The Birch and Swinnerton-Dyer conjectural formula
for modular abelian varieties

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*These slides can be obtained from
http://www.ma.utexas.edu/users/amod/mymath.html

Abstract: The Birch and Swinnerton-Dyer conjectural formula relates a special value coming from the $L$-function of an abelian variety $A$ to certain arithmetic invariants of $A$. We give a formula for the ratio of this special value to the real volume of $A$ for certain quotients of $J_0(N)$, and give numerical and theoretical evidence for the conjecture using this formula.
The Birch and Swinnerton-Dyer conjectural formula

Let

\[ J_0(N) = \text{Jacobian of the modular curve } X_0(N), \]
\[ T = \text{Hecke algebra}, \]
\[ f = \text{a newform}, \]
\[ I_f = \text{Ann}_T f, \text{ an ideal of } T, \text{ and} \]
\[ A = A_f = J_0(N)/I_f J_0(N), \]
\[ \text{the Shimura quotient associated to } f. \]

Conjecture 1 (Birch, Swinnerton-Dyer, Tate).

If \( L(A, 1) \neq 0 \), then

\[ \frac{L(A, 1)}{\Omega(A)} = \frac{\# \Sha(A) \cdot \prod_{p \mid N} c_p(A)}{\#A(\mathbb{Q}) \cdot \#A^\vee(\mathbb{Q})}, \]

where

\[ \Omega(A) = \text{volume of } A(\mathbb{R}) \text{ w.r.t. Néron differentials}, \]
\[ \Sha(A) = \text{Shafarevich-Tate group of } A, \text{ and} \]
\[ c_p(A) = \text{order of the arithmetic component group of } A \text{ at } p. \]
A formula for \( L(A, 1) / \Omega(A) \)

Theorem 2.

\[
\frac{L(A, 1)}{\Omega(A)} = \frac{[\Phi(H_1(X_0(N), \mathbb{Z})^+) : \Phi(T_e)]}{c_A \cdot c_\infty(A)},
\]

where the terms are as follows:

Let \( f_1, f_2, \ldots, f_d = \) Galois conjugates of \( f \).

Then \( \Phi : H_1(X_0(N), \mathbb{Q})^+ \rightarrow \mathbb{C}^d \)

is given by \( \gamma \mapsto \{ \int_\gamma f_1, \ldots, \int_\gamma f_d \} \).

We have

\[
H_1(X_0(N), \mathbb{R}) \cong \text{Hom}_C(H^0(X_0(N), \Omega^1), \mathbb{C})
\]
given by \( \gamma \mapsto \omega \mapsto \int_\gamma \omega \)

Definition: \( e \leftrightarrow \omega \mapsto -\int_{\{0, i\infty\}} \omega \)

Manin-Drinfeld \( \Rightarrow T_e \subseteq H_1(X_0(N), \mathbb{Q})^+ \).

\( \Phi(H_1(X_0(N), \mathbb{Z})^+) \), \( \Phi(T_e) \subseteq \mathbb{R}^d \) are lattices; \( [\Phi(H_1(X_0(N), \mathbb{Z})^+) : \Phi(T_e)] = \) lattice index.

\( c_\infty(A) = \# \) of connected components of \( A(\mathbb{R}) \).

Let \( A = \) Néron model of \( A \) over \( \mathbb{Z} \).

**Definition 3.** The generalized Manin constant of \( A \), denoted \( c_A \), is the index of \( H^0(A, \Omega_{A/\mathbb{Z}}) \) in \( S_2(\Gamma_0(N), \mathbb{Z})[I_f] \).
Some remarks on the formula

\[
\frac{L(A, 1)}{\Omega(A)} = \frac{\Phi(H_1(X_0(N), \mathbb{Z})^+: \Phi(Te))}{c_A \cdot c_\infty(A)}.
\]

1. Was conjectured by W. Stein.
2. Already known for elliptic curves (e.g., see Cremona).
3. Right-hand side of the formula above is a rational number and can be computed using rational arithmetic up to the constant \(c_A\) using modular symbols (rationality of \(L(A, 1)/\Omega(A)\) was already known).
4. Can be used to check if \(L(A, 1) = 0\).
5. Can get information on the order of \(\Sha(A)\) predicted by the Birch and Swinnerton-Dyer conjectural formula

\[
\frac{L(A, 1)}{\Omega(A)} \approx \frac{\#\Sha(A) \cdot \prod_{p|N} c_p(A)}{\#A(\mathbb{Q}) \cdot \#A^\vee(\mathbb{Q})},
\]

up to knowledge of \(c_A\) (Stein).
6. We have good control on the primes dividing \(c_A\):

**Theorem 4 (Mazur, Stein).**

*If \(p\) is a prime that divides \(c_A\), then either \(p^2|N\), or \(p = 2\).*
Some theoretical evidence for the Birch and Swinnerton-Dyer conjectural formula

Corollary 5 (AA, Stein). Suppose $L(A, 1) \neq 0$. Let $x$ be the image of $(0) - (\infty) \in J_0(N)(\mathbb{Q})$ in $A(\mathbb{Q})$ and let $n$ be the order of $x$ in $A(\mathbb{Q})$. Then

$$c_\infty(A) \cdot c_A \cdot \frac{L(A, 1)}{\Omega(A)} \in \frac{1}{n} \mathbb{Z}.$$ 

In particular,

$$c_\infty(A) \cdot c_A \cdot \#A(\mathbb{Q}) \cdot \#A^\vee(\mathbb{Q}) \cdot \frac{L(A, 1)}{\Omega(A)} \in \mathbb{Z}.$$ 

Remarks:

1. The Birch and Swinnerton-Dyer conjecture predicts:

$$\#A(\mathbb{Q}) \cdot \#A^\vee(\mathbb{Q}) \cdot \frac{L(A, 1)}{\Omega(A)} \in \mathbb{Z}.$$ 

2. $c_\infty(A)$ is a power of 2, and we have control on the primes dividing $c_A$ (in particular, if $N$ is a prime, $c_A$ is a power of 2).
Computational evidence for the Birch and Swinnerton-Dyer conjectural formula

Proposition 6. Suppose $L(A, 1) \neq 0$. Let $p$ be a prime that does not divide the degree of the canonical polarization of $A^\vee$. Then the order of the $p$-primary part of $\Sha(A)$ is a perfect square.

Theorem 7 (Mazur, Stein). Suppose $B$ is another Shimura quotient of $J_0(N)$ of Mordell-Weil rank $> 0$. Let $p$ be a prime such that $B^\vee[p] \subseteq A^\vee(\subseteq J_0(N)^\vee)$. Then, under certain mild conditions, there is a non-trivial element of order $p$ in $\Sha(A^\vee)$. Hence, if $L(A, 1) \neq 0$, then $p|\#\Sha(A)$. 


Prime levels with \( L(A, 1) \neq 0 \) and \( \#\Sha(A) > 1 \) (Calculated by William Stein)

\( \#\Sha(A) = \) order of the Shafarevich–Tate group as predicted by the BSD formula.

Warning: only odd parts of the invariants are shown.

<table>
<thead>
<tr>
<th>A</th>
<th>#\Sha(A)</th>
<th>( \sqrt{\deg(\phi_A)} )</th>
<th>B</th>
</tr>
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<tr>
<td>389E</td>
<td>5(^2)</td>
<td>5</td>
<td>389A</td>
</tr>
<tr>
<td>433D</td>
<td>7(^2)</td>
<td>3 \cdot 7 \cdot 37</td>
<td>433A</td>
</tr>
<tr>
<td>563E</td>
<td>13(^2)</td>
<td>13</td>
<td>563A</td>
</tr>
<tr>
<td>571D</td>
<td>3(^2)</td>
<td>3(^2) \cdot 127</td>
<td>571B</td>
</tr>
<tr>
<td>709C</td>
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<td>709A</td>
</tr>
<tr>
<td>997H</td>
<td>3(^4)</td>
<td>3(^2)</td>
<td>997B</td>
</tr>
<tr>
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<td>151(^2)</td>
<td>61 \cdot 151 \cdot 179</td>
<td>1061B</td>
</tr>
<tr>
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<td>1</td>
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</tr>
<tr>
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<td>11(^2)</td>
<td>3(^4) \cdot 11</td>
<td>1171A</td>
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<tr>
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</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>2333C</td>
<td>83341(^2)</td>
<td>83341</td>
<td>2333A</td>
</tr>
</tbody>
</table>

\( \deg(\phi_A) = \) degree of canonical polarization of \( A^\vee \).

**Example 8.** \( 7 \mid \deg(\phi_{1091C}) \Rightarrow \) highest power of 7 dividing \( \#\Sha(1091C) \) has even exponent.

\( B = \) an optimal quotient of \( J_0(N) \) such that

\( L(B, 1) = 0 \) and if an odd prime \( p \) divides \( \#\Sha(A) \), then \( B^\vee[p] \subseteq A^\vee \).

**Example 9 (Stein).** \( 5^2 \mid \#\Sha_{389E} \).