

# Lecture 4

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December 1, 2008

Recall: functor of points.

If  $K$  is a field and  $a \subset k[x_1, \dots, x_n]$  is an ideal then if  $k \subset L$ ,  $L$  a field. Given  $f_1, \dots, f_n \in k[x_1, \dots, x_n]$  we can ask for the zero in  $L^n$ . the solutions to  $a$  over  $L$  are in bijection with the  $k$ -algebra homomorphisms  $k[x_1, \dots, x_n] \rightarrow L$ ,  $x_i \rightarrow a_i$ .

$k$ -algebra homomorphisms are morphisms of schemes.

$$\text{Spec } k[x_1, \dots, x_n]/a \leftarrow \text{Spec } L$$

In particular, take  $L = \bar{k}$  and letting  $X = Z(a) \subseteq A_{\bar{k}}^n$  we get  $X = \text{hom}_{\text{sch}(k)}(\bar{k}, \text{Spec } A(X))$ . So we can recover  $X$  from it's "associated scheme"  $\text{Spec } A(x)$ . This is a better way of thinking of a variety as a scheme. Also,  $\text{Spec } A(x)$  carries more information than  $X$ . We can also recover solutions over any field extension of  $k$ . This might explain the terminology "scheme". It motivates the following definition:

If  $X$  is a scheme over some field  $k$ , and  $L$  is an extension field, then define:

$$X(L) := \text{hom}_{\text{Sch}(k)}(L, X)$$

called the set of  $L$  valued points of  $X$ .

More generally, if  $X, Y$  are schemes on a scheme  $S(\text{base})$ .

Remark: We should distinguish this (even when  $L = \bar{k}$ ) from the points of the scheme.

Remark: If  $R$  is a ring then there exists a unique ring homomorphism  $\mathbb{Z} \Rightarrow R$ .  $\text{Spec } \mathbb{Z} \Leftarrow \text{Spec } R$ . More generally one can show that any scheme has a unique map to  $\text{Spec } \mathbb{Z}$ . So  $\text{Spec } \mathbb{Z}$  is a kind of universal base. Hartshorne 2.2.\*?

If  $X, Y$  are schemes over a scheme  $S$  (base) then we define  $X(y) := \text{hom}_{\text{sch}(s)}(Y, X)$ . These are called the set of  $Y$ -valued points. of  $X$ .

Example: If  $E$  is  $y^2 = x^3 - x$  the elliptic curve defined over  $\mathbb{Q}$  then  $E(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 | (x, y) \in E\}$ . Rational solutions.

Vista: This gives a functor associated to  $X$  denoted  $h_x : \text{Sch}(s) \rightarrow \text{Sets}$ .  $x \mapsto \text{hom}_{\text{Sch}(s)}(Y, X)$ . So there is a functor from  $\text{Sch}(s) \rightarrow \text{Functors}$  from  $\text{Sch}(s)$  to sets.  $x \mapsto \text{hom}_{\text{Sch}(s)}(\cdot, x)$ .

Remark: A functor of the form  $h_x$  is called a representable functor.

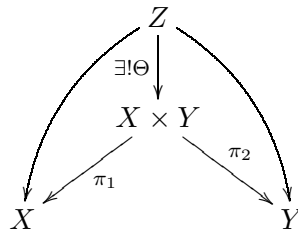
This gives an equivalence of  $\text{Sch}(s)$  with a full subcategory of the category of functors from  $\text{Sch}(s)$  to sets.

Point: Many geometrical constructions, i.e. tangent space, can be carried out using functors. See “Geometry of Schemes”.

Fibered product:

Section 2.3

Recall : If  $C$  is a category and  $X, Y \in \text{Obj}(C)$  then the product  $X \times Y$  in  $C$  is an object that satisfies a certain universal property.

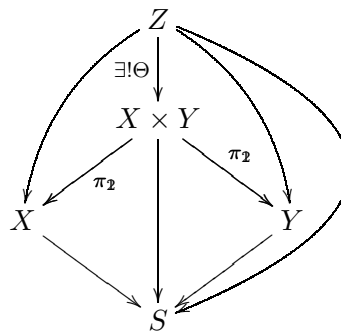


i.e.  $C$  is sets,  $X \times Y$  is the cartesian product. We want to consider products of algebraic sets/schemes and the same over a fixed base.

**Definition 0.1.** The fibered product.

If  $S \in \text{Obj}(C)$  the fibered products in  $C/S := \text{Category of objects over } S$ .

if  $x, y$  is an  $S$ -object  $X \times Y$  with maps  $\phi_1 : X \times_S Y \rightarrow X, \phi_2 : X \times_S Y \rightarrow Y$ .



Remark: If  $C$  has a terminal object  $S$  then the fibered product over  $S$  is the product.

A (fibered) product is unique up to unique isomorphism.

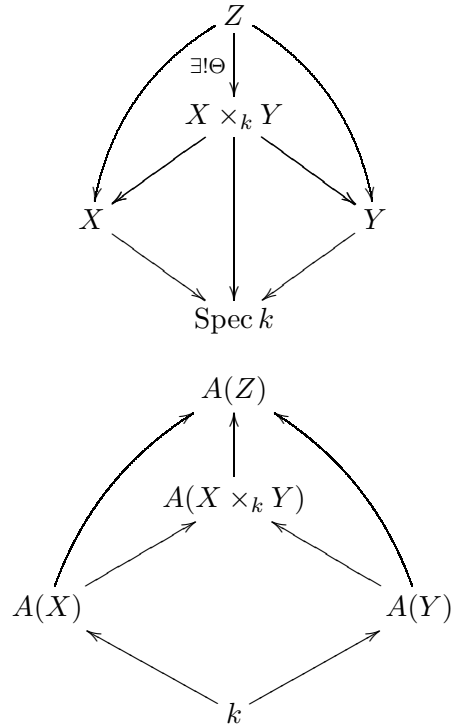
What should be products of algebraic sets? 1st idea: If  $X, Y$  are affine varieties,  $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$  then  $X \times Y \subseteq \mathbb{A}^{m+n}$

We need to check that the Zariski topology is respected.

The same does not work for projective space because the number of coordinates don't match up:  $\mathbb{P}^n \times \mathbb{P}^m \not\cong \mathbb{P}^{m+n}$

What does work: Serge embedding.

Suppose  $X, Y$  are affine varieties over a field  $k$  and  $X \times_k Y$  exists, then There is a correspondence with varieties and finitely generated domains over  $k$ .



Is there a universal object in the category of finitely generated domains over  $k$  which has this universal property satisfied by  $A(X \times_k Y)$ . Answer: Yes.  $A(X) \otimes A(Y)$ . In general in the category of modules over a ring  $R$  there is a fibered product, the tensor product denoted  $M \otimes_R N$

First: Tensor products of modules.

Let  $M, N$  be modules over a ring  $R$ .

**Definition 0.2.** A map  $\phi : M \times N \rightarrow L$  where  $L$  is an  $R$ -module, is said to be  $R$ -bilinear if:  $\phi(a + b, c) = \phi(a, c) + \phi(b, c)$   
 $\phi(a, b + c) = \phi(a, b) + \phi(a, c)$   
 $\phi(ra, b) = \phi(a, rb)$

We want a universal object for such maps, i.e. an object  $M \square_R N$  with a map  $\psi : M \times N \rightarrow M \square_R N$  which fits into the following commutative diagram:

$$\begin{array}{ccc} M \times N & \xrightarrow{\quad} & M \square_R N \\ & \searrow \phi & \swarrow \exists! \theta \\ & & L \end{array}$$

So  $\{R\text{-bilinear maps } M \times N \rightarrow L\} \leftrightarrow \{R\text{-module homomorphisms } M \square_R N \rightarrow L\}$

Idea to construct  $M \square_R N$ :

Consider all pairs  $(m, n)$  and force the conditions.

**Definition 0.3.** We will call  $M \square_R N = M \otimes_R N$ , or the tensor product of  $M$  and  $N$  over  $R$ . If the ring  $R$  is clear from context we may omit  $R$ . The tensor product  $M \otimes N$  is the quotient of the free abelian group  $M \times N$  by the subgroup generated by

$$(a + b, c) = (a, c) + (b, c)$$

$$(a, b + c) = (a, b) + (a, c)$$

The image of  $(m, n)$  is denoted  $m \otimes n$ .  $M \otimes N$  is an  $R$ -module via  $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$ .

Let  $R$  be a subring of a ring  $S$ . (Then  $S$  is an  $R$ -module in a natural way.)  $S \otimes_R R$  is an  $S$ -module via

$$s \left( \sum_{i=1}^k s_i \otimes r_i \right) = \sum_{i=1}^k s s_i \otimes r_i$$

**Lemma 0.4.** The map  $S \otimes_R R \rightarrow S$  is well defined and is an isomorphism of  $S$ -modules.

$$s \otimes r \rightarrow rs$$

*Proof.* Consider the map  $\Theta : S \times R \rightarrow S$ ,  $(s, r) \mapsto sr$ . This is  $R$ -bilinear. This gives the map  $S \otimes_R R \rightarrow S$  above.

Consider the map  $\phi : S \rightarrow S \otimes_R R$ ,  $s \mapsto s \otimes 1$ .

Then:

$$(\phi \circ \Theta)\left(\sum_{i=1}^k s_i \otimes r_i\right) = \phi\left(\sum_{i=1}^k s_i r_i\right) = \left(\sum_{i=1}^k s_i r_i\right) \otimes 1 = \sum_{i=1}^k s_i \otimes r_i$$

$(\Theta \circ \phi)(s) = \Theta(s \otimes 1) = s$ . So this completes the lemma.  $\square$

Corollary of HW: If  $R$  is a subring of a ring  $S$  then  $R^n \otimes_R S \cong (R \otimes_R S)^n \cong S^n$  via  $(r_1, \dots, r_n) \otimes s \mapsto (r_1 \otimes s, \dots, r_n \otimes s) \mapsto (r_1 s, \dots, r_n s)$

Eg. If  $d$  is a field and  $V$  is a vector space over  $k$  and  $L$  is a field containing  $k$  then  $V \cong k^n$ ,  $V \otimes_k L \cong L^n$ . The  $n$  dimensional vectorspace over  $L$ . If  $v_1, \dots, v_n$  is a basis for  $V$  over  $k$  then  $v_1 \otimes 1, \dots, v_n \otimes 1$  is a basis for  $V \otimes_k L$  over  $L$ . This operation is called changing the base of  $V$  from  $k$  to  $L$ .

**Definition 0.5.** If  $A, B$  are algebras over  $R$  (a ring) then in particular they are modules over  $R$ . Then  $A \otimes_R B$  is an  $R$ -module and can be made into an  $R$ -algebra via  $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$  and extending this definition  $R$ -linearly.

Claim: This is well defined:

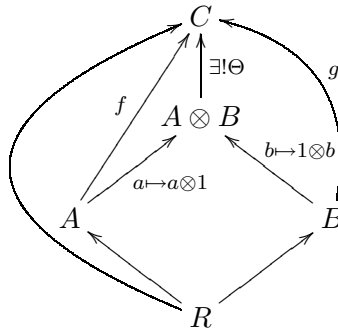
*Proof.* (sketch): We want a map  $A \otimes B \times A \otimes B \rightarrow A \otimes B$ . Consider  $A \otimes B \times A \otimes B \rightarrow A \otimes B$ ,  $(a, b, a', b') \mapsto (aa') \otimes (bb')$ . Check: This is  $R$ -bilinear. Therefore we get a map:  $(A \otimes B) \otimes (A \otimes B) \rightarrow A \otimes B$ . We know that this comes from:

$$\begin{array}{ccc} M \times N & \xrightarrow{\quad} & M \otimes_R N \\ & \searrow & \swarrow \\ & & L \end{array}$$

$(A \otimes B) \times (A \otimes B) \rightarrow A \otimes B$  which takes  $(a \otimes b, a' \otimes b') \mapsto aa' \otimes bb'$   $\square$

Fact:  $A \otimes_R B$  with maps  $A \rightarrow A \otimes B$ ,  $a \mapsto a \otimes 1$  and  $B \rightarrow A \otimes B$ ,

$b \mapsto 1 \otimes b$  satisfies the universal property:



Also the fibered product exists for affine schemes.  $\text{Spec } A \otimes \text{Spec } B = \text{Spec}(A \otimes_R B)$  eg.

We know that  $\mathbb{A}_k^n \leftrightarrow \text{Spec } k[x_1, \dots, x_n]$ . What is  $\mathbb{A}_k^n \times \mathbb{A}_k^m$  as a scheme.  $\mathbb{A}_k^n \times \mathbb{A}_k^m \leftrightarrow \text{Spec}[x_1, \dots, x_n, y_1, \dots, y_m] \leftrightarrow \mathbb{A}_k^{m+n}$

Warning: The Zariski topology on  $\mathbb{A}^{n+m}$  is not the product topology. For general schemes we glue the constructions above. Thm 3.3 in Hartshorne. For any two schemes  $x, y$  over a base  $S$  the fibered product  $X \times_S Y$  exists.