GRV II : HW 11

1. Let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$ and let bars denote passage to the quotient $F[x]/(f(x))$. Prove that for each $\overline{g(x)}$ there is a unique polynomial $g_0(x)$ of degree $\leq n-1$ such that $\overline{g(x)} = \overline{g_0(x)}$ (equivalently, the elements $\overline{1}, \overline{x}, \ldots, \overline{x^{n-1}}$ are a *basis* of the vector space $F[x]/(f(x))$ over F — in particular, the dimension of this space is n). [Use the Division Algorithm]

2. Let $f(x)$ be a polynomial in $F[x]$. Prof that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible. [Use Proposition 6, Section 8.2.]

3. Determine the greatest common divisor of $a(x) = x^3 - x^2 + x - 1$ and $b(x) = x^2 - 1$ in $\mathbf{Q}[x]$ and write it as a linear combination (in $\mathbf{Q}[x]$) of $a(x)$ and $b(x)$.

You may turn in the rest of the problems with the next HW.:

4. Let R be an integral domain with quotient field F and let $p(x)$ be a monic polynomial in $R[x]$. Assume that $p(x) = a(x)b(x)$ where $a(x)$ and $b(x)$ are monic polynomials in $F[x]$ of smaller degree than $p(x)$. Prove that if $a(x) \notin R[x]$, then R is not a UFD. Deduce that $\mathbf{Z}[2\sqrt{2}]$ is not a UFD.

5. Prove that if $f(x)$ and $g(x)$ are polynomials with rational coefficients whose product $f(x)g(x)$ has integer coefficients, then the product of any coefficient of $g(x)$ with any coefficient of $f(x)$ is an integer.