WEIGHTED CHERN-MATHER CLASSES AND MILNOR CLASSES OF HYPERSURFACES

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Abstract. We introduce a class extending the notion of Chern-Mather class to possibly nonreduced schemes, and use it to express the difference between Schwartz-MacPherson's Chern class and the class of the virtual tangent bundle of a singular hypersurface of a nonsingular variety. Applications include constraints on the possible singularities of a hypersurface and on contacts of nonsingular hypersurfaces, and multiplicity computations.

§0. Introduction

The notion of Chern-Mather class was introduced by Robert MacPherson in [10], as one of the main ingredients in his definition of functorial Chern classes for possibly singular complex varieties. An equivalent notion had in fact already been given by Wentsün Wu; the two notions are compared in [15]. One way to think about Mather's class of Y as defined by MacPherson is the following: blow-up Y so that the pull-back of its sheaf of differentials is locally free modulo torsion; then mod out the torsion, dualize, and take Chern classes. The operation can in fact be performed for any sheaf; this is worked out in [9].

This definition ignores possible nilpotents on Y. We feel that it would be desirable to have a class in the spirit of Chern-Mather class, but in some way sensitive to possible nonreduced structures on Y: first, this is natural from the algebro-geometric standpoint; secondly, as we will see, a natural candidate carries useful information when applied to the *singularity subscheme* of a hypersurface (for which possibly non-reduced scheme structures play a fundamental rôle).

Our candidate is introduced in §1. Its definition is a suitable weighted sum of 'conventional' Chern-Mather classes of subvarieties of Y. The subvarieties are the supports of the components of the (intrinsic) normal cone of Y, and the weights are the lengths of the components of this cone. The class we obtain (trivially) agrees with Mather's if Y is a reduced local complete intersection.

If Y is the singularity subscheme of a hypersurface, we can relate the weighted Chern-Mather class with other natural classes defined in this case. For example, in [1] we have defined and studied a *μ-class* associated with the singularity subscheme

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of a hypersurface; in this paper, we answer a question which we could not address previously: how to give a reasonable definition for arbitrary schemes Y, from which the μ-class could be recovered if Y is the singularity subscheme of a hypersurface X. The weighted Chern-Mather class is precisely such a class (Corollary 1.4). We hope that this viewpoint will eventually give us the right hint on how to define a μ-class for the singularities of more general varieties X.

The main application of weighted Chern-Mather classes is to the computation of the difference between Schwartz-MacPherson's class of a hypersurface and the class of its virtual tangent bundle. A formula for the difference, in terms of the μ-class, is proved 'numerically' in [3], and at the level of Chow groups in [2] (Theorem I.5). Such differences have been named 'Milnor classes', as they generalize the fact that, for local complete intersections with isolated singularities, the Milnor *number* computes the difference between the (topological) Euler characteristic and the degree of the class of the virtual tangent bundle (see [13], [14], [12], [5], and references therein).

Weighted Chern-Mather classes allow us to recast the formula from [2]. We state this in §1 (Theorem 1.2), together with other facts about weighted Chern-Mather classes, such as their relation vis-a-vis a class appearing in [12] or their behavior under blow-ups. Proofs of these statements are sketched in §2, together with a few general considerations regarding Milnor classes. Theorem 1.2 is proved in full in §2.

The expression for the μ-class in terms of weighted Chern-Mather classes allows us in principle to compute the former for a wide class of examples. We give a couple of applications in this direction in §3, in the spirit of the examples worked out in [1], §4. For example, we prove that if two nonsingular hypersurfaces M_1, M_2 of degrees d_1, d_2 in projective space are tangent along a positive dimensional subvariety, then $d_1 = d_2$. This fact was proved in [2], but with a strong additional hypothesis on the contact locus of M_1 and M_2; the new formula for the μ-class shows that the extra hypothesis is unnecessary. We also collect in §3 a few explicit computations of weighted Chern-Mather classes.

The core of this paper is little more than a rewriting of a part of [12]. In that reference, Adam Parusiński and Piotr Pragacz give an alternative proof of the formula in [2] by a local computation of multiplicities, which relates it to a formula from [6] (over $\mathbb{C}$, and in homology) for the characteristic cycle of a hypersurface. For singularities of a hypersurface, a complex geometry analog of weighted Chern-Mather classes is introduced in [12]; the classes are compared here in Theorem 1.5. The proof of Theorem 1.2 given in §2 owes much to the approach of Parusiński and Pragacz: it is my attempt to produce a proof in the style of [12], but in a set-up closer to intersection theory in algebraic geometry (hence valid for rational equivalence; and potentially more amenable to algebraic generalizations, e.g., to positive characteristic). The reference to [6] is bypassed by an explicit computation of local Euler obstructions.

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§1. Weighted Chern-Mather classes.

All schemes in this note are of finite type over an algebraically closed field of characteristic 0, and (for simplicity) embeddable in an ambient nonsingular variety, which we will denote by M.

Assume that Y is reduced and irreducible, of dimension k. The *Chern-Mather* class of Y can be defined as follows. Let $G_k(TM)$ denote the Grassmann bundle whose fiber over $p \in M$ consists of the Grassmannian of k-planes in TM, and let Y° be the nonsingular locus in Y. Consider the map

$$Y^\circ \to G_k(TM)$$

defined by sending $p \in Y^\circ$ to $T_pY \subset T_pM$. The *Nash blow-up* of Y is the closure $\widetilde{Y}$ of the image of this map; it comes equipped with a proper map ν to Y, and with the restriction T of the tautological subbundle over $G_k(TM)$. This data is easily checked to be independent of the ambient variety M. The Chern-Mather class of Y is defined by

$$c_{\mathrm{Ma}}(Y) := \nu_*\left(c(T) \cap [\widetilde{Y}]\right)$$

in the Chow group A_*Y of Y. This class of course agrees with the total ('homology') class of the tangent bundle of Y if Y happens to be nonsingular to begin with.

Note that this definition assumes that Y is reduced, as it needs Y to be nonsingular at the general point, and ignores by construction the presence of nilpotents along subvarieties of Y. Our task is to modify this notion to take account of possible nilpotents on Y.

Let then $Y \subset M$ be arbitrary. We consider the normal cone C_YM of Y in M, and associate with Y the set $\{(Y_i, m_i)\}_i$, where the $Y_i \overset{j_i}{\hookrightarrow} Y$ are the supports of the irreducible components C_i of C_YM, and m_i denotes the geometric multiplicity of C_i in C_YM (so $[C_i] = m_i[(C_i)_{\mathrm{red}}]$).

Lemma 1.1. *The data $\{(Y_i, m_i)\}$ is intrinsic of Y, i.e., independent of the ambient nonsingular variety.*

Proof. (Cf. [7], Example 4.2.6.) It is enough to compare embeddings $Y \hookrightarrow M$, $Y \hookrightarrow M'$, where both M, M' are nonsingular, and M is smooth over M'. In this case there is an exact sequence of cones

$$0 \to T_{M'|M} \to C_YM \to C_YM' \to 0$$

(where $T_{M'|M}$ is the relative tangent bundle) in the sense of [7], Example 4.1.6, and it follows that the supports of the irreducible components of the two cones coincide, as well as the geometric multiplicities of the components. $\square$

By Lemma 1.1, the following definition is also intrinsic of Y:

Definition. The *weighted Chern-Mather class* of Y is

$$c_{\mathrm{wMa}}(Y) := \sum_i (-1)^{\dim Y - \dim Y_i} m_i j_{i*} c_{\mathrm{Ma}}(Y_i) \quad \text{in } A_*Y.$$

(Warning: we will henceforth neglect to indicate 'obvious' push-forwards such as j_{i*}, and pull-backs.)

Note that if Y is a reduced irreducible local complete intersection, then its normal cone is reduced and irreducible, so the class defined here agrees with the Chern-Mather class of Y. In particular, if Y is nonsingular then $c_{\mathrm{wMa}}(Y) = c(TY) \cap [Y]$ is the total homology class of the tangent bundle of Y.

A few examples of computations of weighted Chern-Mather classes can be found in §3. Our main motivation in introducing the class $c_{\mathrm{wMa}}(Y)$ is that we can prove it is particularly well-behaved if Y is the *singularity scheme of a hypersurface* X in a nonsingular variety M. By hypersurface here we mean the zero-scheme of a nonzero section of a line-bundle $\mathcal{L}$ on M; the singularity subscheme of X is the subscheme locally defined by the partial derivatives of an equation for X. (This scheme structure is independent of the ambient variety M.) In the rest of this section we survey a few facts about $c_{\mathrm{wMa}}(Y)$ under the hypothesis that Y is the singularity subscheme of a hypersurface. Proofs are given in §2.

Our motivation is to highlight apparently different contexts in which the class $c_{\mathrm{wMa}}(Y)$ manifests itself. Although these contexts will invoke other characters of the play, remember that $c_{\mathrm{wMa}}(Y)$ is a class *intrinsic of* Y, and which is defined regardless of whether Y is the singularity subscheme of a hypersurface. The challenge is to find extensions of these results which do not assume that Y is the singularity subscheme of a hypersurface.

For the first fact, let $c_{\mathrm{SM}}(X) \in A_*X$ denote Schwartz-MacPherson's Chern class of X, and let $c_{\mathrm{F}}(X) \in A_*X$ denote the class of its virtual tangent bundle; the subscript F is to remind us that this class agrees with the class introduced (for much more general schemes) by William Fulton, cf. Example 4.2.6 of [7].

Theorem 1.2. *Let $\mathcal{L} = \mathcal{O}(X)$, and let Y be the singularity subscheme of X. Then*

$$c_{\mathrm{wMa}}(Y) = (-1)^{\dim X - \dim Y} c(\mathcal{L}) \cap (c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X)) \quad \textit{in } A_*(X).$$

That is, $c_{\mathrm{wMa}}(Y)$ essentially measures the difference between the functorial homology Chern class $c_{\mathrm{SM}}(X)$ and the class of the virtual tangent bundle of X. The functoriality of the class $c_{\mathrm{SM}}(X)$ was proved by Robert MacPherson [10]; the class was later shown to agree with the class previously defined by Marie-Hélène Schwartz. For a treatment of Schwartz-MacPherson's classes over any algebraically closed field of characteristic 0, see [8]; this is the context we assume here. Also, we let $c_{\mathrm{SM}}(X) = c_{\mathrm{SM}}(X_{\mathrm{red}})$; with this proviso, Theorem 1.2 holds for nonreduced hypersurfaces X—remarkably, the drastic change in c_{F} when some component of X is replaced by a multiple is precisely compensated by the change in the weighted Mather class of the singularity subscheme.

For the next result, it is convenient to employ the following notations (a variation on the notations used in [2], [3]): for $a \in A_p$ and $\mathcal{L}$ a line bundle, set

$$a_\vee = (-1)^p a \quad , \quad a_{\mathcal{L}} = c(\mathcal{L})^p \cap a \quad .$$

(So $a_{\mathcal{L}} = c(\mathcal{L})^n \cap (a \otimes \mathcal{L})$, where the term in () uses the definition in [3], and n is the dimension of the ambient scheme). These notations behave well with respect to several natural operations, similarly to the notations introduced in [3]. For example, the formula on the right defines an action of Pic on the Chow group: that is, $a_{\mathcal{L}_1 \otimes \mathcal{L}_2} = (a_{\mathcal{L}_1})_{\mathcal{L}_2}$ for line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proposition 1.3. *Let Y be the singularity subscheme of a section of a line bundle $\mathcal{L}$ on a nonsingular variety M. Then*

$$c_{\mathrm{wMa}}(Y) = (-1)^{\dim Y} \left(c(T^*M \otimes \mathcal{L}) \cap s(Y, M)\right)_{\vee \mathcal{L}} \quad \textit{in } A_*Y.$$

Here $s(Y, M)$ denotes the *Segre class* of Y in M, in the sense of [7], Chapter 4. Note that this equality is completely false unless Y is a singularity subscheme of a hypersurface in M. However, if Y *is* a singularity subscheme of a hypersurface in M, then the right-hand-side must be independent of M: this was proved directly in [1], Corollary 1.7, and follows again as the left-hand-side is intrinsic of Y. Proposition 1.3 is significant in view of the consequence:

Corollary 1.4.

$$\mu_{\mathcal{L}}(Y) = (-1)^{\dim Y} c_{\mathrm{wMa}}(Y)_{\vee \mathcal{L}} \quad .$$

The class $\mu_{\mathcal{L}}(Y)$ is the 'μ-class' defined and studied in [1]; it carries a notable amount of information about X, with applications to duality and to the study of contacts of hypersurfaces. Corollary 1.4 solves a puzzle left open in [1] (p. 326): to define a class for arbitrary schemes, specializing to $\mu_{\mathcal{L}}(Y)$ for singular schemes of hypersurfaces. It also clarifies the dependence of the μ-class on the line bundle $\mathcal{L}$: it follows from Corollary 1.4 that if $\mathcal{L}_1$, $\mathcal{L}_2$ are line bundles, then

$$\mu_{\mathcal{L}_2}(Y) = \mu_{\mathcal{L}_1}(Y)_{\mathcal{L}_1^\vee \otimes \mathcal{L}_2}$$

(this does *not* follow formally from the expression for the μ-class in terms of the Segre class of Y.) For applications of Proposition 1.3 and Corollary 1.4, see Examples 3.4, 3.5.

The next fact we list also requires some notations. We now assume that X is a reduced hypersurface, over $\mathbb{C}$. The question is whether, in this particularly 'geometric' case, $c_{\mathrm{wMa}}(Y)$ can be recovered from numerical invariants of X. The answer comes from [12]: define a function $\mu : Y \to \mathbb{Z}$ by setting $\mu(y) = (-1)^{\dim X}(\chi(y) - 1)$, where $\chi(y)$ is the Euler characteristic of the Milnor fiber of X at y; μ is a constructible function on Y, so we can apply to it MacPherson's transformation c_{SM} (that is, write μ as a linear combination of characteristic functions 1_Z for subvarieties Z of Y, then replace each 1_Z in this combination by $c_{\mathrm{SM}}(Z)$).

Theorem 1.5.

$$c_{\mathrm{wMa}}(Y) = (-1)^{\dim Y} c_{\mathrm{SM}}(\mu) \quad \textit{in } A_*Y.$$

Equivalently, write μ as a linear combination of local Euler obstructions (also an ingredient in [10]): $\mu = \sum \ell_i \mathrm{Eu}_{Y_i}$; then the content of Theorem 1.5 is that in this situation the Y_i's are precisely the supports of the components of the normal cone of Y, and the numbers ℓ_i determined by μ agree (up to sign) with the multiplicities m_i used to define $c_{\mathrm{wMa}}(Y)$. Again, we would be very interested in extensions of this result to more general Y: what numerical invariants of a space X (not necessarily a hypersurface) determine the multiplicities of the components of the normal cone of its singularity subscheme? Can these multiplicities be computed for an arbitrary scheme Y, by a similar 'Milnor fiber' approach? Once more, note that the left-hand-side in Theorem 1.5 is defined for arbitrary Y; to what extent can the right-hand-side also be defined for arbitrary Y? We know of several problems in enumerative geometry for which finding these multiplicities is one of the main computational ingredients. For an explicit computation (not directly related to enumerative geometry) see Example 3.6.

Finally, it would be interesting to have results on the functoriality of the class $c_{\mathrm{wMa}}(Y)$; little is known about the functoriality of the ordinary Chern-Mather class. Again, something can be said if Y is the singularity subscheme of a hypersurface X (over an arbitrary algebraically closed field of characteristic zero, and possibly nonreduced). Let Z be a nonsingular subvariety of $Y \subset X \subset M$, and consider the blow-up $\widetilde{M}$ of M along Z:

$$\begin{tikzcd} & Y' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\rho"] & \widetilde{M} \arrow[d, "\pi"] \\ Z \arrow[r] & Y \arrow[r] & X \arrow[r] & M \end{tikzcd} .$$

Here $X' = \pi^{-1}X$ is the scheme-theoretic inverse image of X, a hypersurface of $\widetilde{M}$, and Y' is the singularity subscheme of X'.

Proposition 1.6. *Assume Z has codimension d in M. Then*

$$\rho_* c_{\mathrm{wMa}}(Y') = (-1)^{\dim X - \dim Y} c_{\mathrm{wMa}}(Y) - (d-1) c_{\mathrm{wMa}}(Z) \quad \textit{in } A_* X.$$

Here of course $c_{\mathrm{wMa}}(Z) = c(TZ) \cap [Z]$, as Z is nonsingular. Also note that by assumption X is singular along Z, hence Y' contains the exceptional divisor in $\widetilde{M}$.

Proofs of the statements made in this section are sketched in §2, with emphasis on Theorem 1.2, which relates the weighted Chern-Mather class of the singularity of a hypersurface with its *Milnor class.*

§2. The Milnor class of a hypersurface.

As is well known, for a compact complex hypersurface X with isolated singularities the sum of the Milnor numbers of the singularities measures the difference between the topological Euler characteristic of X and that of a nonsingular hypersurface linearly equivalent to X (if there is such a hypersurface). To my knowledge, the first who used this fact to define and study a generalization of the Milnor number to non-isolated hypersurface singularities is Adam Parusiński, [11].

Now, the functoriality of Schwartz-MacPherson's class implies that, for a hypersurface X as above, the Euler characteristic of X equals the degree of the (zero-dimensional component of the) class $c_{\mathrm{SM}}(X)$. On the other hand, the Euler characteristic of a nonsingular hypersurface linearly equivalent to X equals the degree of the class of the virtual tangent bundle of X (that is, of $c_{\mathrm{F}}(X)$ with notations as in §1). That is, Parusiński's Milnor number equals (up to a sign), the degree of the difference between the two classes:

$$\int (c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X)) \quad .$$

It is natural then to study the whole class $c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X)$; this (or slight variations of it) has been named the *Milnor class* of X by some authors (see [5], [12], [14]).

Note that nothing in the definition of the class $c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X)$ requires X to be a hypersurface: both Schwartz-MacPherson's and Fulton's classes can be defined for arbitrary varieties. For reduced compact complex local complete intersections, the Milnor class is computed in homology in [5] in terms of vector fields on X, an approach reminiscent of Schwartz's definition of $c_{\mathrm{SM}}(X)$.

In fact the class makes sense for arbitrary schemes X over any algebraically closed field of characteristic 0, and naturally lives in the Chow group A_*Y of the singular locus of X. We would like to pose the following question:

—*To what extent is the Milnor class of X determined by the singularity subscheme Y of X?* or, in more ambitious terms:

—*Is there a natural definition of a class on an arbitrary scheme Y, from which the Milnor class of X can be computed if Y is the singularity subscheme of X?*

In view of the results collected in §1, the situation is clear for hypersurfaces. The singular locus of a hypersurface has a natural scheme structure, given by the partial derivatives of local equations of X. Theorem 1.2 then asserts that (for arbitrary hypersurfaces X over an algebraically closed field of characteristic 0, and writing $\mathcal{L} = \mathcal{O}(X)_{|Y}$)

$$c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X) = (-1)^{\dim X - \dim Y} c(\mathcal{L})^{-1} \cap c_{\mathrm{wMa}}(Y) \quad \text{in } A_*X:$$

that is, *if two hypersurfaces have the same singularity subscheme Y and their line bundles restrict to the same bundle on Y, then they have the same Milnor class; and, further, this can be recovered from the class $c_{\mathrm{wMa}}(Y)$, which can be defined for arbitrary schemes Y.*

Therefore, Theorem 1.2 answers the two questions posed above, for hypersurfaces. To our knowledge, the questions are completely open for more general schemes X. Milnor classes of local complete intersections (for which the singular locus also carries a natural scheme structure) have been studied in [5], but from a different viewpoint, which does not seem to address questions such as the ones posed above.

Theorem 1.2 could be deduced from results in the existing literature (particularly from [12] or [2]). However, while the main result in [2] is at the level of generality at which we are aiming, its proof is rather unenlightening. The approach in [12] is much more cogent, but it is stated in homology and relies on the complex geometry of the situation—for example, in [12] the hypersurface is assumed to be reduced and compact. The argument given below works for possibly nonreduced hypersurfaces, over arbitrary algebraically closed fields of characteristic 0, and gives the formula in rational equivalence; it only relies on the basic formalism of Schwartz-MacPherson's classes (as developed in [8]). We would like to stress that, anyway, at its core is a multiplicity computation we learned from [12].

Proof of Theorem 1.2. We consider the blow-up $\widetilde{M} \xrightarrow{\pi} M$ along Y, and let $\mathcal{X}$, $\mathcal{Y}$ be the pull-back of X and the exceptional divisor, respectively. Note that $\mathcal{Y} \subset \mathcal{X}$, so[1] there is an effective Cartier divisor in $\widetilde{M}$ whose cycle equals $\mathcal{X} - \mathcal{Y}$; we will denote this divisor by $\mathcal{X} - \mathcal{Y}$. Now let p be a point of X. We have $\pi^{-1}(p) \subset \mathcal{X} - \mathcal{Y}$, so it makes sense to consider the Segre class of $\pi^{-1}(p)$ in $\mathcal{X} - \mathcal{Y}$.

[1] A note of warning to non-algebraic geometers: here and in the following we are using common set-theoretic notations (such as $\subset$, $-$, etc.) in their scheme-theoretic sense. For example, $\mathcal{Y} \subset \mathcal{X}$ means that the ideal sheaf of $\mathcal{X}$ is contained in the ideal sheaf of $\mathcal{Y}$. Since both $\mathcal{X}$ and $\mathcal{Y}$ are Cartier divisors, this just says that local equations for $\mathcal{X}$ are multiples of local equations for $\mathcal{Y}$. This is necessary for the statement that follows.

Claim 2.1. *Denoting degree by $\int$,*

$$\int \frac{s(\pi^{-1}(p), \mathcal{X} - \mathcal{Y})}{1 + \mathcal{X} - \mathcal{Y}} = 1 \quad .$$

A preliminary result is in order before we prove this claim. We have

$$\pi^{-1}(p) \hookrightarrow (\mathcal{X} - \mathcal{Y}) \hookrightarrow \widetilde{M} \quad ,$$

where the second embedding is regular. We claim that

$$s(\pi^{-1}(p), \mathcal{X} - \mathcal{Y}) = c(N_{\mathcal{X}-\mathcal{Y}}\widetilde{M}) \cap s(\pi^{-1}(p), \widetilde{M}) \quad .$$

Note that this is *not* automatic in this situation, cf. Example 4.2.8 in [7]. In our case, it will follow from the following lemma:

Lemma 2.2. *Let D, E be hypersurfaces in a variety V. Assume that $D - E$ is positive and has no components in common with E. Then $s(E, D) = c(N_D V) \cap s(E, V)$.*

Proof of the lemma. By the hypothesis and Lemma 4.2 in [7],

$$\begin{aligned} s(E, D) &= s(E, E) + s(E \cap (D - E), D - E) = [E] + \frac{E \cdot (D - E)}{1 + E} \\ &= \frac{([E] + E \cdot E) + E \cdot (D - E)}{1 + E} = (1 + D) \cap \frac{[E]}{1 + E} \\ &= c(N_D V) \cap s(E, V) \quad . \quad \square \end{aligned}$$

Proof of Claim 2.1. We apply Lemma 2.2 to the normalized blow-up V of $\widetilde{M}$ along $\pi^{-1}(p)$, with $E=$the exceptional divisor, and $D=$the inverse image of $\mathcal{X} - \mathcal{Y}$. To see that the hypotheses are satisfied, we have to show that every component of E appears with the same multiplicity in E and D.

For this[2], let $\gamma(t)$ be a germ of a nonsingular curve centered at the general point of a component of E, let $\tilde{\gamma}(t)$ be the composition to M, and let F be a local equation for X at p; also, choose local parameters $x_1, \dots, x_n$ for M at p. The ideal of E is the pull-back of $(x_1, \dots, x_n)$ to V, so the multiplicity m_E of the component in E equals the order of vanishing of the pull-back $x_i(t) = \tilde{\gamma}^* x_i$ of a generic local parameter. The multiplicity m_D in D equals $m_{\mathcal{X}} - m_{\mathcal{Y}}$, where $m_{\mathcal{X}}$, $m_{\mathcal{Y}}$ are respectively the multiplicities in the pull-backs of $\mathcal{X}$, $\mathcal{Y}$.

Now $m_{\mathcal{X}}$ is the order of vanishing of

$$\tilde{\gamma}^* F = F(x_1(t), \dots, x_n(t)) \quad ,$$

while $m_{\mathcal{Y}}$ is the order of vanishing of the pull-back of

$$\left(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right) \quad ,$$

[2]This computation is essentially lifted from an analogous computation in the proof of Proposition 2.2 in [12].

that is, the order of vanishing of $\tilde{\gamma}^* \frac{\partial F}{\partial x_i}$ for a generic local parameter x_i. Now taking the derivative with respect to t gives (by the chain rule!)

$$m_{\mathcal{X}} - 1 = m_{\mathcal{Y}} + m_E - 1 \quad ,$$

from which the desired equality $m_E = m_D$ follows.

Applying Lemma 2.2, we get

$$s(E, D) = (1 + \mathcal{X} - \mathcal{Y}) \cap s(E, V) \quad ,$$

hence

$$s(\pi^{-1}(p), \mathcal{X} - \mathcal{Y}) = (1 + \mathcal{X} - \mathcal{Y}) \cap s(\pi^{-1}(p), \widetilde{M})$$

by the birational invariance of Segre classes ([7], Proposition 4.2). From this,

$$\pi_* \frac{s(\pi^{-1}, \mathcal{X} - \mathcal{Y})}{1 + \mathcal{X} - \mathcal{Y}} = s(p, M) = [p] \quad ,$$

again by the birational invariance of Segre classes, and the claim follows by taking degrees. □

We are finally ready to prove Theorem 1.2. Identify $\mathcal{Y}$ with the projective normal cone of Y in M, let $\mathcal{Y}_i$ be the reduced components of $\mathcal{Y}$, and let Y_i be their support in Y. Then

$$\begin{cases} \mathcal{X} = \widetilde{X} + \sum n_i \mathcal{Y}_i \\ \mathcal{Y} = \sum m_i \mathcal{Y}_i \end{cases}$$

for suitable m_i, n_i. By Claim 2.1,

$$\begin{aligned} 1 &= \int \frac{s(\pi^{-1}(p), \mathcal{X} - \mathcal{Y})}{1 + \mathcal{X} - \mathcal{Y}} \\ &= \int \frac{s(\pi^{-1}(p) \cap \widetilde{X}, \widetilde{X}) + \sum (n_i - m_i) s(\pi^{-1}(p) \cap \mathcal{Y}_i, \mathcal{Y}_i)}{1 + \mathcal{X} - \mathcal{Y}} \end{aligned}$$

by Lemma 4.2 in [7]

$$= \mathrm{Eu}_X(p) + \sum (n_i - m_i)(-1)^{\dim M + 1 - \dim Y_i} \mathrm{Eu}_{Y_i}(p)$$

using the formula for Euler obstructions due to Gonzalez-Sprinberg and Verdier, as computed in [8], Lemma 2 (as pointed out in [2], §1.3 and in [12], §3, the divisor $\mathcal{X} - \mathcal{Y}$ can be embedded in $\mathbb{P}(T^*M)$, and $1 + \mathcal{X} - \mathcal{Y}$ is then the restriction of the class of the tautological bundle in $\mathbb{P}(T^*M)$). Now, every relation between constructible functions yields a relation for characteristic classes. Here, this gives (using the formula for Mather's classes in [8], Lemma 1, going back to Claude Sabbah):

$$\begin{aligned} c_{\mathrm{SM}}(X) &= c_{\mathrm{Ma}}(X) + \sum (n_i - m_i)(-1)^{\dim M + 1 - \dim Y_i} c_{\mathrm{Ma}}(Y_i) \\ &= c(TM) \cap \pi_* \left(\frac{[\widetilde{X}]}{1 + \mathcal{X} - \mathcal{Y}} + \sum (n_i - m_i) \frac{[\mathcal{Y}_i]}{1 + \mathcal{X} - \mathcal{Y}} \right) \\ &= c(TM) \cap \pi_* \left(\frac{[\mathcal{X}]}{1 + \mathcal{X}} - \frac{1}{1 + \mathcal{X}} \sum m_i \frac{[\mathcal{Y}_i]}{1 + \mathcal{X} - \mathcal{Y}} \right) \\ &= c_{\mathrm{F}}(X) + c(\mathcal{L})^{-1} \cap \sum m_i (-1)^{\dim M - \dim Y_i} c_{\mathrm{Ma}}(Y_i) \\ &= c_{\mathrm{F}}(X) + (-1)^{\dim M - \dim Y} c(\mathcal{L})^{-1} \cap c_{\mathrm{wMa}}(Y) \end{aligned}$$

which is the desired formula. $\square$

As observed in the proof, $\mathcal{X} - \mathcal{Y}$ can be naturally embedded in $\mathbb{P}(T^*M)$. The content of Claim 2.1 is that $\mathcal{X} - \mathcal{Y}$ gives then the *characteristic cycle* of X (corresponding to the characteristic function 1_X of X in M).

The other statements in §1 now follow easily, either by comparing the expression for c_{SM} with the expressions in [2] and [12], or by direct manipulations that can be extracted from those sources. The argument given here re-proves Theorem I.3 in [2]/Theorem 3.1 in [12]; and for example, §1 in [2] shows how to go directly from this form of the result to expressions in terms of Segre classes or μ-classes (thus proving Proposition 1.3, Corollary 1.4).

The details are left to the reader. Theorem 1.5 is our reading of Theorem 2.3 (iii) from [12]. The blow-up formula of Proposition 1.6 follows from Proposition IV.2 in [2].

§3. Examples and applications.

Normal cones behave well with respect to proper finite maps and with respect to flat maps, cf. Proposition 4.2 in [7]. For example, assume that Y is irreducible, and $\widetilde{M} \xrightarrow{\pi} M$ is a surjective birational map on the ambient space. Then there is an induced surjective map from the cone of $\pi^{-1}Y$ to the cone of Y. This can be used to obtain the data $\{(Y_i, m_i)\}$ of §1, for example by suitably blowing up an ambient space; this can lead to direct computations of weighted Chern-Mather classes.

Example 3.1. Suppose Y consists of a curve C, with an embedded multiple planar point at a point p. More precisely, assume C, Y have local ideals respectively $\mathcal{I}_C$, $\mathcal{I}_C \cdot (x,y)^m$, $m \geq 1$, near p in a nonsingular ambient surface S with local parameters x, y. Also, assume that C has multiplicity r at p. Then

$$c_{\mathrm{wMa}}(Y) = c_{\mathrm{Ma}}(C) - (m+r)[p] \quad .$$

Indeed, blow-up S at p; the total transform of Y consists of the proper transform of C, and of $(m+r)$ times the exceptional divisor. Therefore, the normal cone of Y contains a component with multiplicity $m+r$ over p. (But note there is no such component if $m = 0$.)

For example, take Y to be the union of two lines ℓ_1, ℓ_2 in $\mathbb{P}^2$, with an embedded planar point at the intersection $p = \ell_1 \cap \ell_2$; then $c_{\mathrm{wMa}}(Y) = [\ell_1] + [\ell_2] + [p]$. If the embedded point is on one of the lines, but not at p, then $c_{\mathrm{wMa}}(Y) = [\ell_1] + [\ell_2] + 2[p]$. If each line comes with multiplicity r, and the embedded point is at p, then the class is

$$r c_{\mathrm{Ma}}(\ell_1) + r c_{\mathrm{Ma}}(\ell_2) - (1+2r)[p] = r[\ell_1] + r[\ell_2] + (2r-1)[p] \quad .$$

Example 3.2. Example 3.1 can be easily generalized to the situation in which Y is a subscheme of a given ambient space M, and the residual to a Cartier divisor D in Y is a known scheme Y'. Then $c_{\mathrm{wMa}}(Y)$ can be written in terms of $c_{\mathrm{Ma}}(D)$, $c_{\mathrm{wMa}}(Y')$, and the multiplicity of D along the distinguished components of Y'; details are left to the reader. A very different expression can be obtained if Y' is the singularity subscheme of a hypersurface X in M, and D is the r-th multiple of X ($r \geq 0$).

Claim 3.1. *Let $\mathcal{L} = \mathcal{O}(X)_{|Y'}$. Then*

$$c_{\mathrm{wMa}}(Y) = r\, c_{\mathrm{F}}(X) + (-1)^{\dim X - \dim Y'} \frac{c(\mathcal{L}^{\otimes(r+1)})}{c(\mathcal{L})} \cap c_{\mathrm{wMa}}(Y') \quad .$$

The proof is an easy application of the results in §1, and is also left to the reader.

To contrast the two approaches, take again the example of the union of two lines ℓ_1, ℓ_2 in $\mathbb{P}^2$, each coming with multiplicity r, with an embedded planar point at the intersection. Since the planar point is the singularity subscheme of the union of two (simple) lines, Claim 3.1 computes the weighted Chern-Mather class of this scheme as

$$\begin{aligned} r\, c_{\mathrm{F}}(X) + (-1)^{\dim X - \dim Y'} \frac{c(\mathcal{L}^{\otimes(r+1)})}{c(\mathcal{L})} \cap c_{\mathrm{wMa}}(Y') &= r \frac{c(T\mathbb{P}^2)}{c(\mathcal{O}_{\mathbb{P}^2}(2))} \cap ([\ell_1] + [\ell_2]) - [p] \\ &= r([\ell_1] + [p]) + r([\ell_2] + [p]) - [p] \end{aligned}$$

with the same result as before, but by a very different route.

It would be useful to have formulas such as Claim 3.1, but with less stringent hypotheses on X.

Example 3.3. If $X = X_1 \cup \cdots \cup X_r$ is a divisor with normal crossings, with all X_i supported on nonsingular hypersurfaces $(X_i)_{\mathrm{red}}$, and Y is its singularity subscheme, then

$$c_{\mathrm{wMa}}(Y) = \pm c(TM) \cap \left(1 - \frac{1 + [X]}{(1 + (X_1)_{\mathrm{red}}) \cdots (1 + (X_r)_{\mathrm{red}})}\right) \cap [M] \quad ,$$

taking the sign $+$, resp. $-$ according to whether X is reduced or not. The expression is interpreted by expanding it, which leaves a class naturally supported on Y; it follows from Proposition 1.3 and [2], §2.2 (Lemma II.2 in [2] computes the Segre class if X is reduced, and the computation in the proof of Lemma II.1 is used to cover the non-reduced case).

Example 3.4. What do we learn about hypersurfaces by studying their Milnor classes?

As shown in [1], the μ-class of a hypersurface X packs a good amount of information about X: for example, the multiplicity of X as a point of the discriminant of a linear system and the dimension of this discriminant can be recovered very easily from the μ-class (hence from the Milnor class). In the classical language, the μ-classes of hyperplane sections of an embedded nonsingular projective variety M give a localized analog of the *ranks* of M, and provide a natural tool to study projective duality.

In a different direction, the good behavior of the μ-class can be used to put restrictions on the possible singularities of a hypersurface in a given ambient space. Several examples of this phenomenon are illustrated in [1], §3, where the main tool was the observation that if the singularity subscheme Y of a hypersurface X is *nonsingular,* then

$$\mu_{\mathcal{L}}(Y) = c(T^*Y \otimes \mathcal{L}) \cap [Y] \quad .$$

Now, Corollary 1.4 from §1:

$$\mu_{\mathcal{L}}(Y) = (-1)^{\dim Y} c_{\mathrm{wMa}}(Y)_{\vee \mathcal{L}}$$

is a substantial upgrade of this formula, and this allows us to extend some of those results.

Claim 3.2. *If two smooth hypersurfaces of degree d_1, d_2 in projective space are tangent along a positive dimensional set, then $d_1 = d_2$.*

More generally, if two smooth hypersurfaces M_1, M_2 of a variety V are tangent along an irreducible (for simplicity) set Z, and $\dim Z > 0$, then we claim that

$$r\, M_1 \cdot [Z] = r\, M_2 \cdot [Z] \quad ,$$

where r is the order of tangency of M_1 and M_2 (for example, $r = 1$ if M_1, M_2 have simple contact). This is essentially Proposition IV.7 in [2], with all hypotheses on the contact locus (except the positive dimensionality) removed. The stronger statement given above follows from the results in §1. Indeed, in the situation of the statement, let $X = M_1 \cap M_2$; then X is a hypersurface in two distinct ways: with respect to $\mathcal{L}_2 = \mathcal{O}(M_2)_{|M_1}$ in M_1, and with respect to $\mathcal{L}_1 = \mathcal{O}(M_1)_{|M_2}$ in M_2. The contact locus is $Y = \operatorname{Sing} X$ (with the scheme structure specified in §1), and $[Y] = r[Z]$. By Theorem 1.2

$$c(\mathcal{L}_2)^{-1} \cap c_{\mathrm{wMa}}(Y) = c(\mathcal{L}_1)^{-1} \cap c_{\mathrm{wMa}}(Y) \quad ,$$

implying

$$c_1(\mathcal{L}_1) \cap [Y] = c_1(\mathcal{L}_2) \cap [Y] \quad ,$$

which is the statement.

Example 3.5. We say that a hypersurface X of a nonsingular variety M is (analytically) 'homogeneous at p' if the equation of X is homogeneous for some choice of system of parameters in the completion of the local ring for M at p. We are going to consider degree-d hypersurfaces X in $\mathbb{P}^n$, whose singular scheme Y has a connected component supported on a nonsingular curve C of genus g and degree r; we assume that Y has the reduced structure at all but finitely many points $q_1, \dots, q_s$, and that X is homogeneous at each of the q_i. In particular, X has multiplicity 2 at all other points of C; we let m_i be the multiplicity of X at q_i.

How constrained is this situation? Examples 3.4—3.6 in [1] deal with the case in which the singular scheme is reduced, that is, there are no points 'q_i' as above. This situation is then very rigid: for example, one sees that only quadrics can have singular scheme equal to a line, and no hypersurface in projective space can have singular scheme equal to a twisted cubic (cf. p. 347 in [1]).

The natural expectation would be that letting the singular scheme be nonreduced should allow many more examples. For instance, cones over nodal plane curves give examples of hypersurfaces in $\mathbb{P}^3$ of arbitrary degree ≥ 2 and singular scheme generically reduced, but with an embedded homogeneous point (at the vertex). However, the results in this paper show that the situation is still quite rigid:

Claim 3.3. *Under the hypotheses detailed above, $(n-1)$ must divide $4(g+r-1)$. In fact, necessarily*

$$(n-1)\left((d-2)r - \sum (m_i - 2)\right) = 4(g+r-1) \quad .$$

For example, twisted cubics can support singularity subschemes as above only in dimensions $n = 3, 5, 9$, regardless of the number of embedded points allowed on them. (We do not know if such examples do exist.) The only situation in unconstrained dimension is for $g+r-1=0$, that is, $g=0$ and $r=1$: lines are the only nonsingular curves in projective space which may support a generically reduced singularity subscheme in all dimensions (under the local homogeneity assumption). Further, if Y is supported on a line and only has *one* embedded homogeneous point, then the formula implies that the multiplicity of X at this point is d; therefore, X is necessarily a cone in this case.

For $\sum(m_i-2)=0$, the formula in Claim 3.3 recovers the formula at p. 347 of [1] (that is, the reduced case). For $n=2$, the hypotheses imply that X is a plane curve consisting of a double component C and a residual curve of degree $(d-2r)$; the formula then follows from the genus formula and Bézout's theorem. In higher dimensions, the following argument is the only proof we know.

Proof of the claim. We compute directly the weighted Chern-Mather class of Y and the Segre class $s(Y,\mathbb{P}^n)$. Proposition 1.3 gives a relation between these two classes, and the formula follows by taking degrees.

Explicitly, blow-up $\mathbb{P}^n$ at the 'special' points $q_1,\dots,q_s$, and then along the proper transform of the curve C. The homogeneity hypothesis implies that the (scheme-theoretic) inverse image of Y in the top blow-up is a Cartier divisor, with a component of multiplicity 1 dominating C, and s components with multiplicity $(m_1-1),\dots,(m_s-1)$ dominating the q_i's. The Segre class of Y in $\mathbb{P}^n$ is then computed by using the birational invariance of Segre classes, and we get

$$i_*s(Y,\mathbb{P}^n)=r[\mathbb{P}^1]+\left(s(n-1)+2-2g-r(n+1)+\sum_i((m_i-1)^n-n(m_i-1))\right)[\mathbb{P}^0]$$

(where $i:Y\hookrightarrow\mathbb{P}^n$ is the inclusion).

On the other hand, the component dominating q_i maps to a corresponding component of the projective normal cone to Y in $\mathbb{P}^n$; computing differentials, we see that this map has degree $(m_i-1)^{n-1}-1$. This allows us to compute the weighted Chern-Mather class of Y:

$$c_{\mathrm{wMa}}(Y)=c_{\mathrm{Ma}}(C)-\sum_i\left((m_i-1)^{n-1}-1\right)(m_i-1)c_{\mathrm{Ma}}(q_i)\quad,$$

from which

$$i_*c_{\mathrm{wMa}}(Y)=r[\mathbb{P}^1]+\left(2-2g-\sum_i((m_i-1)^n-(m_i-1))\right)[\mathbb{P}^0]\quad.$$

Now let h denote the hyperplane class in $\mathbb{P}^n$. The expression for the Segre class gives

$$\begin{aligned}i_*c(T^*\mathbb{P}^n\otimes\mathcal{O}(d))\cap s(Y,\mathbb{P}^n)&=i_*\frac{(1+(d-1)h)^{n+1}}{1+dh}\cap s(Y,\mathbb{P}^n)\\&=r[\mathbb{P}^1]+\left((s+rd-2r)(n-1)+2-2g-4r+rd+\sum_i((m_i-1)^n-n(m_i-1))\right)[\mathbb{P}^0]\end{aligned}$$

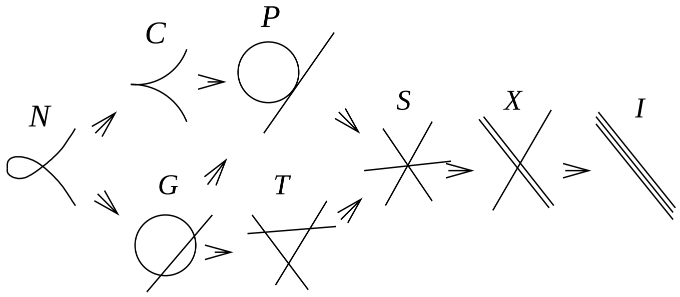
and therefore

$$\begin{aligned}&i_*(-1)^{\dim Y}\left(c(T^*M\otimes\mathcal{L})\cap s(Y,M)\right)_{\vee\mathcal{L}}=r[\mathbb{P}^1]\\&\quad+\left((2r-dr-s)(n-1)-2+2g+4r-\sum_i((m_i-1)^n-n(m_i-1))\right)[\mathbb{P}^0].\end{aligned}$$

By Proposition 1.3, this class must equal $i_*c_{\mathrm{wMa}}(Y)$. Equating the two expressions gives the formula in the statement. □

Example 3.6. Finally, we give an example of the use of weighted Chern-Mather classes in the computation of the multiplicities of components of a normal cone. Such multiplicities are important for enumerative applications, and it would be very useful to develop tools to compute them. For singularity subschemes of hypersurfaces, the connection between weighted Chern-Mather classes and Milnor classes often lets us recover these multiplicities from computations of MacPherson's classes and local Euler obstructions. It would be interesting to extend such techniques to more general schemes.

Let D be the hypersurface of $\mathbb{P}^9$ parametrizing *singular* plane cubics, and let Y be its singularity subscheme. The following picture represents the natural stratification of D (with arrows denoting specialization):



The scheme Y is supported on the union of the closures $\overline{C}$, $\overline{G}$ of the loci parametrizing cuspidal cubics and binodal cubics. What are the multiplicities of the components of the normal cone of Y in $\mathbb{P}^9$? The point here is that we can compute $c_{\mathrm{wMa}}(Y)$ without knowing these multiplicities:

Claim 3.4. *Denote by i the inclusion of Y in $\mathbb{P}^9$. Then*

$$i_* c_{\mathrm{wMa}}(Y) = 69[\mathbb{P}^7]+120[\mathbb{P}^6]+210[\mathbb{P}^5]+252[\mathbb{P}^4]+210[\mathbb{P}^3]+120[\mathbb{P}^2]+45[\mathbb{P}^1]+10[\mathbb{P}^0].$$

Proof. This follows from Theorem 1.2 and the computations of characteristic classes for D in §4 of [4]. □

Now the task is to find the coefficients expressing the weighted Chern-Mather class of Y as a combination of the Chern-Mather classes of the loci C, G, etc. We first find the constructible function ν corresponding to $c_{\mathrm{wMa}}(Y)$ under MacPherson's transformation. For this, we use the result of the computation from [4] of Chern-Schwartz-MacPherson's classes of the strata of D. Writing $c_{\mathrm{wMa}}(Y) = c_{\mathrm{SM}}(\nu) = \nu(C)\cdot c_{\mathrm{SM}}(1_C) + \nu(G)\cdot c_{\mathrm{SM}}(1_G) + \dots$ and solving the resulting system of linear equations, we find

$$\nu(C) = 2;\ \nu(G) = 1;\ \nu(P) = 0;\ \nu(T) = 1;\ \nu(S) = 3;\ \nu(X) = 1;\ \nu(I) = 1.$$

(The paragraph preceding the statement of Theorem 1.5 gives a geometric interpretation of $\mu = -\nu$.) As pointed out in the discussion following Theorem 1.5, to find the multiplicities we now need to express this constructible function as a combination of local Euler obstructions of the strata. These are easy to compute in codimension one, and we proceed to the computation of the multiplicities for the components dominating the loci $\overline{C}$, $\overline{G}$, $\overline{P}$, $\overline{T}$. For these loci, we only need to observe that $\overline{C}$, $\overline{G}$ are nonsingular along P, and $\overline{G}$ has multiplicity 3 along T (these follows from easy local computations). As the local Euler obstruction agrees with the multiplicity in codimension one, this gives

$$\mathrm{Eu}_C = \begin{cases} & \cdots \\ 0 & T \\ 1 & P \\ 0 & G \\ 1 & C \end{cases} \quad , \quad \mathrm{Eu}_G = \begin{cases} & \cdots \\ 3 & T \\ 1 & P \\ 1 & G \\ 0 & C \end{cases} \quad ,$$

where we indicate the value of the function at the general point of the listed locus. Therefore

$$\nu = 2\mathrm{Eu}_C + \mathrm{Eu}_G - 3\mathrm{Eu}_P - 2\mathrm{Eu}_T + \dots$$

from which we read that the multiplicities of the components of the normal cone are: 2 over C, 1 over G, 3 over P, 2 over T.

Finding the multiplicities over the remaining three loci S, X, I requires computing the local Euler obstructions for all the strata of D. We leave this to the motivated reader.

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