Algebraic geometry : HW 9

1. (bonus) Let k be a field, and R and S be two k-algebras. Show that a morphism Spec $S \to$ Spec R of schemes is a morphism of schemes over k (recall that Spec k is often denoted by just k) if and only if the corresponding homomorphism of rings $R \to S$ is a k-algebra homomorphism.

2. (bonus) (a) Let $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ be two affine varieties defined over a field k. Recall that a morphism of algbraic sets from X to Y is said to be defined over k if the corresponding map on the affine coordinate rings is a k-algbera homomorphism. Suppose $\phi : X \to Y$ is a map such that for each i from 1 to m, the i-th coordinate of ϕ (i.e., $x_i \circ \phi$, where x_i denotes the i-th coordinate function on $\mathbf{A}^m$) is a polynomial with coefficients in k. Then show that ϕ is defined over k in the sense above.
(b) Let E be the elliptic curve $y^2 = x^3 - x$ defined over $\mathbf{Q}$. Show that the map $E \to E$ given by $(x, y) \mapsto (x, -y)$ is defined over $\mathbf{Q}$ (in the sense of part (a)).