The Birch and Swinnerton-Dyer formula for modular abelian varieties *

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*These transperencies can be obtained from http://www.math.berkeley.edu/~amod/mymath.html

The Birch and Swinnerton-Dyer conjectural formula

$$\begin{array}{rcl} J_0(N) & = & \text{Jacobian of the modular curve } X_0(N) \\ \mathbf{T} & = & \text{Hecke algebra} \\ f & = & \text{a newform s.t. } L(f,1) \neq 0 \\ I_f & = & \mathrm{Ann}_{\mathbf{T}} f\text{, an ideal of } \mathbf{T} \\ A = A_f & = & J_0(N)/I_f J_0(N), \\ & & \text{the Shimura quotient associated to } f \end{array}$$

Conjecture 1 (Birch, Swinnerton-Dyer, Tate, Gross).

$$\frac{L(A,1)}{\Omega(A)} = \frac{\#Ш(A) \cdot \prod_{p \mid N} c_p(A)}{\#A(\mathbf{Q}) \cdot \#A^{\vee}(\mathbf{Q})},$$

where

$$\begin{array}{rcl} \Omega(A) & = & \textit{real volume of } A \textit{ w.r.t.} \\ & & \textit{Néron differentials} \\ Ш(A) & = & \textit{Shafarevich-Tate group of } A \\ c_p(A) & = & \textit{order of the arithmetic component} \\ & & \textit{group of } A \textit{ at } p \end{array}$$

A formula for $L(A,1)/\Omega(A)$

$f_1, f_2, \ldots, f_d =$ Galois conjugates of f

$$\begin{array}{rcll} \Phi : H_1(X_0(N),\mathbf{Q})^+ & \to & \mathbf{C}^d & \text{is given by} \\ \gamma & \mapsto & \{\int_\gamma f_1, \ldots, \int_\gamma f_d\} & \end{array}$$

Theorem 2 (Conjectured by Stein).

$$\frac{L_A(1)}{\Omega_A} = \frac{[\Phi(H_1(X_0(N),\mathbf{Z})^+) : \Phi(\mathbf{T}e)]}{c_A \cdot c_\infty(A)},$$

where

$e \in H_1(X_0(N),\mathbf{Q})^+$ *is the winding element*

$[\Phi(H_1(X_0(N),\mathbf{Z})^+) : \Phi(\mathbf{T}e)] =$ *lattice index*

$$\begin{array}{rcl} c_A & = & \text{the (generalized) Manin constant} \\ & = & \text{index of the subgroup of} \\ & & \text{Néron differentials in } S_2(\Gamma_0(N),\mathbf{Z})[I_f] \\ c_\infty(A) & = & [A(\mathbf{R}) : A^0(\mathbf{R})] \end{array}$$

Remark 3. The right-hand side of the formula above is a rational number and can be computed up to the constant c_A using modular symbols; thus one can extend some of the BSD calculations of Cremona for elliptic curves to higher dimensional quotients.

The Manin constant

Theorem 4 (Mazur, Stein, Abbes, Ullmo, A. A.).
If p is a prime s.t. $p \mid c_A$, then
either $p^2 \mid N$,
or $p = 2$ and 2 divides the order of the kernel of the composite $A^\vee \hookrightarrow J_0(N) \twoheadrightarrow A$.

$$\begin{array}{rcl} I & = & \text{an ideal of } \mathbf{T} \text{ s.t. } \mathbf{T}/I \text{ is torsion-free} \\ B & = & J_0(N)/IJ_0(N) \\ \mathcal{B} & = & \text{Néron model of } B \text{ over } \mathbf{Z} \end{array}$$

Definition 5.
The (generalized) Manin constant is the order of the cokernel of the map $H^0(\mathcal{B}, \Omega_{\mathcal{B}/\mathbf{Z}}) \to S_2(\Gamma_0(N), \mathbf{Z})[I]$.

Conjecture 6 (Stein, A. A.). *If B is an optimal quotient of the new quotient of $J_0(N)$, then its generalized Manin constant is 1.*

Remark 7. The hypothesis above is necessary: $c_{J_0(33)} = 3$ (Edixhoven).

Visible elements of the Shafarevich-Tate group

$C =$ an abelian subvariety of $J_0(N)$

Definition 8 (Mazur).
The visible part of $\text{Ш}(C)$ *is the kernel of the map* $\text{Ш}(C) \to \text{Ш}(J_0(N))$*.*

Theorem 9 (Mazur, Stein).
Let C *be an abelian subvariety of* $J_0(N)$ *of rank* 0*.*
Suppose D *is another abelian subvariety of* $J_0(N)$ *of rank* > 0*.*
Let p *be a prime s.t.* $D[p] \subseteq C$*.*
Then, under certain mild conditions, p *divides the order of the visible part of* $\text{Ш}(C)$*.*

Let B be a quotient of the new part of $J_0(N)$. Consider the composite $B^\vee \hookrightarrow J_0(N) \twoheadrightarrow B$, which is an isogeny; call it ϕ_B.

Lemma 10 (Mazur, A. A.).
Every visible element of $\text{Ш}(B^\vee)$ *is killed by multiplication by the exponent of* $\ker \phi_B$*.*

Prime levels with $\#Ш_{\mathrm{an}}(A_f) > 1$

(Calculated by William Stein)
Warning: only odd parts of the invariants are shown.

$\mathbf{A_f}$	$\#Ш_{\mathrm{an}}(A_f)$	$\sqrt{deg(\phi_{A_f})}$	$\mathbf{B_g}$
389E	5^2	5	**389A**
433D	7^2	$3 \cdot 7 \cdot 37$	**433A**
563E	13^2	13	**563A**
571D	3^2	$3^2 \cdot 127$	**571B**
709C	11^2	11	**709A**
997H	3^4	3^2	**997B**
1061D	151^2	$61 \cdot 151 \cdot 179$	**1061B**
1091C	7^2	1	NONE
1171D	11^2	$3^4 \cdot 11$	**1171A**
1283C	5^2	$5 \cdot 41 \cdot 59$	NONE
...			
2111B	211^2	1	NONE
...			
2333C	83341^2	83341	**2333A**
...			

$\#Ш_{\mathrm{an}}(A_f) =$ order of the Shafarevich-Tate group as predicted by the BSD formula.

$B_g =$ an optimal quotient of $J_0(N)$ of positive analytic rank s.t. if an odd prime p divides $\#Ш_{\mathrm{an}}(A_f)$, then $B_g^\vee[p] \subseteq A_f^\vee$.

Example 11 (Stein). $5^2 \mid \#Ш_{389E}$.

Congruence primes and the modular degree

$E =$ strong modular elliptic curve of level N
$\phi_E : X_0(N) \to E$ is its modular parametrization.
Modular degree of $E = \deg\phi_E$
Congruence number of $E =$
$r_E =$largest integer r s.t. $\exists$ a modular form g with integral Fourier coefficients, orthogonal to f w.r.t. Petersson inner product, and satisfying $g \equiv f \mod r$.

Theorem 12 (Ribet[Zagier]). $\deg\phi_E | r_E$*; moreover, if* N *is prime, then* $\deg\phi_E = r_E$*. If* p *is a prime s.t.* $p | \frac{r_E}{\deg\phi_E}$*, then* $p | N$*.*

Question 13 (Frey, Muller). Is $\deg\phi_E = r_E$?

Answer: No.
E.g. (Stein), when $E = 54B1$, $\deg\phi_E = 2$ and $3 | r_E$.

Question 14. If p is a prime such that $p^2 \nmid N$, then is it true that $p \nmid \frac{r_E}{\deg\phi_E}$?

Theorem 15. *If* p *is a prime such that* $p^2 \nmid N$*, and* $p | \frac{r_E}{\deg\phi_E}$*, then* $p | \deg\phi_E$*.*

Corollary 16. *If* $p^2 \nmid N$ *and* $p | r_E$*, then* $p | \deg\phi_E$*.*