

# Visibility and the Birch and Swinnerton-Dyer conjecture for analytic rank zero

Amod Agashe \*

February 10, 2012

## Abstract

Let  $E$  be an optimal elliptic curve over  $\mathbf{Q}$  of conductor  $N$  having analytic rank zero, i.e., such that the  $L$ -function  $L_E(s)$  of  $E$  does not vanish at  $s = 1$ . Suppose there is another optimal elliptic curve over  $\mathbf{Q}$  of the same conductor  $N$  whose Mordell-Weil rank is greater than zero and whose associated newform is congruent to the newform associated to  $E$  modulo an integer  $r$ . The theory of visibility then shows that under certain additional hypotheses,  $r$  divides the product of the order of the Shafarevich-Tate group of  $E$  and the orders of the arithmetic component groups of  $E$ . We extract an explicit integer factor from the the Birch and Swinnerton-Dyer *conjectural* formula for the product mentioned above, and under some hypotheses similar to the ones made in the situation above, we show that  $r$  divides this integer factor. This provides theoretical evidence for the second part of the Birch and Swinnerton-Dyer conjecture in the analytic rank zero case.

## 1 Introduction

Let  $N$  be a positive integer. Let  $X_0(N)$  be the modular curve over  $\mathbf{Q}$  associated to  $\Gamma_0(N)$ , and let  $J = J_0(N)$  denote the Jacobian of  $X_0(N)$ , which is an abelian variety over  $\mathbf{Q}$ . Let  $\mathbf{T}$  denote the Hecke algebra, which is the subring of endomorphisms of  $J_0(N)$  generated by the Hecke operators (usually denoted  $T_\ell$  for  $\ell \nmid N$  and  $U_p$  for  $p \mid N$ ). If  $f$  is a newform of weight 2 on  $\Gamma_0(N)$ , then let  $I_f = \text{Ann}_{\mathbf{T}} f$  and let  $A_f$  denote the associated *newform quotient*  $J/I_f J$ , which is an abelian variety over  $\mathbf{Q}$ . Let  $\pi$  denote the quotient map  $J \rightarrow J/I_f J = A_f$ . By the *analytic rank* of  $f$ , we mean

---

\*This material is based upon work supported by the National Science Foundation under Grant No. 0603668.

the order of vanishing at  $s = 1$  of  $L(f, s)$ . The analytic rank of  $A_f$  is then the analytic rank of  $f$  times the dimension of  $A_f$ . Now suppose that the newform  $f$  has integer Fourier coefficients. Then  $A_f$  is an elliptic curve, and we denote it by  $E$  instead. Since  $E$  has dimension one, its analytic rank is the same as that of  $f$ .

Now suppose that  $L_E(1) \neq 0$  (i.e.,  $f$  has analytic rank zero). Then by [KL89],  $E$  has Mordell-Weil rank zero, and the Shafarevich-Tate group  $\text{III}(E)$  of  $E$  is finite. Let  $\mathcal{E}$  denote the Néron model of  $E$  over  $\mathbf{Z}$  and let  $\mathcal{E}^0$  denote the largest open subgroup scheme of  $\mathcal{E}$  in which all the fibers are connected. Let  $\Omega_E$  denote the volume of  $E(\mathbf{R})$  with respect to the measure given by a generator of the rank one  $\mathbf{Z}$ -module of invariant differentials on  $\mathcal{E}$ . If  $p$  is a prime number, then the group of  $\mathbf{F}_p$ -valued points of the quotient  $\mathcal{E}_{\mathbf{F}_p}/\mathcal{E}_{\mathbf{F}_p}^0$  is called the (arithmetic) component group of  $A$  and its order is denoted  $c_p(A)$ . Throughout this article, we use the symbol  $\stackrel{?}{=}$  to denote a conjectural equality.

Considering that  $L_E(1) \neq 0$ , the second part of the Birch and Swinnerton-Dyer conjecture says the following:

**Conjecture 1.1 (Birch and Swinnerton-Dyer).**

$$\frac{L_E(1)}{\Omega_E} \stackrel{?}{=} \frac{|\text{III}(E)| \cdot \prod_{p|N} c_p(E)}{|E(\mathbf{Q})|^2}. \quad (1)$$

It is known that  $L_E(1)/\Omega_E$  is a rational number. The importance of the second part of the Birch and Swinnerton-Dyer conjecture is that it gives a conjectural value of  $|\text{III}(E)|$  in terms of the other quantities in (1) (which can often be computed). Let us denote this conjectural value of  $|\text{III}(E)|$  by  $|\text{III}(E)|_{\text{an}}$  (where “an” stands for “analytic”). The theory of Euler systems has been used to bound  $|\text{III}(E)|$  from above in terms  $|\text{III}(A_f)|_{\text{an}}$  as in the work of Kolyvagin and of Kato (e.g., see [Rub98, Thm 8.6]). Also, the Eisenstein series method is being used by Skinner-Urban (as yet unpublished) to try to show that  $|\text{III}(A_f)|_{\text{an}}$  divides  $|\text{III}(E)|$ . In both of the methods above, one may have to stay away from certain primes.

The conjectural formula (1) may be rewritten as follows:

$$|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E} \stackrel{?}{=} |\text{III}(E)| \cdot \prod_{p|N} c_p(E). \quad (2)$$

We shall refer to the formula above as the Birch and Swinnerton-Dyer conjectural formula.

Now suppose that  $f$  is congruent modulo a prime  $p$  to another newform  $g$  that has integer Fourier coefficients and whose associated elliptic curve has positive Mordell-Weil rank. Let  $r$  denote the highest power of  $p$  modulo which this congruence holds. Then the theory of visibility (e.g., as in [CM00]) often shows that  $r$  divides  $|\text{III}(E)| \cdot \prod_{p|N} c_p(E)$ , the right side of the Birch and Swinnerton-Dyer conjectural formula (2); we give precise results along these lines in Section 2. When this happens, the conjectural formula (2) says that  $r$  should also divide the left side of (2), which is  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$  (since it is not known that the rational number  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$  is an integer, what we mean here and henceforth is that the order at  $p$  of this rational number is at least  $\text{ord}_p r$ ). In Section 3, we show that this does happen, in fact under milder hypotheses. In Section 4, we give the proof of our main result (Theorem 3.2); in the proof, we actually extract an explicit integer factor from  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$ , and under certain hypotheses, we show that  $r$  divides this integer factor. Section 5 is an appendix that may be read independently of the rest of the article, and may be of independent interest. The reader who is interested in seeing only the precise statements of our main results may read Sections 2 and 3, skipping proofs. In each section, we continue to use the notation introduced in earlier sections (unless mentioned otherwise).

We remark that the results of this article are very analogous to the results obtained in [Aga09], where we treated the case where  $E$  had analytic rank one. Also, our results for the case where  $r = p$  (i.e., if  $f$  and  $g$  are congruent modulo  $p$ , but not modulo  $p^2$ ), are covered to some extent in [Aga]. In fact, the present article arose from our efforts to generalize some of the results in [Aga].

*Acknowledgements:* We are grateful to M. Emerton for pointing out an error in an earlier proof of Proposition 5.1.

## 2 Visibility and the right side of the Birch and Swinnerton-Dyer conjectural formula

Let  $F$  denote the elliptic curve associated to the newform  $g$ . If  $A$  is an abelian variety, then we denote its dual abelian variety by  $A^\vee$ . If  $h$  is a newform of weight 2 on  $\Gamma_0(N)$ , then by taking the dual of the quotient map  $J_0(N) \rightarrow A_h$  and using the self-duality of  $J_0(N)$ , we may view  $A_h^\vee$  as an abelian subvariety of  $J_0(N)$ . In particular, we may view  $E^\vee$  and  $F^\vee$  as abelian subvarieties of  $J_0(N)$ . We say that a maximal ideal  $\mathfrak{m}$  of  $\mathbf{T}$  satisfies

*multiplicity one* if  $J_0(N)[\mathfrak{m}]$  is two dimensional over  $\mathbf{T}/\mathfrak{m}$ . Consider the following hypothesis on  $p$ :

(\*) if  $\mathfrak{m}$  is a maximal ideal of  $\mathbf{T}$  with residue characteristic  $p$  and  $\mathfrak{m}$  is in the support of  $J_0(N)[I_f + I_g]$ , then  $\mathfrak{m}$  satisfies multiplicity one.

The following lemma is Lemma 2.1 from [Aga09]; we repeat the statement here since we shall refer to it several times.

**Lemma 2.1.** *Suppose  $p$  is odd, and either*

(i)  $p \nmid N$  or

(ii)  $p \mid N$  and  $E[p]$  or  $F[p]$  is irreducible.

*Then  $p$  satisfies hypothesis (\*).*

**Proposition 2.2.** (i) *Suppose that  $p$  is coprime to*

$$N \cdot |(J_0(N)/F^\vee)(\mathbf{Q})_{\text{tor}}| \cdot |F(\mathbf{Q})_{\text{tor}}| \cdot \prod_{\ell \mid N} c_\ell(F).$$

*Then  $r$  divides  $|\text{III}(E)| \cdot \prod_{p \mid N} c_p(E)$ , the right hand side of the Birch and Swinnerton-Dyer formula (2).*

(ii) *Suppose that  $f$  is congruent to  $g$  modulo an odd prime  $q$  such that  $E[q]$  and  $F[q]$  are irreducible and  $q$  does not divide*

$$N \cdot |(J_0(N)/F^\vee)(\mathbf{Q})_{\text{tor}}| \cdot |F(\mathbf{Q})_{\text{tor}}|.$$

*Also, assume that  $f$  is not congruent modulo  $q$  to a newform of a level dividing  $N/\ell$  for some prime  $\ell$  that divides  $N$  (for Fourier coefficients of index coprime to  $Nq$ ), and either  $q \nmid N$  or for all primes  $\ell$  that divide  $N$ ,  $q \nmid (\ell - 1)$ . Then  $q$  divides  $|\text{III}(E/\mathbf{Q})|$ .*

*Proof.* The proof is identical to the proof of Proposition 3.1 in [Aga09] with  $K$  replaced by  $\mathbf{Q}$ ; we repeat the proof here for the convenience of the reader. Both results follow essentially from Theorem 3.1 of [AS02]. For the first part, take  $A = E^\vee$ ,  $B = F^\vee$ , and  $n = r$  in [AS02, Thm. 3.1], and note that  $F^\vee[r] \subseteq E^\vee$  by Lemma 2.1 and [Aga09, Lemma 2.2] (considering that that  $p \nmid N$  by hypothesis), and that the rank of  $E^\vee(\mathbf{Q})$  is less than the rank of  $F^\vee(\mathbf{Q})$ . For the second part, take  $A = E^\vee$ ,  $B = F^\vee$ , and  $n = q$  in [AS02, Thm. 3.1], and note that the congruence of  $f$  and  $g$  modulo  $q$  forces  $F^\vee[q] = E^\vee[q]$  by [Rib90, Thm. 5.2] (cf. [CM00, p. 20]), and that the hypotheses imply that  $q$  does not divide  $c_\ell(E)$  or  $c_\ell(F)$  for any prime  $\ell$  that divides  $N$ , as we now indicate. By [Eme03, Prop. 4.2], if  $q$  divides  $c_\ell(E)$  for some prime  $\ell$  that divides  $N$ , then for some maximal ideal  $\mathfrak{m}$  of  $\mathbf{T}$  having characteristic  $q$  and containing  $I_f$ , either  $\rho_{\mathfrak{m}}$  is finite or reducible (here,

$\rho_{\mathfrak{m}}$  is the canonical two dimensional representation associated to  $\mathfrak{m}$ , e.g., as in [Rib90, Prop. 5.1]). Since  $E[q]$  is irreducible, this can happen only if  $\rho_{\mathfrak{m}}$  is finite. But this is not possible by [Rib90, Thm. 1.1], in view of the hypothesis that  $f$  is not congruent modulo  $q$  to a newform of a level dividing  $N/\ell$  for any prime  $\ell$  that divides  $N$  (for Fourier coefficients of index coprime to  $Nq$ ), and either  $q \nmid N$  or for all primes  $\ell$  that divide  $N$ ,  $q \nmid (\ell - 1)$ . Thus  $q$  does not divide  $c_\ell(E)$  for any prime  $\ell$  that divides  $N$ . Similarly,  $q$  does not divide  $c_\ell(F)$  for any prime  $\ell$  that divides  $N$ , considering that the hypothesis that  $f$  is not congruent modulo  $q$  to a newform of a level dividing  $N/\ell$  for any prime  $\ell$  that divides  $N$  (for Fourier coefficients of index coprime to  $Nq$ ) applies to  $g$  as well, since  $g$  is congruent to  $f$  modulo  $q$ . This finishes the proof of the proposition.  $\square$

See [CM00] or [AS05] for examples where the theory of visibility proves the existence of non-trivial elements of the Shafarevich-Tate group of an elliptic curve of analytic rank zero.

### 3 Congruences and the left side of the Birch and Swinnerton-Dyer conjectural formula

Considering that under certain hypotheses, the theory of visibility (more precisely Proposition 2.2(i)) implies that  $r$  divides  $|\text{III}(E)| \cdot \prod_{p|N} c_p(E)$ , the right hand side of the Birch and Swinnerton-Dyer conjectural formula (2), under similar hypotheses, one should be able to show that  $r$  also divides  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$ , which is the left hand side of (2). The theory of Euler systems says under certain hypotheses that the order of  $\text{III}(E)$  divides its Birch and Swinnerton-Dyer conjectural order (e.g., as in the work of Kolyvagin and Kato). Thus, in conjunction with Proposition 2.2(i), the theory of Euler systems shows that under certain additional hypotheses,  $r$  does divide  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$ . For example, we have the following:

**Proposition 3.1.** *Suppose that  $p$  is coprime to*

$$2 \cdot N \cdot |(J_0(N)/F^\vee)(K)_{\text{tor}}| \cdot |F(K)_{\text{tor}}| \cdot \prod_{\ell|N} c_\ell(F).$$

*Assume that the image of the absolute Galois group of  $\mathbf{Q}$  acting on  $E[p]$  is isomorphic to  $\text{GL}_2(\mathbf{Z}/p\mathbf{Z})$ . Then  $r$  divides  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$ .*

*Proof.* Proposition 2.2(i) which uses the theory of visibility, implies that  $r$  divides  $|\text{III}(E)| \cdot \prod_{p|N} c_p(E)$ . The result now follows by [SW08, Theorem 13], which is an extension of a theorem of Kato.  $\square$

The pullback of a generator of the rank one  $\mathbf{Z}$ -module of invariant differentials on the Néron model of  $E$  to  $X_0(N)$  (under the modular parametrization) is a multiple of the differential  $2\pi if(z)dz$  by a rational number; this number is called the Manin constant of  $E$ , and we shall denote it by  $c_E$ . It is conjectured that  $c_E$  is one, and one knows that  $c_E$  is an integer, and that if  $p$  is a prime such that  $p^2 \nmid 4N$ , then  $p$  does not divide  $c_E$  (by [Maz78, Cor. 4.1] and [AU96, Thm. A]).

**Theorem 3.2.** *Suppose that  $p$  is odd and satisfies the hypothesis (\*). Assume that either  $p^2 \nmid N$  or that the Manin constant  $c_E$  is one (as is conjectured). Then  $r$  divides  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$ , the left side of the Birch and Swinnerton-Dyer conjectural formula (2).*

We shall prove this theorem in Section 4.

**Corollary 3.3.** *Suppose  $p$  is odd, and either*

(a)  $p \nmid N$  or

(b)  $p \mid N$  and  $E[p]$  or  $F[p]$  is irreducible.

*Then  $r$  divides  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$ , which is the left side of the Birch and Swinnerton-Dyer conjectural formula (2).*

*Proof.* This follows from the theorem above, considering that the hypothesis (\*) is satisfied, in view of Lemma 2.1.  $\square$

Corollary 3.3 may be compared to the similar Proposition 3.1 that uses the theory of visibility and the theory of Euler systems. Note that in Corollary 3.3, we do not assume the following hypotheses of Proposition 3.1:  $p \nmid N$  (although we do need that  $p^2 \nmid N$ ),  $p$  does not divide  $|F(K)_{\text{tor}}| \cdot \prod_{\ell \mid N} c_\ell(F)$ , and the image of the absolute Galois group of  $\mathbf{Q}$  acting on  $E[p]$  is isomorphic to  $\text{GL}_2(\mathbf{Z}/p\mathbf{Z})$ . Also, our proof of Theorem 3.2 does not use the theory of visibility or the theory of Euler systems, and is much more elementary than either theories. In fact, our approach may be considered an alternative to the theory of Euler systems in the context where the theory of visibility predicts non-triviality of Shafarevich-Tate groups for analytic rank zero.

**Corollary 3.4.** *Suppose  $q$  is an odd prime such that  $q^2 \nmid N$  and  $E[q]$  and  $F[q]$  are irreducible. Let  $m$  denote the highest power of  $q$  modulo which  $f$  and  $g$  are congruent. Then  $m$  divides  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$ . If moreover,  $f$  is not congruent modulo  $q$  to a newform of a level dividing  $N/\ell$  for some prime  $\ell$  that divides  $N$  (for Fourier coefficients of index coprime to  $Nq$ ), and either  $q \nmid N$  or for all primes  $\ell$  that divide  $N$ ,  $q \nmid (\ell - 1)$ , then  $m$  divides the Birch and Swinnerton-Dyer conjectural order of  $\text{III}(E)$ .*

*Proof.* Take  $p = q$  in Theorem 3.2. By Lemma 2.1,  $q$  satisfies hypothesis (\*). Hence by Theorem 3.2,  $m$  divides  $|E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E}$ . As explained in the proof of Proposition 2.2(ii), the hypotheses imply that  $q$  does not divide  $c_\ell(E)$  for any prime  $\ell$ . Hence, by (2),  $m$  divides the Birch and Swinnerton-Dyer conjectural order of  $\text{III}(E)$ .  $\square$

In view of Proposition 2.2, Corollaries 3.3 and 3.4 provide theoretical evidence towards the Birch and Swinnerton-Dyer conjectural formula (2). We remark that Corollary 3.3 is to be compared to part (i) of Proposition 2.2 and Corollary 3.4 to part (ii) of Proposition 2.2.

## 4 Proof of Theorem 3.2

We work in slightly more generality in the beginning and assume that  $f$  and  $g$  are any newforms (whose Fourier coefficients need not be integers), with  $f$  having analytic rank zero and  $g$  having analytic rank greater than zero. Thus the associated newform quotients  $A_f$  and  $A_g$  need not be elliptic curves, but we will still denote them by  $E$  and  $F$  (respectively) for simplicity of notation.

Recall that  $I_g = \text{Ann}_{\mathbf{T}} g$ . Let  $J' = J/(I_f \cap I_g)J$  and let  $\pi''$  denote the quotient map  $J \rightarrow J'$ . Then the quotient map  $J \xrightarrow{\pi} E$  factors through  $J'$ ; let  $\pi'$  denote the map  $J' \rightarrow E$  in this factorization. Let  $F'$  denote the kernel of  $\pi'$ . Let  $E'$  denote the image of  $E^\vee \subseteq J$  in  $J'$  under the quotient map  $\pi'' : J \rightarrow J'$ . Let  $B$  denote the kernel of the projection map  $\pi : J \rightarrow E$ ; it is the abelian subvariety  $I_f J$  of  $J$ . We have the following diagram, in which the two sequences of four arrows are exact (one horizontal and one upwards diagonal):

$$\begin{array}{ccccccc}
& & F^\vee & & E^\vee & & 0 \\
& & \searrow & & \downarrow & \searrow \sim & \\
0 & \longrightarrow & B & \longrightarrow & J & \xrightarrow{\pi} & E \longrightarrow 0 \\
& & \downarrow & & \downarrow \pi'' & \nearrow \pi' & \\
& & F' & & J' & & F \\
& & \nearrow & & \searrow & & \\
0 & & & & & & 
\end{array}$$

Now  $F'$  is connected, since it is a quotient of  $B$  (as a simple diagram chase above shows) and  $B$  is connected. Thus, by looking at dimensions, one sees that  $F'$  is the image of  $F^\vee$  under  $\pi''$ . Since the composite  $F^\vee \hookrightarrow J \rightarrow J' \rightarrow F$  is an isogeny, the quotient map  $J' \rightarrow F$  induces an isogeny  $\pi''(F^\vee) \sim F$ , and hence an isogeny  $F' \sim F$ . Let  $E'$  denote  $\pi''(E^\vee)$ . Since  $\pi$  induces an isogeny from  $E^\vee$  to  $E$ , we see that  $\pi'$  also induces an isogeny from  $E'$  to  $E$ . In particular, we have:

**Lemma 4.1.**  $\dim E' = \dim E$  and  $\dim F' = \dim F$ .

Let  $\mathfrak{S}$  denote the annihilator, under the action of  $\mathbf{T}$ , of the divisor  $(0) - (\infty)$ , considered as an element of  $J_0(N)(\mathbf{C})$ . We have an isomorphism

$$H_1(X_0(N), \mathbf{Z}) \otimes \mathbf{R} \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{C}}(H^0(X_0(N), \Omega^1), \mathbf{C}),$$

obtained by integrating differentials along cycles (see [Lan95, § IV.1]). Let  $e$  be the element of  $H_1(X_0(N), \mathbf{Z}) \otimes \mathbf{R}$  that corresponds to the map  $\omega \mapsto -\int_{\{0, i\infty\}} \omega$  under this isomorphism. It is called the *winding element*. By [Maz77, II.18.6], we have  $\mathfrak{S}e \subseteq H_1(X_0(N), \mathbf{C}) = H_1(J_0(N), \mathbf{C})$  (note that in loc. cit., the definition of  $\mathfrak{S}$  is different and  $N$  is assumed to be prime; but the only essential property of  $\mathfrak{S}$  that is used in the proof is that  $\mathfrak{S}$  annihilates the divisor  $(0) - (\infty)$ , and the assumption that  $N$  is prime is not used). If  $\phi$  is a map of abelian varieties over  $\mathbf{Q}$ , then we denote the induced map on the first homology groups by  $\phi_*$ .

**Lemma 4.2.**  $\pi'_*(\mathfrak{S}e) \subseteq H_1(E', \mathbf{Z})$ .

*Proof.* Since  $J'$  is isogenous to  $E' \oplus F'$ , we have  $H_1(J', \mathbf{Z}) \otimes \mathbf{Q} \cong H_1(E', \mathbf{Z}) \otimes \mathbf{Q} \oplus H_1(F', \mathbf{Z}) \otimes \mathbf{Q}$ . Viewing  $\pi_*''(\mathfrak{S}e)$  as a subset of  $H_1(J', \mathbf{Z}) \otimes \mathbf{Q}$ , it suffices to show that  $\pi_*''(\mathfrak{S}e) \cap (H_1(F', \mathbf{Z}) \otimes \mathbf{Q}) = 0$ . Suppose  $x \in \pi_*''(\mathfrak{S}e) \cap (H_1(F', \mathbf{Z}) \otimes \mathbf{Q})$ ; we need to show that then  $x = 0$ . For some integer  $n$ , we have  $nx \in H_1(F', \mathbf{Z})$ , and for some  $t \in \mathfrak{S}$ , we have  $t\pi_*''(e) = nx$ . Let  $\omega$  be a differential over  $\mathbf{Q}$  on  $F'$ , which we may view as a differential on  $J'$ . Then  $\pi_*''(\omega)$ , when viewed as a differential on  $X_0(N)$ , is of the form  $2\pi i h(z) dz$  for some  $h$  in  $S_2(\Gamma_0(N), \mathbf{Q})[I_g]$ . Thus  $\int_{nx} \omega = \int_{t\pi_*''(e)} \omega = \int_{te} 2\pi i h(z) dz = \int_e 2\pi i (th)(z) dz$ . Now  $th \in S_2(\Gamma_0(N), \mathbf{Q})[I_g]$ , and so  $th$  is a  $\mathbf{Q}$ -linear combination of the Galois conjugates of  $g$ . Hence  $\int_e 2\pi i (th)(z) dz$  is a  $\mathbf{Q}$ -linear combination of  $\int_e 2\pi i g^\sigma(z) dz = L(g^\sigma, 1)$  for various conjugates  $g^\sigma$  of  $g$ , where  $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ . Since  $g$  has positive analytic rank,  $L(g, 1) = 0$ , and so  $L(g^\sigma, 1) = 0$  for all  $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ , e.g., by [GZ86, Cor. V.1.3]. Thus, by the discussion above, we see that  $\int_{nx} \omega = 0$  for every differential  $\omega$  over  $\mathbf{Q}$  on  $F'$ , and so  $nx = 0$  in  $H_1(F', \mathbf{Z}) \otimes \mathbf{Q}$ . Hence  $x = 0$ , as was to be shown.  $\square$

There is a complex conjugation involution acting on  $H_1(X_0(N), \mathbf{C})$ , and if  $G$  is a group on which it induces an involution, then by  $G^+$  we mean the subgroup of elements of  $G$  fixed by the involution. It is easy to see that  $e$  is fixed by the complex conjugation involution, and so by Lemma 4.2, we have  $\pi_*''(\mathfrak{S}e) \subseteq H_1(E', \mathbf{Z})^+$ . The following is an analog of [Aga, Theorem 3.2]:

**Proposition 4.3.** *Up to a power of 2,*

$$c_E \cdot c_\infty(E) \cdot \frac{L_E(1)}{\Omega_E} = \frac{\left| \frac{H_1(J', \mathbf{Z})^+}{H_1(F', \mathbf{Z})^+ + H_1(E', \mathbf{Z})^+} \right| \cdot \left| \frac{H_1(E', \mathbf{Z})^+ + H_1(F', \mathbf{Z})^+}{\pi_*''(\mathfrak{S}e) + H_1(F', \mathbf{Z})^+} \right|}{|\pi_*(\mathbf{T}e)/\pi_*(\mathfrak{S}e)|}. \quad (3)$$

*Proof.* By [Aga, Thm. 2.1], we have

$$\frac{L_E(1)}{\Omega_E} = \frac{[H_1(A_f, \mathbf{Z})^+ : \pi_*(\mathbf{T}e)]}{c_E \cdot c_\infty(E)}, \quad (4)$$

where  $[H_1(A_f, \mathbf{Z})^+ : \pi_*(\mathbf{T}e)]$  denotes the absolute value of the determinant of an automorphism of  $H_1(A_f, \mathbf{Q})$  that takes the lattice  $H_1(A_f, \mathbf{Z})^+$  isomorphically onto the lattice  $\pi_*(\mathbf{T}e)$ . Now  $\pi_*''$  and  $\pi_*'$  are both surjective, since the kernels of  $\pi''$  and  $\pi'$  (respectively) are connected. Thus  $H_1(E, \mathbf{Z}) = \pi_*'(H_1(J', \mathbf{Z}))$ . Putting this in (4), and considering that  $\pi_*''(\mathfrak{S}e) \subseteq H_1(J', \mathbf{Z})^+$  (since  $\mathfrak{S}e \subseteq H_1(J_0(N), \mathbf{C})^+$ ), we get

$$c_E \cdot c_\infty(E) \cdot \frac{L_E(1)}{\Omega_E} = [\pi_*'(H_1(J', \mathbf{Z}))^+ : \pi_*(\mathbf{T}e)] = \frac{|\pi_*'(H_1(J', \mathbf{Z}))^+ / \pi_*'(\pi_*''(\mathfrak{S}e))|}{|\pi_*(\mathbf{T}e)/\pi_*(\mathfrak{S}e)|}. \quad (5)$$

The long exact sequence of homology associated to the short exact sequence  $0 \rightarrow F' \rightarrow J' \rightarrow E \rightarrow 0$  is:

$$\dots \rightarrow H_1(F', \mathbf{Z}) \rightarrow H_1(J', \mathbf{Z}) \xrightarrow{\pi'_*} H_1(E, \mathbf{Z}) \rightarrow 0 \rightarrow \dots$$

Thus  $H_1(F', \mathbf{Z}) \subseteq \ker(\pi'_*)$ .

*Claim:*  $H_1(F', \mathbf{Z}) = \ker(\pi'_*)$ .

*Proof.* Since  $H_1(F', \mathbf{Z})$  is saturated in  $H_1(J', \mathbf{Z})$ , it suffices to show that  $H_1(F', \mathbf{Z}) \otimes \mathbf{Q} = \ker(\pi'_*) \otimes \mathbf{Q}$ , i.e., that the free abelian groups  $H_1(F', \mathbf{Z})$  and  $\ker(\pi'_*)$  have the same rank. But

$$\begin{aligned} \text{rank}(\ker(\pi'_*)) &= 2 \cdot \dim J' - 2 \cdot \dim E \\ &= 2 \cdot \dim_{\mathbf{Q}} S_2(\Gamma_0(N), \mathbf{Q})[I_f \cap I_g] - 2 \cdot \dim_{\mathbf{Q}} S_2(\Gamma_0(N), \mathbf{Q})[I_f] \\ &= 2 \cdot \dim_{\mathbf{Q}} S_2(\Gamma_0(N), \mathbf{Q})[I_g] = 2 \cdot \dim_{\mathbf{Q}} F' = \text{rank}(H_1(F', \mathbf{Z})). \end{aligned}$$

This proves the claim.  $\square$

The kernel of the natural map  $H_1(J', \mathbf{Z}) \rightarrow \pi'_*(H_1(J', \mathbf{Z}))/\pi'_*(\pi''_*(\mathfrak{S}e))$  is  $\ker(\pi'_*) + \pi''_*(\mathfrak{S}e) = H_1(F', \mathbf{Z}) + \pi''_*(\mathfrak{S}e)$ , by the claim above. Thus up to a power of 2,

$$|\pi_*(H_1(J', \mathbf{Z}))^+ / \pi'_*(\pi''_*(\mathfrak{S}e))| = \left| \frac{H_1(J', \mathbf{Z})^+}{H_1(F', \mathbf{Z})^+ + \pi''_*(\mathfrak{S}e)} \right|. \quad (6)$$

In view of Lemma 4.2,

$$\left| \frac{H_1(J', \mathbf{Z})^+}{H_1(F', \mathbf{Z})^+ + \pi''_*(\mathfrak{S}e)} \right| = \left| \frac{H_1(J', \mathbf{Z})^+}{H_1(F', \mathbf{Z})^+ + H_1(E', \mathbf{Z})^+} \right| \cdot \left| \frac{H_1(E', \mathbf{Z})^+ + H_1(F', \mathbf{Z})^+}{\pi''_*(\mathfrak{S}e) + H_1(F', \mathbf{Z})^+} \right|. \quad (7)$$

Putting (7) in (6), and then putting the result in (5), we get the formula in the proposition.  $\square$

**Lemma 4.4.** *Suppose  $p$  satisfies hypothesis (\*), and assume that  $f$  and  $g$  have integer Fourier coefficients, so that  $E'$  and  $F'$  are elliptic curves (by Lemma 4.1). Then  $E'[r] = F'[r]$ , and both are direct summands of  $E' \cap F'$  as  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -modules.*

*Proof.* The proof is an adaptation of the proof of [Aga09, Lemma 2.2]. Let  $\mathbf{T}' = \mathbf{T}/(I_f \cap I_g)$ . We call a maximal ideal of  $\mathbf{T}'$  *nice* if its inverse image in  $\mathbf{T}$  satisfies multiplicity one. Let  $\mathfrak{m}'$  be a nice maximal ideal, and let  $\mathfrak{m}$  denote its inverse image in  $\mathbf{T}$ . Then  $\mathfrak{m}$  satisfies multiplicity one, and so using Proposition 5.1 with  $I = I_f \cap I_g$  ( $= \text{Ann}_{\mathbf{T}}(S_{[f]} \oplus S_{[g]})$ ), where the notation

is as in Section 5), we see that  $\mathfrak{m}'$  is good for  $J'$ , in the sense of [Eme03, p. 437] (which is recalled in Section 5). Hence, by [Eme03, Cor. 2.5], if  $I'$  is a saturated ideal of  $\mathbf{T}'$ , then the  $\mathfrak{m}'$ -adic completion of the group of connected components of  $J'[I']$  is trivial.

If  $L \rightarrow M$  is a homomorphism of two  $\mathbf{T}'$ -modules, then we say that  $L = M$  *away from* a given set of maximal ideals of  $\mathbf{T}'$  if the induced map on the  $\mathfrak{m}'$ -adic completions is an isomorphism for all maximal ideals  $\mathfrak{m}'$  that are not in the prescribed set. Let  $I'_f$  and  $I'_g$  denote the images of  $I_f$  and  $I_g$  in  $\mathbf{T}'$ . Then  $I'_f$  and  $I'_g$  are both saturated ideals of  $\mathbf{T}'$ . Hence, by the previous paragraph and by a consideration of dimensions, we see that the inclusions  $E' \subseteq J'[I'_f]$  and  $F' \subseteq J'[I'_g]$  are equalities away from maximal ideals of  $\mathbf{T}'$  that are not nice.

*Claim:* The inclusion  $E' \cap F' \subseteq J'[I'_f + I'_g]$  is an equality away from maximal ideals of  $\mathbf{T}'$  that are not nice.

*Proof.* Consider the natural map  $F' \cap J'[I'_f] \rightarrow J'[I'_f]/E'$ . Its kernel is  $F' \cap J'[I'_f] \cap E' = F' \cap E'$ , and hence we have an injection:

$$\frac{F' \cap J'[I'_f]}{F' \cap E'} \hookrightarrow \frac{J'[I'_f]}{E'}. \quad (8)$$

Also, the natural map  $J'[I'_f + I'_g] = J'[I'_g][I'_f] \rightarrow J'[I'_g]/F'$  has kernel  $F' \cap J'[I'_g][I'_f] = F' \cap J'[I'_f]$ , and hence we have an injection

$$\frac{J'[I'_f + I'_g]}{F' \cap J'[I'_f]} \hookrightarrow \frac{J'[I'_g]}{F'}. \quad (9)$$

The claim follows from equations (8) and (9), considering that the Hecke modules on the right sides of the two equations are supported on the set of maximal ideals of  $\mathbf{T}'$  that are not nice (by the statement just before the claim).  $\square$

Let  $m$  denote the largest integer such that  $f$  and  $g$  are congruent modulo  $m$ . Then  $m$  is the order of the group

$$\frac{S_2(\Gamma_0(N), \mathbf{Z})[I_f \cap I_g]}{S_2(\Gamma_0(N), \mathbf{Z})[I_f] + S_2(\Gamma_0(N), \mathbf{Z})[I_g]} \cong \frac{\mathbf{T}/(I_f \cap I_g)}{I_f/(I_f \cap I_g) + I_g/(I_f \cap I_g)} = \frac{\mathbf{T}'}{I'_f + I'_g},$$

where the first isomorphism comes from the perfect pairing  $S_2(\Gamma_0(N), \mathbf{Z}) \times \mathbf{T} \rightarrow \mathbf{Z}$  that takes  $(f, T)$  to the first Fourier coefficient of  $T(f)$  (cf. [ARS06, Lemma 3.3]). On  $E'$ ,  $\mathbf{T}'$  acts via  $\mathbf{T}'/I'_f \cong \mathbf{T}/I_f$ , which is isomorphic to  $\mathbf{Z}$  by

the map that takes  $T_\ell$  to  $a_\ell(f)$  for all primes  $\ell$ . By the statement just before the previous one, we see that the image of  $\frac{I'_f + I'_g}{I'_f}$  under this map is a subgroup of index  $m$  in  $\mathbf{Z}$ , hence is  $m\mathbf{Z}$ . Thus  $E'[I'_f + I'_g] = E' \left[ \frac{I'_f + I'_g}{I'_f} \right] = E'[m]$ . Now  $E' \cap F' \subseteq E'[I'_f + I'_g] \subseteq J'[I'_f + I'_g]$ , and by the claim above, we see that the inclusion  $E' \cap F' \subseteq E'[I'_f + I'_g] = E'[m]$  is an equality away from the maximal ideals of  $\mathbf{T}'$  in the support of  $J'[I'_f + I'_g]$  that are not nice. Similarly  $E' \cap F' \subseteq F'[I'_f + I'_g] = F'[m]$  is an equality away from the maximal ideals of  $\mathbf{T}'$  in the support of  $J'[I'_f + I'_g]$  that are not nice. Finally, suppose  $\mathfrak{m}'$  is a maximal ideal of  $\mathbf{T}'$  in the support of  $J'[I'_f + I'_g]$ . Then  $\mathfrak{m}'$  contains  $I'_f + I'_g$ , and so the inverse image  $\mathfrak{m}$  of  $\mathfrak{m}'$  in  $\mathbf{T}$  contains  $I_f + I_g$ . Hence by hypotheses (\*) on  $p$ ,  $\mathfrak{m}$  satisfies multiplicity one, and so  $\mathfrak{m}'$  is nice. Thus all the maximal ideals of  $\mathbf{T}'$  in the support of  $J'[I'_f + I'_g]$  are nice. From the discussion above, and by the definition of  $r$ , it follows that  $(E' \cap F')[p^\infty] = E'[r] = F'[r]$ . Thus  $E'[r]$  and  $F'[r]$  are identical and are direct summands of  $E' \cap F'$  as  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -modules.  $\square$

*Proof of Theorem 3.2.* Note that since  $F$  has positive Mordell-Weil rank,  $g$  has positive analytic rank (by [KL89]), and so the discussion of this section applies. By Proposition 4.3, we see that up to a power of 2,

$$\begin{aligned} & |E(\mathbf{Q})|^2 \cdot \frac{L_E(1)}{\Omega_E} \\ &= \frac{\left| \frac{H_1(J', \mathbf{Z})^+}{H_1(F', \mathbf{Z})^+ + H_1(E', \mathbf{Z})^+} \right| \cdot \left| \frac{H_1(E', \mathbf{Z})^+ + H_1(F', \mathbf{Z})^+}{\pi_*''(\mathfrak{S}e) + H_1(F', \mathbf{Z})^+} \right|}{c_E \cdot c_\infty(E)} \cdot \frac{|E(\mathbf{Q})|}{|\pi_*(\mathbf{T}e)/\pi_*(\mathfrak{S}e)|} \cdot |E(\mathbf{Q})|. \end{aligned} \tag{10}$$

By Lemma 4.4, we see that  $r$  divides  $|E' \cap F'|$ . By [Aga, Lemma 4.1], we have  $\left| \frac{H_1(J', \mathbf{Z})}{H_1(F', \mathbf{Z}) + H_1(E', \mathbf{Z})} \right| = |E' \cap F'|$ . Hence  $r$  divides the term  $\left| \frac{H_1(J', \mathbf{Z})^+}{H_1(F', \mathbf{Z})^+ + H_1(E', \mathbf{Z})^+} \right|$  on the right side of (10) (considering that  $\left| \frac{H_1(J', \mathbf{Z})^+}{H_1(F', \mathbf{Z})^+ + H_1(E', \mathbf{Z})^+} \right|$  differs from  $\left| \frac{H_1(J', \mathbf{Z})}{H_1(F', \mathbf{Z}) + H_1(E', \mathbf{Z})} \right|$  by a power of 2 and that  $r$  is odd). The theorem now follows from equation (10), in view of the facts that  $|\pi_*(\mathbf{T}e)/\pi_*(\mathfrak{S}e)|$  divides  $|E(\mathbf{Q})|$  (by [Aga, Lemma 3.3]),  $c_E$  is coprime to  $p$  if  $p^2 \nmid N$  (by [Maz78, Cor. 4.1]), and  $c_\infty(E)$  is a power of 2, hence coprime to  $r$ .  $\square$

**Remark 4.5.** We would like to take the chance to make a correction to our earlier paper [Aga09]. In the fourth paragraph of Section 5 of loc. cit., we claimed that “since  $E^\vee[r] = F^\vee[r]$  and both are direct summands of  $E^\vee \cap F^\vee$  as  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -modules, on applying  $\pi''$  we find that  $E'[r] = F'[r]$  and both are direct summands of  $E' \cap F'$  as  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -modules”. Since it

may not be true that  $\pi''(E^\vee[r]) = E'[r]$  or  $\pi''(F^\vee[r]) = F'[r]$ , our reasoning was incomplete. However, the quoted claim above holds by Lemma 4.4, in whose proof the analytic or Mordell-Weil ranks of  $f$  and  $g$  do not play any role.

## 5 Appendix

In this section, we prove Proposition 5.1 below, which is used in the proof of Lemma 4.4. The results of this section may be of independent interest.

If  $G$  is an abelian variety over  $\mathbf{Q}$  and  $R$  is a finite flat commutative  $\mathbf{Z}$ -algebra that acts on  $G$ , then following [Eme03], we say that a maximal ideal  $\mathfrak{m}$  of  $R$  is *good* for  $G$  if  $G_{\mathfrak{m}}^\vee$  is a free  $R_{\mathfrak{m}}$  module, where  $G_{\mathfrak{m}}$  is the  $\mathfrak{m}$ -divisible group of  $G$ , the superscript  $\vee$  always denotes the Pontryagin dual, and  $R_{\mathfrak{m}}$  is the completion of  $R$  at  $\mathfrak{m}$ . The significance of the notion of being good is that if  $I$  is a saturated ideal of  $R$ , then by [Eme03, Cor. 2.5], the component group of  $G[I]$  is supported at maximal ideals of  $R$  that are not good. In the situation above, we will denote by  $T_{\mathfrak{m}}(G)$  the usual  $\mathfrak{m}$ -adic Tate module of  $G$ .

Recall that  $N$  is a positive integer,  $J_0(N)$  denotes the Jacobian of the modular curve  $X_0(N)$ , and  $\mathbf{T}$  denotes the Hecke algebra, which is the subring of endomorphisms of  $J_0(N)$  generated by the Hecke operators (usually denoted  $T_\ell$  for  $\ell \nmid N$  and  $U_p$  for  $p \mid N$ ). Recall that a maximal ideal  $\mathfrak{m}$  of  $\mathbf{T}$  is said to satisfy *multiplicity one* if  $J_0(N)[\mathfrak{m}]$  is two dimensional over  $\mathbf{T}/\mathfrak{m}$ .

In this paragraph, the symbol  $g$  stands for a newform of some level  $N_g$  dividing  $N$ . Let  $S'_g$  denote the subspace of  $S_2(\Gamma_0(N), \mathbf{C})$  spanned by the forms  $g(dz)$  where  $d$  ranges over the divisors of  $N/N_g$ . Let  $[g]$  denote the Galois orbit of  $g$ , and let  $S_{[g]}$  denote the  $\mathbf{Q}$ -subspace of forms in  $\bigoplus_{h \in [g]} S'_h$  with rational Fourier coefficients. We have  $S_2(\Gamma_0(N), \mathbf{Q}) = \bigoplus_{[g]} S_{[g]}$ , where the sum is over Galois conjugacy classes of newforms of some level dividing  $N$ . Let  $X$  be a subset of the set of Galois conjugacy classes of newforms of some level dividing  $N$ , and let  $I = \text{Ann}_{\mathbf{T}}(\bigoplus_{[g] \in X} S_{[g]})$ .

**Proposition 5.1.** *Let  $\mathfrak{m}$  be a maximal ideal of  $\mathbf{T}$  that satisfies multiplicity one and contains  $I$ , and let  $\mathfrak{m}'$  denote the image of  $\mathfrak{m}$  in  $\mathbf{T}/I$ . Then  $\mathfrak{m}'$  is good for  $J/IJ$ .*

Before giving the proof, we state two lemmas that will be used in the proof. If  $g$  is a newform of some level dividing  $N$ , then  $S_{[g]}$  is preserved by  $\mathbf{T}$ ; let  $\mathbf{T}_{[g]}$  denote the image of  $\mathbf{T}$  acting on  $S_{[g]}$ . Then the natural map

$$\phi : \mathbf{T} \otimes \mathbf{Q} \rightarrow \bigoplus_{[g]} \mathbf{T}_{[g]}$$

is an isomorphism of  $\mathbf{T} \otimes \mathbf{Q}$  algebras, where  $[g]$  ranges over all Galois conjugacy classes of newforms of level dividing  $N$  (see, e.g., [Par99, Thm. 3.5]). Let  $\widehat{I}$  denote  $\text{Ann}_{\mathbf{T}}(I)$ .

**Lemma 5.2.** (i) *The image of  $I \otimes \mathbf{Q}$  under  $\phi$  is  $\bigoplus_{[g] \notin X} \mathbf{T}_{[g]}$ , and the image of  $\widehat{I} \otimes \mathbf{Q}$  is  $\bigoplus_{[g] \in X} \mathbf{T}_{[g]}$ . Thus  $\mathbf{T} \otimes \mathbf{Q} \cong I \otimes \mathbf{Q} \oplus \widehat{I} \otimes \mathbf{Q}$  as  $\mathbf{T} \otimes \mathbf{Q}$ -modules.*  
(ii) *As  $\mathbf{T} \otimes \mathbf{Q}$ -modules,  $S_2(\Gamma_0(N), \mathbf{Q}) \cong IS_2(\Gamma_0(N), \mathbf{Q}) \oplus \widehat{I}S_2(\Gamma_0(N), \mathbf{Q})$ . Also,  $\widehat{I}S_2(\Gamma_0(N), \mathbf{Q}) = S_2(\Gamma_0(N), \mathbf{Q})[I]$ .*

*Proof.* We have the decomposition

$$\bigoplus_{[g]} \mathbf{T}_{[g]} = \left( \bigoplus_{[g] \in X} \mathbf{T}_{[g]} \right) \oplus \left( \bigoplus_{[g] \notin X} \mathbf{T}_{[g]} \right). \quad (11)$$

It is clear that the image of  $I \otimes \mathbf{Q}$  under  $\phi$  is  $\bigoplus_{[g] \notin X} \mathbf{T}_{[g]}$ . As for the image of  $\widehat{I} \otimes \mathbf{Q}$ , it clearly contains  $\bigoplus_{[g] \in X} \mathbf{T}_{[g]}$ . Conversely, if  $x \in \widehat{I} \otimes \mathbf{Q}$ , then it annihilates the element  $(0, 1)$  in the decomposition of (11) (since  $(0, 1)$  is in the image of  $I \otimes \mathbf{Q}$  under  $\phi$ ), so the image of  $x \cdot (0, 1)$  in  $\bigoplus_{[g]} \mathbf{T}_{[g]}$  must be zero. Thus  $x \in \bigoplus_{[g] \in X} \mathbf{T}_{[g]}$ , which finishes the proof of Part (i) of the lemma. For Part (ii), since  $S_2(\Gamma_0(N), \mathbf{Q})$  is free of rank one as a  $\mathbf{T} \otimes \mathbf{Q}$  module, we get the decomposition  $S_2(\Gamma_0(N), \mathbf{Q}) \cong IS_2(\Gamma_0(N), \mathbf{Q}) \oplus \widehat{I}S_2(\Gamma_0(N), \mathbf{Q})$ . For the second statement of Part (ii), note that clearly  $\widehat{I}S_2(\Gamma_0(N), \mathbf{Q}) \subseteq S_2(\Gamma_0(N), \mathbf{Q})[I]$ . Conversely, if  $h \in S_2(\Gamma_0(N), \mathbf{Q})[I]$ , then as  $(0, 1) \in \phi(I \otimes \mathbf{Q})$ , we have  $h = (1, 0)h + (0, 1)h = (1, 0)h \in \widehat{I}S_2(\Gamma_0(N), \mathbf{Q})$ , which shows that  $S_2(\Gamma_0(N), \mathbf{Q})[I] \subseteq \widehat{I}S_2(\Gamma_0(N), \mathbf{Q})$ .  $\square$

There is a perfect pairing

$$\mathbf{T} \times S_2(\Gamma_0(N), \mathbf{Z}) \rightarrow \mathbf{Z} \quad (12)$$

which associates to  $(T, f)$  the first Fourier coefficient  $a_1(f|T)$  of the modular form  $f|T$  (see, e.g., [Rib83, (2.2)]); this induces a pairing

$$\psi : (\mathbf{T}/I) \otimes \mathbf{Q} \times S_2(\Gamma_0(N), \mathbf{Q})[I] \rightarrow \mathbf{Z}.$$

of  $(\mathbf{T}/I) \otimes \mathbf{Q}$  modules. Note that  $\mathbf{T}/I$  is torsion-free, i.e.,  $I$  is saturated in  $\mathbf{T}$ .

**Lemma 5.3.**  *$\psi$  is a perfect pairing.*

*Proof.* It suffices to show that the induced maps  $S_2(\Gamma_0(N), \mathbf{Q})[I] \rightarrow \text{Hom}(\mathbf{T} \otimes \mathbf{Q}/I \otimes \mathbf{Q}, \mathbf{Q})$  and  $\mathbf{T} \otimes \mathbf{Q}/I \otimes \mathbf{Q} \rightarrow \text{Hom}(S_2(\Gamma_0(N), \mathbf{Q})[I], \mathbf{Q})$  are injective. The injectivity of the first map follows from the perfectness of the pairing (12). Suppose the image of  $T \in \mathbf{T}$  in  $(\mathbf{T}/I) \otimes \mathbf{Q}$  is in the kernel of the map

$\mathbf{T} \otimes \mathbf{Q}/I \otimes \mathbf{Q} \rightarrow \text{Hom}(S_2(\Gamma_0(N), \mathbf{Q})[I], \mathbf{Q})$ . Then if  $h \in S_2(\Gamma_0(N), \mathbf{Q})[I]$ , we have  $a_1(h|T) = 0$ . But then  $a_n(h|T) = a_1((h|T)|T_n) = a_1((h|T_n)|T) = 0$  for all  $n$  (considering that  $h|T_n \in S_2(\Gamma_0(N), \mathbf{Q})[I]$ ), and hence  $h|T = 0$  for all  $h \in S_2(\Gamma_0(N), \mathbf{Q})[I]$ . Hence  $T$  annihilates  $S_2(\Gamma_0(N), \mathbf{Q})[I] \cong \widehat{I} \otimes \mathbf{Q}$  (as  $\mathbf{T} \otimes \mathbf{Q}$ -modules). From the decomposition  $\mathbf{T} \otimes \mathbf{Q} \cong I \otimes \mathbf{Q} \oplus \widehat{I} \otimes \mathbf{Q}$ , we see that then  $T \in I$  (keeping in mind that  $I$  is saturated in  $T$ ), which proves the injectivity.  $\square$

*Proof of Proposition 5.1.* Let  $J = J_0(N)$  and let  $J' = J/IJ$ . Since  $\mathfrak{m}$  satisfies multiplicity one, by a standard argument,  $T_{\mathfrak{m}}(J)$  is free of rank two over  $\mathbf{T}_{\mathfrak{m}}$  (e.g., see [Til97, p. 333]). The complex analytic description of  $J(\mathbf{C})$  is Hecke equivariant, and from it we see that  $T_{\mathfrak{m}}(J) \cong H_1(J, \mathbf{Z})_{\mathfrak{m}}$  as Hecke modules. Also, the complex analytic description of  $J'(\mathbf{C})$  fits into the following commutative diagram:

$$\begin{array}{ccc} J(\mathbf{C}) & \longrightarrow & J'(\mathbf{C}) \\ \uparrow & & \uparrow \\ \frac{H^0(J, \Omega_{J/\mathbf{C}})^{\vee}}{H_1(J, \mathbf{Z})} & \longrightarrow & \frac{H^0(J', \Omega_{J'/\mathbf{C}})^{\vee}}{H_1(J', \mathbf{Z})} \end{array}$$

and hence is also seen to be Hecke equivariant. Thus  $T_{\mathfrak{m}'}(J') \cong H_1(J', \mathbf{Z})_{\mathfrak{m}'}$  as  $(\mathbf{T}/I)_{\mathfrak{m}'}$  modules.

*Claim:*  $H_1(J', \mathbf{Z})_{\mathfrak{m}'} = H_1(J', \mathbf{Z})_{\mathfrak{m}}$  is free over  $(\mathbf{T}/I)_{\mathfrak{m}'} \cong \mathbf{T}_{\mathfrak{m}}/I$ .

*Proof.* Since the kernel  $IJ$  of the map  $J \rightarrow J'$  is connected, we see that  $H_1(J', \mathbf{Z}) \cong H_1(J, \mathbf{Z})/H_1(IJ, \mathbf{Z})$ . Since  $\mathbf{T}_{\mathfrak{m}}/I$  is torsion-free, it is a flat  $\mathbf{Z}$ -module, and so we see that  $H_1(J', \mathbf{Z})_{\mathfrak{m}} \cong H_1(J, \mathbf{Z})_{\mathfrak{m}}/H_1(IJ, \mathbf{Z})_{\mathfrak{m}}$ . Now  $IH_1(J, \mathbf{Z}) \subseteq H_1(IJ, \mathbf{Z})$ , so we have an induced map

$$\phi : H_1(J, \mathbf{Z})_{\mathfrak{m}}/IH_1(J, \mathbf{Z})_{\mathfrak{m}} \rightarrow H_1(J, \mathbf{Z})_{\mathfrak{m}}/H_1(IJ, \mathbf{Z})_{\mathfrak{m}}.$$

We will show that  $\phi$  is an isomorphism; assuming this,  $H_1(J', \mathbf{Z})_{\mathfrak{m}} \cong H_1(J, \mathbf{Z})_{\mathfrak{m}}/IH_1(J, \mathbf{Z})_{\mathfrak{m}} \cong T_{\mathfrak{m}}(J)/IT_{\mathfrak{m}}(J) \cong \mathbf{T}_{\mathfrak{m}}^2/IT_{\mathfrak{m}}^2 \cong (\mathbf{T}_{\mathfrak{m}}/I)^2$ , which would prove our claim.

It thus remains to show that  $\phi$  is an isomorphism. Now  $\phi$  is clearly surjective, so it suffices to show that  $\phi$  is injective. Now  $\mathbf{T}_{\mathfrak{m}}$  is finite as a  $\mathbf{Z}_p$ -module, where  $p$  is the characteristic of  $\mathbf{T}/\mathfrak{m}$ . Hence  $\mathbf{T}_{\mathfrak{m}} \otimes \mathbf{Q}$  is a finite dimensional vector space over  $\mathbf{Q}_p$ . Thus  $(H_1(J, \mathbf{Z})_{\mathfrak{m}}/IH_1(J, \mathbf{Z})_{\mathfrak{m}}) \otimes \mathbf{Q}$  has dimension twice the dimension of  $(\mathbf{T}_{\mathfrak{m}}/I) \otimes \mathbf{Q}$  over  $\mathbf{Q}_p$ . Now  $H_1(J, \mathbf{Z})/H_1(IJ, \mathbf{Z}) \cong H_1(J', \mathbf{Z})$  is torsion free. Also,  $\text{rank}(H_1(J, \mathbf{Z})/H_1(IJ, \mathbf{Z})) = 2 \cdot \dim J - 2 \cdot \dim(IJ) = 2 \cdot \dim(J/IJ) = 2 \cdot \dim_{\mathbf{Q}} S_2(\Gamma_0(N), \mathbf{Q})[I] = 2 \cdot \dim_{\mathbf{Q}}(\mathbf{T}/I \otimes \mathbf{Q})$ ,

where the last equality follows by Lemma 5.3. Thus  $H_1(J, \mathbf{Z})/H_1(IJ, \mathbf{Z})$  is free of rank two over  $\mathbf{T}/I$  (recall that  $\mathbf{T}/I$  is torsion-free) and so  $(H_1(J, \mathbf{Z})_{\mathfrak{m}}/H_1(IJ, \mathbf{Z})_{\mathfrak{m}}) \otimes \mathbf{Q}$  also has dimension twice the dimension of  $(\mathbf{T}_{\mathfrak{m}}/I) \otimes \mathbf{Q}$  over  $\mathbf{Q}_p$ . So  $\phi \otimes \mathbf{Q}$  is a surjective map of vector spaces of the same dimension over  $\mathbf{Q}_p$ , and hence is injective. Thus  $\ker(\phi) \otimes \mathbf{Q} = \ker(\phi \otimes \mathbf{Q})$  is trivial. But the domain of  $\phi$  is  $H_1(J, \mathbf{Z})_{\mathfrak{m}}/IH_1(J, \mathbf{Z})_{\mathfrak{m}} \cong (\mathbf{T}_{\mathfrak{m}}/I)^2$ , which is torsion free (since  $\mathbf{T}/I$  is torsion free). Thus  $\ker(\phi)$  is also torsion-free, and hence is also trivial. Thus  $\phi$  is injective, as was left to be shown.  $\square$

Let  $\mathbf{T}' = \mathbf{T}/I$ . By the claim above,  $T_{\mathfrak{m}'}(J')$  is free of rank two over  $\mathbf{T}'_{\mathfrak{m}'}$ . Then  $J'[\mathfrak{m}']$  is free of rank two over  $\mathbf{T}'/\mathfrak{m}'$ . Suppose  $(J'_{\mathfrak{m}'})^{\vee} \otimes \mathbf{Q}$  were free of rank two over  $\mathbf{T}'_{\mathfrak{m}'} \otimes \mathbf{Q}$ . Then by a standard argument that uses Nakayama's lemma,  $(J'_{\mathfrak{m}'})^{\vee}$  would be free of rank two over  $\mathbf{T}'_{\mathfrak{m}'}$  (e.g., see [Til97, p. 341]), and thus prove the lemma. So it suffices to prove that  $(J'_{\mathfrak{m}'})^{\vee} \otimes \mathbf{Q}$  is free of rank two over  $\mathbf{T}'_{\mathfrak{m}'} \otimes \mathbf{Q}$ .

We now follow part of the proof of Theorem A on page 449 of [Eme03]. As mentioned there, the  $\mathbf{T}$ -module  $(J'_{\mathfrak{m}'})^{\vee}$  is isomorphic to the  $\mathfrak{m}'$ -adic component of the cohomology space  $H^1(J', \mathbf{Q}_p)$ , and so it suffices to show that  $H^1(J', \mathbf{Q}_p)$  is free of rank two over  $(\mathbf{T}/I) \otimes \mathbf{Q}_p$ . For this, we may replace  $\mathbf{Q}_p$  by  $\mathbf{R}$  (or any other field of characteristic zero). As described on page 448-449 of [Eme03], there is a natural isomorphism of  $\mathbf{T} \otimes \mathbf{R}$ -modules between  $S_2(\Gamma_0(N), \mathbf{C})$  and  $H_1(J, \mathbf{R})$ , which in turn gives a natural isomorphism of  $(\mathbf{T}/I) \otimes \mathbf{R}$ -modules between  $S_2(\Gamma_0(N), \mathbf{C})/IS_2(\Gamma_0(N), \mathbf{C})$  and  $H_1(J, \mathbf{R})/IH_1(J, \mathbf{R}) = H_1(J', \mathbf{R})$ . Thus, as  $(\mathbf{T}/I) \otimes \mathbf{R}$ -modules,  $H^1(J', \mathbf{R})$  is dual to  $S_2(\Gamma_0(N), \mathbf{C})/IS_2(\Gamma_0(N), \mathbf{C})$ . By Lemma 5.2(ii), as  $(\mathbf{T}/I) \otimes \mathbf{Q}$ -modules,  $S_2(\Gamma_0(N), \mathbf{Q}) \cong IS_2(\Gamma_0(N), \mathbf{Q}) \oplus S_2(\Gamma_0(N), \mathbf{Q})[I]$ . Hence, as  $(\mathbf{T}/I) \otimes \mathbf{R}$ -modules,  $H^1(J', \mathbf{R})$  is dual to  $S_2(\Gamma_0(N), \mathbf{C})[I]$ . Then by Lemma 5.3, we see that  $(\mathbf{T}/I) \otimes \mathbf{R}$ -modules,  $H^1(J', \mathbf{R}) \cong (\mathbf{T}/I) \otimes \mathbf{C}$ , which is of rank two over  $(\mathbf{T}/I) \otimes \mathbf{R}$ , and hence so is  $H^1(J', \mathbf{R})$ , as was left to be shown.  $\square$

## References

- [Aga] A. Agashe, *A visible factor of the special L-value*, to appear in J. Reine Angew. Math. (Crelle's journal), available at arXiv:0810.2477 or <http://www.math.fsu.edu/~agashe/math.html>.
- [Aga09] A. Agashe, *Visibility and the Birch and Swinnerton-dyer conjecture for analytic rank one*, Int. Math. Res. Not. (2009), doi:

10.1093/imrn/rnp036 (electronic; print version to appear); available at arXiv:0810.2487 or <http://www.math.fsu.edu/~agashe/math.html>.

- [ARS06] A. Agashe, K. Ribet, and W. A. Stein, *The modular degree, congruence primes, and multiplicity one*, preprint (2006), available at <http://www.math.fsu.edu/~agashe/math.html>.
- [AS02] A. Agashe and W. A. Stein, *Visibility of Shafarevich-Tate groups of abelian varieties*, J. Number Theory **97** (2002), no. 1, 171–185.
- [AS05] A. Agashe and W. A. Stein, *Visible evidence for the Birch and Swinnerton-Dyer conjecture for modular abelian varieties of analytic rank zero*, Math. Comp. **74** (2005), no. 249, 455–484.
- [AU96] Ahmed Abbes and Emmanuel Ullmo, *À propos de la conjecture de Manin pour les courbes elliptiques modulaires*, Compositio Math. **103** (1996), no. 3, 269–286.
- [CM00] J. E. Cremona and B. Mazur, *Visualizing elements in the Shafarevich-Tate group*, Experiment. Math. **9** (2000), no. 1, 13–28.
- [Eme03] Matthew Emerton, *Optimal quotients of modular Jacobians*, Math. Ann. **327** (2003), no. 3, 429–458.
- [GZ86] B. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), no. 2, 225–320.
- [KL89] V. A. Kolyvagin and D. Y. Logachev, *Finiteness of the Shafarevich-Tate group and the group of rational points for some modular abelian varieties*, Algebra i Analiz **1** (1989), no. 5, 171–196.
- [Lan95] S. Lang, *Introduction to modular forms*, Springer-Verlag, Berlin, 1995, With appendixes by D. Zagier and W. Feit, Corrected reprint of the 1976 original.
- [Maz77] B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 33–186 (1978).
- [Maz78] B. Mazur, *Rational isogenies of prime degree (with an appendix by D. Goldfeld)*, Invent. Math. **44** (1978), no. 2, 129–162.

- [Par99] P. Parent, *Bornes effectives pour la torsion des courbes elliptiques sur les corps de nombres*, J. Reine Angew. Math. **506** (1999), 85–116.
- [Rib83] Kenneth A. Ribet, *Mod  $p$  Hecke operators and congruences between modular forms*, Invent. Math. **71** (1983), no. 1, 193–205.
- [Rib90] K. A. Ribet, *On modular representations of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  arising from modular forms*, Invent. Math. **100** (1990), no. 2, 431–476.
- [Rub98] K. Rubin, *Euler systems and modular elliptic curves*, Galois representations in arithmetic algebraic geometry (Durham, 1996), Cambridge Univ. Press, Cambridge, 1998, pp. 351–367.
- [SW08] W. Stein and C. Wuthrich, *Computations about Tate-Shafarevich groups using Iwasawa theory*, preprint (2008), available at <http://modular.math.washington.edu/papers>.
- [Til97] Jacques Tilouine, *Hecke algebras and the Gorenstein property*, Modular forms and Fermat’s last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 327–342.