# Algebraic geometry : HW 4 solution

$1^*$ (bonus problem). Let $S_d$ denote the polynomials in of $k[x_0, \dots, x_n]$ that are $k$-linear combinations of monomials of degree $d$. Prove that an ideal $\mathfrak{a}$ of $k[x_0, \dots, x_n]$ is homogeneous (i.e., generated by homogeneous polynomials) if and only if $\mathfrak{a} = \sum_d \mathfrak{a} \cap S_d$.

*Sketch of proof:* The "only if" part is easy: just pick generators for $\mathfrak{a}$; then their degree $d$ pieces for various $d$'s give homogeneous generators for $\mathfrak{a}$. We now prove the "if" part. Let $g_1, \dots, g_r$ be homogeneous generators for $\mathfrak{a}$. Let $f \in \mathfrak{a}$ have degree $m$. Then for some $a_1, \dots, a_r$ in $k[x_0, \dots, x_n]$, we have $f = \sum_{i=1}^r a_i g_i$. Recall that if $g$ is a polynomial, then $g_d$ denotes the sum of the monomials of degree $d$ in $g$. Considering degrees and the fact that each $g_i$ is a sum of monomials of the same degree (which may depend on $i$), we see that that $f_m = \sum_{i=1}^r (a_i)_{m-\deg g_i} g_i$. This shows that $f_m \in \mathfrak{a}$. Now replace $f$ by $f - f_m$, which has degree less than $m$, and use induction on $m$.