Algebraic geometry : HW 4 solution

1* (bonus problem). Let S_d denote the polynomials in of $k[x_0, \dots, x_n]$ that are k-linear combinations of monomials of degree d. Prove that an ideal $\mathfrak{a}$ of $k[x_0, \dots, x_n]$ is homogeneous (i.e., generated by homogeneous polynomials) if and only if $\mathfrak{a} = \sum_d \mathfrak{a} \cap S_d$.

Sketch of proof: The "only if" part is easy: just pick generators for $\mathfrak{a}$; then their degree d pieces for various d's give homogeneous generators for $\mathfrak{a}$. We now prove the "if" part. Let $g_1, \dots, g_r$ be homogeneous generators for $\mathfrak{a}$. Let $f \in \mathfrak{a}$ have degree m. Then for some $a_1, \dots, a_r$ in $k[x_0, \dots, x_n]$, we have $f = \sum_{i=1}^r a_i g_i$. Recall that if g is a polynomial, then g_d denotes the sum of the monomials of degree d in g. Considering degrees and the fact that each g_i is a sum of monomials of the same degree (which may depend on i), we see that that $f_m = \sum_{i=1}^r (a_i)_{m-\deg g_i} g_i$. This shows that $f_m \in \mathfrak{a}$. Now replace f by $f - f_m$, which has degree less than m, and use induction on m.