ON ASSOCIATED GRADED RINGS OF NORMAL IDEALS

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1. Introduction

This paper was inspired by a theorem concerning the depths of associated graded rings of normal ideals recently appearing in [HH]. That theorem was in turn partially motivated by a vanishing theorem proved by Grauert and Riemenschneider [GR] and the following formulation of it in the Cohen-Macaulay case due to Sancho de Salas:

Theorem 1.1. ([S, Theorem 2.8]) *Let R be a Cohen-Macaulay local ring which is essentially of finite type over $\mathbb{C}$ and let I be an ideal of R. If $\operatorname{Proj} R[It]$ is nonsingular then $G(I^n)$ is Cohen-Macaulay for all large values of n. (Here $G(I^n)$ denotes the associated graded ring of R with respect to I^n.)*

The proof of this theorem, because it uses results from [GR], is dependent on complex analysis. Some natural questions are: Is there an 'algebraic' proof of Thereom 1.1? Can the assumptions on R be relaxed? What about those on $\operatorname{Proj} R[It]$? It is known that Theorem 1.1 fails if the nonsingular assumption on $\operatorname{Proj} R[It]$ is replaced by a normality assumption; Cutkosky [C, §3] gave an example showing this for the ring $R = \mathbb{C}[[x, y, z]]$ (see also [HH, Theorem 3.12]). In the two-dimensional case, however, we show that it is possible to replace the assumption on $\operatorname{Proj} R[It]$ with the condition that I (or some power of I) is a normal ideal (see Corollary 3.5). Moreover, the theorem is valid for arbitrary two-dimensional Cohen-Macaulay rings, not just those which are essentially of finite type over $\mathbb{C}$. This had been established by Huckaba and Huneke [HH, Corollary 3.8] (and previously outlined by Huneke in [Hun]) in the case $\dim R/I = 0$.

We now describe in more detail the connections between some of these results. The Grauert-Riemenschneider vanishing theorem of [GR] has the following dual form ([HO, Proposition 2.2]):

> Let R be a local ring which is essentially of finite type over $\mathbb{C}$ and $X = \operatorname{Proj} R[It]$ a desingularization of $\operatorname{Spec} R$. Let Y be the closed fiber of the morphism $f\colon X \to \operatorname{Spec} R$. Then $\mathrm{H}^i_Y(X, \mathcal{O}_X) = 0$ for $i < \dim R$.

Lipman proved this dual version for arbitrary two-dimensional normal local domains R [L1, Theorem 2.4]. Moreover, he showed that in this situation one can relax the nonsingular condition on $\operatorname{Proj} R[It]$ to just requiring $\operatorname{Proj} R[It]$ to be normal.

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Sancho de Salas showed that when R and X are Cohen-Macaulay (but not necessarily nonsingular), the vanishing of $\mathrm{H}^i_Y(X, \mathcal{O}_X)$ for $i < \dim R$ is equivalent to the associated graded ring $G(I^n)$ being Cohen-Macaulay for n sufficiently large (see also [L2, Theorem 4.3]). Theorem 1.1 follows as a corollary, as does a generalized two-dimensional version of it by invoking Lipman's result.

As a generalization along different lines, Huckaba and Huneke proved that if (R, m) is a two-dimensional Cohen-Macaulay local ring and I is an m-primary normal ideal (i.e., all powers of I are integrally closed), then $G(I^n)$ is Cohen-Macaulay for large n ([HH, Corollary 3.8]). Their theorem is a consequence of a more general result concerning normal ideals integral over regular sequences of length at least two ([HH, Theorem 3.1]).

Our main results (Theorem 3.4 and its corollaries) generalize those in [HH] in that they hold for any normal ideal. In particular, we give an algebraic proof of the following version of Theorem 1.1:

Corollary 3.9. *Let R be a two-dimensional Cohen-Macaulay local ring and I an ideal of R such that* $\operatorname{Proj} R[It]$ *is normal. Suppose that either* $\operatorname{ht} I > 0$ *or R is analytically unramified. Then $G(I^n)$ is Cohen-Macaulay for all n sufficiently large.*

The techniques we use are completely different from those of [HH] or [L1]. Our methods are based on an analysis of the graded local cohomology modules of the Rees algebra and associated graded ring of I. There are two ingredients of our proof that are worth special mention. One is a result due to Itoh ([It, Lemma 5]) concerning the vanishing of the degree zero component of $\mathrm{H}^2_J(R[It, t^{-1}])$ where I is a normal ideal and J is a homogeneous ideal satisfying certain conditions (see Lemma 3.1). The second important tool is the concept of generalized depth, studied first (although not using that terminology) by Brodmann [Br] and Faltings [F], and exploited later in [TI] and [HM] to study relationships between the depths of associated graded rings and Rees algebras. We use generalized depth in this paper to bound the degrees of the nonzero graded components of the local cohomology modules of associated graded rings and Rees algebras.

The paper is arranged into four sections. Section 2 consists mainly of background material. In particular, we recall some basic properties of $\operatorname{Proj} R[It]$ and discuss the notion of generalized depth. Section 3 contains our main results on normal ideals. After proving Theorem 3.4 and its immediate corollaries, we present a few results related to the Cohen-Macaulay property of $R[I^n t]$ for large n where I is an ideal of linear type. These are built around a curious result (Proposition 3.11) about the integer $a_d(R[It])$, where R is a d-dimensional Gorenstein local ring and I is of linear type. We show that this integer must be negative, and use it to give some conditions under which $R[I^n t]$ is Cohen-Macaulay for sufficiently large n (Corollaries 3.12 and 3.13).

Section 4 is devoted to an analysis of the depths of $R[I^n t, t^{-1}]$ for large n. The results in this section can be used to give another proof of a version of Theorem 3.4 (Theorem 4.7), but also help clarify what needs to be done to prove Theorem 1.2 for Cohen-Macaulay local rings of higher dimensions. For instance, we show (see Corollary 4.5) that if R and $\operatorname{Proj} R[It]$ are CM and R is the homomorphic image of a Gorenstein ring, then $G(I^n)$ is Cohen-Macaulay for large n if and only if $\mathrm{H}^i_{M^*}(R[It, t^{-1}])_0 = 0$ for $i < \dim R + 1$ (where

M^* is the homogenenous maximal ideal of $R[It, t^{-1}]$).

2. Preliminaries

In this section we will summarize some basic terminology and results concerning Proj of graded rings and its connection with generalized depth. All rings in the paper are assumed to be commutative and possessing an identity element.

Let $A = \oplus A_n$ be a graded (by which we always mean $\mathbf{Z}$-graded unless explicitly stated otherwise) Noetherian ring. If U is a multiplicatively closed subset of homogeneous elements of A and M is a graded A-module then the localization M_U is a graded A_U-module whose grading is induced by $\deg(\frac{m}{w}) = \deg m - \deg w$. The homogeneous localization $M_{(U)}$ is the degree zero component of M_U. Following standard notation (see [HIO, Chapter 2] for example), if Q is a homogeneous prime ideal of A and U is the set of all homogeneous elements of A not in Q then $M_{(U)}$ is denoted by $M_{(Q)}$. It is easily seen that the ring $A_{(Q)}$ is a local ring. If $x \in A$ is homogeneous and $U = \{x^n\}_{n\geq 1}$ then $M_{(U)}$ is denoted by $M_{(x)}$.

Recall that as a set $\operatorname{Proj} A$ is defined to be the set of homogeneous prime ideals of A that do not contain the set $A_+ = \bigoplus_{n\geq 1} A_n$. Usually, $\operatorname{Proj} A$ is defined for positively graded rings (in which case A_+ is an ideal of A), but here we wish to allow for the consideration of $\mathbf{Z}$-graded rings. For k a nonnegative integer let $\mathcal{R}_k$ and $\mathcal{S}_k$ denote the Serre conditions: A satisfies $\mathcal{R}_k$ if A_p is a regular local ring for all prime ideals p of A with $\operatorname{ht} p \leq k$, and A satisfies $\mathcal{S}_k$ if $\operatorname{depth} A_p \geq \min\{k, \operatorname{ht} p\}$ for all prime ideals p of A.

The following result is well-known. For convenience we sketch the proof.

Proposition 2.1. *Suppose A is a Noetherian graded ring and let $x_1, \ldots, x_n \in A_1$ be elements which generate the ideal $(A_+)A$ up to radical. Then the following statements are equivalent.*

(a) *$A_{(Q)}$ satisfies $\mathcal{R}_k$ (respectively $\mathcal{S}_k$, respectively is Gorenstein) for all $Q \in \operatorname{Proj} A$.*
(b) *A_Q satisfies $\mathcal{R}_k$ (respectively $\mathcal{S}_k$, respectively is Gorenstein) for all $Q \in \operatorname{Proj} A$.*
(c) *$A_{(x_i)}$ satisfies $\mathcal{R}_k$ (respectively $\mathcal{S}_k$, respectively is Gorenstein) for $1 \leq i \leq n$.*
(d) *A_{x_i} satisfies $\mathcal{R}_k$ (respectively $\mathcal{S}_k$, respectively is Gorenstein) for $1 \leq i \leq n$.*

Proof. First note that if U is a multiplicatively closed subset of homogeneous elements of A, and U contains an element of A_1, then $A_U = A_{(U)}[t, t^{-1}]$ where t is an indeterminate (see [E, Exercise 2.17], for example).

$(a) \Leftrightarrow (b)$: The inclusion $A_{(Q)} \to A_Q$ is faithfully flat with regular fibers (as A_Q is a localization of $A_{(Q)}[t, t^{-1}]$). The implications now follow from [Mat, Theorems 23.4 and 23.9].

$(c) \Leftrightarrow (d)$: This also follows from [Mat, Theorems 23.4 and 23.9] since $A_{x_i} = A_{(x_i)}[t, t^{-1}]$ for $1 \leq i \leq n$.

$(c) \Rightarrow (b)$: If $Q \in \operatorname{Proj} A$ then $x_i \notin Q$ for some i. Thus $A_Q \cong (A_{x_i})_Q$.

$(b) \Rightarrow (c)$: We prove the $\mathcal{R}_k$ property; the proofs for $\mathcal{S}_k$ and Gorenstein are similar. Let P be a prime ideal of A_{x_i} having height at most k and let P^* denote the (prime) ideal generated by the homogeneous elements contained in P. Then $(A_{x_i})_P$ is a regular local ring if and only if $(A_{x_i})_{P^*}$ is a regular local ring ([BH, Exercise 2.2.24]). Thus, we may assume

P is homogenous. Let $Q \in \operatorname{Spec} A$ such that $P = Q_{x_i}$. Then $(Q^*)_{x_i} = (Q_{x_i})^* = P^* = P$; hence, Q is also homogeneous. As $x_i \notin Q$, $Q \in \operatorname{Proj} A$. It follows that $(A_{x_i})_P = A_Q$ is regular since $\operatorname{ht} Q = \operatorname{ht} P \leq k$. □

Given a graded ring A (usually the Rees algebra, extended Rees algebra, or associated graded ring of an ideal), we are interested in studying arithmetic properties of A that are induced by assumptions on $\operatorname{Proj} A$. We spell out the relevant terminology. If $\mathcal{P}$ is a local ring-theoretic property (e.g., regularity, normality, the Cohen-Macaulay property), then the scheme $\operatorname{Proj} A$ has property $\mathcal{P}$ if $A_{(Q)}$ has this same property for all $Q \in \operatorname{Proj} A$. By the preceding lemma if $\mathcal{P}$ is one of the Serre conditions or the Gorenstein property then $\operatorname{Proj} A$ satisfies $\mathcal{P}$ if and only if A_Q satisfies $\mathcal{P}$ for all $Q \in \operatorname{Proj} A$. This applies to any property which can be defined in terms of the Serre conditions (e.g, normality).

Next we want to collect some results from [HM] concerning the concept of generalized depth. We first recall the definition. Let $(R, \mathfrak{m})$ be a local ring, I an ideal of R and M a finitely generated R-module. The *generalized depth of M with respect to I* is defined by

$$\text{g-depth}_I M := \max\{k \geq 0 \mid I \subseteq \sqrt{\operatorname{Ann}_R H^i_{\mathfrak{m}}(M)} \text{ for all } i < k\}.$$

If A is a Noetherian graded ring which has a unique homogeneous maximal ideal $\mathfrak{q}$ and M is a finitely generated graded A-module, we define $\text{g-depth}\, M := \text{g-depth}_{(A_+)A_{\mathfrak{q}}} M_{\mathfrak{q}}$.

This integer was first studied by Brodmann [Br] and Faltings [F] in connection with their study of finiteness of local cohomology modules. The term 'generalized depth' is reflective of its close connection to ordinary depth. It is clear from the definition that $\operatorname{depth} M \leq \text{g-depth}_I M$ for any ideal I. If $\text{g-depth}_I M = \dim M$ then M is said to be generalized Cohen-Macaulay with respect to I ([TI]). In [HM] the authors used g-depth to help prove statements comparing $\operatorname{depth} R[It]$ with $\operatorname{depth} G(I)$ for an ideal I in a local ring R (see Lemma 3.3). We now isolate some important properties of generalized depth in the graded case.

Proposition 2.2. *Let A be an Noetherian graded ring having a unique homogeneous maximal ideal $\mathfrak{q}$ and which is the homomorphic image of a Gorenstein ring. Let M be a finitely generated graded A-module. Then*

$$\text{g-depth}\, M = \min_{p \in \operatorname{Proj} A} \{\operatorname{depth} M_p + \dim A/p\}.$$

Proof. A straightforward modification of the proof of [HM, Proposition 2.1] gives that

$$\begin{aligned}\text{g-depth}\, M &= \text{g-depth}_{(A_+)A_{\mathfrak{q}}} M_{\mathfrak{q}} \\ &= \min\{\operatorname{depth} M_p + \dim A_{\mathfrak{q}}/pA_{\mathfrak{q}} \mid p \in \operatorname{Spec} A, A_+ \not\subseteq p, \text{ and } p \subseteq \mathfrak{q}\}.\end{aligned}$$

Observe that if p is not homogeneous then

$$\operatorname{depth} M_{p^*} + \dim A_{\mathfrak{q}}/p^* A_{\mathfrak{q}} = \operatorname{depth} M_p + \dim A_{\mathfrak{q}}/pA_{\mathfrak{q}}$$

where p^* is the ideal generated by the homogeneous elements contained in p (see [BH, Section 1.5], for example). Hence we may assume $p \in \operatorname{Proj} A$ and the result follows. □

As a consequence we see that if A is equidimensional and the homomorphic image of a Gorenstein ring then the Cohen-Macaulayness of $\operatorname{Proj} A$ is equivalent to the ring having maximal g-depth.

Corollary 2.3. *Let A be an Noetherian equidimensional graded ring having a unique homogeneous maximal ideal $\mathfrak{q}$ and which is the homomorphic image of a Gorenstein ring. If* $\operatorname{Proj} A$ *satisfies $\mathcal{S}_k$ (for $0 \le k < \dim A$) then* $\operatorname{g-depth} A \ge k+1$. *Moreover,* $\operatorname{Proj} A$ *is Cohen-Macaulay if and only if* $\operatorname{g-depth} A = \dim A$.

proof. Using Proposition 2.2 and that A is equidimensional, we have that

$$\operatorname{g-depth} A = \min_{p \in \operatorname{Proj} A} \{\operatorname{depth} A_p + \dim A - \operatorname{ht} p\} = \min_{p \in \operatorname{Proj} A} \{\dim A - (\dim A_p - \operatorname{depth} A_p)\}.$$

The result readily follows. □

The generalized depth of a graded module is closely related to another cohomological invariant studied in [Mar2]. This invariant, which we denote by $\operatorname{fg}(M)$, is a measure of which local cohomology modules are 'finitely graded'. The definition follows. Again let A be a graded Noetherian ring with unique homogeneous maximal ideal q. If M is a graded A-module, we define

$$\operatorname{fg}(M) := \max\{k \ge 0 \mid \mathrm{H}^i_{\mathfrak{q}}(M)_n = 0 \text{ for all but finitely many } n \text{ and for all } i < k\}.$$

The utility of using $\operatorname{fg}(M)$ in this paper is that it simplifies some of the proofs, and it applies to rings and modules which are not positively graded (important for §4). It is clear from the definitions that $\operatorname{fg}(N) \le \operatorname{g-depth} N$. The following lemma states, however, that in the case A is positively graded we have equality of these two invariants.

Lemma 2.4. *Let A be a positively graded Noetherian ring with a unique homogeneous maximal ideal and M a finitely generated graded A-module. Then* $\operatorname{fg}(M) = \operatorname{g-depth} M$.

proof. The proof is an easy adaptation of the argument used in [TI, Lemma 2.2]. □

If M is a graded A-module and k is a positive integer then the kth Veronese submodule of M is $M^{(k)} := \oplus_n M_{kn}$. Note that if $x_1, \dots, x_n \in A_1$ generate $(A_+)A$ up to radical then $x_1^k, \dots, x_n^k \in A_1^{(k)}$ generate $(A_+^{(k)})A^{(k)}$ up to radical, and $M_{(x_i)} \cong (M^{(k)})_{(x_i^k)}$ for $1 \le i \le n$ and any graded A-module M. In particular, the sets of local rings of $\operatorname{Proj} A$ and $\operatorname{Proj} A^{(k)}$ are the same up to isomorphism. More precisely, $\operatorname{Proj} A$ and $\operatorname{Proj} A^{(k)}$ are isomorphic as schemes. Another important property of the Veronese embedding is that it commutes with local cohomology. Proofs in special cases are given in [GW, Theorem 3.1.1], [HIO, 47.5], and [HHK, Remark 2.4], while a proof of the general case is outlined in [BS, Exercise 12.4.6]. We give another general proof below:

Proposition 2.5. *Let A be a Noetherian graded ring, I a homogeneous ideal and M a graded A-module. Then for all $i \ge 0$, $\mathrm{H}^i_{I^{(k)}}(M^{(k)}) \cong \mathrm{H}^i_I(M)^{(k)}$ as graded $A^{(k)}$-modules.*

Proof. Consider $N = M/M^{(k)}$ as a graded $A^{(k)}$-module. Then $N_{kn} = 0$ for all n. Hence, for any homogeneous element $y \in A^{(k)}$, $(N_y)_{kn} = 0$ for all n. By computing local cohomology using the Čech complex, we see that $\mathrm{H}^i_{I^{(k)}}(N)_{kn} = 0$ for all i, n. Using this fact together with the long exact sequence on local cohomology induced by $0 \to M^{(k)} \to M \to N \to 0$, we see that $\mathrm{H}^i_{I^{(k)}}(M^{(k)}) \cong \mathrm{H}^i_{I^{(k)}}(M)^{(k)}$ for all i. But $\mathrm{H}^i_{I^{(k)}}(M) = \mathrm{H}^i_{I^{(k)}A}(M) = \mathrm{H}^i_I(M)$, because $I^{(k)}A \subseteq I \subseteq \sqrt{I^{(k)}A}$. $\square$

We also note that under mild hypotheses, g-depth M (and hence $\mathrm{fg}(M)$ if A is positively graded) is invariant under the Veronese functor.

Proposition 2.6. *Let A be a Noetherian graded ring having a unique homogeneous maximal ideal and suppose $(A_+)A$ is generated up to radical by A_1. Then for any finitely generated graded A-module M,* $\operatorname{g-depth} M = \operatorname{g-depth} M^{(k)}$ *for all $k \geq 1$.*

proof. As A is integral over $A^{(k)}$, $\operatorname{Proj} A^{(k)} = \{p^{(k)} \mid p \in \operatorname{Proj} A\}$. Furthermore, for $p \in \operatorname{Proj} A$, $A_{(p)} \to A_p$ is faithfully flat and $M_p = M_{(p)} \otimes_{A_{(p)}} A_p$. By [Mat, Theorem 23.3] $\operatorname{depth} M_{(p)} = \operatorname{depth} M_p$. Since for any $k \geq 1$ $(M^{(k)})_{(p^{(k)})} \cong M_{(p)}$, we have that $\operatorname{depth}(M^{(k)})_{p^{(k)}} = \operatorname{depth} M_p$. Also, $\dim A^{(k)}/p^{(k)} = \dim(A/p)^{(k)} = \dim A/p$ since A/p is integral over $(A/p)^{(k)}$. The conclusion now follows from the Proposition 2.2. $\square$

3. Associated graded rings of powers of normal ideals

Through the remainder of this paper (sections 3 and 4), $(R, \mathfrak{m})$ is assumed to be a local ring of dimension d and I an ideal of R. For convenience of notation we often denote the Rees algebra $R[It]$ by $S(I)$ (or simply by S if there is no ambiguity about the ideal I). Similarly, we denote the extended Rees algebra $R[It, t^{-1}]$ by $S^*(I)$ (or simply S^*) and the associated graded ring S/IS of I by $G(I)$ (or G). We let $M = (\mathfrak{m}, It)S$ and $M^* = (\mathfrak{m}, It, t^{-1})S^*$. We remark that G can be considered either as the graded S-module S/IS or as the graded S^*-module $S^*/t^{-1}S^*$, and that $\mathrm{H}^i_M(G) \cong \mathrm{H}^i_{M^*}(G)$ for all i.

Our main goal in this section is to prove a version of [HH, Theorem 3.1] concerning the depths of the associated graded rings of large powers of normal ideals. The proof in [HH] uses the theory of superficial elements and requires that the normal ideal be integral over regular sequence. The proof we give instead relies on techniques of graded local cohomology. This allows us to remove the restriction that the ideal be integral over a regular sequence, although we do need to add the hypothesis that the ring be quasi-unmixed.

Recall that an element $r \in R$ is said to be integral over I if there exists $a_i \in I^i$ such that $r^n + a_1 r^{n-1} + \cdots + a_1 r + a_n = 0$. The *integral closure* of I , denoted by $\overline{I}$, is defined to be the set of elements in R that are integral over I, and is itself an ideal of R. An ideal is said to be *integrally closed* if it equals its integral closure, and is said to be *normal* if all of its powers are integrally closed. We note that if R is a local ring which contains a normal ideal then R must be reduced. For if I is a normal ideal and $x \in R$ is nilpotent, then $x \in \cap \overline{I^n} = \cap I^n = 0$, by Krull's intersection theorem. We will frequently make us of the following well-known fact.

Remark. Suppose R is a local ring but not a field and I is a normal ideal. Then $\operatorname{depth} G(I) \geq 1$.

proof. Since R is reduced but not a field, $\dim R = \dim G \geq 1$. As $\overline{I^n} = I^n$ for all n, it is easy to see that $t^{-1}S^*$ is an integrally closed ideal. Hence every associated prime of $G = S^*/t^{-1}S^*$ is minimal. $\square$

We now turn our attention to the local cohomology of form rings of normal ideals. We will make critical use of the fact that the second local cohomology module of the extended Rees algebra of a normal ideal is concentrated in positive degrees. This is largely a consequence of the following result due to Itoh:

Lemma 3.1. ([It, Lemma 5]) *Let R be a Noetherian ring and I a normal ideal. Let J be a homogeneous ideal of $S^*(I)$ such that $t^{-1} \in J$ and $\operatorname{ht} J\overline{G(I)_{\mathrm{red}}} \geq 2$. Then $\mathrm{H}^2_J(S^*)_n = 0$ for $n \leq 0$. (Here, $\overline{G(I)_{\mathrm{red}}}$ is the integral closure of $G(I)/\mathfrak{a}$, where $\mathfrak{a}$ is the nilradical of $G(I)$.)*

Recall that a local ring $(R, \mathfrak{m})$ is said to be *quasi-unmixed* (also known as *formally equidimensional*) if all minimal primes in the $\mathfrak{m}$-adic completion $\hat{R}$ have the same dimension. A ring R is said to satisfy the *second chain condition for prime ideals* if for every minimal prime p of R and for any integral extension T of R/p, the length of any maximal chain of prime ideals in T is equal to $\dim R$ ([Na, section 34]).

As a consequence of Itoh's lemma, we get the following:

Proposition 3.2. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring of dimension at least two and I a normal ideal of R. Let $S^* = S^*(I)$ and $G = G(I)$. Then*

(a) $\mathrm{H}^2_{M^*}(S^*)_n = 0$ *for* $n \leq 0$.
(b) $\mathrm{H}^1_{M^*}(G)_n = 0$ *for* $n < 0$.

Proof. Let $N = M^*G$, the homogeneous maximal ideal of G. The first statement will follow from Lemma 3.1 provided we show that $\operatorname{ht} N\overline{G_{\mathrm{red}}} \geq 2$. By [Ra, Theorem 3.8], G is equidimensional, and hence $(G_{\mathrm{red}})_N$ is equidimensional as well. Thus $(G_{\mathrm{red}})_N$ is quasi-unmixed by [Mat, 31.6]. Consequently, $(G_{\mathrm{red}})_N$ satisfies Nagata's second chain condition for prime ideals by [Na, Theorem 34.6]. This implies that $\operatorname{ht} N\overline{(G_{\mathrm{red}})_N} = \operatorname{ht} N \geq 2$. But $\operatorname{ht} N\overline{(G_{\mathrm{red}})_N} = \operatorname{ht} N(\overline{G_{\mathrm{red}}})_W)$ where $W = \overline{G_{\mathrm{red}}} \setminus N$, and therefore $\operatorname{ht} N\overline{G_{red}} \geq 2$. This proves (a).

Since I is normal, $\operatorname{depth} G \geq 1$ and consequently, $\operatorname{depth} S^* \geq 2$. Therefore the long exact sequence on local cohomology derived from

$$0 \to S^*(1) \xrightarrow{t^{-1}} S^* \to G \to 0$$

yields the exact sequence

$$0 \to \mathrm{H}^1_{M^*}(G)_n \to \mathrm{H}^2_{M^*}(S^*)_{n+1}$$

for all n. It follows that $\mathrm{H}^1_{M^*}(G)_n = 0$ for $n \leq -1$. $\square$

We will need to use a result from [HM] concerning the generalized depths of $S(I)$ and $G(I)$:

Lemma 3.3. ([HM, Proposition 3.2]) *Let R be a local ring and I and ideal of R. Then* $\text{g-depth}\, S(I) = \text{g-depth}\, G(I) + 1$, *or equivalently (by Lemma 2.4),* $\operatorname{fg}(S(I)) = \operatorname{fg}(G(I)) + 1$.

We now prove our main result, which is a generalization (except for the quasi-unmixed assumption) of [HH, Theorem 3.1]. In particular, the ideal I is not required to be integral over a regular sequence.

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and I a nonzero normal ideal of R. Then the following statements are equivalent:*

(a) $\operatorname{depth} R \geq 2$.
(b) $\operatorname{depth}_{S(I^n)} I^n S(I^n) \geq 3$ *for* $n \gg 0$.
(c) $\operatorname{depth} G(I^n) \geq 2$ *for* $n \gg 0$.

Proof. Suppose $\operatorname{depth} R \geq 2$. Since I is normal ideal we have that $\operatorname{depth} G \geq 1$ and, by [HM, Propsition 3.6 and Theorem 3.10], $\operatorname{depth} S \geq 2$. By Proposition 3.2 we obtain that $\mathrm{H}^1_M(G)_n = 0$ for all $n < 0$ and so $\operatorname{fg}(G) \geq 2$. Consequently, $\operatorname{fg}(S) \geq 3$ by Lemma 3.3. This means there exists $t \geq 0$ such that

$$\mathrm{H}^2_M(S)_n = 0 \text{ for } |n| > t.$$

By Proposition 2.5, if $n > t$ then

$$\mathrm{H}^2_{M^{(n)}}(S^{(n)})_k = \mathrm{H}^2_M(S)_{kn} = 0 \text{ for } k \neq 0.$$

Since $S^{(n)} = R[I^n t^n]$ we may replace I with I^n and assume that $\mathrm{H}^2_M(S)_k = 0$ for $k \neq 0$. We will show that $\operatorname{depth}_S IS \geq 3$. Using the exact sequence

$$0 \to IS(-1) \to S \to R \to 0$$

and the fact that $\operatorname{depth} S \geq 2$, we have $\mathrm{H}^0_M(IS) = \mathrm{H}^1_M(IS) = 0$. Since $\operatorname{depth} R \geq 2$ this yields that

$$0 \to \mathrm{H}^2_M(IS)_{k-1} \to \mathrm{H}^2_M(S)_k$$

is exact for all k. Therefore $\mathrm{H}^2_M(IS)_k = 0$ for $k \neq -1$. On the other hand, we have the exact sequence

$$(*) \qquad 0 \to \mathrm{H}^1_M(G)_k \to \mathrm{H}^2_M(IS)_k \to \mathrm{H}^2_M(S)_k$$

obtained from the short exact sequence $0 \to IS \to S \to G \to 0$. We conclude that $\mathrm{H}^2_M(IS)_{-1} = 0$ since $\mathrm{H}^1_M(G)_{-1} = 0$ (by Proposition 3.2) and $\mathrm{H}^2_M(S)_{-1} = 0$. Therefore $\mathrm{H}^2_M(IS) = 0$ and $\operatorname{depth}_S IS \geq 3$.

For $(b) \Rightarrow (c)$, it suffices to prove the statement for $n = 1$. As I is normal we have $\mathrm{H}^0_M(G) = 0$, and from the exact sequence $(*)$ it follows that $\mathrm{H}^1_M(G) = 0$. Therefore $\operatorname{depth} G \geq 2$.

Finally, $(c) \Rightarrow (a)$ follows from [Mar2, Theorem 3.4(b)]. □

The next result generalizes [HH, Corollary 3.8] to any normal ideal.

Corollary 3.5. *Let $(R, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring and I a normal ideal of R. Then $G(I^n)$ is Cohen-Macaulay for $n \gg 0$.*

Proof. This is immediate from Theorem 3.4. □

Next we wish to look at Theorem 3.4 in light of conditions on $\operatorname{Proj} S$. We first investigate the relationship between the normality of $\operatorname{Proj} S$ and I being a normal ideal. The following result is well-known:

Proposition 3.6. *Let R be a Noetherian ring and I an ideal of R having positive grade. If* $\operatorname{Proj} S(I)$ *is normal then all large powers I are normal.*

Proof. By applying [RR,(2.3.2)] $\operatorname{grade} G(I^k)_+ \geq 1$ for $k \gg 0$. Since $\operatorname{Proj} S(I^k)$ is normal for all $k \geq 1$, it suffices to show that if I is an ideal such that $\operatorname{Proj} S$ is normal and $\operatorname{grade} G_+ \geq 1$ (where $S = S(I)$, $G = G(I)$) then I is integrally closed. We claim that IS is an integrally closed ideal; this will imply that $I = IS \cap R$ is also integrally closed. Let $Q \in \operatorname{Ass} S/IS$. If $It \subseteq Q$ then $\operatorname{grade} G_+ = 0$ since $It(S/IS) = G_+$. Therefore $It \not\subseteq Q$ which implies IS_Q is principal. Also, S_Q is a normal domain by the assumption on $\operatorname{Proj} S$. As principal ideals of integrally closed domains are integrally closed, IS_Q is integrally closed for all $Q \in \operatorname{Ass} S/IS$, completing the proof. $\square$

As noted earlier, if R is a local ring possessing a normal ideal then R must be reduced. In fact, if R has an m-primary normal ideal then R is analytically unramified ([Re]). There are examples of reduced rings R and ideals I such that $\operatorname{grade} I = 0$ and $\operatorname{Proj} S(I)$ is normal. (E.g., $R = k[[x, y]]/(xy)$ and $I = (x)R$.) Are large powers of such ideals integrally closed? The answer is yes in the case R is analytically unramified.

Proposition 3.7. *Suppose R is an analytically unramified local ring and* $\operatorname{Proj} S(I)$ *is normal for some ideal I of R. Then I^n is integrally closed for all large n.*

proof. Let $\overline{S} = \oplus \overline{I^n}$ be the integral closure of $S = S(I)$ in $R[t]$. As R is analytically unramified, $\overline{S}$ and $T = \overline{S}/S$ are finite as S-modules ([Re]). Since $\operatorname{Proj} S$ is normal, $T_Q = 0$ for all $Q \in \operatorname{Proj} S$. Consequently $\operatorname{Ann}_S T \not\subseteq Q$ for all $Q \in \operatorname{Proj} S$. In other words, $S_+ \subseteq \sqrt{\operatorname{Ann}_S T}$. As T is a finitely generated graded S-module, this implies that $T_n = 0$ for n sufficiently large. Hence $\overline{I^n} = I^n$ for large n. $\square$

Combining these results with Theorem 3.4, we get the following:

Corollary 3.8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring and I an ideal such that* $\operatorname{Proj} S(I)$ *is normal. Suppose that either* $\operatorname{ht} I > 0$ *or R is analytically unramified. Then* $\operatorname{depth} G(I^n) \geq 2$ *for $n \gg 0$.*

Proof. By Propositions 3.6 and 3.7, some power of I is normal. Now use Theorem 3.4. $\square$

Applying 3.8 in the two-dimensional case leads to the following generalization of Theorem 1.2 (see also [L1, Theorem 2.4]):

Corollary 3.9. *Let $(R, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring and I an ideal such that* $\operatorname{Proj} S(I)$ *is normal. Suppose that either* $\operatorname{ht} I > 0$ *or R is analytically unramified. Then $G(I^n)$ is Cohen-Macaulay for $n \gg 0$.*

We now turn to some results concerning the integers $a_i(A)$ for a positively graded Noetherian ring A with a unique homogeneous maximal ideal $\mathfrak{q}$. For $0 \leq i \leq \dim A$ these integers are defined by

$$a_i(A) := \max\{n \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{q}}(A)_n \neq 0\}.$$

If $d = \dim A$ then $a_d(A)$ is called the *a-invariant* of A (see [GW]). In the case where A_0 is Artinian, and A is generated by one-forms over A_0, $\max\{a_i(A)\}$ is an upper bound for the largest n for which the Hilbert function $H(A,n)$ and Hilbert polynomial $P(A,n)$ do not coincide (see [BH, Theorem 4.3.5] in the field case, and [Mar1, Lemma 1.3] in the Artinian case). It is also known that the a_i's play a role in bounding the reduction number of A_+ (see [T]), which in turn can lead to information about the depth of A.

The following proposition gives a condition under which high powers of an ideal yield a Cohen-Macaulay Rees algebra. Compare with [KN] and [G], where the negativity of the a_i's was linked to the Cohen-Macaulayness of $R[It]$ in special cases.

Proposition 3.10. *Let $(R,\mathfrak{m})$ be a d-dimensional equidimensional local ring which is the homomorphic image of a Gorenstein ring, I an ideal of R having positive height, and $S=S(I)$. Suppose that $a_i(S)<0$ for $0\le i\le d$. Then* $\operatorname{Proj} S$ *is Cohen-Macaulay if and only if $S(I^n)$ is Cohen-Macaulay for $n\gg 0$.*

Proof. As R is equidimensional and $\operatorname{ht} I>0$, S is equidimensional. By Corollary 2.3 and Lemma 2.4, $\operatorname{Proj} S$ is Cohen-Macaulay if and only if $\mathrm{H}^i_M(S)_n=0$ for all but finitely many n. By the assumptions on $a_i(S)$, $\mathrm{H}^i_M(S)_0=0$ for $0\le i\le d$. Thus, $\operatorname{Proj} S$ is Cohen-Macaulay if and only if $\mathrm{H}^i_{M^{(n)}}(S^{(n)})=\mathrm{H}^i_M(S)^{(n)}=0$ for $n\gg 0$. $\square$

The next result gives a bound on $a_d(S(I))$, where R is assumed to be Gorenstein and I is an ideal of linear type. Recall that an ideal $I=(x_1,\dots,x_m)$ of a local ring R is of *linear type* if the kernel of the homomorphism $\phi\colon R[T_1,\dots,T_m]\to S(I)$ defined by $\phi(f(T_1,\dots,T_n))=f(x_1t,\dots,x_nt)$ can be generated by polynomials linear in the T_i's; i.e., $S(I)$ is isomorphic to the symmetric algebra of I.

Proposition 3.11. *Let $(R,\mathfrak{m})$ be a d-dimensional Gorenstein local ring and I an ideal of R having linear type and positive height. Then $a_d(S(I))<0$.*

Proof. Without loss of generality we may assume that R is complete. Suppose $I=(x_1,\dots,x_n)$ and set $T=R[T_1,\dots,T_n]$. Then $S\cong T/J$ where J is the kernel of the canonical map $\phi\colon T\to S$ mentioned above. By assumption J is generated by homogeneous polynomials which are linear in the T_i's. Furthermore $\operatorname{ht} J=\dim T-\dim S=n-1$, and thus there exists homogeneous elements $a_1,\dots,a_{n-1}\in J$ which are linear in the T_i's and form a regular sequence in T. Note that the canonical module of T is $\omega_T\cong T(-n)$ (cf. [BH, section 3.6]). Set $A=T/(a_1,\dots,a_{n-1})T$ and $L=J/(a_1,\dots,a_{n-1})T$. Then A is a Gorenstein ring, L is generated by images of polynomials which are linear in the T_i's, $\operatorname{ht} L=0$, $\omega_A\cong A(-1)$, and $A/L\cong S$.

We need to show that $\mathrm{H}^d_M(S)_n=0$ for all $n\ge 0$. If N is a graded A-module and E is the (graded) injective hull of $R/\mathfrak{m}$, let $N^\vee=\operatorname{Hom}_R(N,E)$. By the graded version of local duality (see [BH, Theorem 3.6.19]), $\mathrm{H}^d_M(S)^\vee\cong\operatorname{Ext}^1_A(A/L,A(-1))$. The grading on $H^d_M(R[It])^\vee$ is given by $(\mathrm{H}^d_M(S)^\vee)_i=(\mathrm{H}^d_M(S)_{-i})^\vee$, and so we have

$$(\mathrm{H}^d_M(S)_n)^\vee=(\mathrm{H}^d_M(S)^\vee)_{-n}=\operatorname{Ext}^1_A(A/L,A(-1))_{-n}.$$

Therefore it suffices to show that $\operatorname{Ext}^1_A(A/L,A(-1))_n=0$ for $n\le 0$. Consider the exact sequence of A-modules

$$A(-1)\to\operatorname{Hom}_A(L,A(-1))\to\operatorname{Ext}^1_A(A/L,A(-1))\to 0,$$

obtained by applying the functor $\mathrm{Hom}_A(-, A(-1))$ to the short exact sequence of A-modules $0 \to L \to A \to A/L \to 0$. Because $A(-1)_n = 0$ for $n \leq 0$ we have

$$\mathrm{Hom}_A(L, A(-1))_n \cong \mathrm{Ext}^1_A(A/L, A(-1))_n$$

for $n \leq 0$. Hence it suffices to show that $\mathrm{Hom}^1_A(L, A(-1))_n = 0$ for $n \leq 0$.

Let $f \in \mathrm{Hom}_A(L, A(-1))_n$ and let L_1 denote the set of homogeneous elements of L having degree 1 in the T_i's. Then $f(L_1) \subseteq A(-1)_{n+1} = A_n$. If $n < 0$ then $A_n = 0$, thus $f(L_1) = 0$. Since L is generated by L_1, $f = 0$ in this case. Therefore it suffices to prove the $n = 0$ case.

We claim that

$$L = (0 \underset{A}{:} (0 \underset{A}{:} L)).$$

Setting $K = (0 : (0 : L))$, it is clear that $K \supseteq L$. Thus, it suffices to show that $L_P = K_P$ for all $P \in \mathrm{Ass}\, A/L = \mathrm{Ass}\, S$. Since R is Cohen-Macaulay and $\mathrm{ht}\, I > 0$ the associated primes of S all have dimension equal to $\dim S$ ([V, Proposition 1.1]). Hence A_P is a zero-dimensional Gorenstein local ring for all $P \in \mathrm{Ass}\, A/L$. Thus $L_P = K_P$ (see [BH, Exercise 3.2.15], for example) for all $P \in \mathrm{Ass}\, A/L$, which proves the claim.

We can now prove the $n = 0$ case. Observe that $f(L) \subseteq L$. For if $w \in L$ then $w(0 : L) = 0$, which implies $f(w)(0 : L) = f(w(0 : L)) = 0$. Therefore (using the above claim) $f(w) \in L$. In particular $f(L_1) \subseteq A_0 \cap L = 0$ (because $n = 0$). As L_1 generates L it follows that $f = 0$, completing the proof. $\square$

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a d-dimensional Gorenstein local ring and I an ideal of R having linear type and positive height. Assume further that* $\mathrm{Proj}\, S(I)$ *is Cohen-Macaulay. Then* $\mathrm{depth}\, G(I^m)) \geq d-1$ *for some m if and only if $S(I^n)$ is Cohen-Macaulay for $n \gg 0$.*

Proof. If $\mathrm{depth}\, G(I^m) \geq d - 1$ then by [HM, Theorem 3.10] we have $\mathrm{depth}\, S^{(m)} = \mathrm{depth}\, S(I^m) \geq d$. Therefore, $\mathrm{H}^i_M(S)_0 = \mathrm{H}^i_{M^{(m)}}(S^{(m)})_0 = 0$ for $i \leq d-1$. Further, $a_d(S) < 0$ by Proposition 3.11; consequently, $\mathrm{H}^d_M(S)_0 = 0$. By Corollary 2.3 and Lemma 2.4, $\mathrm{H}^i_M(S)_k = 0$ for $i \leq d$ and all but finitely many k. Hence $\mathrm{H}^i_{M^{(n)}}(S^{(n)}) = 0$ for $i \leq d$ and n sufficiently large. For the converse, if $S^{(n)}$ is Cohen-Macaulay then by [HM, Lemma 3.3(b)] $\mathrm{depth} G(I^n) \geq d - 1$. $\square$

We end this section with the following consequence of Theorem 3.4 and Corollary 3.12:

Corollary 3.13. *Let $(R, \mathfrak{m})$ be a 3-dimensional Gorenstein local ring and I an ideal of positive height and linear type. If* $\mathrm{Proj}\, S(I)$ *is both normal and Cohen-Macaulay then $S(I^n)$ is Cohen-Macaulay for $n \gg 0$.*

Proof. By Corollary 3.8 there exists $m > 0$ such that $\mathrm{depth}\, G(I^m) \geq 2$. The conclusion now follows from Corollary 3.12. $\square$

4. The depths of extended Rees algebras of powers of an ideal

The purpose of this section is to characterize $\mathrm{depth}\, S^*(I^n)$ for large n in terms of the local cohomology of $S^*(I)$ (Proposition 4.4 and Corollary 4.5). This characterization can be used to give a proof of a version of Theorem 3.4 (see Theorem 4.7), but also may be of independent interest. We adopt the notation from section 3 for the Rees algebra, extended Rees algebra and the associated graded ring of an ideal. We begin with an analysis of $\mathrm{H}^i_M(S^*)_n$ when $|n|$ is large.

Proposition 4.1. *Let I be an ideal of a local ring (R, m). Let $S = S(I)$ and $S^* = S^*(I)$. Then*

(a) $\mathrm{H}^i_{M^*}(S^*)_n = 0$ *for $n \ll 0$ and $i < \mathrm{fg}(S)$.*
(b) *If $i = \mathrm{fg}(S)$ and $k \in \mathbb{Z}$ is given, then there exists $n < k$ such that $\mathrm{H}^i_{M^*}(S^*)_n \neq 0$.*
(c) $\mathrm{H}^i_{M^*}(S^*)_n = 0$ *for $n \gg 0$ and $i < \operatorname{depth} R + 1$.*
(d) $\mathrm{H}^i_{M^*}(S^*)_n \neq 0$ *for $n \gg 0$ and $i = \operatorname{depth} R + 1$.*

Proof. From the short exact sequence of S^*-modules

$$0 \to S^*(1) \xrightarrow{t^{-1}} S^* \to G \to 0,$$

we get the exact sequence

$$\mathrm{H}^{i-1}_{M^*}(S^*)_n \to \mathrm{H}^{i-1}_{M^*}(G)_n \to \mathrm{H}^i_{M^*}(S^*)_{n+1} \xrightarrow{t^{-1}} \mathrm{H}^i_{M^*}(S^*)_n$$

for all i and n. If $i < \mathrm{fg}(S) = \mathrm{fg}(G) + 1$ then $\mathrm{H}^{i-1}_{M^*}(G)_n = 0$ for $n \ll 0$. Thus, t^{-1} is not a zero-divisor on $\mathrm{H}^i_{M^*}(S^*)_n$ for $n \ll 0$. Part (a) now follows since every element of $\mathrm{H}^i_{M^*}(S^*)$ is annihilated by some power of t^{-1}. If $i = \mathrm{fg}(S)$ then $i - 1 = \mathrm{fg}(G)$. This means that given any $k \in \mathbb{Z}$ there exists $n < k$ such that $\mathrm{H}^{i-1}_{M^*}(G)_n \neq 0$ (because $\mathrm{H}^{i-1}_{M^*}(G)_n = 0$ for $n \gg 0$). Using that $\mathrm{H}^{i-1}_{M^*}(S^*)_n = 0$ for $n \ll 0$, along with the previous sentence, we obtain (b).

The short exact sequence of graded S-modules

$$0 \to S \to S^* \to S^*/S \to 0$$

leads to the exact sequence

$$\mathrm{H}^{i-1}_M(S^*/S)_n \to \mathrm{H}^i_M(S)_n \to \mathrm{H}^i_M(S^*)_n \to \mathrm{H}^i_M(S^*/S)_n$$

for all i and n. Since every element of S^*/S is annihilated by a power of S_+, $\mathrm{H}^i_M(S^*/S) \cong \mathrm{H}^i_{mS}(S^*/S) \cong \oplus_{n<0} \mathrm{H}^i_m(R)t^n$. Thus $\mathrm{H}^i_M(S)_n \cong \mathrm{H}^i_M(S^*)_n$ for all i and $n \geq 0$. Hence $\mathrm{H}^i_M(S^*)_n = 0$ for $n \gg 0$ and all i.

Now, from [It, Appendix 2] we have the following exact sequence of graded S^*-modules:

$$\mathrm{H}^{i-1}_M(S^*) \to \mathrm{H}^{i-1}_M(R[t^{-1}, t]) \to \mathrm{H}^i_{M^*}(S^*) \to \mathrm{H}^i_M(S^*)$$

for all i. Using that $\mathrm{H}^i_M(S^*)_n = 0$ for $n \gg 0$ and $\mathrm{H}^{i-1}_M(R[t^{-1}, t])_n \cong \mathrm{H}^{i-1}_m(R)$ for all n, parts (c) and (d) readily follow. $\square$

Here is a consequence:

Corollary 4.2. *Let (R, m) be a local ring, I an ideal of R, $S^* = S^*(I)$ and $S = S(I)$. Then*

(a) $\operatorname{fg}(S^*) = \min\{\operatorname{fg}(S), \operatorname{depth} R + 1\}$.
(b) $\operatorname{fg}(S^*(I^n)) = \operatorname{fg}(S^*)$ *for all* $n \geq 1$.

proof. The first statement follows immediately from Proposition 4.1. To prove the second statement it suffices by part (a) to show $\operatorname{fg}(S) = \operatorname{fg}(S(I^n))$. As $\operatorname{fg}(S) = \operatorname{fg}(S \otimes_R \hat{R})$ where $\hat{R}$ is the m-adic completion of R, we may assume S is the homomorphic image of a Gorenstein ring. Then by Lemma 2.4 and Proposition 2.6, $\operatorname{fg}(S) = \operatorname{g-depth} S = \operatorname{g-depth} S(I^n) = \operatorname{fg}(S(I^n))$.

We also have:

Corollary 4.3. *Suppose (R, m) is an equidimensional local ring which is the homomorphic image of a Gorenstein ring, and let I be an ideal of R. If* $\operatorname{depth} R = k$ *and* $\operatorname{Proj} S(I)$ *satisfies $\mathcal{S}_k$ then* $\operatorname{fg}(S^*(I)) = k + 1$.

Proof. From the hypotheses, we have that $S^*(I)$ is equidimensional and the homomorphic image of a Gorenstein ring. The result now follows immediately from Corollary 4.2, Corollary 2.3, and Lemma 2.4. □

We now obtain a characterization for the depth of $S^*(I^n)$ for large n:

Proposition 4.4. *Let (R, m) be a local ring, I an ideal of R and $S^* = S^*(I)$. Then for $n \gg 0$*

$$\operatorname{depth} S^*(I^n) = \max\{k \leq \operatorname{fg}(S^*) \mid \mathrm{H}^i_{M^*}(S^*)_0 = 0 \textit{ for all } i < k\}.$$

Proof. Let p denote the right-hand side of the above equation. Since $S^*(I^n) \cong (S^*)^{(n)}$, we have by Proposition 2.5

$$\mathrm{H}^i_{M_n^*}(S^*(I^n))_k \cong \mathrm{H}^i_{M^*}(S^*)_{nk}$$

for all k, where M_n^* is the homogeneous maximal ideal of $S^*(I^n)$. Thus for $i < \operatorname{fg}(S^*)$ and n large, $\mathrm{H}^i_{M_n^*}(S^*(I^n))_k = 0$ for all $k \neq 0$. Hence $\operatorname{depth} S^*(I^n) \geq p$ for n large. For the reverse inequality, just note that $\operatorname{depth} S^*(I^n) \leq \operatorname{fg}(S^*(I^n)) = \operatorname{fg}(S^*)$ (by Corollary 4.2(b)) and $\mathrm{H}^i_{M_n^*}(S^*(I^n))_0 \cong \mathrm{H}^i_{M^*}(S^*)_0$ for all i. □

As a special case of this proposition, we get the following:

Corollary 4.5. *Let (R, m) be a Cohen-Macaulay local ring which is the homomorphic image of a Gorenstein ring and I an ideal of R such that* $\operatorname{Proj} S(I)$ *is Cohen-Macaulay. Then for $n \gg 0$*

$$\operatorname{depth} S^*(I^n) = \max\{k \leq \dim S^* \mid \mathrm{H}^i_{M^*}(S^*)_0 = 0 \textit{ for all } i < k\}.$$

Proof. By Corollary 4.3, $\operatorname{fg}(S^*) = d + 1 = \dim S^*$. □

Consequently, in the situation above, $G(I^n)$ is Cohen-Macaulay for $n \gg 0$ if and only if $\mathrm{H}^i_{M^*}(S^*)_0 = 0$ for all $i < \dim S^*$. Note that as t^{-1} is a regular element on S^*, we always have $\mathrm{H}^0_{M^*}(S^*) = 0$. In addition, we have:

Proposition 4.6. *Let (R, m) be a local ring, I an ideal of R and $S^* = S^*(I)$. Then*

(a) $H^1_{M^*}(S^*)_0 = 0$.
(b) $\operatorname{depth} S^*(I^n) \geq 2$ *for* $n \gg 0$ *if and only if* $\operatorname{depth} R \geq 1$.

proof. As $H^0_{M^*}(G)$ has finite length, $\operatorname{fg}(G) \geq 1$. Thus, $\operatorname{fg}(S) \geq 2$ by Lemma 3.3. This implies by Proposition 4.1(a) that $\mathrm{H}^1_{M^*}(S^*)_n = 0$ for all $n \ll 0$. Since $\mathrm{H}^1_{M^*}(S^*(I^k))_0 \cong \mathrm{H}^1_{M^*}(S^*)_0$ for all $k \geq 1$, we may assume, by replacing I by a large enough power, that $\mathrm{H}^1_{M^*}(S^*)_n = 0$ for $n < 0$. Applying the long exact sequence

$$0 \to \mathrm{H}^0_{M^*}(G)_n \to \mathrm{H}^1_{M^*}(S^*)_{n+1} \xrightarrow{t^{-1}} \mathrm{H}^1_{M^*}(S^*)_n,$$

we see that $\mathrm{H}^1_{M^*}(S^*)_0 = 0$.

For part (b), note that by Corollary 4.2(a), $\operatorname{fg}(S^*(I^n)) \geq 2$ for any n if and only if $\operatorname{depth} R \geq 1$. The result now follows by part (a) and Proposition 4.4. □

Finally, we give another proof of a version of Theorem 3.4, this one using the extended Rees algebra.

Theorem 4.7. *Suppose that R is a quasi-unmixed and the homomorphic image of a Gorenstein ring. Let I be an ideal of positive grade such that* $\operatorname{Proj} S$ *is normal. Then the following are equivalent:*

(a) $\operatorname{depth} R \geq 2$.
(b) $\operatorname{depth} S^*(I^n) \geq 3$ *for* $n \gg 0$.
(c) $\operatorname{depth} G(I^n) \geq 2$ *for* $n \gg 0$.

proof. Suppose $\operatorname{depth} R \geq 2$. By Proposition 3.6, some power of I is normal. Proposition 3.2 (applied to a power of I) gives that $\mathrm{H}^2_{M^*}(S^*)_0 = 0$. By Proposition 4.6, $\mathrm{H}^1_{M^*}(S^*)_0 = 0$. Now R is equidimensional by [Mats, Theorem 31.5]. Applying Corollary 4.3, we obtain that $\operatorname{fg}(S^*) \geq 3$. Hence $\operatorname{depth} S^*(I^n) \geq 3$ by Proposition 4.4. The implication $(b) \Rightarrow (c)$ is trivial, while $(c) \Rightarrow (a)$ follows from [Mar2, Theorem 3.4(b)]. □

Note that, as in Corollary 3.8, the hypothesis on the grade of I can be removed if one assumes R to be analytically unramified.

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