

Review of “Mod- p reducibility, the torsion subgroup, and the Shafarevich–Tate group”

1. General comments

The result in the paper is interesting and it deserves to be published in a good journal. However, I do not quite find it up to the standards of PAMS. The main reasons:

- Although the result of this paper (in this form) does not appear in the literature, the main ideas are straightforward generalizations of results of Mazur from his Eisenstein ideal paper. For instance, the core of the argument is an easy generalization of Mazur’s result (II.3.16) from the celebrated paper on the Eisenstein ideal (if \mathfrak{m} is a maximal that is Eisenstein, then $\text{III}(J_0(N)/\mathbb{Q})[\mathfrak{m}] = 0$).
- As the author points out, it is not possible to use the suggested methods to establish optimal upper bounds on the order of III in the reducible case for arbitrary conductor N (not just prime conductor). It’s not at all clear what the right statement is in this case. In that sense, the result is rather special.
- The author provides no comments on how often an elliptic curve of prime conductor has a reducible mod p -representation for some odd prime p . It seems to me that this does not happen that often and if I recall correctly, it might almost never happen. Before publishing the paper, the author should carefully research this question. I recommend starting with the paper of Mestre–Oesterle *Courbes de Weil semi-stables*.

2. Specific comments

- *everywhere*: $\text{III}(E)$ should be replaced by $\text{III}(E/\mathbb{Q})$ or the author should mention what $\text{III}(E)$ means.
- *p.2, line -12*: $\text{III}(E/\mathbb{Q}) \otimes_{\mathbb{T}_p} \mathbb{T}_{\mathfrak{m}}$ should be $\text{III}(E/\mathbb{Q}) \otimes_{\mathbb{T}_{\mathfrak{m}}} \mathbb{T}_p$. Indeed, \mathbb{T}_p is a $\mathbb{T}_{\mathfrak{m}}$ -module.
- *p.3, acknowledgements*: You should replace *sage* by *SAGE*.
- *p.3, eqn. (1)*: All of a sudden, you change the notation to III_E ; it should be consistent throughout.
- *p.4, line 1*: I would specify “Theorem B(iv) or Theorem B(v)”.
- *p.6–7, Section 3*: This section should be revised - it’s good that data is provided, but Question 3.1 is not supported well by computational evidence (as the author points out at the end of the section. The question is rather speculative and should be omitted. However, the data is interesting and should be provided in the paper with the only purpose of showing that the statement of Theorem 1.3 does not hold when N is composite.
- *p.7, line -6-7*: Revise sentence “We denote the map induced by η on ...”.
- *p.7, line -2-1*: I am not sure I understand precisely (just by reading the sentence) what *modulo* Δ means? The author should clarify this statement.
- *p.8, line -16*: The author should check whether the specified reference by Emerton is Thm.4.13 or Thm.4.12. I have Thm. 4.13 in my version, but it may not be the latest one - this is why I

MOD- p REDUCIBILITY

recommend that the author double checks.