Plumbing Graphs for Normal Surface-Curve Pairs [1]

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Abstract

Consider the set of surface-curve pairs $(X, \mathcal{C})$, where X is a normal surface and $\mathcal{C}$ is an algebraic curve. In this paper, we define a family $\mathcal{F}$ of *normal* surface-curve pairs, which is closed under coverings, and which contains all *smooth* surface-curve pairs $(X, \mathcal{C})$, where X is smooth and $\mathcal{C}$ has smooth irreducible components with normal crossings. We give a modification of W. Neumann's definition of plumbing graphs, their associated 3-dimensional graph manifolds, and intersection matrices, and use this construction to describe rational intersection matrices and boundary manifolds for regular branched coverings.

1. Introduction. Let $(X, \mathcal{C})$ be a surface-curve pair, consisting of a normal surface X and an algebraic curve $\mathcal{C} \subset X$. The boundary manifold of a regular neighborhood $M(X, \mathcal{C})$ of $\mathcal{C}$ in X can be described simply by taking a smooth model $(\widetilde{X}, \widetilde{\mathcal{C}})$ of $(X, \mathcal{C})$ using W. Neumann's associated plumbing graphs $\Gamma_{\text{plumb}}(X, \mathcal{C})$ (see [Neu]). The *intersection matrix* $S(X, \mathcal{C})$ of a surface-curve pair $(X, \mathcal{C})$ is the matrix with entries the pairwise rational intersections of irreducible components of $\mathcal{C}$ with respect to some ordering. When $(X, \mathcal{C})$ is a *smooth* surface-curve pair, where X is smooth and $\mathcal{C}$ has smooth irreducible components with normal crossings, the intersection matrix $S(X, \mathcal{C})$ only depends on the combinatorics of $\mathcal{C}$, and thus is also determined by $\Gamma_{\text{plumb}}(X, \mathcal{C})$. Neumann defines the intersection matrix $S(\Gamma_{\text{plumb}})$ for the plumbing graph of a smooth surface-curve pair $(X, \mathcal{C})$, so that $S(X, \mathcal{C}) = S(\Gamma_{\text{plumb}}(X, \mathcal{C}))$.

A modified definition of plumbing graphs is useful for dealing with branched coverings. A *(regular) covering of surface-curve pairs*

$$\rho : (Y, \mathcal{D}) \to (X, \mathcal{C})$$

is a finite surjective morphism

$$\rho : Y \to X$$

so that $\mathcal{D} = \rho^{-1}(\mathcal{C})$ and the restriction

$$\rho : Y \setminus \mathcal{D} \to X \setminus \mathcal{C}$$

is a (regular) unbranched covering. Even if $(X, \mathcal{C})$ is a smooth surface-curve pair, the covering $(Y, \mathcal{D})$ of $(X, \mathcal{C})$ need not be smooth.

[1] Dedicated to Peter Orlik on his 60th birthyear

Let $\mathcal{S}$ be the collection of smooth surface-curve pairs. We will define a family $\mathcal{F}$ of *normal* surface-curve pairs, which contains $\mathcal{S}$ and is closed under coverings, in the sense that: if $(X, \mathcal{C}) \in \mathcal{F}$, and $\rho : (Y, \mathcal{D}) \to (X, \mathcal{C})$ is a covering of surface-curve pairs, then $(Y, \mathcal{D}) \in \mathcal{F}$. We modify Neumann's definition of plumbing graphs and their intersection matrices to describe the local topology of surface-curve pairs in $\mathcal{F}$ and their intersection matrices. This gives a method for studying coverings and computing intersection matrices without having to pass to smooth models, and generalizes the results of [Hir1] and [Hir2], where formulas for intersection matrices of abelian coverings are given.

The reader is reminded of basic definitions and properties of graphs of groups and complexes in Section 2. The modified definition of plumbing graphs, and their associated 3-manifolds and coverings are given in Sections 3. Section 4 contains a definition of normal surface-curve pairs, their associated plumbing graphs, and associated intersection matrices. Formulas for invariants of the plumbing graph of a covering of a normal surface-curve pair from covering data are given in Section 5.

2. Graphs of groups and complexes. The concept of plumbing graph comes out of a more general construction by which finite CW-complexes and finitely generated groups are described in terms of information attached to the nodes and vertices of a graph. We give the basics of these definitions in this section.

By a *graph* Γ we mean a collection of vertices $\mathcal{V}(\Gamma)$ and oriented edges $\mathcal{Y}(\Gamma)$. For any $y \in \mathcal{Y}(\Gamma)$, we write $o(y)$ for the initial point and $t(y)$ for the terminal point. We will always assume that graphs are *finite* and *connected*. Furthermore, given $y \in \mathcal{Y}(\Gamma)$, we will assume $\overline{y} \in \mathcal{Y}(\Gamma)$, where

$$\begin{aligned} o(\overline{y}) &= t(y), \qquad \text{and} \\ t(\overline{y}) &= o(y). \end{aligned}$$

For any vertex $v \in \mathcal{V}(\Gamma)$, denote by $d(v)$ the degree of the graph Γ at v.

A *graph* of groups $G(\Gamma)$ over Γ is a collection of groups

$$\begin{aligned} &G_v, \quad v \in \mathcal{V}(\Gamma), \\ &G_y, \quad y \in \mathcal{Y}(\Gamma), \end{aligned}$$

so that $G_y = G_{\overline{y}}$; and monomorphisms

$$h : G_y \to G_{t(y)},$$

for each $y \in \mathcal{Y}(\Gamma)$.

A *path* on Γ is an ordered, possibly empty, collection

$$c = (y_1, \ldots, y_k),$$

where

$$y_i \in \mathcal{Y}(\Gamma) \qquad \text{for } i = 1, \ldots, k,$$

and

$$t(y_i) = o(y_{i+1}), \qquad \text{for } i = 1, \ldots, k-1.$$

Given a path $c = (y_1, \ldots, y_k)$ on Γ and collection $r = (r_0, \ldots, r_k)$, where $r_0 \in G_{o(y_1)}$, and $r_i \in G_{t(y_i)}$, for $i = 1, \ldots, k$. Let $|c, r|$ be the word

$$r_0 y_1 r_1 \ldots y_k r_k.$$

Let $F(G(\Gamma))$ be the group of words $|c, r|$ subject to the relations in the vertex and edge groups G_v and G_y, and the relation

$$y r \overline{y} = r_1,$$

if and only if $r = h_y(r_1)$.

The *fundamental group* $\pi_1(G(\Gamma))$ can be defined in two ways. The first is in terms of a basepoint $v_0 \in \mathcal{V}(\Gamma)$. A path $c = (y_1, \ldots, y_k)$ is a *closed circuit based at* v_0, where $v_0 \in \mathcal{V}(\Gamma)$, if

$$v_0 = o(y_1) = t(y_k).$$

The *fundamental group* $\pi_1(G(\Gamma), v_0)$ is defined to be the set of words $|c, r|$, where c is a closed circuit based at v_0.

The second way to describe the fundamental group $\pi_1(G(\Gamma))$ is in terms of a maximal tree inside Γ. A *maximal tree* $\mathcal{T}$ in Γ is a subgraph containing all vertices of Γ, and such that, given any two distinct vertices $v_1, v_2 \in \mathcal{V}(\Gamma)$, there is a unique path $c = (y_1, \ldots, y_k)$ in $\mathcal{T}$ so that

$$y_i \neq \overline{y_{i+1}},$$

for $i = 1, \ldots, k-1$, and $v_1 = o(y_1)$, $v_2 = t(y_k)$. The *fundamental group* $\pi_1(G(\Gamma), \mathcal{T})$ is the group $F(G(\Gamma))$ modulo the normal subgroup generated by the edges in $\mathcal{Y}(\mathcal{T})$ thought of as elements of $F(G(\Gamma))$.

Lemma 1 ([Ser], p. 43) *The natural homomorphism*

$$\pi_1(G(\Gamma), v_0) \to \pi_1(G(\Gamma), \mathcal{T}),$$

given by including $\pi_1(G(\Gamma), v_0)$ in $F(G(\Gamma))$ and then taking the quotient by the normal subgroup generated by $\mathcal{Y}(\mathcal{T})$, is an isomorphism.

Given a maximal tree $\mathcal{T}$ of Γ, there are natural maps

$$\psi_v : G_v \to \pi_1(G(\Gamma), \mathcal{T})$$

induced by the natural inclusion of G_v in $F(G(\Gamma))$.

Lemma 2 ([Ser], Theorem 11, Corollary 1) *The maps ψ_v are monomorphisms.*

The fundamental group of $G(\Gamma)$ can also be considered as the fundamental group of a naturally associated finite CW-complex. A graph of complexes $\Sigma(\Gamma)$, is a collection of finite CW-complexes

$$X_v, \quad v \in \mathcal{V}(\Gamma),$$

and subcomplexes

$$X_y \subset X_{t(y)}, \quad y \in \mathcal{Y}(\Gamma),$$

such that the induced maps

$$\pi_1(X_y) \to \pi_1(X_{t(y)})$$

are injective, with homeomorphisms

$$h_y : X_{\overline{y}} \to X_y,$$

so that $h_{\overline{y}} = h_y^{-1}$.

Given a graph of complexes $\Sigma(\Gamma)$, the associated *graph complex*, which we will also denote by $\Sigma(\Gamma)$, is the CW-complex obtained by gluing together the X_v along the X_y according to the identifications h_y. Setting $G_v = \pi_1(X_v)$, for $v \in \mathcal{V}(\Gamma)$, and $G_y = \pi_1(X_y)$, for $y \in \mathcal{Y}(\Gamma)$, gives a corresponding graph of groups $G_\Sigma(\Gamma)$.

Theorem 3 ([Hem], Theorem 2.1) *The fundamental group of $G_\Sigma(\Gamma)$ is isomorphic to the fundamental group of $\Sigma(\Gamma)$.*

A *morphism* between graphs of complexes

$$\Psi : \Sigma'(\Gamma') \to \Sigma(\Gamma)$$

is a morphism of graphs

$$\Psi_\Gamma : \Gamma' \to \Gamma$$

and cellular maps

$$\begin{array}{rcll} \Psi_v & : & X_v \to X_{\Psi_\Gamma(v)}, & v \in \mathcal{V}(\Gamma'), \qquad \text{and} \\ \Psi_y & : & X_y \to X_{\Psi_\Gamma(y)}, & y \in \mathcal{Y}(\Gamma'), \end{array}$$

so that

$$\begin{tikzcd} X_y \arrow[r, "\Psi_y"] \arrow[d, "h_y"'] & X_{\Psi_\Gamma(y)} \arrow[d, "h_{\Psi_\Gamma(y)}"'] \\ X_{t(y)} \arrow[r, "\Psi_{t(y)}"] & X_{\Psi_\Gamma(t(y))} \end{tikzcd}$$

commutes, for all $y \in \mathcal{Y}(\Gamma')$.

An *(unbranched) covering*

$$\rho : \Sigma'(\Gamma') \to \Sigma(\Gamma)$$

is a morphism of graph complexes so that

$$\rho_\Gamma : \Gamma' \to \Gamma$$

is onto, and

$$\begin{array}{rcll} \rho_v & : & X_v \to X_{\rho_\Gamma(v)}, & v \in \mathcal{V}(\Gamma'), \qquad \text{and} \\ \rho_y & : & X_y \to X_{\rho_\Gamma(y)}, & y \in \mathcal{Y}(\Gamma') \end{array}$$

are unbranched coverings. Note that if ρ is an unbranched covering, then the induced map

$$G_{\Sigma'}(\Gamma') \to G_\Sigma(\Gamma)$$

on graphs of groups induces a monomorphism of groups

$$\rho_* : \pi_1(G_{\Sigma'}(\Gamma), v_0) \to \pi_1(G_\Sigma(\Gamma), \rho(v_0)),$$

for any $v_0 \in \Gamma'$.

An unbranched covering

$$\rho : \Sigma'(\Gamma') \to \Sigma(\Gamma)$$

is *regular* if the maps ρ_v and ρ_y are regular coverings, for all $v \in \mathcal{V}(\Gamma')$ and all $y \in \mathcal{Y}(\Gamma')$. Regular coverings $\Sigma'(\Gamma')$ of $\Sigma(\Gamma)$ correspond to epimorphisms

$$\psi : \pi_1(G_\Sigma(\Gamma)) \to F,$$

where F is a finite group.

Fix a maximal tree in Γ. A *lift*

$$\ell : \mathcal{T} \to \Gamma'$$

of $\mathcal{T}$ in the covering graph Γ', is a morphism of graphs so that

$$\begin{array}{rcll} \rho_\Gamma(\ell(v)) & = & v, & v \in \mathcal{V}(\mathcal{T}), \qquad \text{and} \\ \rho_\Gamma(\ell(y)) & = & y, & y \in \mathcal{Y}(\mathcal{T}). \end{array}$$

Identify $G_v = \pi_1(X_v)$ and $G_y = \pi_1(X_y)$ with the corresponding subgroups of

$$\pi_1(G_\Sigma(\Gamma)) = \pi_1(G_\Sigma(\Gamma), \mathcal{T}).$$

For each $v \in \mathcal{V}(\Gamma)$, let ψ_v be the restriction of ψ to G_v, and, for each $y \in \mathcal{Y}(\Gamma)$, let ψ_y be the restriction of ψ to G_y.

Let

$$\begin{aligned} F_v &= \psi_v(G_v), \quad v \in \mathcal{V}(\Gamma), \qquad \text{and} \\ F_y &= \psi_y(G_y), \quad y \in \mathcal{Y}(\Gamma). \end{aligned}$$

Note that the conjugacy classes of G_v and G_y, and hence F_v and F_y don't depend on the choice of maximal tree $\mathcal{T}$.

For $y \in \mathcal{Y}(\Gamma)$, let $s(y) = \psi(y)$, where we identify $\mathcal{Y}(\Gamma)$ with its natural image in $\pi_1(G_\Sigma(\Gamma), \mathcal{T})$.

The following propositions and corollaries follow from elementary properties of coverings.

Proposition 4 *For $v \in \mathcal{V}(\Gamma)$, the identification*

$$[\alpha F_v] = \alpha\ell(v)$$

gives a one-to-one correspondence between elements in the preimage $\rho^{-1}(v)$ cosets of F_v in F. Furthermore, for $v' \in \rho^{-1}(v)$, the covering

$$\Sigma'_{v'} \to \Sigma_v$$

has defining map

$$\psi_v : \pi_1(\Sigma_v) = G_v \to F_v.$$

Corollary 5 *The number of vertices in $\rho^{-1}(v)$ is*

$$\#|\rho^{-1}(v)| = [F : F_v]$$

where $[F : F_v]$ is the index of F_v in F. For $v' \in \rho^{-1}(v)$, the degree of the covering

$$\Sigma'_{v'} \to \Sigma_v$$

is $\#|F_v|$, the order of F_v.

Similarly, for the edges, we have the following.

Proposition 6 *For $y \in \mathcal{Y}(\Gamma)$, the identification*

$$\alpha\ell(y) = [\alpha F_y]$$

gives a one to one correspondence between the edges in $\rho^{-1}(y)$ *and cosets of* F_y *in* F *so that*

$$t(\alpha\ell(y)) = \alpha(s(y)\ell(t(y)) = [\alpha s(y)F_{t(y)}].$$

Furthermore, the covering

$$\Sigma'_{y'} \to \Sigma_y$$

has defining map

$$\psi_y : \pi_1(\Sigma_y) = G_y \to F_y.$$

Corollary 7 *For* $y \in \mathcal{Y}(\Gamma)$,

$$\#|\rho^{-1}(y)| = [F : F_y];$$

for $y' \in \rho^{-1}(y)$, *the covering*

$$\Sigma'_{y'} \to \Sigma_y$$

has degree $\#|F_y|$; *and, if* $t(y) = v$ *and* $v' \in \rho^{-1}(v)$, *we have*

$$\#\{y' \in \rho^{-1}(y) \ : \ t(y') = v'\} = \frac{\#F_v}{\#F_y}.$$

3. Plumbing graphs. In [Wal], F. Waldhausen defines a 3-dimensional *graph manifold* to be a manifold with a torus decomposition into Seifert fibered pieces, noting that this gives the manifold an underlying graph structure. Neumann distills the information using plumbing graphs in [Neu], and develops a calculus for determining the topological equivalence of two graph manifolds. In this section, we review the part of his definition of graph manifold which applies to smooth surface-curve pairs, and then define a modification which we later show applies to normal surface-curve pairs.

A *plumbing graph* $\Gamma_{\mathrm{plumb}} = \langle \Gamma, g, e\rangle$ is a finite connected graph Γ, together with maps

$$\begin{aligned} g &: \mathcal{V}(\Gamma) \to \mathbb{Z}_{\geq 0} \\ e &: \mathcal{V}(\Gamma) \to \mathbb{Z} \end{aligned}$$

Given a plumbing graph Γ_{plumb}, there is an associated graph of complexes $M(\Gamma_{\mathrm{plumb}})$ given as follows. For each vertex $v \in \mathcal{V}(\Gamma)$, let S_v be an oriented surface of genus $g(v)$, with $d(v)$ boundary components, labeled by the edges $y \in \mathcal{Y}(\Gamma)$, where $t(y) = v$; and let $f_v : M_v \to S_v$ be an S^1-bundle map, with trivializations at the boundary components of S_v, so that f_v has euler number $e(v)$.

Let $h : S^1 \times S^1 \to S^1 \times S^1$ be the automorphism defined by $h(a,b) = (b,a)$. We can think of h as being induced by the action of

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

on $\pi_1(S^1 \times S^1)$, with respect to the natural identification

$$\pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}.$$

Let $T_y \in M_{t(y)}$ be the boundary component of $M_{t(y)}$ associated to the oriented edge y. The local trivialization of f_v at T_y, canonically identifies T_y with $S^1 \times S^1$ so that $f_v|_{T_y}$ is projection onto the second component.

The graph of complexes associated to Γ_{plumb} consists of the manifolds

$$\begin{array}{rcll} X_v & = & M_v, \quad v \in \mathcal{V}(\Gamma), & \qquad \text{and} \\ X_y & = & T_y, \quad y \in \mathcal{Y}(\Gamma). & \end{array}$$

with gluing maps

$$\begin{array}{ccc} T_{\overline{y}} & \xrightarrow{h_y} & T_y \\ \| & & \| \\ S^1 \times S^1 & \xrightarrow{h} & S^1 \times S^1 \end{array}$$

The graph of complexes $M(\Gamma_{\text{plumb}})$ is a graph manifold.

Let $\mathrm{Fin}(S^1 \times S^1)$ be the set of finite unbranched coverings of $S^1 \times S^1$ to itself. A *modified plumbing graph* $\Gamma^{\text{m}}_{\text{plumb}} = \langle \Gamma, g, e, m \rangle$ is a plumbing graph with maps

$$m : \mathcal{Y}(\Gamma) \to \mathrm{Fin}(S^1 \times S^1)$$

so that

(1) the induced maps

$$m(y)_* : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

are non-negative upper triangular matrices in $M_2(\mathbb{Z})$,

$$m(y)_* = \begin{bmatrix} a(y) & b(y) \\ 0 & c(y) \end{bmatrix},$$

where $0 \le b(y) < a(y)$ and $c(y) > 0$; and

(2) the matrices $m(y)_*$ and $Hm(\overline{y})_*$ have the same image in $\mathbb{Z} \oplus \mathbb{Z}$.

Given a modified plumbing graph $\Gamma^{\mathrm{m}}_{\mathrm{plumb}} = \langle \Gamma, g, e, m \rangle$, we define an associated graph manifold $M(\Gamma^{\mathrm{m}}_{\mathrm{plumb}})$ to have vertex and edge manifolds as for $\Gamma_{\mathrm{plumb}} = \langle \Gamma, g, e \rangle$, except that we identify T_y with $S^1 \times S^1$ so that if R is the element of $\mathrm{GL}(2, \mathbb{Z})$ giving

$$m(y)_* R = H m(\overline{y})_*,$$

then $h_y : T_{\overline{y}} \to T_y$ is the map induced by R. We thus have a commutative diagram

$$\begin{tikzcd} T_{\overline{y}} \arrow[r, "h_y"] \arrow[d, "m(\overline{y})"'] & T_y \arrow[d, "m(y)"'] \\ S^1 \times S^1 \arrow[r, "h"] & S^1 \times S^1 \end{tikzcd} .$$

Since $h = h^{-1}$, it follows that $h_{\overline{y}} = h_y^{-1}$.

Morphisms and coverings of modified plumbing graphs are morphisms and coverings of the associated graph manifolds

$$\Psi : M(\Gamma^{\mathrm{m}}_{\mathrm{plumb}}) \to M({\Gamma^{\mathrm{m}}_{\mathrm{plumb}}}')$$

such that the following diagram commutes:

$$\begin{tikzcd} T_y \arrow[r, "m(y)"] \arrow[d, "\Psi_y"'] & S^1 \times S^1 \arrow[d, equal] \\ T_{\Psi_\Gamma(y)} \arrow[r, "m(\Psi_\Gamma(y))"] & S^1 \times S^1 \end{tikzcd} .$$

Given a plumbing graph Γ_{plumb}, one can associate a modified plumbing graph $\Gamma^{\mathrm{m}}_{\mathrm{plumb}}$, by setting all maps $m(y)$ to be the identity. One can easily verify that, in this case, the definitions for the associated graph manifold, and morphisms are the same.

4. Normal surface-curve pairs. Let X be a normal complex projective surface, and let $\mathcal{C} \subset X$ be an algebraic curve. We will assume for simplicity that $\mathcal{C}$ is connected. Let $|\mathcal{C}|$ be the set of irreducible curves in $\mathcal{C}$, and let $\mathcal{P} = \mathrm{Sing}(\mathcal{C})$. Let $\mathcal{F}$ be the family of surface-curve pairs $(X, \mathcal{C})$ satisfying the following conditions:

(1) each $C \in |\mathcal{C}|$ is unibranched;

(2) $\mathrm{Sing}(X) \cap \mathcal{C} \subset \mathcal{P}$; and

(3) for each $p \in \mathrm{Sing}(\mathcal{C})$, there is a locally defined finite covering of surface-curve pairs

$$\mu_p : (X, \mathcal{C}) \to (\mathbb{C}^2, \{x = 0\} \cup \{y = 0\})$$

defined near the germ (X, p).

A surface-curve pair $(X, \mathcal{C}) \in \mathcal{F}$ is call a *normal surface-curve pair.*

The following is immediate.

Lemma 8 *The family of normal surface-curve pairs is closed under coverings of surface-curve pairs.*

The fundamental group $\pi_1(\mathbb{C}^2 \setminus \{x = 0\} \cup \{y = 0\})$ is canonically isomorphic to the integer lattice $\mathbb{Z} \oplus \mathbb{Z}$, and finite coverings correspond to lattices of finite index. Given $p \in \mathcal{P}$, and $C, D \in \mathcal{C}$ containing p, let a, b, c be non-negative integers so that $(a, 0)$ and (b, c) generate the sublattice, and $0 \leq b < a$. Note that the numbers a, b, c are uniquely determined given the ordering of C and D. Changing the ordering corresponds to changing the order of the canonical basis for $\mathbb{Z} \oplus \mathbb{Z}$, and hence corresponds to switching columns of the matrix

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix},$$

and column-reducing to get

$$\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} R,$$

where $0 \leq b' < a$ and $R \in \mathrm{GL}(2, \mathbb{Z})$.

The matrix R can be obtained from a continued fraction expansion $[m_1, \ldots, m_k]$ for $\frac{a}{b}$, where

$$\frac{a}{b} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{\cdots \cfrac{1}{m_k}}}.$$

Lemma 9 *The matrix R is given by*

$$R = HM_1H \cdots HM_kH,$$

where

$$M_i = (M_i)^{-1} = \begin{bmatrix} 1 & m_i \\ 0 & -1 \end{bmatrix},$$

for $i = 1, \ldots, k$. Furthermore,

$$R^{-1} = HM_kH \cdots HM_1H.$$

One proof of this lemma comes from a study of the singularity (X, p) (see Theorem 11).

Theorem 10 ([Lauf], [Hir2]) *The germ (X, p) is smooth if and only if $b = 0$. In this case, $\mathcal{C}$ must have a normal crossing at p. Otherwise, the germ (X, p) can be desingularized by replacing p by exceptional curves $E_1, \dots, E_k$, with self-intersections*

$$E_i^2 = -m_i, \qquad \text{for } i = 1, \dots, k,$$

where $[m_1, \dots, m_k]$ is the continued fraction expansion for $\frac{a}{b}$.

Note that reversing the order of the pair of curves C and D passing through to p simply reverses the order of $E_1, \dots, E_k$. The exceptional curves and the proper transforms of C and D are arranged as in the graph of Figure 1,

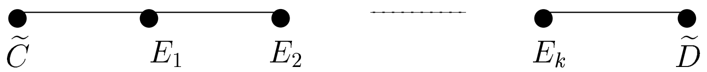

Figure 1.

where all edges in the graph correspond to normal crossing intersections.

Given a surface-curve pair $(X, \mathcal{C}) \in \mathcal{F}$, with specified maps μ_p for $p \in \mathrm{Sing}(X)$, there is a canonically associated modified plumbing graph $\Gamma^{\mathrm{m}}_{\mathrm{plumb}} = \Gamma^{\mathrm{m}}_{\mathrm{plumb}}(X, \mathcal{C})$ given as follows. Let $(\widetilde{X}, \widetilde{\mathcal{C}})$ be a minimal desingularization of $(X, \mathcal{C})$ obtained from the μ_p as in [Lauf]. For each $C \in |\mathcal{C}|$, let $\widetilde{C} \in \mathcal{C}'$ be the proper transform of C.

(1) The graph Γ for $\Gamma^{\mathrm{m}}_{\mathrm{plumb}}$ has vertices and edges

$$\begin{aligned} V(\Gamma) &= \{ v_C \ : \ C \in |\mathcal{C}| \}, \qquad \text{and} \\ \mathcal{Y}(\Gamma) &= \{ y_{p,C} \ : \ p \in \mathcal{P} \cap \mathcal{C} \}, \end{aligned}$$

where $t(y_{p,C}) = v_C$; and for each $p \in \mathcal{P}$, if $C, D \in |\mathcal{C}|$ is the pair of curves so that $p \in C \cap D$, then we have

$$\overline{y_{p,C}} = y_{p,D};$$

(2) for each $C \in |\mathcal{C}|$, let

$$\begin{aligned} g(v_C) &= g(\widetilde{C}) = g(C), \qquad \text{and} \\ e(V_C) &= e(\widetilde{C}) = \widetilde{C}^2; \end{aligned}$$

and

(3) for each $y = y_{p,C} \in \mathcal{Y}(\Gamma)$, let

$$m(y_{p,C}) : S^1 \times S^1 \to S^1 \times S^1$$

be the finite unbranched covering induced by the matrix

$$\begin{bmatrix} a(y) & b(y) \\ 0 & c(y) \end{bmatrix},$$

where $(a(y), 0)$ and $(b(y), c(y))$ generate the image of

$$(\mu_p)_* : \pi_1(X \setminus C \cup D) \to \pi_1(\mathbb{C}^2 \setminus \{x = 0\} \cup \{y = 0\}),$$

and $0 \leq b(y) < a(y)$, $0 < c(y)$.

Let $M(X, \mathcal{C})$ be the boundary of a regular neighborhood of $\mathcal{C}$ in X.

Theorem 11 *The graph manifold $M(\Gamma_{\text{plumb}}(X, \mathcal{C}))$ is homeomorphic to $M(X, \mathcal{C})$.*

Proof. For the case when X is smooth see [Neu], p. 333. When X has a singularity at p, since $(X, \mathcal{C})$ is a normal surface-curve pair, there are exactly two curves $C, D \in \mathcal{C}$ so that $p \in C \cap D$. The link S_p of the singularity (X, p) is a lens space, and $X \setminus \mathcal{C}$ looks locally like a cone over $S^3 \setminus L$ near p, where L is an oriented Hopf link. Let T_C and T_D be the torus boundary components of M_C and M_D near p. Then

$$M_p = S_p \setminus U(C_p),$$

where $U(C_p)$ is a regular neighborhood of $\mathcal{C}$ in X, is homeomorphic to a thickened torus with boundary components T_C and T_D. Identifying M_p with the product of a torus and an interval determines a homeomorphism of T_C to T_D, which we will now describe.

Let $y = y_{p,C}$ (so we have $o(y) = D$ and $t(y) = C$), and suppose

$$m(y)_* = \begin{bmatrix} a(y) & b(y) \\ 0 & c(y) \end{bmatrix}.$$

Then (X, p) can be desingularized as in Figure 1.

Give T_C and T_D trivializations so that M_C and M_D have euler number equal to the self intersections of the proper transforms $\widetilde{C}$ and $\widetilde{D}$ in the minimal desingularization $(\widetilde{X}, \widetilde{C})$ of $(X, \mathcal{C})$.

Consider the plumbing graph of $(\widetilde{X}, \widetilde{\mathcal{C}})$ over p, which is shown in Figure 1. The vertices corresponding to the E_i have corresponding vertex manifolds which are thickened tori with two boundary components. If we give these boundary components trivializations so that the euler number of the associated S^1-bundle is $-m_i$, then the boundary components are identified via the product structure by the map

$$S^1 \times S^1 \to S^1 \times S^1$$

corresponding to M_i.

The gluing map

$$h_y : T_C \to T_D$$

can be thought of as a composition of the gluing maps for the plumbing graph of $(\widetilde{X}, \widetilde{\mathcal{C}})$ over p. Thus, h_y is the map corresponding to

$$(h_y)_* = HM_1H \cdots HM_kH$$

as in Lemma 9.

By the construction,

$$h \circ m(\overline{y}) = m(y) \circ h_y,$$

and it is also easy to see that $M_i = M_i^{-1}$, for $i = 1, \ldots, k$, and

$$(h_{\overline{y}})_* = HM_kH \cdots HM_1H.$$

■

Given a non-modified plumbing graph Γ_{plumb}, and an ordering of the vertices $v_1, \ldots, v_k \in \mathcal{V}(\Gamma)$, the associated intersection matrix $S(\Gamma_{\text{plumb}})$ is the $k \times k$ matrix with entries $a_{i,j}$, where

$$a_{i,j} = \begin{cases} e(v_i) & \text{if } i = j \\ n(i,j) & \text{otherwise} \end{cases}$$

where $n(i,j)$ is the number of $y \in \mathcal{Y}(\Gamma)$, with $o(y) = v_i$ and $t(y) = v_j$.

When $\Gamma^{\text{m}}_{\text{plumb}}$ is modified, then we define the intersection matrix $S(\Gamma^{\text{m}}_{\text{plumb}})$ to be the matrix with entries $a_{i,j}$ given by

$$a_{i,j} = \begin{cases} e(v_i) + \displaystyle\sum_{\substack{y \in \mathcal{Y}(\Gamma) \\ t(y) = v_i}} \frac{b(y)}{a(y)} & \text{if } i = j \\[2em] \displaystyle\sum_{\substack{y \in \mathcal{Y}(\Gamma) \\ o(y) = v_i, t(y) = v_j}} \frac{\gcd(a(y), b(y))}{a(y)} & \text{otherwise} \end{cases}$$

Note that the intersection matrices for the modified and non-modified plumbing graphs agree if and only if $b(y) = 0$ for all $y \in \mathcal{Y}(\Gamma)$.

Theorem 12 *If $(X, \mathcal{C})$ is a normal surface-curve pair, then the intersection matrix $S(\Gamma^{\mathrm{m}}_{\mathrm{plumb}}(X, \mathcal{C}))$ equals $S(X, \mathcal{C})$.*

Proof. The formula for intersection numbers of distinct pairs follows directly from [Hir2] (see Lemma 3.5 and Lemma 3.7). For the self intersections, recall that, for any $C \in |\mathcal{C}|$, the pull-back $\overline{C}$ of C in the minimal desingularization is defined to be the divisor equal to the proper transform $\widetilde{C}$ of C plus the unique rational multiples of the exceptional curves, determined by the condition that

$$\overline{C}.E = 0,$$

for any exceptional curve E (see [Mum]). This implies that for each $p \in \mathcal{P} \cap C$, we need only be concerned with the coefficient r_p of the unique exceptional curve E_p over p which intersects $\widetilde{C}$. That is,

$$\begin{aligned} C^2 &= (\overline{C})^2 \\ &= \widetilde{C}.\left(\widetilde{C} + \sum_{p \in \mathcal{P} \cap C} r_p E_p\right) \\ &= (\widetilde{C})^2 + \sum_{p \in \mathcal{P} \cap C} r_p. \end{aligned}$$

The rest follows from the calculations in [Hir2] (see Lemma 3.7).

■

5. Applications to computations on coverings. Let $(X, \mathcal{C})$ be a normal surface-curve pair, and let $\Gamma^{\mathrm{m}}_{\mathrm{plumb}} = \Gamma^{\mathrm{m}}_{\mathrm{plumb}}(X, \mathcal{C})$ be its modified plumbing graph. Let

$$\rho : (Y, \mathcal{D}) \to (X, \mathcal{C})$$

be a regular covering defined by the epimorphism

$$\phi : \pi_1(X, \mathcal{C}) \to F.$$

In this section, we describe the intersection matrix and modified plumbing data for the covering $(Y, \mathcal{D})$ in terms of $\Gamma^{\mathrm{m}}_{\mathrm{plumb}}$, and the induced defining map

$$\psi : \pi_1(G(\Gamma^{\mathrm{m}}_{\mathrm{plumb}}), \mathcal{T}) \to F$$

where $\mathcal{T}$ is a maximal tree in Γ.

Let $F_v = \psi(G_v)$, $F_y = \psi(G_y)$, and let $I_v = \psi(Z_v)$, where Z_v is the subgroup of $G_v = \pi_1(M_v)$ generated by the fiber of the S^1-bundle M_v. For each $y \in \mathcal{Y}(\Gamma)$, let

$s(y) = \psi(y)$, where y is considered as an element of $\pi_1(G(\Gamma^{\mathrm{m}}_{\mathrm{plumb}}), \mathcal{T})$. (This $s(y)$ is called the *twisting data* in [Hir1] and [Hir2])

Let Γ' be the graph consisting of vertices

$$\mathcal{V}(\Gamma') = \{[\alpha F_v] \ : \ v \in \mathcal{V}(\Gamma), \ \alpha \in F\}$$

and edges

$$\mathcal{Y}(\Gamma') = \{[\alpha F_y] \ : \ y \in \mathcal{Y}(\Gamma), \ \alpha \in F\};$$

where, if $y' = [\alpha F_y]$, let $\overline{y'} = [\alpha F_{\overline{y}}]$, and let $t(y') = v'$ where $v' = [\alpha s(y) F_v]$.

Lemma 13 *The graph Γ' is the underlying graph of the covering, and the map*

$$\rho_\Gamma : \Gamma' \to \Gamma$$

is given by

$$\begin{aligned} \rho_\Gamma([\alpha F_v]) &= v, \quad v \in \mathcal{V}(\Gamma), \qquad \text{and} \\ \rho_\Gamma([\alpha F_y]) &= y, \quad y \in \mathcal{Y}(\Gamma). \end{aligned}$$

Note that this presentation of the graph Γ' contains within it a natural lifting of a maximal tree $\mathcal{T}$ in Γ. Giving an identification of $\mathcal{V}(\Gamma')$ with $|\mathcal{D}|$ requires some extra information. Choose a *section*

$$\tau : \mathcal{T} \to M(X, \mathcal{C}).$$

This amounts to choosing base-points in M_v and M_y, for all $v \in \mathcal{V}(\Gamma)$ and $y \in \mathcal{Y}(\Gamma)$, and connecting paths, for each $y \in \mathcal{Y}(\Gamma)$, connecting the base-point in M_y to the base-point in $M_{t(y)}$. The section τ lifts to the boundary manifold $M(Y, \mathcal{D})$ and gives a natural identification of $|\mathcal{D}|$ with the vertices in $\mathcal{V}(\Gamma')$ so that the lift of $\tau(v)$ lies on $D \in |\mathcal{D}|$ if and only if

$$v_{\alpha D} = [\alpha F_{v_{\rho(D)}}],$$

for all $\alpha \in |\mathcal{D}|$. We will call such an identification a *compatible* identification of $|\mathcal{D}|$ with $\mathcal{V}(\Gamma')$.

The genus associated to vertices in Γ', and hence to the components of $\mathcal{D}$ are given as follows.

Lemma 14 *For $v' \in \mathcal{V}(\Gamma')$, and $\rho_\Gamma(v') = v$, the genus $g(v')$ is given by*

$$g(v') = \frac{1}{2}\left(2 - \frac{\#F_v}{\#I_v}(2 - 2g(v) - d(v)) - \sum_{\substack{y \in \mathcal{Y}(\Gamma) \\ t(y) = v}} \frac{\#F_v}{\#F_y}\right).$$

Proof. The formula follows from additive properties of the topological euler characteristic, Corollary 5, and Corollary 7.

■

The map $m : \mathcal{Y}(\Gamma') \to \mathrm{Fin}(S^1 \times S^1)$ can also be written in terms of the covering data and the modified plumbing graph of the base.

Lemma 15 *For $y' \in \mathcal{Y}(\Gamma')$, and $y = \rho_\Gamma(y')$, $m(y')$ is the composition*

$$m(y') = m(y) \circ \rho_y,$$

where $\rho_y \in \mathrm{Fin}(S^1 \times S^1)$ is the unique map induced by

$$\psi_y : \mathbb{Z} \oplus \mathbb{Z} = G_y \to F_y,$$

such that

$$m(y')_* = \begin{bmatrix} a'(y) & b'(y) \\ 0 & c'(y) \end{bmatrix},$$

where $0 \leq b'(y) < a'(y)$ and $0 < c'(y)$.

Proof. This lemma is a consequence of the definitions of modified plumbing graphs and Proposition 6, noting that the form of $m(y')$ can be arranged by composing with an automorphism of the domain of ψ_y.

■

Lemma 15, leads to the following formulas for intersection matrices of coverings, generalizing the results of [Hir2].

Theorem 16 *The intersection matrix $S(Y, \mathcal{D})$, with respect to a compatible identification of $|\mathcal{D}|$ with $\mathcal{V}(\Gamma')$, is given by*

$$[\alpha F_v].[\beta F_w] = \sum_{\substack{y \in \mathcal{Y}(\Gamma) \\ o(y)=v \\ t(y)=w}} \frac{\#\left(\alpha F_v \cap \beta s(y)^{-1} F_w\right)}{\#(I_v + I_w)} \frac{\gcd(a'(y), b'(y))}{a'(y)},$$

when $v, w \in \mathcal{V}(\Gamma)$ are distinct pairs, and

$$[\alpha F_v].[\beta F_v] = \frac{\#\left(\alpha F_v \cap \beta F_v\right)}{(\# I_v)^2}.$$

Proof. The first formula follows from Theorem 12, and Proposition 4, while the second formula follows from [Hir2] (see Lemma 3.3).

■

The second formula in Theorem 16 leads to the following formula for the euler numbers attached to vertices of Γ'.

Lemma 17 *Given $v' \in \mathcal{V}(\Gamma')$, and $v = \rho(v')$, the euler number $e(v')$ is given by*

$$e(v') = \frac{\#F_v}{(\#I_v)^2} - \sum_{\substack{y \in \mathcal{Y}(\Gamma) \\ t(y) = v}} \frac{b'(y)}{a'(y)} \frac{\#F_v}{\#F_y}.$$

Proof. The formula follows from Theorem 12 and Theorem 16.

■

This completes the description of the covering modified plumbing graph.

Note that the above formulas depend only on the map ψ restricted to G_v, G_y, Z_v, and $\mathcal{Y}(\Gamma)$. This leads to the question of which defining maps for the boundary manifold ψ are induced by global defining maps on $\pi_1(X \setminus \mathcal{C})$, and thus to the question of the relation between $\pi_1(X \setminus \mathcal{C})$ and $\pi_1(M(X \setminus \mathcal{C})$.

In general (when $\mathcal{C}$ supports an ample divisor), the fundamental group of the boundary manifold of $\mathcal{C}$ in X surjects onto the fundamental group of $X \setminus \mathcal{C}$ under the map induced by inclusion. To understand the kernel of this map is a harder problem and includes the problem of understanding the effect of locations of singularities on $\mathcal{C}$ on the fundamental group of the complement.

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