A generalization of Kronecker’s first limit formula

Amod Agashe

Abstract

Kronecker’s first limit formula gives the polar and constant terms of the Laurent series expansion of the Eisenstein series for SL(2, Z) at s = 1. In this article, we generalize the formula to certain maximal parabolic Eisenstein series associated to SL(n, Z) for n ≥ 2.

1 Introduction and results

Let n ≥ 2 be an integer. Let τ be in the generalized upper half-plane \( \mathfrak{H}^n \), which consists of \( n \times n \) matrices with real number entries that are the product of an upper triangular matrix with 1’s along the diagonal and a diagonal matrix with positive diagonal entries such that the lowermost diagonal entry is 1. When \( n = 2 \), one can identify \( \mathfrak{H}^n \) with the usual complex upper half plane. For details, see, e.g., [Gol06, §1.2].

In the following, \( m_1, \ldots, m_n \) denote integers with perhaps some added restrictions as noted; in particular, we follow the convention that in any sum over a subset of \( m_1, \ldots, m_n \), if a term has denominator zero for some values of \( m_1, \ldots, m_n \), then the term is to be skipped in the sum.

Consider the maximal parabolic Eisenstein series

\[
E_n(\tau, s) = \sum_{(m_1, \ldots, m_n) = 1} \frac{(\det \tau)^s}{\| (m_1 \ldots m_n) \tau \|^{ns/2}},
\]

where \( \| (m_1 \ldots m_n) \tau \| \) denotes the norm of the row vector that is the product of the row vector \( (m_1 \ldots m_n) \) and the matrix \( \tau \).

Let

\[
E'_n(\tau, s) = \zeta(ns) E(\tau, s) = \sum_{m_1, \ldots, m_n} \frac{(\det \tau)^s}{\| (m_1 \ldots m_n) \tau \|^{ns/2}}.
\]

Note that in the case where \( n = 2 \), if \( \tau \) corresponds to the point \( z = x + iy \) in the complex upper half plane, then

\[
E'_2(\tau, s) = \sum_{m_1, m_2} y^s |m_1 z + m_2|^{2s},
\]

which is often denoted by the same symbol without the prime superscript (see, e.g., [Lan87, §20.4]).

The classical Kronecker’s first limit formula gives the first two terms of the Laurent expansion of \( E'_2(\tau, s) \) at \( s = 1 \) (see, e.g., [Lan87, §20.4]):

\[
E'_2(\tau, s) = \pi \left( \frac{1}{s-1} + (2\gamma - \log 4 - \log y - 4 \log |\eta(z)|) + O(s-1) \right),
\]

\[\text{(1)}\]

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where \( \eta(z) \) is the Dedekind eta-function. Note that sometimes Kronecker’s first limit formula is stated for Dedekind zeta functions or for Epstein zeta functions, but such formulas can be deduced from the formula above.

Kronecker’s first limit formula was generalized to the case \( n = 3 \) in [BG84] and [Efr92]. In this article, we generalize the formula to arbitrary \( n \geq 2 \) (see Theorem 1.1). A generalization of Kronecker’s first limit formula to arbitrary \( n \geq 2 \) is also given in [Ter73] for Epstein zeta functions.

We also consider the function

\[
E^*_n(\tau, s) = \pi^{-ns/2} \Gamma(ns/2) \zeta(ns) E(\tau, s) = \pi^{-ns/2} \Gamma(ns/2) \sum_{m_1, \ldots, m_n} \frac{(\det \tau)^s}{\| (m_1 \ldots m_n) \tau \|^{ns/2}}.
\]

(2)

Let \( y_1, \ldots, y_{n-1} \) denote the unique positive real numbers such that for \( i \geq 1 \), we have \( \tau_{n-i,n-i} = \prod_{j=1}^{i-1} y_j \).

**Theorem 1.1.** If \( m_1, \ldots, m_n \) are integers, then for \( j = 1, \ldots, n \), let \( b_j = \sum_{i=1, \ldots, j-1} m_i \tau_{i,j} \) and \( c_j = \tau_{j,j} \); also let \( m \) be the nonnegative real number such that \( m^2 = m_1^2/c_1^2 + m_2^2/c_2^2 + \cdots + m_n^2/c_n^2 \) and \( d = b_n m_n/c_n + b_{n-1} m_{n-1}/c_{n-1} + \cdots + b_2 m_2/c_2 \). Let \( \tau' \) be \( \tau \) with toppmost row and leftmost column removed and let

\[
g(\tau) = \exp \left\{ -\frac{1}{4} \left( \prod_{i=1}^{n-1} y_i^{\frac{1}{2} \tau_{i,i}} \right) E^*_n \left( \frac{\tau'}{n-1}, \frac{n}{n-1} \right) + \sum_{m_1 \neq 0} \frac{1}{m_1} \sum_{\left( m_{2}, \ldots, m_{n} \right) \neq (0, \ldots, 0)} \exp \left( 2\pi id - 2\pi |m_1| m \prod_{i=1}^{n-1} y_i \right) \right\}.
\]

Then

\[
E^*_n(\tau, s) = \frac{2/n}{s-1} + \left( \gamma - \log 4\pi - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i^\tau \right) - 4 \log g(\tau) \right) + O(s-1), \quad \text{and}
\]

\[
E'_n(\tau, s) = \pi \left( \frac{2/n}{s-1} + \left( 2\gamma - \log 4 - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i^\tau \right) - 4 \log g(\tau) \right) + O(s-1) \right).
\]

If we put \( n = 2 \) in the formula above for \( E'_n(\tau, s) \), then we get the classical Kronecker’s first limit formula (1), with \( g(\tau) = |\eta(\tau)| \) (this last equality follows from the correctness of our formula, but also because our proof is identical to the classical proof in the case \( n = 2 \) as given in [Lan87, §20.4]). Thus our function \( g(\tau) \) for arbitrary \( n \) is a generalization of \(|\eta(\tau)|\). If we put \( n = 3 \), then we recover the formula for \( E^*_n(\tau, s) \) given in [Efr92, Theorem 1].

We suspect that the expression for \( E^*_n(\tau, s) \) above can be used to show that \( g(\tau) \) is automorphic and that \( \log(g(\tau)) \) is a harmonic function on \( \mathbb{H}^n \) (i.e., is annihilated by invariant differential operators), as is done for \( n = 2 \) (see, e.g., [Sie80, §I.2] for automorphy) and \( n = 3 \) (see [Efr92, §3]). This may be the approach to answer the question raised at the end of §1 and §3 in [Ter73]. However, we shall not pursue these issues in the present article.

The rest of this article, which consists of just one more section, is devoted to the proof of Theorem 1.1. The methods used are very elementary: one only needs some basic Calculus and the Poisson summation formula, which is recalled in equation (10). Our proof is a generalization of the proof of the classical Kronecker’s first limit formula given in [Lan87, §20.4], and the key observation is to perform the sum in (2) over \( m_1, \ldots, m_n \) conveniently (we first sum over \( \{m_1\} \) and then over \( \{m_2, \ldots, m_n\} \); when \( n = 2 \), there is not much of a choice) and to apply the Poisson summation formula in the correct order (over \( m_n \) first, followed by \( m_{n-1} \), and so on, up to \( m_2 \)).
Even in the case $n = 3$, our proof differs in some key steps with that in [Efr92] (who introduces complex coordinates on $\mathbb{H}^3$, while we don’t) and that in [BG84] (who use minimal parabolic Eisenstein series). Our proof techniques are similar to those used in [Ter73] (about which we learned only after a first draft of this article was written), but are more direct and elementary (the main goal of [Ter73] is to prove the functional equation of the Epstein zeta function using generalizations of the Selberg-Chowla formula).

2 Proof of Theorem 1.1

We first prove the formula for $E_n'(\tau, s)$ and deduce from it the formula for $E_n''(\tau, s)$. In formula (2), the term corresponding to $m_1 = 0$ is

$$S_1 = \pi^{-ns/2} \Gamma(ns/2) \sum_{m_2, \ldots, m_n} \frac{(\prod_{i=1}^{n-1} y_i^{n-i}s)}{\| (m_2 \ldots m_n) \tau' \|^{|ns/2|}}$$

$$= \pi^{-ns/2} \Gamma(ns/2) \cdot \left( \prod_{i=1}^{n-1} y_i^{(n-i)} \right) \cdot \sum_{m_2, \ldots, m_n} \frac{(\prod_{i=1}^{n-2} y_i^{n-i-1})(\frac{n-s}{n-1})}{\| (m_2 \ldots m_n) \tau' \|^{|n-1)(\frac{n-s}{n-1})/2|}}$$

$$= \left( \prod_{i=1}^{n-1} y_i^{(n-i)} \right) \cdot \pi^{-ns/2} \Gamma(ns/2) \cdot E_{n-1} \left( \tau', \frac{n}{n-1} \right)$$

$$= \left( \prod_{i=1}^{n-1} y_i^{(n-i)} \right) \cdot \pi^{-(n-1)(\frac{n-s}{n-1})/2} \Gamma \left( (n-1) \left( \frac{n}{n-1} \right) / 2 \right) \cdot E_{n-1} \left( \tau', \frac{n}{n-1} \right)$$

$$= \left( \prod_{i=1}^{n-1} y_i^{(n-i)} \right) \cdot E_{n-1}^* \left( \tau', \frac{n}{n-1} \right). \tag{3}$$

Let

$$S_2 = \sum_{m_1 \neq 0} \sum_{m_2, \ldots, m_n} \frac{\pi^{-ns/2} \Gamma(ns/2)}{\| (m_1 \ldots m_n) \tau \|^{|ns/2|}},$$

so that

$$E_n''(\tau, s) = S_1 + \left( \prod_{i=1}^{n-1} y_i^{n-i} \right) \cdot S_2. \tag{4}$$

Our next goal is to find a suitable expression for $S_2$, which will not be achieved till equation (15) below. We use the formula

$$\frac{\pi^{-s} \Gamma(s)}{a^s} = \int_0^\infty \exp(-\pi at)t^s dt \tag{5}$$

with $a = \| (m_1 \ldots m_n) \tau \|$, and $s$ replaced by $ns/2$ to get

$$S_2 = \sum_{m_1 \neq 0} \sum_{m_2, \ldots, m_n} \int_0^\infty \exp(-\pi t \| (m_1 \ldots m_n) \tau \|) t^{ns/2} dt \tag{6}$$

Now

$$\| (m_1 \ldots m_n) \tau \| = (m_1 \tau_{1,1})^2 + \cdots + \left( \sum_{i=1, \ldots, n-1} m_i \tau_{i,n-1} \right)^2 + \left( \sum_{i=1, \ldots, n-1} m_i \tau_{i,n} + m_n \tau_{n,n} \right)^2$$

$$= a_n + (b_n + c_n m_n)^2, \tag{7}$$
where
\[ a_n = (m_1 \tau_{1,1})^2 + \cdots + \left( \sum_{i=1,\ldots,n-1} m_i \tau_{i,n-1} \right)^2, \] (8)
\[ b_n = \sum_{i=1,\ldots,n-1} m_i \tau_{i,n}, \text{ and } c_n = \tau_{n,n}. \] Putting (7) in (6), we get
\[ S_2 = \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_{n-1}} \int_0^\infty \exp(-\pi t a_n) \exp(-\pi t (b_n + c_n m_n)^2) \frac{t^{ns/2} dt}{t} \]
\[ = \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_{n-1}} \int_0^\infty \exp(-\pi t a_n) \exp(-\pi t (b_n + c_n m_n)^2) \frac{t^{ns/2} dt}{t} \] (9)
The Poisson summation formula says that for real numbers \( t, b, c, \) with \( c \neq 0, \)
\[ \sum_{m \in \mathbb{Z}} \exp(-\pi t (b + cm)^2) = \frac{1}{c\sqrt{t}} \sum_{m \in \mathbb{Z}} \exp(2\pi i bm/c) \exp(-\pi m^2/tc^2). \] (10)
Using this with \( m = m_n, \) \( b = b_n, \) and \( c = c_n, \) and noting that \( c_n, \) being a diagonal entry of \( \tau, \) is always positive, we get
\[ \sum_{m_n} \exp(-\pi t (b_n + c_n m_n)^2) = \frac{1}{c_n\sqrt{t}} \sum_{m_n} \exp(-\pi t a_n) \sum_{m_n} \exp(2\pi i b_m m_n/c_n) \exp(-\pi m_n^2/tc_n^2). \]
Putting this in (9), we get
\[ S_2 = \frac{1}{c_n} \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_{n-1}} \int_0^\infty \exp(-\pi t a_n) \sum_{m_n} \exp(2\pi i b_m m_n/c_n) \exp(-\pi m_n^2/tc_n^2) t^{ns/2-1/2} \frac{dt}{t} \]
\[ = \frac{1}{c_n} \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_{n-1}} \exp(2\pi i b_m m_n/c_n) \int_0^\infty \exp(-\pi a_n t + \pi m_n^2/tc_n^2) t^{ns/2-1/2} \frac{dt}{t} \]
\[ = \frac{1}{c_n} \sum_{m_1 \neq 0} \exp(2\pi i b_m m_n/c_n) \]
\[ \cdot \sum_{m_2,\ldots,m_{n-2}} \sum_{m_{n-1}} \exp(-\pi a_n t) t^{ns/2-1/2} \frac{dt}{t} \] (11)
Now from equation (8),
\[ a_n = (m_1 \tau_{1,1})^2 + \cdots + \left( \sum_{i=1,\ldots,n-2} m_i \tau_{i,n-2} \right)^2 + \left( \sum_{i=1,\ldots,n-2} m_i \tau_{i,n-1} + m_{n-1} \tau_{n-1,n-1} \right)^2, \]
\[ = a_{n-1} + (b_{n-1} + c_{n-1} m_n)^2, \]
where \( a_{n-1} = (m_1 \tau_{1,1})^2 + \cdots + \left( \sum_{i=1,\ldots,n-2} m_i \tau_{i,n-2} \right)^2, \)
\( b_{n-1} = \sum_{i=1,\ldots,n-2} m_i \tau_{i,n-1}, \) and \( c_{n-1} = \tau_{n-1,n-1}. \) So using formula (10), noting that \( c_{n-1}, \) being a diagonal entry of \( \tau, \) is always positive, we have
\[ \sum_{m_{n-1}} \exp(-\pi a_n t) \]
\[ = \exp(-\pi t a_{n-1}) \sum_{m_{n-1}} \exp(-\pi t (b_{n-1} + c_{n-1} m_{n-1})^2) \]
\[ = \frac{1}{c_{n-1}\sqrt{t}} \exp(-\pi t a_{n-1}) \sum_{m_{n-1}} \exp(2\pi i b_m m_{n-1}/c_{n-1}) \exp(-\pi m_{n-1}^2/tc_{n-1}^2). \]
Putting this in (11),

\[ S_2 = \frac{1}{c_{n-1}c_n} \sum_{m_1 \neq 0,m_n} \exp(2\pi i b_n m_n/c_n) \int_0^\infty \exp(-\pi m_n^2/tc_n^2) \sum_{m_2,\ldots,m_{n-2}} \frac{1}{\sqrt{t}} \exp(-\pi t\alpha_{n-1}) \]

\[ \cdot \sum_{m_{n-1}} \exp(2\pi i (b_n m_n/c_n + b_{n-1} m_{n-1}/c_{n-1})) \int_0^\infty \exp(-\pi (m_n^2/c_n^2 + m_{n-1}^2/c_{n-1}^2)/t) \sum_{m_2,\ldots,m_{n-2}} \exp(-\pi t\alpha_{n-1}) t^{ns/2-2/2} \frac{dt}{t} \]

\[ = \frac{1}{c_{n-1}c_n} \sum_{m_1 \neq 0,m_n,m_{n-1}} \exp(2\pi i (b_n m_n/c_n + b_{n-1} m_{n-1}/c_{n-1})) \]

\[ \cdot \int_0^\infty \exp(-\pi (m_n^2/c_n^2 + m_{n-1}^2/c_{n-1}^2)/t) \sum_{m_2,\ldots,m_{n-2}} \sum_{m_{n-3}} \exp(-\pi t\alpha_{n-1}) t^{ns/2-2/2} \frac{dt}{t} \]  

(12)

Looking at equations (11) and (12), we see that repeated use of Poisson summation gives

\[ S_2 = \frac{1}{\prod_{i=2}^n \Gamma(c_i)} \sum_{m_1 \neq 0,m_2,\ldots,m_n} \exp(2\pi i d) \int_0^\infty \exp(-\pi m_n^2/t) \exp(-\pi t\alpha_1) t^{ns/2-(n-1)/2} \frac{dt}{t}. \]

Now \( a_1 = m_2 y^2 \) where \( y = \tau_{1,1} = y_1 y_2 \cdots y_{n-1} \). So

\[ \left( \prod_{i=2}^n c_i \right) S_2 = \sum_{m_1 \neq 0,m_2,\ldots,m_n} \exp(2\pi i d) \int_0^\infty \exp(-\pi (m_1 y)^2 t + \pi m_n^2/t) t^{ns/2+1/2} \frac{dt}{t}. \]

If \( m_2 = \cdots = m_n = 0 \), then the corresponding term becomes

\[ S_2' = \sum_{m_1 \neq 0} \int_0^\infty \exp(-\pi (m_1 y)^2 t) t^{n(s-1)/2+1/2} \frac{dt}{t}, \]

which, using formula (5), with \( s \) replaced by \( n(s-1)/2 + 1/2 \) becomes

\[ S_2' = \sum_{m_1 \neq 0} \frac{\pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2)}{(m_1 y)^{2(n(s-1)/2+1/2)}} \]

\[ = y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2) \cdot 2 \sum_{m_1 \geq 0} \frac{1}{m_1^{n(s-1)+1}} \]

\[ = 2y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2) \zeta(n(s-1) + 1) \]  

(13)

Let

\[ S_2'' = \left( \prod_{i=2}^n c_i \right) S_2 - S_2' \]

(14)

\[ = \sum_{m_1 \neq 0} \sum_{(m_2,\ldots,m_n) \neq (0,\ldots,0)} \exp(2\pi i d) \int_0^\infty \exp(-\pi (m_1 y)^2 t + \pi m_n^2/t) t^{n(s-1)/2+1/2} \frac{dt}{t}. \]
For $a$ and $b$ positive real numbers, recall the function

$$K_s(a, b) = \int_0^\infty \exp(-(a^2 t + b^2/t)) t^{s-1} dt$$

Noting that $m \neq 0$ if not all $m_2, \ldots, m_n$ are zero,

$$S_2'' = \sum_{m_1 \neq 0} \sum_{(m_2, \ldots, m_n) \neq (0, \ldots, 0)} \exp(2\pi id) K_n(s-1)/2+1/2(\sqrt{\pi}|m_1|y, \sqrt{\pi}|m|)$$

From equations (3), (4), (13), (14), and (15), we finally get an expression for $E_n^*(\tau, s)$:

$$E_n^*(\tau, s) = \left( \prod_{i=1}^{n-1} y_i \left( \frac{i-1}{n-1} \right) \right)^s E_{n-1}^* \left( \tau', \frac{n}{n-1}s \right)$$

$$+ 2 \left( \prod_{i=1}^{n-1} y_i^{-i} \right)^s \prod_{i=2}^n c_i \sum_{m_1 \neq 0} \sum_{(m_2, \ldots, m_n) \neq (0, \ldots, 0)} \exp(2\pi id) K_n(s-1)/2+1/2(\sqrt{\pi}|m_1|y, \sqrt{\pi}|m|)$$

The good thing about the formula above is that it is easy to read off the polar part and the constant term in each of the summands above, which is what we do now. It is known that $K_s$ is an entire function of $s$, and so all the functions appearing in the expression above are holomorphic at $s = 1$ except $\zeta(2s-1)$, which has a simple pole at $s = 1$, and perhaps $E^*_{n-1} \left( \tau', \frac{n}{n-1}s \right)$. By induction, $E^*_{n-1} \left( \tau', \frac{n}{n-1}s \right)$ is also holomorphic except perhaps when $\left( \frac{n}{n-1} \right) s = 1$, and in particular is homomorphic at $s = 1$. So the first and last summands on the right side of equation (16) are holomorphic at $s = 1$; using the fact that $K_{1/2}(a, b) = \frac{\sqrt{\pi} a}{\exp(-2\pi ab)}$, their sum is

$$\left( \prod_{i=1}^{n-1} y_i^{-i} \right)^s E_n^* \left( \tau', \frac{n}{n-1} \right) + \sum_{m_1 \neq 0} \sum_{(m_2, \ldots, m_n) \neq (0, \ldots, 0)} \exp(2\pi id) \frac{1}{|m_1|} \exp(-2\pi|m_1||m|y) + O(s-1),$$

which is $-4 \log g(\tau) + O(s-1)$.

In order to deal with the second summand on the right side of equation (16), note that

$$\zeta(n(s-1) + 1) = \frac{1}{n(s-1)} + \gamma + O(s-1),$$

$$\Gamma(n(s-1)/2 + 1/2) = \sqrt{\pi} \left( 1 + \frac{n}{2} (\gamma - \log 4)(s-1) + O(s-1)^2 \right),$$

$$\pi^{- \left( n(s-1)/2 + 1/2 \right)} = \frac{1}{\sqrt{\pi}} \left( 1 - \frac{n}{2} \log \pi (s-1) + O(s-1)^2 \right),$$

$$y^{- (n(s-1) + 1)} = y^{-1} \left( 1 - n \log y(s-1) + O(s-1)^2 \right),$$

and

$$\left( \prod_{i=1}^{n-1} y_i^{-i} \right)^s = \left( \prod_{i=1}^{n-1} y_i^{-i} \right)^{(s-1)+1} = \left( \prod_{i=1}^{n-1} y_i^{-i} \right) \left( 1 + \log \left( \prod_{i=1}^{n-1} y_i^{-i} \right) \right) (s-1) + O(s-1)^2).$$
Using the formulas above, the second summand in on the right side of equation (16) becomes

\[ \frac{2/n}{s-1} + \left( \gamma - \log 4\pi - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i \right) \right) + O(s-1) \]

Using the formulas obtained above for the three summands on the right side of equation (16), we get the desired formula for \( E^*_n(\tau, s) \).

Now

\[ E'_n(\tau, s) = \pi^s \Gamma(s)^{-1} E^*_n(\tau, s), \]
\[ \pi^s = \pi(1 + \log \pi(s-1) + O(s-1)^2), \text{ and} \]
\[ \Gamma(s)^{-1} = (1 + \gamma(s-1) + O(s-1)^2). \]

Using the equations above and the formula for \( E^*_n(\tau, s) \) gives the desired formula for \( E'_n(\tau, s) \).

References


