MAS 4303 BRIEF SOLUTIONS TO MIDTERM 3 Spring 2012

Name: ______________________________

Justify all answers. No credit will be given without proper justification. The exam is out of 32 points. Good luck!

1. (4 points) Recall that $M(\mathbf{R})$ denotes the ring of 2×2 matrices with entries in $\mathbf{R}$. What is the dimension of $M(\mathbf{R})$ as a vector space over $\mathbf{R}$? Be sure to justify your answer.

Claim

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is a basis for $M(\mathbb{R}) / \mathbb{R}$.

So dimension = 4

Proof of claim:

Span: Given $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(\mathbb{R})$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear indep:

If $c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$.

then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So $c_1 = c_2 = c_3 = c_4 = 0$

2. (4 points) Show that if F is a field and K is a field extension of F whose dimension over F is a prime number, then there are no fields between F and K, i.e., if E is a field such that $F \subseteq E \subseteq K$, then $E = F$ or $E = K$.

let $p = [K:F]$, a prime.

Then $p = [K:F] = [K:E][E:F]$

So $[K:E] = 1$ or p

If $[K:E] = 1$, then $E = K$

If $[K:E] = p$, then $[E:F] = 1$ by, so $E = F$

3. (4 points) Find a basis for $\mathbf{Q}(\sqrt{3})$ over $\mathbf{Q}$ (you may think $\sqrt{3}$ being an element of $\mathbf{R}$, and of $\mathbf{Q}(\sqrt{3})$ as the subfield of $\mathbf{R}$ obtained by adjoining $\sqrt{3}$ to $\mathbf{Q}$).

The min poly for $\sqrt{3}$ is $x^2 - 3$, which has degree 2.

So by a theorem in the book,

a basis is $\{1, (\sqrt{3})^{2-1}\}$ ie $\{1, \sqrt{3}\}$.

(the theorem says that a basis for $\mathbb{Q}(u)$

where u has min poly of degree n

is $\{1, u, \ldots, u^{n-1}\}$)

4. (4 points) Show that the minimal polynomial of $2^{1/3}$ (the unique cube root of 3 in $\mathbf{R}$) is $x^3 - 2$. If you use something that we have not proved in class or in a homework, then be sure to prove it.

Write $\sqrt[3]{2}$ for $2^{1/3}$.

Now $\sqrt[3]{2}$ satisfies x^3-2, which is monic.

So it suffices to show that x^3-2 is irreducible.

Suppose not.

Then $x^3-2 = f(x)g(x)$ with $\deg f, \deg g > 1$

$\therefore 3 = \deg f + \deg g$

So either $\deg f = 1$, $\deg g = 2$

or $\deg f = 2$, $\deg g = 1$

Wlog, suppose $\deg f = 1$.

Then f has a root in $\mathbb{Q}$, call it α.

Then $\alpha^3 - 2 = f(\alpha)g(\alpha) = 0$ by

So $\alpha^3 = 2$ with $\alpha \in \mathbb{Q}$

Claim: This cannot happen

Proof: Optional, but is similar to showing $\sqrt{2} \notin \mathbb{Q}$:

Suppose $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ relatively prime, $q \neq 0$

Then $\frac{p^3}{q^3} = 2$ so $p^3 = 2q^3$

So $2 \mid p^3$ $\therefore 2 \mid p$ since p is prime.

Say $p = 2k$ Then $8k^3 = 2q^3$ so $4k^3 = q^3$

$\therefore 2 \mid q^3$ $\therefore 2 \mid q$ contradicts p & q relatively prime $\square$

5. (4 points) What is the degree (dimension) of $\mathbf{Q}(\sqrt{3}, 2^{1/3})$ over $\mathbf{Q}$? You may assume the statement of problem 4 even if you have not proved it.

$[\mathbb{Q}(\sqrt{3}):\mathbb{Q}] = 2$

$>$ relatively prime

$[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}] = 3$

So by a HW problem,

$[\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}):\mathbb{Q}] = 2 \cdot 3 = 6.$

Note: $[\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}):\mathbb{Q}]$

$$= \underbrace{[\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}):\mathbb{Q}(\sqrt[3]{2})]}_{\uparrow} \underset{\substack{\| \\ 3}}{[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}]}$$

We know this is ≤ 2 since $\sqrt{3}$ satisfies $x^2-3 \in \mathbb{Q}(\sqrt[3]{2})[x]$

but you have to show it is 2

eg by showing $\sqrt{3} \notin \mathbb{Q}(\sqrt[3]{2})$

6. (4 points) Prove or disprove: if F is a field, K is a field extension of F, and $u, v \in K$ are algebraic over F, then $F(u, v) = F(u) \cup F(v)$.

False : eg $u = \sqrt{2}$, $v = \sqrt{3}$, $F = \mathbb{Q}$

We know (eg from class)

that a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{2} \cdot \sqrt{3}\}$

& for $\mathbb{Q}(\sqrt{2})$ is $\{1, \sqrt{2}\}$

& for $\mathbb{Q}(\sqrt{3})$ is $\{1, \sqrt{3}\}$

But $\sqrt{2} \cdot \sqrt{3}$ is not of the form $a + b\sqrt{2}$ or $c + d\sqrt{3}$ with $a, b, c, d \in \mathbb{Q}$.

so it cannot be in $\mathbb{Q}(\sqrt{2}) \cup \mathbb{Q}(\sqrt{3})$

7. (4 points) (a) Is $(x+1)$ a maximal ideal in $\mathbf{Q}[x]$? Is it a prime ideal? (be sure to answer both questions!).

$x+1$ is irreducible in $\mathbb{Q}[x]$.

Proof (optional) If not, $(x+1) = f(x)g(x)$
& $1 = \deg f + \deg g$
So one of f, g is a (non-zero) constant & ∴ a unit □

∴ $(x+1)$ is maximal (since $\mathbb{Q}[x]$ is a PID)
∴ $(x+1)$ is prime

(b) (4 points) Is $(x+1)$ a maximal ideal in $\mathbf{Z}[x]$? Is it a prime ideal? (be sure to answer both questions!).

Note: $\mathbb{Z}[x]$ is not a PID, so the strategy in (a) need not work. Instead:

Claim $\mathbb{Z}[x]/(x+1) \cong \mathbb{Z}$.

$\mathbb{Z}$ is not a field, so $(x+1)$ is not maximal
$\mathbb{Z}$ is a domain, so $(x+1)$ is prime

Proof of claim: Consider the map $f: \mathbb{Z}[x] \longrightarrow \mathbb{Z}$
$f(x) \mapsto f(-1)$

Check that f is a surjective homomorphism with kernel $(x+1)$ [If $f(x) \mapsto 0$, then let $f(x) = (x+1)q(x) + r(x)$ with $\deg r<1$; putting $x=-1$, we see $r(-1)=0$
∴ $r(x)=0$ ∴ $(x+1) \mid f(x)$]
The first isom thm for rings gives the claim □