is understandable since it involves universal quantifiers on three potentially large sets. To indicate some limits on its applicability, note that while AIT was used in the proof of Theorem 1.9 to settle the imbeddability question for $\prod R_a$, where each R_a is zero-dimensional, we have not been able to use it to determine whether $\prod R_a/\sum \oplus R_a$ is imbeddable in a zero-dimensional ring. Roger Wiegand originally raised this question for $\prod_1^\infty(\mathbb{Z}/p^n\mathbb{Z})/\sum_1^\infty \oplus(\mathbb{Z}/p^n\mathbb{Z})$, and in time we were able to provide a negative answer ([GH8, Cor. 3.3]) to his question, but our proof did not use AIT, and the question for an arbitrary family $\{R_a\}$ of zero-dimensional rings remains open.

Rings imbeddable in a von Neumann regular ring were determined in Theorem 1.3. What about the analogous question for Artinian rings? In the spirit of AIT we can say that if R is imbeddable in an Artinian ring S, then the zero ideal of R is a finite intersection of strongly primary ideals of R; this follows by contracting such an intersection representation in S to R. Conversely, suppose $(0) = \cap_{i=1}^n Q_i$ in the ring T, where Q_i is a strongly primary ideal with radical P_i. Then T is imbedded in $\prod_{i=1}^n S_i$, where $S_i = (T/Q_i)_{P_i/Q_i}$ is zero-dimensional and quasilocal. Thus a sufficient condition for T to be imbeddable in an Artinian ring is that each S_i is so imbeddable. This observation has tended to focus attention in the Artinian imbedding question on the case of a zero-dimensional quasilocal ring (S, M), where $M^k = (0)$ for some $k \in \mathbb{Z}^+$. In this vein we prove:

Proposition 1.10. ([GH1, Thm. 2.10]) *Suppose $(0) = \cap_{i=1}^n Q_i$ in the ring T, where Q_i is P_i-primary and $P_i^2 \subseteq Q_i$. Then T is imbeddable in an Artinian ring.*

Proof. As observed above, it suffices to consider the case where (T, M) is quasilocal

[1] A primary ideal Q is *strongly primary* if Q contains a power of its radical.

and $M^2 = (0)$. If M is finitely generated, T is Artinian. If M is not finitely generated, let $\{m\} \cup \{m_\lambda\}_{\lambda\in\Lambda}$ be a basis for the (T/M)-vector space M. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a set of indeterminates over T, let $R = T(\{X_\lambda\})$, and let $S = R/I$, where I is the ideal of R generated by the set $\{m_\lambda - mX_\lambda\}_{\lambda\in\Lambda}$. As is well-known, MR is the unique maximal ideal of R. Moreover, $(MR)^2 = (0)$, and hence R is zero-dimensional. The definition of I shows that $m + I$ generates the ideal MR/I of S. The ring S is Noetherian by Cohen's Theorem, and hence S is Artinian. (In fact, S is a zero-dimensional local PIR.) To complete the proof, we show that T is imbedded in S by showing that $T\cap I = (0)$. In doing so, there is no loss of generality in assuming that Λ is finite. Thus, we show that if I_k is the ideal of $T(X_1, \dots, X_k)$ generated by $\{m_i - mX_i\}_{i=1}^k$, then $I_k \cap T = (0)$. We consider separately the cases $k = 1$ and $k > 1$.

If $k = 1$, choose $t \in I_1 \cap T$ and assume that $t \neq 0$. Write $t = f(m_1 - mX_1)/g$, where $f, g \in T[X_1]$ and g has a unit coefficient. The equation $tg = f(m_1 - mX_1)$ shows that $t \in M$ and $f(m_1 - mX_1) \neq 0$. Because $M^2 = (0)$, it follows that f also has a unit coefficient, but then the Dedekind-Mertens Lemma [G, Thm. 28.1] implies that tT is the content ideal of tg while $(m_1, m)T$ is the content ideal of $f(m_1 - mX_1)$. This contradicts the fact that $\{m, m_1\}$ is linearly independent over T/M. Therefore $T \cap I_1 = (0)$, as we wished to show.

If $k > 1$, we again assume that $T \cap I_k$ contains a nonzero element t and we obtain an equation $tg = \sum_{i=1}^k (m_i - mX_i)f_i$, where $g, f_1, \dots, f_k \in T[X_1, \dots, X_k]$ and g has a unit coefficient. Since $M^2 = (0)$, we also assume that each nonzero coefficient of each nonzero polynomial f_i is a unit of T. Among the nonzero monomials occurring in some f_i, let G be one of lowest total degree and let H be one of highest total degree — say H occurs in f_j. The coefficient of G in $\sum_1^k (m_i - mX_i)f_i = \sum_1^k m_i f_i - \sum_1^k mX_i f_i$ is a nonzero element u of the subspace of M generated by $\{m_i\}_1^k$, while the coefficient of X_jH in the same polynomial is a nonzero element of Tm. This leads to essentially the same contradiction we obtained in the case where $k = 1$: the content ideal of tg is a one-dimensional subspace of the (T/M)-vector space M, but the content ideal of $\sum_1^k (m_i - mX_i)f_i$ has dimension at least 2 over T/M. We conclude that $T \cap I_k = (0)$. ∎

In (1.12) we present an example, motivated by the paper [K], which shows that Proposition 1.10 fails under the hypothesis that $P_i^3 \subseteq Q_i$ for each i.

In Example 1.12 and in imbedding problems in general, the concept of a universally contracted ideal is useful. This was defined by Gilmer and McAdam in [GM] and considered further by Heinzer and Lantz in [HL2]. The definition is as follows. An ideal I of a ring R is *universally contracted* if I is contracted from each extension ring of R; equivalently, I is universally contracted if $IS \cap R = I$ for each extension ring S of R. The following result concerning universally contracted ideals is routine to establish, and we omit a proof.

Proposition 1.11. *Let $U(R)$ be the set of universally contracted ideals of the ring R.*

(1) $eR \in U(R)$ for each idempotent $e \in R$.

(2) $U(R)$ is closed under taking radicals, arbitrary intersections, and directed unions.

(3) If $I \in U(R)$, then $I : J \in U(R)$ for each ideal J of R.

(4) If $I \in U(R)$, then each minimal prime of I is also in $U(R)$.

Taking $I = (0)$ in (3) and (4) of (1.11), we obtain the well-known results that annihilator ideals and minimal primes of a ring are universally contracted. The fact that annihilator ideals are universally contracted implies that if R is imbeddable in an Artinian ring S, then R satisfies both a.c.c. and d.c.c. on annihilator ideals. In fact, since S has finite length, there is an upper bound on the number of distinct terms in any chain of annihilator ideals of R. The ring of Example 1.12 is not imbeddable in an Artinian because it fails to satisfy d.c.c. on annihilator ideals.

Example 1.12. Let $X = \{X_i\}_{i=1}^{\infty}$ be a set of indeterminates over the field F, let $D = F[X]$, $P = (X)$, $I = (\{X_i^3, X_iX_j \;:\; i,j \in \mathbb{Z}^+ \text{ and } i \neq j\})$, and $R = D/I$. Then R is a zero-dimensional quasilocal ring with maximal ideal $M = P/I$, where $M^3 = (0)$. If $x_i = X_i + I$ for each i, it is easy to see, however, that $x_{n+1} \in \mathrm{Ann}(x_1, \ldots, x_n) - \mathrm{Ann}(x_1, \ldots, x_{n+1})$ for each $n \in \mathbb{Z}^+$, and hence $\mathrm{Ann}(x_1) > \mathrm{Ann}(x_1, x_2) > \ldots$. Therefore R is not imbeddable in an Artinian ring.

2. Zero-Dimensional Subrings

Let R be a ring and denote by $\mathcal{Z}(R)$, $\mathcal{A}(R)$, and $\mathcal{V}(R)$, respectively, the sets of zero-dimensional, Artinian, and von Neumann regular subrings of R. Several properties of these sets have been considered in the literature, including the property (analogous to Section 1 considerations) of being nonempty. Rather than the existence question, however, we choose to focus here on behavior properties of $\mathcal{Z}(R)$, $\mathcal{A}(R)$, and $\mathcal{V}(R)$, particularly under intersection. A primary result (Theorem 2.1) is that $\mathcal{Z}(R)$ is closed under arbitrary intersection if $\mathcal{Z}(R)$ is nonempty.

Theorem 2.1. **([GH5, Thm. 2.4])** *Suppose R is a ring that admits a zero-dimensional subring. Then $\mathcal{Z}(R)$ is closed under arbitrary intersection.*

Proof. Let $\{R_a\}_{a\in A}$ be a nonempty family of zero-dimensional subrings of R and let $S = \cap_{a\in A} R_a$. To show that S is zero-dimensional, we show that some power of xS is idempotent for each $x \in S$. If $a \in A$, then $x^t R_a$ is idempotent for some $t \in \mathbb{Z}^+$. We show that $x^t R_b$ is idempotent for each $b \in A$. The ideal $x^n R_b$ is idempotent for some $n \in \mathbb{Z}^+$, and if $t \geq n$, then $x^t R_b$ is also idempotent. Suppose $t < n$. To show $x^t R_b$ is idempotent, it suffices to show $x^t \in x^n R_b$. If $x^n = x^{2n} r$, where $r \in R_b$, then $e = x^n r$ is an idempotent generator of $x^n R_b$. Hence $x^n R_b$ is contracted from R by part (1) of Proposition 1.11, and it suffices to show that $x^t \in x^n R$; this is a consequence of the idempotency of $x^t R_a$. Let y_a be the pointwise inverse of x^t in R_a and let y_b be the pointwise inverse of x^t in R_b. Because x^t has a unique pointwise inverse in R, it follows that $y_a = y_b$. Since $b \in B$ is arbitrary, $y_a \in S$, so the ideal $x^t S$ is idempotent. Therefore S is zero-dimensional. ■

Corollary 2.2. *Suppose $\{R_a\}_{a\in A}$ is a nonempty family of zero-dimensional subrings of a ring and let $S = \cap_{a\in A} R_a$. If S is reduced, then S is von Neumann regular; in particular, S is von Neumann regular if some R_a is von Neumann regular. The set $\mathcal{V}(R)$ is closed under arbitrary intersection if $\mathcal{V}(R)$ is nonempty.*

Because a von Neumann regular ring with only finitely many idempotents is a finite direct sum of fields, and hence is Artinian, the next result follows from Corollary 2.2.

Corollary 2.3. *Suppose $\{R_a\}_{a\in A}$ is a nonempty family of zero-dimensional subrings of a ring R and let $S = \cap_{a\in A} R_a$. If S is reduced and has only finitely many idempotents, then S is Artinian. In particular, S is Artinian if some R_a is an Artinian reduced ring. If R is reduced and $\mathcal{A}(R)$ is nonempty, then $\mathcal{A}(R)$ is closed under arbitrary intersection.*

In general the intersection of two Artinian subrings of a ring R need not be Artinian. An example related to the classical $(D+M)$-construction can be obtained as follows. Let K be a field that admits subfields K_1 and K_2 such that $[K : K_1]$ and $[K : K_2]$ are finite, but $[K : K_1 \cap K_2]$ is infinite. (A classical example of this form is $K = F(X)$, $K_1 = F(X^2)$, and $K_2 = F(X + X^2)$, where F is a field of characteristic 0 and X is an indeterminate over F.) Let V be a rank-one discrete valuation domain of the form $K + tV$, where tV is the maximal ideal of V, and let $R = V/t^2V$. By abuse of notation we write $R = K + M$, where $M = tV/t^2V$ and K denotes the canonical image of K in R. Thus R is a local PIR. The subrings $R_1 = K_1 + M$ and $R_2 = K_2 + M$ are Artinian (by Eakin's Theorem or Cohen's Theorem, for example), but $R_1 \cap R_2 = (K_1 \cap K_2) + M$ is not Artinian since $[K : K_1 \cap K_2]$ is infinite.

The construction in the preceding paragraph can be modified to yield an example of a local PIR R such that $\mathcal{A}(R)$ is nonempty, $\mathcal{A}(R)$ is closed under finite intersection, but $\mathcal{A}(R)$ is not closed under arbitrary intersection. Because of space considerations we confine ourselves to a statement of Example 2.4; the interested reader is referred to [GH7, Sect. 4] for details. It is worth noting here that both the statement of Example 2.4, and its verification in [GH7], depend on use of the notion of the Steinitz number of an absolutely algebraic field of nonzero characteristic (see [GH7, p. 304] for the definition).

Example 2.4. Let K be an absolutely algebraic field of nonzero characteristic, let $V = K + tV$ be a rank-one discrete valuation domain with maximal ideal tV, and let $R_K = V/t^2V$. For any K, the ring R_K is a local PIR and $\mathcal{A}(R_K)$ is closed under finite intersection. The set $\mathcal{A}(R_K)$ is closed under arbitrary intersection if and only if only finitely many coordinates of the Steinitz number of K are nonzero.

Suppose R is a subring of the ring T and denote by $\mathcal{Z}(R,T)$ the set of zero-dimensional subrings of T that contain R. Theorem 1.6 shows that $\mathcal{Z}(R,T) \neq \emptyset$ if and only if a power of xT is idempotent for each $x \in R$. One interesting consequence of Theorem 2.1 is that if $\mathcal{Z}(R,T)$ is nonempty, then there exists a unique minimal zero-dimensional subring $R^0(T)$ of T containing R. Theorem 2.5, an amalgamation of results from [Ar2] and [GH5, Sect. 3], lists some properties of $R^0(T)$ as an extension ring of R.

Theorem 2.5. *Let the notation be as in the preceding paragraph, and assume that $\mathcal{Z}(R,T)$ is nonempty.*

(1) For $x \in R$, choose $m(x) \in \mathbb{Z}^+$ so that $x^{m(x)}T$ is idempotent. Let t_x be the pointwise inverse of $x^{m(x)}$ in T. Then $R^0 = R^0(T) = R[\{t_x : x \in R\}]$.

(2) If $P \in \mathrm{Spec}(R^0)$, then R^0/P is the quotient field of $R/(R \cap P)$.

(3) A primary ideal of R^0 is uniquely determined by its contraction to R.

(4) If $P \in \mathrm{Spec}(R)$, then P is contracted from R^0 if and only if P is contracted from T.

Proof. In (1), the proof of Theorem 1.6 shows that $S = R[\{t_x : x \in R\}]$ is zero-dimensional, and hence $S \supseteq R^0(T) = R^0$. If U is any zero-dimensional subring of T containing R and if $x \in R$, the proof of Theorem 2.1 shows that $x^{m(x)}U$ is an idempotent ideal of U. Thus, the pointwise inverse of $x^{m(x)}$ in U is necessarily t_x, the unique pointwise inverse of $x^{m(x)}$ in T. Consequently, $t_x \in U$, $S \subseteq U$, and hence $S \subseteq R^0$.

(2) follows from (1) and from the proof of Theorem 1.6.

(3): Suppose Q_1 and Q_2 are primary ideals of R^0 with the same contraction Q to R. We wish to show that $Q_1 = Q_2$. Let $P_i = \operatorname{rad}(Q_i)$ and let $P = \operatorname{rad}(Q)$; thus $P_1 \cap R = P_2 \cap R = P$. We first show that, for $x \in R$, Q_1 contains t_x if and only if Q_2 contains t_x. Assume $t_x \in Q_1$. Since $x^{m(x)}$ and t_x generate the same ideal of R^0, it follows that $x^{m(x)} \in Q_1 \cap R = Q \subseteq Q_2$. Hence $t_x \in Q_2$, and our assertion follows by symmetry. Next consider an arbitrary element w of Q_1. There exists a finite subset $\{x_i\}_{i=1}^n$ of R such that $w \in R[t_{x_1}, \ldots, t_{x_n}]$, say $w = \sum_1^k r_i m_i$, where each $r_i \in R$ and each m_i is a monomial in $t_{x_1}, \ldots, t_{x_n}$. In showing that $w \in Q_2$ we may assume without loss of generality that $t_{x_i} \notin Q_1$ for $1 \leq i \leq n$. The equation $t_{x_i}(1 - t_{x_i} x_i^{m(x_i)}) = 0$ shows that $e_i = 1 - t_{x_i} x_i^{m(x_i)} \in P_1 \cap P_2$, and since the element e_i is idempotent, $e_i \in Q_1 \cap Q_2$. We conclude that for $1 \leq i \leq n$, $t_{x_i} x_i^{m(x_i)} \equiv 1 \pmod{Q_1 \cap Q_2}$. Consequently, there exists a monomial $y = x_1^{u_1} \cdots x_n^{u_n}$ in $x_1, \ldots, x_n$ such that, for $1 \leq i \leq k$, $m_i y$ is congruent modulo $Q_1 \cap Q_2$ to an element of R. Hence $wy = \sum_1^k r_i m_i y \equiv r \pmod{Q_1 \cap Q_2}$ for some element $r \in R$. But $w \in Q_1$ implies $r \in Q_1 \cap R = Q$. Therefore $wy \in Q_2$, and since no x_i is in P_2, it follows that $w \in Q_2$. We have shown that $Q_1 \subseteq Q_2$, and the reverse inclusion follows by symmetry.

In (4), it is clear that P is contracted from R^0 if it is contracted from T. The converse holds because each prime of R^0 is contracted from T since R^0 is zero-dimensional. ∎

If R is imbeddable in a zero-dimensional ring, it is easy to see that rings $R^0(T_1)$ and $R^0(T_2)$, for different extension rings T_1 and T_2, need not be isomorphic. For example, $\mathbb{Z}$ is imbedded in $\mathbb{Q}$ and $\mathbb{Z}^0(\mathbb{Q}) = \mathbb{Q}$, but $\mathbb{Z}$ is also imbedded in $S = \prod\{\mathbb{Z}/p\mathbb{Z} : p \text{ is prime}\}$, and $\mathbb{Z}^0(S)$ is a ring with infinitely many idempotents. The ring $\mathbb{Z}^0(S)$ is von Neumann regular, and it turns out that $\mathbb{Z}^0(S)$ is what Olivier in [O] calls the *regular ring associated to* $\mathbb{Z}$. The definition is as follows. Given a ring R, Olivier shows that there exists a von Neumann regular ring R^* and a homomorphism $f : R \to R^*$ with the following universal mapping property. If $g : R \to U$ is any homomorphism, where U is von Neumann regular, then there exists a unique homomorphism $h : R^* \to U$ such that $h \circ f = g$. The ring R^*

is *the regular ring associated to* R, it is unique up to isomorphism, and the map f is an imbedding if and only if R is reduced. In [PV2, Sect. 3], Popescu and Vraciu show that R^* may be constructed as follows. Let $\mathrm{Spec}(R) = \{P_i\}_{i \in I}$, let K_i be the quotient field of R/P_i, and let f be the canonical diagonal imbedding of R into $T = \prod_{i \in I} K_i$. Popescu and Vraciu show that $R^* \cong f(R)^0(T)$, the minimal zero-dimensional subring of T containing $f(R)$.

In general, Gilmer and Heinzer show in [GH5, Thm. 4.1] that if R is imbeddable in a zero-dimensional ring, then all minimal zero-dimensional extensions of R are isomorphic if and only if R itself is zero-dimensional.

We conclude this section with some comments about other questions concerning zero-dimensional subrings that have been considered in the literature. The primary source for these comments is [GH7].

Suppose R is a ring. One is interested in magnitudes of the sets $\mathcal{Z}(R)$, $\mathcal{A}(R)$, and $\mathcal{V}(R)$. In particular, under what conditions is a selected one of these sets nonempty? Obviously this question reduces to the characteristic-zero case, and certainly $\mathcal{Z}(R)$ is nonempty if $\mathcal{A}(R)$ or $\mathcal{V}(R)$ is nonempty. We remark that the von Neumann regular ring $S = \prod\{\mathbb{Z}/p\mathbb{Z} \ : \ p \text{ is prime}\}$ has no Artinian subrings (cf. [III, Cor. 2]), while the ring $T_p = \prod_{n=1}^{\infty}(\mathbb{Z}/p^n\mathbb{Z})$, where p is prime, has no zero-dimensional subrings. A proof that $\mathcal{Z}(T_p)$ is empty can be based on the fact that $\pi + I$, where π is the prime subring of T_p and I is the direct sum ideal of T_p, is not imbeddable in a zero-dimensional ring. (The proof of Theorem 1.9 can be modified to obtain this property of $\pi + I$.)

Other questions that have been considered include determination of conditions under which $\mathcal{A}(R)$, for example, is directed or, more restrictively, is closed under taking compositum. Also, what rings R are expressible as a directed union of Artinian subrings (cf. [I, Thm. 13])? The preceding question, in the case where R is von Neumann regular, was the subject of Andy Magid's talk at the 1994 Barrett Lectures Conference (see [Mg]).

3. Two Problems

In closing we mention two problems that are related to material in this paper. The first, labelled (E1), concerns zero-dimensional extension rings. The second, (S1), has to do with zero-dimensional subrings.

(E1) *In relation to the question "What rings are imbeddable in a zero-dimensional ring?", seek conditions that are easier to apply than are those of* AIT.

It may be in the nature of the beast that there is no clean characterization of rings imbeddable in a zero-dimensional ring, for in seeking such a characterization one is, in effect, trying to identify a set W of conditions such that (1) each subring of a zero-dimensional ring satisfies the conditions of W and (2) the conditions of W imply imbeddability in a zero-dimensional ring. The conditions of W must obviously be satisfied by each zero-dimensional ring, and when one thinks in terms of properties inherited by subrings, it is natural to think in terms of intersections of ideals. Recognizing that (A2) of AIT is a consequence of the fact that, in a zero-dimensional ring, a power of each principal ideal is idempotent, the condition does not seem so exotic. I'm suggesting that, in the general case, there may not be significantly better criteria for imbeddability than those given in AIT. But perhaps there are conditions that are easier to apply in the case of a more restricted class of rings. A test case here would be a set of conditions that would apply to $\prod R_a / \sum \oplus R_a$, where $\{R_a\}_{a \in A}$ is a set of zero-dimensional rings such that $\prod_{a \in A} R_a$ is infinite-dimensional. As mentioned in Section 1, Bill and I were able to show that $\prod R_a / \sum \oplus R_a$ is not imbeddable in a zero-dimensional ring (in fact, in a finite-dimensional ring) if each R_a is a PIR, but our proof did not entail a modified version of AIT, nor did it apply to the case of an arbitrary family of zero-dimensional rings.

(S1) *Suppose R is a zero-dimensional ring with nilradical N. If R/N is expressible as a directed union of Artinian subrings, is R so expressible?*

Question (S1) has been raised explicitly in Remark 6.10 of [GH4]. For reasons that tend to be specialized, we know that (S1) has an affirmative answer in the case where R is semiquasilocal, and in the case where R is a direct product of zero-dimensional rings. Because a homomorphic image of a ring expressible as a directed union of Artinian subrings inherits this property, a general affirmative answer to (S1) would reduce consideration of zero-dimensional rings R expressible as a directed union of Artinian subrings to the case where R is von Neumann regular.