Elliptic curves : HW 3

Brief solutions

Let C denote the curve $y = x^2$ in $\mathbf{A}^2$.

1. Check that C is smooth.

Let $f(x, y) = y - x^2$. Then $f_y = 1$, so the vector (f_x, f_y) is never $(0, 0)$. Since C is the zero locus of $f(x, y)$, C is smooth.

2. (bonus) Show that at a point $P = (x_0, y_0)$ on C, a uniformizer is $x - x_0$ (recall that a uniformizer is a generator of the maximal ideal of the local ring at P). Also, if $x_0 \neq 0$, then $y - y_0$ is also a uniformizer at P. Added later: assume that the characteristic is not 2.

The maximal ideal is generated by $(x - x_0, y - y_0)$. Now $y - y_0 = x^2 - x_0^2 = (x - x_0)(x + x_0)$. So $x - x_0$ generates the maximal ideal, and hence is a uniformizer. If $x_0 \neq 0$ then at (x_0, y_0), $x + x_0 = 2x_0 \neq 0$, and so $y - y_0 = \frac{x - x_0}{x + x_0}$ also generates the maximal ideal, and hence is a uniformizer as well.

3. Find the order of vanishing of y at $O = (0, 0)$.

$y = x^2$, and x is a uniformizer at O, so the order of vanishing of y is 2.

4. Find the divisor of the function y on C.

The only zero of y on C is at O, which has order 2 by part 3, and there is no pole. So the divisor is $2O$.

5. Find the divisor of the differential dy on C.

Consider a point $P(x_0, y_0)$. Now $dy = 2xdx = 2xd(x - x_0)$, and $x - x_0$ is a uniformizer at P , so the order of vanishing of dy at P is the same as the order of vanishing of x at P, which is 0, unless $P = O$, where the order of vanishing is 1. Hence the divisor of dy is O.

Let C' denote the curve $yz = x^2$ in $\mathbf{P}^2$ (the projectivization of C).

5. (bonus) Check that C' is nonsingular at the point at infinity (denote this point by ∞).

The homogenized equation is $yz = x^2$, and the point at infinity is obtained by setting $z = 0$, whence $x = 0$ and y is arbitrary. So the point at infinity is $\infty = [0 : 1 : 0]$. So we can put $y = 1$ in $yz = x^2$ near ∞, which becomes $z = x^2$. Now just like in part 1 this curve is non-singular everywhere.

6. (bonus) Find a uniformizer at ∞ on C'.

The maximal ideal is generated by x and z, and since $z = x^2$, it is generated by x. So x is a uniformizer.

7. (bonus) Show that the divisor of the differential $d(x/z)$ on C' is -2∞.

Over the affine piece, at a point $P = [x_0 : y_0 : 1]$, $d(x/z) = dx = d(x - x_0)$, and $x - x_0$ is a uniformizer; so the order of vanishing of $d(x/z)$ is 0 at any such point. At ∞, $d(x/z) = d(x/x^2) = -x^{-2}dx$, and since x is a unformizer, the order of vanishing is -2. Hence the divisor is -2∞.

8. Show that the genus of C' is 0.

The genus is the same as the dimension of $\mathcal{L}(\text{div } \omega)$, for any differential ω. Taking $\omega = d(x/z)$, $f \in \mathcal{L}(\text{div } d(x/z)) = \mathcal{L}(-2\infty)$ iff $\text{div } f \geq 2\infty$, but since $\text{div } f$ has degree 0, this is not possible. So the genus is 0.