# Lecture 4

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Recall: functor of points.
If $K$ is a field and $a \subset k[x_1, ..., x_n]$ is an ideal then if $k \subset L$, $L$ a field. Given $f_1, ..., f_n \in k[x_1, ..., x_n]$ we can ask for the zero in $L^n$. the solutions to $a$ over $L$ are in bijection with the $k$-algebra homomorphisms $k[x_1, ..., x_n] \to L$, $x_i \to a_i$.

$k$-algebra homomorphisms are morphisms of schemes.

$$\operatorname{Spec} k[x_1, \ldots, x_n]/a \leftarrow \operatorname{Spec} L$$

In particular, take $L = \bar{k}$ and letting $X = Z(a) \subseteq A^n_{\bar{k}}$ we get $X = \hom_{sch(k)}(\bar{k}, \operatorname{Spec} A(X))$. So we can recover $X$ from it's "associated scheme" $\operatorname{Spec} A(x)$. This is a better way of thinking of a variety as a scheme. Also, $\operatorname{Spec} A(x)$ carries more information than $X$. We can also recover solutions over any field extension of $k$. This might explain the terminology "scheme". It motivates the following definition:

If $X$ is a scheme over some field $k$, and $L$ is an extension field, then define:

$$X(L) := \hom_{\mathrm{Sch}(k)}(L, X)$$

called the set of $L$ *valued points* of X.

More generally, if $X, Y$ are schemes on a scheme $S$(base).

Remark: We should distinguish this (even when $L = \bar{k}$) from the points of the scheme.

Remark: If $R$ is a ring then there exists a unique ring homomorphism $\mathbb{Z} \Rightarrow R$. $\operatorname{Spec} Z \Leftarrow \operatorname{Spec} R$. More generally one can show that any scheme has a unique map to $\operatorname{Spec} Z$. So $\operatorname{Spec} \mathbb{Z}$ is a kind of universal base. Hartshorne 2.2.*?

If $X, Y$ are schemes over a scheme $S$ (base) then we define $X(y) := \hom_{sch(s)}(Y, X)$. These are called the set of Y-valued points. of X.

Example: If $E$ is $y^2 = x^3 - x$ the elliptic curve defined over $\mathbb{Q}$ then $E(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 | (x, y) \in E\}$. Rational solutions.

Vista: This gives a functor associated to $X$ denoted $h_x : \mathrm{Sch}(s) \to Sets$. $x \mapsto \hom_{\mathrm{Sch}\, s}(Y, X)$. So there is a functor from $\mathrm{Sch}(s) \to Functors$ from $\mathrm{Sch}(s)$ to sets. $x \mapsto \hom_{\mathrm{Sch}(s)}(., x)$.

Remark: A functor of the form $h_x$ is called a representable functor.

This gives an equivalence of $\mathrm{Sch}(s)$ with a full subcategory of the caretgory of functors from $\mathrm{Sch}(s)$ to sets.

Point: Many geometrical constructions, i.e. tangent space, can be carried out using functors. See "Geometry of Schemes".

Fibered product:

Section 2.3

Recall : If $C$ is a category and $X, Y \in \mathrm{Obj}(C)$ then the product $X \times Y$ in $C$ is an object that satisfies a certain universal property.

$$\begin{tikzcd}
 & Z \arrow[d, "\exists!\Theta"'] \arrow[dl, bend right] \arrow[dr, bend left] & \\
 & X \times Y \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & \\
X & & Y
\end{tikzcd}$$

i.e. $C$ is sets, $X \times Y$ is the cartesian product. We want to consider products of algebraic sets/schemes and the same over a fixed base.

**Definition 0.1.** The fibered product.

If $S \in \mathrm{Obj}(C)$ the fibered products in $C/S :=$ Category of objects over S.

if $x, y$ is an $S$-object $X \times Y$ with maps $\phi_1 : X \times_s Y \to X, \phi_2 : X \times_S Y \to Y$.

$$\begin{tikzcd}
 & Z \arrow[d, "\exists!\Theta"'] \arrow[dl, bend right] \arrow[dr, bend left] \arrow[ddd, bend left=90] & \\
 & X \times Y \arrow[dl, "\pi_2"] \arrow[dr, "\pi_2"] \arrow[dd] & \\
X \arrow[dr] & & Y \arrow[dl] \\
 & S &
\end{tikzcd}$$

Remark: If $C$ has a terminal object $S$ then the fibered product over $S$ is the product.

A (fibered) product is unique up to unique isomorphism.

What should be products of algebraic sets? 1st idea: If $X, Y$ are affine varieties, $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ then $X \times Y \subseteq \mathbb{A}^{m+n}$

We need to check that the Zariski topology is respected.

The same does not work for projective space because the number of coordinates don't match up: $\mathbb{P}^n \times \mathbb{P}^m \ncong \mathbb{P}^{m+n}$

What does work: Serge embedding.

Suppose $X, Y$ are affine varieties over a field $k$ and $X \times_k Y$ exists, then There is a correspondence with varieties and finitely generated domains over $k$.

$$\begin{tikzcd}
 & Z \arrow[d, "\exists!\Theta"'] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & X \times_k Y \arrow[dl] \arrow[dr] \arrow[dd] & \\
X \arrow[dr] & & Y \arrow[dl] \\
 & \operatorname{Spec} k &
\end{tikzcd}$$

$$\begin{tikzcd}
 & A(Z) & \\
 & A(X \times_k Y) \arrow[u] & \\
A(X) \arrow[ur] \arrow[uur, bend left] & & A(Y) \arrow[ul] \arrow[uul, bend right] \\
 & k \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Is there a universal object in the category of finitely generated domains over $k$ which has this unversal property satisfied by $A(X \times_k Y)$. Answer: Yes. $A(X) \otimes A(Y)$. In general in the category of modules over a fing $R$ there is a fibered product, the tensor product denoted $M \otimes_R N$

First: Tensor products of modules.

Let $M, N$ be modules over a ring $R$.

**Definition 0.2.** A map $\phi : M \times N \to L$ where $L$ is an $R$-module, is said to be $R$-billinear if: $\phi(a+b, c) = \phi(a, c) + \phi(b, c)$
$\phi(a, b+c) = \phi(a, b) + \phi(a, c)$
$\phi(ra, b) = \phi(a, rb)$

We want a universal object for such maps, i.e. an object $M\Box_R N$ with a map $\psi : M \times N \to M\Box_R N$ which fits into the following commutative diagram:

$$\begin{tikzcd} M \times N \arrow[rr] \arrow[dr, "\phi"'] & & M\Box_R N \arrow[dl, "\exists! \theta"] \\ & L & \end{tikzcd}$$

So $\{R$-bilinear maps $M \times N \to L\ \} \leftrightarrow \{\ R$-module homomorphisms $M\Box_R N \to L$

Idea to construct $M\Box N$:

Consider all pairs $(m, n)$ and force the conditions.

**Definition 0.3.** We will call $M\Box_R N M \otimes_R N$, or the tensor product of $M$ and $N$ over $R$. If the ring $R$ is clear from context we may omit $R$. The tensor product $M \otimes N$ is the quotient of the free abelian group $M \times N$ by the subgroup generated by

$(a+b, c) = (a, c) + (b, c)$
$(a, b+c) = (a, b) + (a, c)$

The image of $(m, n)$ is denoted $m \otimes n$. $M \otimes N$ is an $R$-module via $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$.

Let $R$ be a subring of a ring $S$. (Then $S$ is an $R$-module in a natural way.) $S \otimes_R R$ is an $S$-module via

$$s\left(\sum_{i=1}^{k} s_i \otimes r_i\right) = \sum_{i=1}^{k} s s_i \otimes r_i$$

**Lemma 0.4.** *The map $S \otimes_R R \to S$ is well defined and is an isomorphism of $S$-modules.*
$s \otimes r \to rs$

*Proof.* Consider the map $\Theta : S \times R \to S$, $(s, r) \mapsto sr$ . This is $R$-billinear. This gives the map $S \otimes_R R \to S$ above.

Consider the map $\phi : S \to S \otimes_R R$, $s \mapsto s \otimes 1$.

Then:

$$(\phi \circ \Theta)(\sum_{i=1}^{k} s_i \otimes r_i) = \phi(\sum_{i=1}^{k} s_i r_i) = \left(\sum_{i=1}^{k} s_i r_i\right) \otimes 1 = \sum_{i=1}^{k} s_i \otimes r_i$$

$(\Theta \circ \phi)(s) = \Theta(s \otimes 1) = s$. So this completes the lemma. $\square$

Corollary of HW: If $R$ is a subring of a ring $S$ then $R^n \otimes_R S \cong (R \otimes_R S)^n \cong S^n$ via $(r_1, \ldots, r_n) \otimes s \mapsto (r_1 \otimes s, \ldots, r_n \otimes s) \mapsto (r_1 s, \ldots, r_n s)$

Eg. If $d$ is a field and $V$ is a vector space over $k$ and $L$ is a field containing $k$ then $V \cong k^n$, $V \otimes_k L \cong L^n$. The $n$ dimentional vectorspace over L. If $v_1, \ldots, v_n$ is a basis for $V$ over $k$ then $v_1 \otimes 1, \ldots, v_n \otimes 1$ is a basis for $V \otimes_k L$ over $L$. This operation is called changing the base of $V$ from $k$ to $L$.

**Definition 0.5.** If $A, B$ are algebras over $R$ (a ring) then in particular they are modules over $R$. Then $A \otimes_R B$ is an $R$-module and can be made into an $R$-algebra via $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$ and extending this definition $R$-linearly.

Claim: This is well defined:

*Proof.* (sketch): We want a map $A \otimes B \times A \times B \to A \otimes B$. Consider $A \times B \times A \times B \to A \otimes B$, $a, b, a', b') \mapsto (aa') \otimes (bb')$. Check: This is $R$-billinear. Therefore we get a map: $(A \otimes B) \otimes (A \otimes B) \to A \otimes B$. We know that this comes from:

$$\begin{tikzcd} M \times N \arrow[rr] \arrow[dr] & & M \otimes_R N \arrow[dl] \\ & L & \end{tikzcd}$$

$(A \otimes B) \times (A \otimes B) \to A \otimes B$ which takes $(a \otimes b, a' \otimes b') \mapsto aa' \otimes bb'$ $\square$

Fact: $A \otimes_R B$ with maps $A \to A \otimes B$, $a \mapsto a \otimes 1$ and $B \to A \otimes B$,

$b \mapsto 1 \otimes b$ satisfies the universal property:

$$\begin{tikzcd}
 & C & \\
 & A \otimes B \arrow[u, "\exists ! \Theta"'] & \\
A \arrow[ur, "a \mapsto a \otimes 1"'] \arrow[uur, "f"] & & B \arrow[ul, "b \mapsto 1 \otimes b"'] \arrow[uul, bend right, "g"'] \\
 & R \arrow[ul] \arrow[ur] \arrow[uuu, bend left=90] &
\end{tikzcd}$$

Also the fibered product exists for affine schemes. $\operatorname{Spec} A \otimes \operatorname{Spec} B = \operatorname{Spec}(A \otimes_R B)$ eg.

We know that $\mathbb{A}_k^n \leftrightarrow \operatorname{Spec} k[x_1, \ldots, x_n]$. What is $\mathbb{A}_k^n \times \mathbb{A}_k^m$ as a scheme. $\mathbb{A}_k^n \times \mathbb{A}_k^m \leftrightarrow \operatorname{Spec}[x_1, \ldots, x_n, y_1, \ldots, y_m] \leftrightarrow \mathbb{A}_k^{m+n}$

Warning: The Zariski topology on $\mathbb{A}^{n+m}$ is not the product topology. For general schemes we glue the constructions above. Thm 3.3 in Hartshorne. For any two schemes $x, y$ over a base $S$ the fibered product $X \times_S Y$ exists.