The Birch and Swinnerton-Dyer conjectural formula for modular abelian varieties *

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*These slides can be obtained from http://www.ma.utexas.edu/users/amod/mymath.html

Abstract: The Birch and Swinnerton-Dyer conjectural formula relates a special value coming from the L-function of an abelian variety A to certain arithmetic invariants of A. We give a formula for the ratio of this special value to the real volume of A for certain quotients of $J_0(N)$, and give numerical and theoretical evidence for the conjecture using this formula.

The Birch and Swinnerton-Dyer conjectural formula

Let

$$
\begin{aligned}
J_0(N) &= \text{Jacobian of the modular curve } X_0(N),\\
\mathbf{T} &= \text{Hecke algebra},\\
f &= \text{a newform},\\
I_f &= \mathrm{Ann}_{\mathbf{T}} f, \text{ an ideal of } \mathbf{T}, \text{ and}\\
A = A_f &= J_0(N)/I_f J_0(N),\\
&\quad\ \text{the Shimura quotient associated to } f.
\end{aligned}
$$

Conjecture 1 (Birch, Swinnerton-Dyer, Tate). *If $L(A,1) \neq 0$, then*

$$\frac{L(A,1)}{\Omega(A)} = \frac{\#\text{Ш}(A) \cdot \prod_{p\mid N} c_p(A)}{\#A(\mathbf{Q}) \cdot \#A^{\vee}(\mathbf{Q})},$$

where

$$
\begin{aligned}
\Omega(A) &= \textit{volume of } A(\mathbf{R}) \textit{ w.r.t. Néron differentials},\\
\text{Ш}(A) &= \textit{Shafarevich-Tate group of } A, \textit{ and}\\
c_p(A) &= \textit{order of the arithmetic component group of } A \textit{ at } p.
\end{aligned}
$$

A formula for $L(A,1)/\Omega(A)$

Theorem 2.

$$\frac{L(A,1)}{\Omega(A)} = \frac{[\Phi(H_1(X_0(N),\mathbf{Z})^+) : \Phi(\mathbf{T}e)]}{c_A \cdot c_\infty(A)},$$

where the terms are as follows:

Let $f_1, f_2, \ldots, f_d =$ Galois conjugates of f.

$$\begin{array}{rcl} \text{Then } \Phi : H_1(X_0(N),\mathbf{Q})^+ & \to & \mathbf{C}^d \\ \text{is given by } \gamma & \mapsto & \{\int_\gamma f_1, \ldots, \int_\gamma f_d\}. \end{array}$$

We have

$$\begin{array}{rcl} H_1(X_0(N),\mathbf{R}) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbf{C}}(H^0(X_0(N),\Omega^1),\mathbf{C}) \\ \text{given by } \gamma & \mapsto & \omega \mapsto \int_\gamma \omega \\ \text{Definition: } e & \leftrightarrow & \omega \mapsto -\int_{\{0,i\infty\}} \omega \end{array}$$

Manin-Drinfeld $\Rightarrow \mathbf{T}e \subseteq H_1(X_0(N),\mathbf{Q})^+$.

$\Phi(H_1(X_0(N),\mathbf{Z})^+)$, $\Phi(\mathbf{T}e) \subseteq \mathbf{R}^d$ are lattices;
$[\Phi(H_1(X_0(N),\mathbf{Z})^+) : \Phi(\mathbf{T}e)] =$ lattice index.

$c_\infty(A) = \#$ of connected components of $A(\mathbf{R})$.

Let $\mathcal{A} =$ Néron model of A over $\mathbf{Z}$.

Definition 3. *The generalized Manin constant of A, denoted c_A, is the index of $H^0(\mathcal{A},\Omega_{\mathcal{A}/\mathbf{Z}})$ in $S_2(\Gamma_0(N),\mathbf{Z})[I_f]$.*

Some remarks on the formula

$$\frac{L(A,1)}{\Omega(A)} = \frac{[\Phi(H_1(X_0(N),\mathbf{Z})^+) : \Phi(\mathbf{T}e)]}{c_A \cdot c_\infty(A)}.$$

1. Was conjectured by W. Stein.
2. Already known for elliptic curves (e.g., see Cremona).
3. Right-hand side of the formula above is a rational number and can be computed using rational arithmetic up to the constant c_A using modular symbols (rationality of $L(A,1)/\Omega(A)$ was already known).
4. Can be used to check if $L(A,1) = 0$.
5. Can get information on the order of $Ш(A)$ predicted by the Birch and Swinnerton-Dyer conjectural formula

$$\frac{L(A,1)}{\Omega(A)} \stackrel{?}{=} \frac{\#Ш(A) \cdot \prod_{p|N} c_p(A)}{\#A(\mathbf{Q}) \cdot \#A^\vee(\mathbf{Q})},$$

up to knowledge of c_A (Stein).
6. We have good control on the primes dividing c_A:

Theorem 4 (Mazur, Stein).
If p is a prime that divides c_A, then either $p^2 \mid N$, or $p = 2$.

Some theoretical evidence for the Birch and Swinnerton-Dyer conjectural formula

Corollary 5 (AA, Stein). *Suppose* $L(A,1) \neq 0$*. Let* x *be the image of* $(0) - (\infty) \in J_0(N)(\mathbf{Q})$ *in* $A(\mathbf{Q})$ *and let* n *be the order of* x *in* $A(\mathbf{Q})$*. Then*

$$c_\infty(A) \cdot c_A \cdot \frac{L(A,1)}{\Omega(A)} \in \frac{1}{n}\mathbf{Z}.$$

In particular,

$$c_\infty(A) \cdot c_A \cdot \#A(\mathbf{Q}) \cdot \#A^\vee(\mathbf{Q}) \cdot \frac{L(A,1)}{\Omega(A)} \in \mathbf{Z}.$$

Remarks:

1. The Birch and Swinnerton-Dyer conjecture predicts:

$$\#A(\mathbf{Q}) \cdot \#A^\vee(\mathbf{Q}) \cdot \frac{L(A,1)}{\Omega(A)} \in \mathbf{Z}.$$

2. $c_\infty(A)$ is a power of 2, and we have control on the primes dividing c_A (in particular, if N is a prime, c_A is a power of 2).

Computational evidence for the Birch and Swinnerton-Dyer conjectural formula

Proposition 6. *Suppose $L(A,1) \neq 0$.*
Let p be a prime that does not divide the degree of the canonical polarization of $A^\vee$.
Then the order of the p-primary part of $Ш(A)$ is a perfect square.

Theorem 7 (Mazur, Stein).
Suppose B is another Shimura quotient of $J_0(N)$ of Mordell-Weil rank > 0.
Let p be a prime such that $B^\vee[p] \subseteq A^\vee (\subseteq J_0(N)^\vee)$.
Then, under certain mild conditions,
there is a non-trivial element of order p in $Ш(A^\vee)$.
Hence, if $L(A,1) \neq 0$, then $p | \# Ш(A)$.

Prime levels with $L(A,1) \neq 0$

and $\#Ш_{\mathrm{an}}(A) > 1$ (Calculated by William Stein)

$\#Ш_{\mathrm{an}}(A) =$ order of the Shafarevich-Tate group as predicted by the BSD formula.
Warning: only odd parts of the invariants are shown.

A	$\#Ш_{\mathrm{an}}(A)$	$\sqrt{\deg(\phi_A)}$	**B**
389E	5^2	5	**389A**
433D	7^2	$3 \cdot 7 \cdot 37$	**433A**
563E	13^2	13	**563A**
571D	3^2	$3^2 \cdot 127$	**571B**
709C	11^2	11	**709A**
997H	3^4	3^2	**997B**
1061D	151^2	$61 \cdot 151 \cdot 179$	**1061B**
1091C	7^2	1	NONE
1171D	11^2	$3^4 \cdot 11$	**1171A**
1283C	5^2	$5 \cdot 41 \cdot 59$	NONE
...			
2333C	83341^2	83341	**2333A**

$\deg(\phi_A) =$ degree of canonical polarization of $A^\vee$.

Example 8. $7 \nmid \deg(\phi_{1091C}) \Rightarrow$ *highest power of 7 dividing $\#Ш(1091C)$ has even exponent.*

$B =$ an optimal quotient of $J_0(N)$ such that $L(B,1) = 0$ and if an odd prime p divides $\#Ш_{\mathrm{an}}(A)$, then $B^\vee[p] \subseteq A^\vee$.

Example 9 (Stein). $5^2 \mid \#Ш_{389E}$.