

THE MODULAR NUMBER, CONGRUENCE NUMBER, AND MULTIPLICITY ONE

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*To Ken Ribet,
on the occasion of his sixtieth birthday*

ABSTRACT. Let N be a positive integer and let f be a newform of weight 2 on $\Gamma_0(N)$. In earlier joint work with K. Ribet and W. Stein, we introduced the notions of the modular number and the congruence number of the quotient abelian variety A_f of $J_0(N)$ associated to the newform f . These invariants are analogs of the notions of the modular degree and congruence primes respectively associated to elliptic curves. We show that if p is a prime such that every maximal ideal of the Hecke algebra of characteristic p that contains the annihilator ideal of f satisfies multiplicity one, then the modular number and the congruence number have the same p -adic valuation.

1 INTRODUCTION AND RESULTS

Let N be a positive integer and let $X_0(N)$ denote the modular curve over \mathbf{Q} associated to the classification of isomorphism classes of elliptic curves with a cyclic subgroup of order N . The Hecke algebra \mathbf{T} of level N is the subring of the ring of endomorphisms of $J_0(N) = \text{Jac}(X_0(N))$ generated by the Hecke operators T_n for all $n \geq 1$. Let f be a newform of weight 2 for $\Gamma_0(N)$ and let I_f denote $\text{Ann}_{\mathbf{T}}(f)$. Then the quotient abelian variety $A_f = J_0(N)/I_f J_0(N)$ is called the newform quotient

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associated to f . If f has integer Fourier coefficients, then A_f is an elliptic curve and in fact by [BCDT01] any elliptic curve over \mathbf{Q} is isogenous to such an elliptic curve for some f . The dual abelian variety A_f^\vee of A_f may be viewed as an abelian subvariety of $J_0(N)$. Recall that the *exponent* of a finite group G is the smallest positive integer n such that multiplication by n annihilates every element of G .

The exponent of the group $A_f^\vee \cap I_f J$ is called the *modular exponent* of A_f and its order is called the *modular number* (see [ARS07, §3]). When f has integer Fourier coefficients, so that A_f is an elliptic curve, we will sometimes denote A_f by E for emphasis. In that case, composing the embedding $X_0(N) \hookrightarrow J_0(N)$ that sends ∞ to 0 with the quotient map $J_0(N) \rightarrow E$, we obtain a surjective morphism of curves $\phi_E : X_0(N) \rightarrow E$, whose degree is called the *modular degree* of E . The modular exponent \tilde{n}_E of E is equal to the modular degree, and the modular number n_E is the square of the modular degree (see [ARS07, §3]). In general, for any newform f , the modular number n_{A_f} is a perfect square (e.g., see [AS05, Lemma 3.14]).

Let $S_2(\mathbf{Z})$ denote the group of cuspforms of weight 2 on $\Gamma_0(N)$ with integral Fourier coefficients, and if G is a subgroup of $S_2(\mathbf{Z})$, let G^\perp denote the subgroup of $S_2(\mathbf{Z})$ consisting of cuspforms that are orthogonal to every g in G with respect to the Petersson inner product. The exponent of the quotient group

$$\frac{S_2(\mathbf{Z})}{S_2(\mathbf{Z})[I_f] + S_2(\mathbf{Z})[I_f]^\perp}$$

is called the *congruence exponent* of A_f (really, that of f) and its order is called the *congruence number* (see [ARS07, §3]). If f has integer Fourier coefficients, so that A_f is an elliptic curve, then r_{A_f} is the largest integer r such that there exists a cuspform $g \in S_2(\mathbf{Z})$ that is orthogonal to f under the Petersson inner product and whose n -th Fourier coefficient is congruent modulo r to the n -th Fourier coefficient of f for all positive integers n . We say that a prime is a *congruence prime for A_f* if it divides the congruence number r_{A_f} .

Congruence primes have been studied by Doi, Hida, Ribet, Mazur and others (see, e.g., [Rib83, §1]), and played an important role in Wiles's work [Wil95] on Fermat's last theorem. Frey and Mai-Murty have observed that an appropriate asymptotic bound on the modular degree is equivalent to the *abc*-conjecture (see [Fre97, p.544] and [Mur99, p.180]). Thus congruence primes and the modular degree are quantities of significant interest. Theorem 3.6 of [ARS07] says that the modular

exponent \tilde{n}_{A_f} divides the congruence exponent \tilde{r}_{A_f} and if p is a prime such that $p^2 \nmid N$, then $\text{ord}_p(\tilde{n}_{A_f}) = \text{ord}_p(\tilde{r}_{A_f})$.

One might wonder if similar relations hold for the modular number r_{A_f} and congruence number n_{A_f} (as opposed to modular/congruence exponents). As mentioned earlier, if A_f is an elliptic curve, then $n_{A_f} = \tilde{n}_{A_f}^2$ and so, considering that $\tilde{n}_{A_f} \mid \tilde{r}_{A_f}$ and $\tilde{r}_{A_f} \mid r_{A_f}$, one sees that $n_{A_f} \mid r_{A_f}^2$. One might wonder if n_{A_f} divides $r_{A_f}^2$ even if A_f is not an elliptic curve (i.e., has dimension more than one). It turns out that the answer is no: as mentioned in [ARS07, Remark 3.7] we have

EXAMPLE 1.1. There is a newform of degree 24 in $S_2(\Gamma_0(431))$ such that

$$n_{A_f} = (2^{11} \cdot 6947)^2 \nmid r_{A_f}^2 = (2^{10} \cdot 6947)^2.$$

We say that a maximal ideal \mathfrak{m} of \mathbf{T} satisfies *multiplicity one* if $J_0(N)[\mathfrak{m}]$ is of dimension two over \mathbf{T}/\mathfrak{m} . The reason one calls this “multiplicity one” is that if the canonical two dimensional representation $\rho_{\mathfrak{m}}$ over \mathbf{T}/\mathfrak{m} attached to \mathfrak{m} (e.g., see [Rib90, Prop. 5.1]) is irreducible, then $J_0(N)[\mathfrak{m}]$ is a direct sum of copies of $\rho_{\mathfrak{m}}$ (e.g., see [Rib90, Thm. 5.2]), and a maximal ideal \mathfrak{m} of \mathbf{T} satisfies *multiplicity one* precisely if the multiplicity of $\rho_{\mathfrak{m}}$ in this decomposition is one. Even if $\rho_{\mathfrak{m}}$ is reducible, the definition of multiplicity one given above is relevant (e.g., see [Maz77, Cor. 16.3]). It was remarked in [ARS07] that concerning Example 1.1 above where $n_{A_f} \nmid r_{A_f}^2$, the level 431 is prime and by [Kil02], mod 2 multiplicity one fails for $J_0(431)$. In this article, we show that multiplicity one is the only obstruction for the divisibility $n_{A_f} \mid r_{A_f}^2$ to fail. In fact, we show something stronger:

THEOREM 1.2. *Let p be a prime such that every maximal ideal \mathfrak{m} with residue characteristic p that contains I_f satisfies multiplicity one. Then $\text{ord}_p(n_{A_f}) = \text{ord}_p(r_{A_f}^2)$.*

The theorem above follows from the more general Theorem 2.1 below. Example 1.1 above shows that the multiplicity one hypothesis cannot be completely removed from the theorem. Also, in the context of Example 1.1, our theorem gives a new proof that mod 2 multiplicity fails for $J_0(431)$ (the original proof being the one in [Kil02]). Note that in [ARS07], the authors found examples of failure of multiplicity one using Proposition 5.9 of loc. cit., which implies that if the modular exponent does not equal the congruence exponent for some newform f , then there is a maximal ideal of \mathbf{T} that not satisfy multiplicity one.

However, we could not have detected the failure of multiplicity one in Example 1.1 by checking if the modular *exponent* equals the congruence *exponent*, since the equality holds in the example for any newform f by [ARS07, Thm. 3.6(b)], considering that the level is prime in the example. At the same time, consideration of the modular *number* and the congruence *number* did detect the failure of multiplicity one. It would be interesting to do more calculations to see when $n_{A_f} \nmid r_{A_f}^2$, as this may give new instances of failure of multiplicity one.

We remark that our theorem gives information about the *order* of a certain intersection of abelian subvarieties of $J_0(N)$ in terms of congruences between modular forms (in fact, we give information in a more general setting in Section 2). We expect that the relation between a particular such intersection and certain congruences will be useful in understanding the “visible factor” in [Aga07] (in loc. cit., we were able to say something about the primes that divide this factor, as opposed to saying something about the entire factor), and hope that such relations will be useful in other contexts as well.

It is known that multiplicity one holds in several situations. We content ourselves by pointing out that by the main theorem in Section 1.2 of [MR91], a maximal ideal \mathfrak{m} with residue characteristic p satisfies multiplicity one if either $p \nmid N$ or $p \mid N$ and $\rho_{\mathfrak{m}}$ is not modular of level N/p . We also have:

PROPOSITION 1.3. *Let p be an odd prime and \mathfrak{m} be a maximal ideal of \mathbf{T} with residue characteristic p such that $\rho_{\mathfrak{m}}$ is irreducible. Assume that either*

- (i) $p \nmid N$ or
- (ii) $p \mid N$ and $I_f \subseteq \mathfrak{m}$ for some newform f .

Then \mathfrak{m} satisfies multiplicity one.

Proof. If $p \nmid N$, then the claim follows from Theorem 5.2(b) of [Rib90], so let us assume that $p \mid N$. Let $X_0(N)_{\mathbf{Z}_p}$ denote the minimal regular resolution of the compactified coarse moduli scheme over \mathbf{Z}_p associated to $\Gamma_0(N)$ as in [DR73, § IV.3] and let $\Omega_{X_0(N)_{\mathbf{Z}_p}/\mathbf{Z}_p}$ denote the relative dualizing sheaf of $X_0(N)_{\mathbf{Z}_p}$ over \mathbf{Z}_p (it is the sheaf of regular differentials as in [MR91, §7]). We denote by $X_0(N)_{\mathbf{F}_p}$ the special fiber of $X_0(N)_{\mathbf{Z}_p}$ at the prime p and by $\Omega_{X_0(N)_{\mathbf{F}_p}/\mathbf{F}_p}$ the relative dualizing sheaf of $X_0(N)_{\mathbf{F}_p}$ over \mathbf{F}_p . It is shown in [ARS07, §5.2.2] that under the hypotheses above, $\dim_{\mathbf{T}/\mathfrak{m}} H^0(X_0(N)_{\mathbf{F}_p}, \Omega_{X_0(N)_{\mathbf{F}_p}/\mathbf{F}_p})[\mathfrak{m}] \leq 1$. Let $J_{\mathbf{Z}_p}$ denote the Néron model of $J_0(N)$ over \mathbf{Z}_p and let $J_{\mathbf{Z}_p}^0$ denote its

identity component. Then the natural morphism $\text{Pic}_{X_0(N)/\mathbf{Z}_p}^0 \rightarrow J_{\mathbf{Z}_p}$ identifies $\text{Pic}_{X_0(N)/\mathbf{Z}_p}^0$ with $J_{\mathbf{Z}_p}^0$ (see, e.g., [BLR90, §9.4–9.5]). Passing to tangent spaces along the identity section over \mathbf{Z}_p , we obtain an isomorphism $H^1(X_0(N)_{\mathbf{Z}_p}, \mathcal{O}_{X_0(N)_{\mathbf{Z}_p}}) \cong \text{Tan}(J_{\mathbf{Z}_p})$. Reducing both sides modulo p and applying Grothendieck duality, we get $\text{Tan}(J_{\mathbf{F}_p}) \cong \text{Hom}(H^0(X_0(N)_{\mathbf{F}_p}, \Omega_{X_0(N)/\mathbf{F}_p}), \mathbf{F}_p)$. Thus from the above discussion, we see that $\text{Tan}(J_{\mathbf{F}_p})/\mathfrak{m}\text{Tan}(J_{\mathbf{F}_p})$ has dimension at most one over \mathbf{T}/\mathfrak{m} . Since $\text{Tan}(J_{\mathbf{Z}_p})$ is a faithful $\mathbf{T} \otimes \mathbf{Z}_p$ -module, we see that $\text{Tan}(J_{\mathbf{F}_p})/\mathfrak{m}\text{Tan}(J_{\mathbf{F}_p})$ is non-trivial, hence it is one dimensional over \mathbf{T}/\mathfrak{m} . With this input, the proof of multiplicity one in Theorem 2.1 of [Wil95], which is in the $\Gamma_1(N)$ context, but is a formal argument involving abelian varieties (apart from the input above), carries over in the $\Gamma_0(N)$ context with the obvious modifications (in particular, replacing $X_1(N/p, p)_{\mathbf{Z}_p}$ in loc. cit. by $X_0(N)_{\mathbf{Z}_p}$) to prove our claim (see p. 487–488 of loc. cit., as well as [Til97], where the input above is the equation (***) on p. 339). \square

We remark that the condition that $p^2 \nmid N$ in condition (ii) of the proposition above cannot be removed, as follows from the counterexamples in [ARS07, §2.2]. From Theorem 1.2 and Proposition 1.3, we obtain:

COROLLARY 1.4. *Let p be an odd prime. Suppose that either*

(i) $p \nmid N$ or

(ii) $p \mid N$ and $A_f^\vee[\mathfrak{m}]$ is irreducible for every maximal ideal \mathfrak{m} of \mathbf{T} with residue characteristic p .

Then $\text{ord}_p(n_{A_f}) = \text{ord}_p(r_{A_f}^2)$.

Proof. The corollary is clear from Theorem 1.2 and Proposition 1.3 in the case where $p \nmid N$, so let us assume that $p \mid N$. By Theorem 1.2 and Proposition 1.3, it suffices to show that $\rho_{\mathfrak{m}}$ is irreducible for every maximal ideal \mathfrak{m} of \mathbf{T} with residue characteristic p such that $I_f \subseteq \mathfrak{m}$.

Let \mathfrak{m} be such a maximal ideal. Then note that $A_f^\vee[\mathfrak{m}]$ is non-trivial since \mathbf{T}/I_f acts faithfully on A_f^\vee . Let D denote the direct sum of $A_f^\vee[\mathfrak{m}]$ and its Cartier dual. Let ℓ be a prime that does not divide Np and let Frob_ℓ denote the Frobenius element of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ at ℓ . As discussed in [Maz77, p. 115], by the Eichler-Shimura relation, the characteristic polynomial of Frob_ℓ acting on D is $(X^2 - a_\ell X + \ell)^d = 0$, where a_ℓ is the image of T_ℓ in \mathbf{T}/\mathfrak{m} and d is the \mathbf{T}/\mathfrak{m} -dimension of $A_f^\vee[\mathfrak{m}]$. But this is also the characteristic polynomial of Frob_ℓ acting on the direct sum of d copies of $\rho_{\mathfrak{m}}$. By the Chebotarev density theorem and the

Brauer-Nesbitt theorem, the semisimplification of D is $\rho_{\mathfrak{m}}^d$. Thus the semisimplification of $A_f^{\vee}[\mathfrak{m}]$ is a direct sum of certain number of copies of $\rho_{\mathfrak{m}}$. But $A_f^{\vee}[\mathfrak{m}]$ is irreducible by hypothesis, so $\rho_{\mathfrak{m}} = A_f^{\vee}[\mathfrak{m}]$. Thus $\rho_{\mathfrak{m}}$ is also irreducible, as was to be shown. \square

The corollary above is the analog of Theorem 3.6(b) of [ARS07], which says that $\text{ord}_p(\tilde{n}_{A_f}) = \text{ord}_p(\tilde{r}_{A_f})$ provided $p^2 \nmid N$, in the setting of modular/congruence *numbers* as opposed to modular/congruence *exponents* (although, note that we have an extra irreducibility hypothesis in our corollary). We remark that the proofs of both results use “multiplicity one for differentials” (as defined in [ARS07, §5.2]).

If the level N is prime, then more can be said. By Prop. II.14.2 and Corollary II.16.3 of [Maz77], every maximal ideal \mathfrak{m} such that $\rho_{\mathfrak{m}}$ is reducible also satisfies multiplicity one. Thus in view of Theorem 1.2 and Proposition 1.3, we obtain the following:

COROLLARY 1.5. *Suppose the level N is prime and let p be an odd prime. Then $\text{ord}_p(n_{A_f}) = \text{ord}_p(r_{A_f}^2)$.*

Also, much is known in this situation if $\rho_{\mathfrak{m}}$ is irreducible and \mathfrak{m} has residue characteristic is 2 – we refer to [Kil02] and the references therein for details. But note that by the examples in [Kil02] or by Example 1.1 and Theorem 1.2, multiplicity one need not hold for a maximal ideal \mathfrak{m} of residue characteristic 2 with $\rho_{\mathfrak{m}}$ irreducible even if the level N is prime.

In Section 2, we describe a more general setup, which includes newform quotients of $J_1(N)$, and state a more general version of Theorem 1.2 (Theorem 2.1 below). In Section 3, we give the proof of Theorem 2.1.

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2 A MORE GENERAL SETUP

For the benefit of the reader, we repeat below some of the discussion in [ARS07, Section 3].

For $N \geq 4$, let Γ be either $\Gamma_0(N)$ or $\Gamma_1(N)$. Let X denote the modular curve over \mathbf{Q} associated to Γ , and let J be the Jacobian of X .

Let J_f denote the standard abelian subvariety of J attached to f by Shimura [Shi94, Thm. 7.14]. Up to isogeny, J is the product of factors $J_f^{e(f)}$ where f runs over the set of newforms of level dividing N , taken up to Galois conjugation, and $e(f)$ is the number of divisors of $N/N(f)$, where $N(f)$ is the level of f . Let A be the sum of $J_f^{e(f)}$ for some set of f 's (taken up to Galois conjugation), and let B be the sum of all the other $J_f^{e(f)}$'s. Clearly $A + B = J$. The J_f 's are simple (over \mathbf{Q}), hence $A \cap B$ is finite. By [ARS07, Lemma 3.1], $\text{End}(J)$ preserves A and B , where if C is an abelian variety over \mathbf{Q} , by $\text{End}(C)$ we mean the ring of endomorphisms of C defined over \mathbf{Q} . If f is a newform of weight 2 on Γ and A_f is its associated newform quotient, then A_f^\vee and $I_f J$ provide an example of A and B respectively as above, as shown in the discussion following Lemma 3.1 in [ARS07].

The *modular exponent* \tilde{n}_A of A is defined as the exponent of $A \cap B$ and the *modular number* n_A of A is its order (see [ARS07, §3]). Note that the definition is symmetric with respect to A and B . In fact, the definition depends on both A and B , unlike what the notation may suggest—we have suppressed the dependence on B for ease of notation, with the understanding that there is a natural choice of B . If f is a newform, then by the modular exponent/number of A_f , we mean that of $A = A_f^\vee$, with $B = I_f J$, which agrees with our earlier definition.

If R is a subring of \mathbf{C} , let $S_2(R) = S_2(\Gamma; R)$ denote the subgroup of $S_2(\Gamma; \mathbf{C})$ consisting of cusp forms whose Fourier expansions at the cusp ∞ have coefficients in R . Let \mathbf{T} denote the Hecke algebra corresponding to the group Γ . There is a \mathbf{T} -equivariant bilinear pairing

$$\mathbf{T} \times S_2(\mathbf{Z}) \rightarrow \mathbf{Z} \quad (1)$$

given by $(t, g) \mapsto a_1(t(g))$, which is perfect (e.g., see [AU96, Lemma 2.1] or [Rib83, Theorem 2.2]). Let \mathbf{T}_A denote the image of \mathbf{T} in $\text{End}(A)$, and let \mathbf{T}_B be the image of \mathbf{T} in $\text{End}(B)$ (since $\mathbf{T} \subset \text{End}(J)$, \mathbf{T} preserves A and B). Since $A + B = J$, the natural map $\mathbf{T} \rightarrow \mathbf{T}_A \oplus \mathbf{T}_B$ is injective, and moreover, its cokernel is finite (since $A \cap B$ is finite).

Let $S_A = \text{Hom}(\mathbf{T}_A, \mathbf{Z})$ and $S_B = \text{Hom}(\mathbf{T}_B, \mathbf{Z})$ be the subgroups of $S_2(\mathbf{Z})$ obtained via the pairing in (1). By [ARS07, Lemma 3.3], we have an isomorphism

$$\frac{S_2(\mathbf{Z})}{S_A + S_B} \cong \frac{\mathbf{T}_A \oplus \mathbf{T}_B}{\mathbf{T}}. \quad (2)$$

By definition [ARS07], the exponent of either of the isomorphic groups in (2) is the *congruence exponent* \tilde{r}_A of A and the order of either group

is the *congruence number* r_A . Note that this definition is also symmetric with respect to A and B , and again, the definition depends on both A and B , unlike what the notation may suggest – we have suppressed the dependence on B with the implicit understanding that B has been chosen (given A). If f is a newform, then by the congruence exponent/number of A_f , we mean that of $A = A_f^\vee$, with $B = I_f J$. In this situation, $\mathbf{T}_A = \mathbf{T}/I_f$ and $S_A = S_2(\mathbf{Z})[I_f]$. Also, $\text{Hom}(\mathbf{T}_B, \mathbf{Z})$ is the unique saturated Hecke-stable complement of $S_2(\mathbf{Z})[I_f]$ in $S_2(\mathbf{Z})$, hence must equal $S_2(\mathbf{Z})[I_f]^\perp$. This shows that the new definition of the congruence number/exponent generalizes our earlier definition for A_f .

Let $I_A = \text{Ann}_{\mathbf{T}}(A)$ and $I_B = \text{Ann}_{\mathbf{T}}(B)$. Theorem 3.6(a) of [ARS07] says that the modular exponent \tilde{n}_A divides the congruence exponent \tilde{r}_A , and Proposition 5.9 of loc. cit. says that if p is a prime such that all maximal ideals \mathfrak{m} of \mathbf{T} containing $I_A + I_B$ satisfy multiplicity one, then $\text{ord}_p(\tilde{r}_A) = \text{ord}_p(\tilde{n}_A)$. Our main theorem deals with the case of modular/congruence numbers as opposed to modular/congruence exponents. In view of the case of newform quotients discussed in Section 1, one would like to understand the relation between the modular number n_A and the *square* of the congruence number r_A . As mentioned earlier, it is not true that n_A divides r_A^2 in general. At the same time, we have:

THEOREM 2.1. *Let p be an odd prime such that every maximal ideal \mathfrak{m} with residue characteristic p that contains $I_A + I_B$ satisfies multiplicity one. Then $\text{ord}_p(n_A) = \text{ord}_p(r_A^2)$.*

This theorem is proved in the next section. It is an analog of Proposition 5.9 of [ARS07] mentioned above in the context of modular/congruence numbers as opposed to modular/congruence exponents. For results on multiplicity one in the $\Gamma = \Gamma_1(N)$ context, see, e.g., [Til97] and the references therein.

3 PROOF OF THEOREM 2.1

We continue to use the notation introduced in previous sections. The following lemma is easily extracted from [Eme03], and is the key input in our proof of Theorem 2.1:

LEMMA 3.1 (Emerton). *Let I be a saturated ideal of \mathbf{T} and let $J[I]^0$ denote the abelian subvariety of J that is the connected component of $J[I]$. Then the quotient $J[I]/J[I]^0$ is supported at maximal ideals of \mathbf{T} that do not satisfy multiplicity one.*

Proof. It is shown in the proof of Theorem A of [Eme03] that if \mathfrak{m} satisfies multiplicity one, then \mathfrak{m} is good for J (in the notation of loc. cit.). The lemma now follows from Corollary 2.3 of loc. cit. (note that [Eme03] is in the $\Gamma_0(N)$ context, but the ideas behind the argument above work in the $\Gamma_1(N)$ situation as well). \square

PROPOSITION 3.2. *The cokernel of the injection $A \cap B \rightarrow J[I_A + I_B]$ is supported at maximal ideals \mathfrak{m} of \mathbf{T} containing $I_A + I_B$ that do not satisfy multiplicity one.*

Proof. Consider the natural map $B \cap J[I_A] \rightarrow J[I_A]/A$. Its kernel is $B \cap J[I_A] \cap A = B \cap A$, and hence we have an injection:

$$\frac{B \cap J[I_A]}{B \cap A} \hookrightarrow \frac{J[I_A]}{A}. \quad (3)$$

Also, the natural map $J[I_A + I_B] = J[I_B][I_A] \rightarrow J[I_B]/B$ has kernel $B \cap J[I_B][I_A] = B \cap J[I_A]$, and hence we have an injection

$$\frac{J[I_A + I_B]}{B \cap J[I_A]} \hookrightarrow \frac{J[I_B]}{B}. \quad (4)$$

Now A is the connected component of $J[I_A]$ and similarly B is the connected component of $J[I_B]$. Thus, by Lemma 3.1, the quotient groups on the right side of (3) and (4) are supported at maximal ideals of \mathbf{T} that do not satisfy multiplicity one. Then, by the injections (3) and (4), the cokernel of the injection $A \cap B \rightarrow J[I_A + I_B]$ is supported at maximal ideals \mathfrak{m} that do not satisfy multiplicity one. Also, any maximal ideal in the support of $J[I_A + I_B]$ contains $I_A + I_B$. Our lemma follows. \square

The following lemma is perhaps known to experts; its proof was indicated to us by K. Ribet.

LEMMA 3.3 (Ribet). *Let I be an ideal of \mathbf{T} of finite index. Suppose that every maximal ideal \mathfrak{m} of \mathbf{T} that contains I satisfies multiplicity one (i.e., $J[\mathfrak{m}]$ has order $|\mathbf{T}/\mathfrak{m}|^2$). Then $J[I]$ has order $|\mathbf{T}/I|^2$.*

Proof. If \mathfrak{m} is a maximal ideal of \mathbf{T} , then let $J_{\mathfrak{m}}$ denote the \mathfrak{m} -divisible group attached to J . It suffices to show that $J_{\mathfrak{m}}[I]$ is of order $|\mathbf{T}_{\mathfrak{m}}/I\mathbf{T}_{\mathfrak{m}}|^2$ for each maximal ideal \mathfrak{m} containing I . Let \mathfrak{m} be such a maximal ideal and let $J_{\mathfrak{m}}^{\vee}$ denote the Pontryagin dual of $J_{\mathfrak{m}}$. Considering that $J[\mathfrak{m}]$ is free of rank 2 over \mathbf{T}/\mathfrak{m} , by a standard argument due to Mazur that uses Nakayama's lemma (e.g., see [Til97, p. 333 and p. 341]), $J_{\mathfrak{m}}^{\vee}$ is free

of rank two over \mathbf{T}_m . Then J_m^\vee/IJ_m^\vee is free of rank two over $\mathbf{T}_m/I\mathbf{T}_m$, and in particular has order $|\mathbf{T}_m/I\mathbf{T}_m|^2$. But $J_m[I]$ is Pontryagin dual to J_m^\vee/IJ_m^\vee , and hence has the same order $|\mathbf{T}_m/I\mathbf{T}_m|^2$, as was to be shown. \square

Proof of Theorem 2.1. Taking $I = I_A + I_B$ in Lemma 3.3, we see that

$$\mathrm{ord}_p\left(|J[I_A + I_B]|\right) = \mathrm{ord}_p\left(\left|\frac{\mathbf{T}}{I_A + I_B}\right|^2\right) = \mathrm{ord}_p(r_A^2), \quad (5)$$

where the last equality follows since we have an isomorphism

$$\frac{\mathbf{T}}{I_A + I_B} \xrightarrow{\simeq} \frac{\mathbf{T}_A \oplus \mathbf{T}_B}{\mathbf{T}}$$

obtained by sending $t \in \mathbf{T}$ to $(\pi_A(t), 0) \in \mathbf{T}_A \oplus \mathbf{T}_B$, where π_A is the projection map $\mathbf{T} \rightarrow \mathbf{T}_A$. Also by Proposition 3.2 and the hypothesis that every maximal ideal \mathfrak{m} with residue characteristic p that contains $I_A + I_B$ satisfies multiplicity one, we have

$$\mathrm{ord}_p\left(|J[I_A + I_B]|\right) = \mathrm{ord}_p(|A \cap B|) = \mathrm{ord}_p(n_A), \quad (6)$$

where the last equality follows by the definition of n_A . The theorem now follows from (5) and (6). \square

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