Visibility of Shafarevich-Tate Groups of Abelian Varieties

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We investigate Mazur's notion of visibility of elements of Shafarevich-Tate groups of abelian varieties. We give a proof that every cohomology class is visible in a suitable abelian variety, discuss the visibility dimension, and describe a construction of visible elements of certain Shafarevich-Tate groups. This construction can be used to give some of the first evidence for the Birch and Swinnerton-Dyer Conjecture for abelian varieties of large dimension. We then give examples of visible and invisible Shafarevich-Tate groups.

Key Words: Visibility, Shafarevich-Tate Group, Birch and Swinnerton-Dyer Conjecture, Modular Abelian Variety

INTRODUCTION

If a genus 0 curve $X$ over $\mathbb{Q}$ has a point in every local field $\mathbb{Q}_p$ and in $\mathbb{R}$, then it has a global point over $\mathbb{Q}$. For genus 1 curves, this "local-to-global principle" frequently fails. For example, the nonsingular projective curve defined by the equation $3x^3 + 4y^2 + 5z^3 = 0$ has a point over each local field and $\mathbb{R}$, but has no $\mathbb{Q}$-point. The Shafarevich-Tate group of an elliptic curve $E$, denoted $\Sha(E)$, is a group that measures the extent to which a local-to-global principle fails for the genus one curves with Jacobian $E$. More generally, if $A$ is an abelian variety over a number field $K$, then the elements of the Shafarevich-Tate group $\Sha(A)$ of $A$ correspond to the torsors for $A$ that have a point everywhere locally, but not globally. In this paper, we study a geometric way of realizing (or "visualizing") torsors corresponding to elements of $\Sha(A)$. 
Let $A$ be an abelian variety over a field $K$. If $i : A \hookrightarrow J$ is a closed immersion of abelian varieties, then the subgroup of $H^1(K, A)$ visible in $J$ (with respect to $i$) is $\ker(H^1(K, A) \rightarrow H^1(K, J))$. We prove that every element of $H^1(K, A)$ is visible in some abelian variety, and give bounds on the smallest size of an abelian variety in which an element of $H^1(K, A)$ is visible. Next assume that $K$ is a number field. We give a construction of visible elements of $\Sha(A)$, which we demonstrate by giving evidence for the Birch and Swinnerton-Dyer conjecture for a certain 20-dimensional abelian variety. We also give an example of an elliptic curve $E$ over $\mathbb{Q}$ of conductor $N$ whose Shafarevich-Tate group is not visible in $J_0(N)$ but is visible in $J_0(Np)$ for some prime $p$.

This paper is organized as follows. Section 1 contains the definition of visibility for cohomology classes and elements of Shafarevich-Tate groups. Then in Section 1.3, we use a restriction of scalars construction to prove that every cohomology class is visible in some abelian variety. Next, in Section 2, we investigate the visibility dimension of cohomology classes. Section 3 contains a theorem that can be used to construct visible elements of Shafarevich-Tate groups. The final section, Section 4, contains examples and applications of our visibility results in the context of modular abelian varieties.

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1. VISIBILITY

In Section 1.1 we introduce visible cohomology classes, then in Section 1.2 we discuss visible elements of Shafarevich-Tate groups. In Section 1.3, we use restriction of scalars to deduce that every cohomology class is visible somewhere.

For a field $K$ and a smooth commutative $K$-group scheme $G$, we write $H^i(K, G)$ to denote the group cohomology $H^i(\text{Gal}(K_s/K), G(K_s))$ where $K_s$ is a fixed separable closure of $K$; equivalently, $H^i(K, G)$ denotes the $i$th étale cohomology of $G$ viewed as an étale sheaf on $\text{Spec}(K)_{\text{ét}}$. 

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1.1. Visible Elements of $H^1(K, A)$

In [Maz99], Mazur introduced the following definition. Let $A$ be an abelian variety over an arbitrary field $K$.

**Definition 1.1.** Let $\iota : A \to J$ be an embedding of $A$ into an abelian variety $J$ over $K$. Then the *visible subgroup of $H^1(K, A)$ with respect to the embedding $\iota$* is

$$\text{Vis}_\iota(H^1(K, A)) = \text{Ker}(H^1(K, A) \to H^1(K, J)).$$

The visible subgroup $\text{Vis}_\iota(H^1(K, A))$ depends on the choice of embedding $\iota$, but we do not include $\iota$ in the notation, as it is usually clear from context.

The Galois cohomology group $H^1(K, A)$ has a geometric interpretation as the group of classes of torsors $X$ for $A$ (see [LT58]). To a cohomology class $c \in H^1(K, A)$, there is a corresponding variety $X$ over $K$ and a map $A \times X \to X$ that satisfies axioms similar to those for a simply transitive group action. The set of equivalence classes of such $X$ forms a group, the Weil-Chatelet group of $A$, which is canonically isomorphic to $H^1(K, A)$.

There is a close relationship between visibility and the geometric interpretation of Galois cohomology. Suppose $\iota : A \to J$ is an embedding and $c \in \text{Vis}_\iota(H^1(K, A))$. We have an exact sequence of abelian varieties $0 \to A \to J \to C \to 0$, where $C = J/A$. A piece of the associated long exact sequence of Galois cohomology is

$$0 \to A(K) \to J(K) \to C(K) \to H^1(K, A) \to H^1(K, J) \to \cdots,$$

so there is an exact sequence

$$0 \to J(K)/A(K) \to C(K) \to \text{Vis}_\iota(H^1(K, A)) \to 0. \quad (1.1)$$

Thus there is a point $x \in C(K)$ that maps to $c$. The fiber $X$ over $x$ is a subvariety of $J$, which, when equipped with its natural action of $A$, lies in the class of torsors corresponding to $c$. This is the origin of the terminology “visible”. Also, we remark that when $K$ is a number field, $\text{Vis}_\iota(H^1(K, A))$ is finite because it is torsion and is the surjective image of the finitely generated group $C(K)$.

1.2. Visible Elements of $\mathcal{V}(A)$

Let $A$ be an abelian variety over a number field $K$. The Shafarevich-Tate group of $A$, which is defined below, measures the failure of the local-to-global principle for certain torsors. The *Shafarevich-Tate group* of $A$ is

$$\mathcal{V}(A) := \text{Ker} \left( H^1(K, A) \to \prod_v H^1(K_v, A) \right),$$

where the product is over all places of $K$.3
Definition 1.2. If \( \iota : A \rightarrow J \) is an embedding, then the visible subgroup of \( \mathfrak{III}(A) \) with respect to \( \iota \) is

\[
\text{Vis}_J(\mathfrak{III}(A)) := \mathfrak{III}(A) \cap \text{Vis}_J(H^1(K, A)) = \ker(\mathfrak{III}(A) \rightarrow \mathfrak{III}(J)).
\]

1.3. Every Element is Visible Somewhere

Proposition 1.3. Every element of \( H^1(K, A) \) is visible in some abelian variety \( J \).

Proof. Fix \( c \in H^1(K, A) \). There is a finite separable extension \( L \) of \( K \) such that \( \text{res}_L(c) = 0 \in H^1(L, A) \). Let \( J = \text{Res}_{L/K}(A_L) \) be the Weil restriction of scalars from \( L \) to \( K \) of the abelian variety \( A_L \) (see [BLR90, §7.6]). Thus \( J \) is an abelian variety over \( K \) of dimension \( [L : K] \cdot \dim(A) \), and for any scheme \( S \) over \( K \), we have a natural (functorial) group isomorphism \( A_L(S_L) \cong J(S) \). The functorial injection \( A(S) \rightarrow A_L(S_L) \cong J(S) \) corresponds via Yoneda's Lemma to a natural \( K \)-group scheme map \( \iota : A \rightarrow J \), and by construction \( \iota \) is a monomorphism. But \( \iota \) is proper and thus is a closed immersion (see [Gro66, §8.11.5]). Using the Shapiro lemma one finds, after a tedious computation, that there is a canonical isomorphism \( H^1(K, J) \cong H^1(L, A) \) which identifies \( \iota_*(c) \) with \( \text{res}_L(c) = 0 \).

Remark 1.4.
1. In [CM00], de Jong gave a totally different proof of the above proposition in the case when \( A \) is an elliptic curve over a number field. His argument actually displays \( A \) as visible inside the Jacobian of a curve.
2. L. Clozel has remarked that the method of proof above is a standard technique in the theory of algebraic groups.

2. THE VISIBILITY DIMENSION

Let \( A \) be an abelian variety over a field \( K \) and fix \( c \in H^1(K, A) \).

Definition 2.1. The visibility dimension of \( c \) is the minimum of the dimensions of the abelian varieties \( J \) such that \( c \) is visible in \( J \).

In Section 2.1 we prove an elementary lemma which, when combined with the proof of Proposition 1.3, gives an upper bound on the visibility dimension of \( c \) in terms of the order of \( c \) and the dimension of \( A \). Then, in Section 2.2, we consider the visibility dimension in the case when \( A = E \) is an elliptic curve. After summarizing the results of Mazur and Klenke on the visibility dimension, we apply a theorem of Cassels to deduce that the visibility dimension of \( c \in \mathfrak{III}(E) \) is at most the order of \( c \).
2.1. A Simple Bound

The following elementary lemma, which the second author learned from Hendrik Lenstra, will be used to give a bound on the visibility dimension in terms of the order of \( c \) and the dimension of \( A \).

**Lemma 2.2.** Let \( G \) be a group, \( M \) be a finite (discrete) \( G \)-module, and \( c \in H^1(G, M) \). Then there is a subgroup \( H \) of \( G \) such that \( \text{res}_H(c) = 0 \) and \( \#(G/H) \leq \#M \).

**Proof.** Let \( f : G \to M \) be a cocycle corresponding to \( c \), so \( f(\tau \sigma) = f(\tau) + \tau f(\sigma) \) for all \( \tau, \sigma \in G \). Let \( H = \ker(f) = \{ \sigma \in G : f(\sigma) = 0 \} \). The map \( \tau H \mapsto f(\tau) \) is a well-defined injection from the coset space \( G/H \) to \( M \).

The following is a general bound on the visibility dimension.

**Proposition 2.3.** The visibility dimension of any \( c \in H^1(K, A) \) is at most \( d \cdot n^{2d} \) where \( n \) is the order of \( c \) and \( d \) is the dimension of \( A \).

**Proof.** The map \( H^1(K, A[n]) \to H^1(K, A)[n] \) is surjective and \( A[n] \) has order \( n^{2d} \), so Lemma 2.2 implies that there is an extension \( L \) of \( K \) of degree at most \( n^{2d} \) such that \( \text{res}_L(c) = 0 \). The proof of Proposition 1.3 implies that \( c \) is visible in an abelian variety of dimension \([L : K] \cdot \dim A \leq dn^{2d} \).

2.2. The Visibility Dimension for Elliptic Curves

We now consider the case when \( A = E \) is an elliptic curve over a number field \( K \). Mazur proved in [Maz99] that every nonzero \( c \in \text{III}(E)[3] \) has visibility dimension 2 (note that Proposition 2.3 only implies that the visibility dimension is \( \leq 3 \)). Mazur’s result is particularly nice because it shows that \( c \) is visible in an abelian variety that is isogenous to the product of two elliptic curves. Using similar techniques, T. Klenke proved in [Kle01] that every nonzero \( c \in H^1(K, E)[2] \) has visibility dimension 2 (note that Proposition 2.3 only implies that the visibility dimension of any \( c \in H^1(K, E)[2] \) is \( \leq 4 \)). It is unknown whether the visibility dimension of every nonzero element of \( H^1(K, E)[3] \) is 2, and it is not known whether elements of \( \text{III}(E)[5] \) must have visibility dimension 2.

When \( c \) lies in \( \text{III}(E) \) we use a classical result of Cassels to strengthen the conclusion of Proposition 2.3.

**Proposition 2.4.** Let \( E \) be an elliptic curve over a number field \( K \) and let \( c \in \text{III}(E) \). Then the visibility dimension of \( c \) is at most the order of \( c \).

**Proof.** Let \( n \) be the order of \( c \). In view of the restriction of scalars construction in the proof of Proposition 1.3, it suffices to show that there is an extension \( L \) of \( K \) of degree \( n \) such that \( \text{res}_L(c) = 0 \). Without the
hypothesis that \( c \) lies in \( III(E) \), such an extension \( L \) might not exist, as Cassels observed in [Cas63]. However, in that same paper, Cassels proved that such an \( L \) exists when \( c \in III(E) \) (see also [O’N01] for another proof).

\[ \]

**Remark 2.5.** In contrast to the case of dimension 1, it seems to be an open problem to determine whether or not elements of \( III(A)[n] \) split over an extension of degree \( n \).

3. CONSTRUCTION OF VISIBLE ELEMENTS

The goal of this section is to state and prove the main result of this paper, which we use to construct visible elements of Shafarevich-Tate groups and sometimes give a nontrivial lower bound for the order of the Shafarevich-Tate group of an abelian variety, thus providing new evidence for the conjecture of Birch and Swinnerton-Dyer (see Section 4.1 and [AS02]). The Tamagawa numbers \( c_{A,v} \) and \( c_{B,v} \) will be defined in Section 3.1 below.

**Theorem 3.1.** Let \( A \) and \( B \) be abelian subvarieties of an abelian variety \( J \) over a number field \( K \) such that \( A \cap B \) is finite. Let \( N \) be an integer divisible by the residue characteristic of primes of bad reduction for \( B \). Suppose \( n \) is an integer such that for each prime \( p \mid n \), we have \( e_p < p - 1 \) where \( e_p \) is the largest ramification of any prime of \( K \) lying over \( p \), and that

\[
\gcd \left( n, N \cdot \#(J/B)(K)_{tor} \cdot \#B(K)_{tor} \cdot \prod_{ \text{all places } v} (c_{A,v} \cdot c_{B,v}) \right) = 1,
\]

where \( c_{A,v} = \#\Phi_{A,v}(\mathbb{F}_v) \) (resp., \( c_{B,v} \)) is the Tamagawa number of \( A \) (resp., \( B \)) at \( v \) (see Section 3.1 for the definition of \( \Phi_{A,v} \)). Suppose furthermore that \( B[n] \subset A \) as subgroup schemes of \( J \). Then there is a natural map

\[
\varphi : B(K)/nB(K) \to \text{Vis}_J(III(A)),
\]

such that \( \ker(\varphi) \subset J(K)/(B(K) + A(K)) \). If \( A(K) \) has rank 0, then \( \ker(\varphi) = 0 \) (more generally, \( \ker(\varphi) \) has order at most \( n^r \) where \( r \) is the rank of \( A(K) \)).

**Remark 3.2.** Mazur has proved similar results for elliptic curves using flat cohomology (unpublished), and discussions with him motivated this theorem.

In Section 3.1 we recall a definition of the Tamagawa numbers of an abelian variety. In Section 3.2 we prove a lemma, which gives a condition under which there is an unramified \( n \)th root of an unramified point. In Section 3.3, we use the snake lemma to produce a map

\[
B(K)/nB(K) \to \text{Vis}_J(H^1(K,A))
\]

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with bounded kernel. Finally, in Section 3.4, we use a local analysis at each place of $K$ to show that the image of the above map lies in $\Pi(A)$.

### 3.1. Tamagawa Numbers

Let $A$ be an abelian variety over a local field $K$ with residue class field $k$, and let $\mathcal{A}$ be the Néron model of $A$ over the ring of integers of $K$. The closed fiber $\mathcal{A}_k$ of $\mathcal{A}$ need not be connected. Let $\mathcal{A}_k^0$ denote the geometric component of $\mathcal{A}$ that contains the identity. The group $\Phi_A = \mathcal{A}_k / \mathcal{A}_k^0$ of connected components is a finite group scheme over $k$. This group scheme is called the component group of $\mathcal{A}$, and the Tamagawa number of $A$ is $c_A = \# \Phi_A(k)$.

Now suppose that $A$ is an abelian variety over a global field $K$. For every place $v$ of $K$, the Tamagawa number of $A$ at $v$, denoted $c_{A,v}$ or just $c_v$, is the Tamagawa number of $A_K$, where $K_v$ is the completion of $K$ at $v$.

### 3.2. Smoothness and Surjectivity

In this section, we recall some well-known lemmas that we will use in Section 3.4 to produce unramified cohomology classes. The authors are grateful to B. Conrad for explaining the proofs of these lemmas.

**Lemma 3.3.** If $G$ is a finite-type smooth commutative group scheme over a strictly henselian local ring $R$ and the fibers of $G$ over $R$ are (geometrically) connected, then the multiplication map $n_G : G(R) \to G(R)$ is surjective when $n \in R^\times$.

**Proof.** Pick an element $g \in G(R)$ and form the cartesian diagram

$$
\begin{array}{ccc}
Y_g & \xrightarrow{\psi} & \text{Spec}(R) \\
\downarrow & & \downarrow \\
G & \xrightarrow{\text{id}} & G \\
\end{array}
$$

We want to prove that $\psi$ has a section. Since $R$ is strictly henselian, by [Gro67, 18.8.1] it suffices to show that $Y_g$ is étale over $R$ with non-empty closed fiber, or more generally that $n_G$ is étale and surjective.

By Lemma 2(b) of [BLR90, §7.3], $n_G$ is étale. The image of the étale $n_G$ must be an open subgroup scheme, and on fibers over $\text{Spec}(R)$ we get surjectivity since an open subgroup scheme of a smooth connected (hence irreducible) group scheme over a field must fill up the whole space [Gro70, VI, A, 0.5].

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Lemma 3.4. Let $A$ be an abelian variety over the fraction field $K$ of a strictly henselian dvr (e.g., $K$ could be the maximal unramified extension a local field). Let $n$ be an integer not divisible by the residue characteristic of $K$. Suppose that $x$ is a point of $A(K)$ whose reduction lands in the identity component of the closed fiber of the Néron model of $A$. Then there exists $z \in A(K)$ such that $nz = x$.

Proof. Let $\mathcal{A}$ denote the Néron model of $A$ over the valuation ring $R$ of $K$, and let $\mathcal{A}^0$ denote the “identity component” (i.e., the open subgroup scheme obtained by removing the non-identity components of the closed fiber of $\mathcal{A}$). The hypothesis on the reduction of $x \in A(K) = A(R)$ says exactly that $x \in \mathcal{A}^0(R)$. Since connected schemes over a field are geometrically connected when there is a rational point [Gro65, Prop. 4.5.13], the fibers of $\mathcal{A}^0$ over $\text{Spec}(K)$ are geometrically connected. The lemma now follows from Lemma 3.3 with $G = \mathcal{A}^0$.

Remark 3.5. M. Baker noted that this argument can also be formulated in terms of formal groups when $R$ is the strict henselization of a complete dvr.

Lemma 3.6. Let $f : \mathcal{J} \to \mathcal{C}$ be a smooth surjective morphism of schemes over a strictly Henselian local ring $R$. Then the induced map $\mathcal{J}(R) \to \mathcal{C}(R)$ is surjective.

Proof. The argument is similar to that of the proof of Lemma 3.3. Pick an element $g \in \mathcal{C}(R)$ and form the cartesian diagram

\[
\begin{array}{ccc}
Y_g & \xrightarrow{\psi} & \text{Spec}(R) \\
\downarrow & & \downarrow \\
\mathcal{J} & \xrightarrow{\phi} & \mathcal{C}
\end{array}
\]

We want to prove that $\psi$ has a section. Since $\phi$ is smooth, $\psi$ is also smooth. By [Gro67, 18.5.17], to show that $\psi$ has a section, we just need to show that the closed fiber of $\psi$ has a section (i.e., a rational point). But this closed fiber is smooth and non-empty (since $\phi$ is surjective); also its base field is separably closed since $R$ is strictly Henselian. Hence by [BLR90, Cor. 2.2.13], the closed fiber has an $R$-rational point.

3.3. Visible Elements of $H^1(K, A)$

In this section, we produce a map $B(K)/nB(K) \to \text{Vis}_J(H^1(K, A))$ with bounded kernel.

Lemma 3.7. Let $A$ and $B$ be abelian subvarieties of an abelian variety $J$ over a number field $K$ such that $A \cap B$ is finite. Suppose $n$ is a natural
number such that

\[ \gcd(n, \#(J/B)(K)_{tor} \cdot \#B(K)_{tor}) = 1 \]

and \( B[n] \subset A \) as subgroup schemes of \( J \). Then there is a natural map

\[
\varphi: B(K)/nB(K) \to \text{Vis}_J(H^1(K, A))
\]

such that \( \ker(\varphi) \subset J(K)/(B(K) + A(K)) \). If \( A(K) \) has rank 0, then \( \ker(\varphi) = 0 \) (more generally, \( \ker(\varphi) \) has order at most \( n^r \) where \( r \) is the rank of \( A(K) \)).

Proof. First we produce a map \( \varphi: B(K)/nB(K) \to \text{Vis}(H^1(K, A)) \) by using that \( B[n] \subset A \) hence a certain map factors through multiplication by \( n \). Then we use the snake lemma and our hypothesis that \( n \) does not divide the orders of certain torsion groups to bound the dimension of the kernel of \( \varphi \).

The quotient \( J/A \) is an abelian variety \( C \) over \( K \). The long exact sequence of Galois cohomology associated to the short exact sequence

\[ 0 \to A \to J \to C \to 0 \]

begins

\[ 0 \to A(K) \to J(K) \to C(K) \xrightarrow{\delta} H^1(K, A) \to \cdots. \]  \hfill (3.1)

Let \( \psi \) be map \( B \to C \) obtained by composing the inclusion \( B \hookrightarrow J \) with the quotient map \( J \to C \). Since \( B[n] \subset A \), we see that \( \psi \) factors through multiplication by \( n \), so the following diagram commutes:

\[
\begin{array}{ccc}
B & \xrightarrow{n} & B \\
\downarrow \psi & & \downarrow \psi \\
A & \to & J & \to & C.
\end{array}
\]

Using that \( B[n](K) = \{0\} \), we obtain the following commutative diagram, all of whose rows and columns are exact:

\[
\begin{array}{ccc}
K_0 & \xrightarrow{n} & K_1 & \xrightarrow{n} & K_2 \\
0 & \to & B(K) & \to & B(K) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & B(K)/nB(K) & \xrightarrow{\varphi} & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & J(K)/A(K) & \to & C(K) \\
\downarrow & & \downarrow & & \downarrow \\
K_3, & & & & & \delta(C(K)) & \to & 0
\end{array}
\]
where $K_0$, $K_1$ and $K_2$ are the indicated kernels and $K_3$ is the indicated cokernel. Exactness of the top row expresses the fact that $B[n](K) = \{0\}$, and the bottom exact row arises from the exact sequence (3.1) above. The first vertical map $B(K) \to J(K)/A(K)$ is induced by the inclusion $B(K) \hookrightarrow J(K)$ composed with the quotient map $J(K) \to J(K)/A(K)$. The second vertical map $B(K) \to C(K)$ exists because the composition $B \to J \to C$ has kernel $B \cap A$, which contains $B[n]$, by assumption. The third vertical map exists because $\pi$ contains $nB(K)$ in its kernel, so that $\pi$ factors through $B(K)/nB(K)$.

The sequence (1.1) on page 3 implies that the image of $\varphi$ is contained in $\text{Vis}_J(H^1(K, A))$. The snake lemma gives an exact sequence

$$K_0 \to K_1 \to K_2 \to K_3.$$

Because $B \to C$ has finite kernel, $K_1 \subset B(K)_{\text{tor}}$. Since $B[n](K) = \{0\}$ and $K_2$ is an $n$-torsion group, the map $K_1 \to K_2$ is the 0 map. Thus $K_2 = \ker(\varphi)$ is isomorphic to a subgroup of $K_3 = J(K)/(A(K) + B(K))$, as claimed.

Any torsion in the quotient $J(K)/B(K)$ is of order coprime to $n$ because $J(K)/B(K)$ is a subgroup of $(J/B)(K)$, and $\gcd(n, \#(J/B)(K)_{\text{tor}}) = 1$, by assumption. Thus if $A(K)$ is a torsion group, $K_3 = (J(K)/B(K))/A(K)$ has no nontrivial torsion of order dividing $n$, so when $A(K)$ has rank zero, $\ker(\varphi) = 0$.

Consider the map $\psi : A(K) \to J(K)/B(K)$. To show that $\ker(\psi)$ has order at most $n^r$, where $r$ is the rank of $A(K)$, it suffices to show that $\text{coker}(\psi)[n]$ has order at most $n^r$. To prove the latter statement, by the structure theorem for finite abelian groups, it suffices to prove it for the case when $n$ is a power of a prime. Moreover, we may assume that $A(K)$ and $J(K)/B(K)$ have no prime-to-$n$ torsion. Then $J(K)/B(K)$ is in fact torsion-free, and so we may also assume $A(K)$ is torsion-free. With these assumptions, the statement we want to prove follows easily by elementary group-theoretic arguments (in particular, by considering of the Smith normal form of the matrix representing $\psi$).

### 3.4. Proof of Theorem 3.1

**Proof of Theorem 3.1.** The proof proceeds in two steps. The first step is to use the hypothesis that $B[n] \subset A$ to produce a map $B(K)/nB(K) \to \text{Vis}_J(H^1(K, A))[n]$. This was done in Section 3.3. The second step is to perform a local analysis at each place $v$ of $K$ in order to prove that the image of this map consists of locally-trivial cohomology classes. We divide this local analysis into three cases:

1. When $v$ is real archimedean, we use that $\gcd(2, n) = 1$. (We know that for any $p \mid n$ we have $p > 2$ because $1 \leq e_p < p - 1$, by assumption.)
2. When $\gcd(\text{char}(v), n) = 1$, we use the result of Section 3.2 and a relationship between unramified cohomology and the cohomology of a component group.

3. When $\gcd(\text{char}(v), n) \neq 1$, for each prime $p \mid n$, the reduction of $J$ is abelian and by hypothesis $e_p < p - 1$, so we can apply an exactness theorem from [BLR90].

We now deduce that the image of $B(K)/nB(K)$ in $H^1(K, A)$ lies in $III(A)$. Fix an element $x ∈ B(K)$. To show that $π(x) ∈ III(A)$, it suffices to show that $\text{res}_v(π(x)) = 0$ for all places $v$ of $K$.

**Case 1.** $v$ real archimedean: At a real archimedean place $v$, the restriction $\text{res}_v(π(x))$ is killed by 2 and the odd $n_v$ hence $\text{res}_v(π(x)) = 0$.

**Case 2.** $\gcd(\text{char}(v), n) = 1$: Suppose that $\gcd(\text{char}(v), n) = 1$. Let $m = \ell_{B,v} = \Phi_{B,v}(F_v)$ be the Tamagawa number of $B$ at $v$. The reduction of $mz$ lies in the identity component of the closed fiber $\mathcal{B}_v$ of the Néron model of $B$ at $v$, so by Lemma 3.4, there exists $z ∈ B(K_v^{ur})$ such that $n_z = mz$. Thus the cohomology class $\text{res}_v(\pi(mx))$ is defined by a cocycle that sends $σ ∈ \text{Gal}(K_v^{ur}/K_v)$ to $σ(z) - z ∈ A(K_v^{ur})$ (see diagram (3.2) for the definition of $π$). In particular, $\text{res}_v(\pi(mx))$ is unramified at $v$. By [Mil86, Prop. 3.8],

$$H^1(K_v^{ur}/K_v, A(K_v^{ur})) = H^1(K_v^{ur}/K_v, Φ_{A,v}(F_v)),$$

where $Φ_{A,v}$ is the component group of $A$ at $v$. The Herbrand quotient of a finite module is 1 (see, e.g., [Ser79, VIII A.8]), so

$$\#Φ_{A,v}(F_v) = \#H^1(K_v^{ur}/K_v, Φ_{A,v}(F_v)).$$

Thus the order of $\text{res}_v(\pi(mx))$ divides both $\#Φ_{A,v}(F_v)$ and $n$. Since by assumption $\gcd(\#Φ_{A,v}(F_v), n) = 1$, it follows that $\text{res}_v(\pi(mx)) = 0$, hence $m \text{res}_v(\pi(x)) = 0$. Again, since the order of $π(x)$ divides $n$, and $\gcd(n, m) = 1$, we have $\text{res}_v(\pi(x)) = 0$.

**Case 3.** $\gcd(\text{char}(v), n) = p \neq 1$: Suppose that $\text{char}(v) = p \mid n$. Let $R$ be the ring of integers of $K_v^{ur}$, and let $A$, $J$, and $C$ be the Néron models of $A$, $J$, and $C$, respectively. Since $e_p < p - 1$ and $J$ has abelian reduction at $v$ (since $p \nmid N$), by [BLR90, Thm. 7.5.4(iii)], the induced sequence $0 → A → J → C → 0$ is exact, which means that $φ$ is faithfully flat and surjective with scheme-theoretic kernel $A$. Since $φ$ is faithfully flat with smooth kernel, $φ$ is smooth (see, e.g., [BLR90, 2.4.8]). By Lemma 3.6, $J(R) → C(R)$ is a surjection; i.e., $J(K_v^{ur}) → C(K_v^{ur})$ is a surjection.

So $\text{res}_v(\pi(x))$ is unramified, and again by [Mil86, Prop. 3.8],

$$H^1(K_v^{ur}/K_v, A) ≃ H^1(K_v^{ur}/K_v, Φ_{A,v}(F_v)).$$

But $H^1(K_v^{ur}/K_v, Φ_{A,v}(F_v)) = \{0\}$, since $Φ_{A,v}(F_v)$ is trivial, as $A$ has good reduction at $v$ (because $p \nmid N$). Thus $\text{res}_v(\pi(x)) = 0$. ■
4. SOME EXAMPLES

This section contains some examples of visible and invisible elements of Shafarevich-Tate groups. Section 4.1 uses Theorem 3.1 to produce nontrivial visible elements of \( \text{III}(A) \), where \( A \) is a 20-dimensional modular abelian variety, thus giving evidence for the BSD conjecture. In Section 4.2 we show that an invisible Shafarevich-Tate group from [CM00] becomes visible at a higher level.

In [AS02], we describe the notation used below (which is standard) and the algorithms that we used to carry out the computations described below. We also report on a large number of similar computations, which were performed using the second author’s modular symbols package, which is part of Magma (see [BCP97]).

4.1. Visibility in an Abelian Variety of Dimension 20

Using the methods described in [AS02], we find that \( S_2(\Gamma_0(389)) \) contains exactly five Galois-conjugacy classes of newforms, and these are defined over extensions of \( \mathbb{Q} \) of degrees 1, 2, 3, 6, and 20. Thus \( J = J_0(389) \) decomposes, up to isogeny, as a product \( A_1 \times A_2 \times A_3 \times A_6 \times A_{20} \) of abelian varieties, where \( d = \dim A_d \) and \( A_d \) is the quotient corresponding to the appropriate Galois-conjugacy class of newforms.

Next we consider the arithmetic of each \( A_d \). Using [AS02], we find that

\[
L(A_1, 1) = L(A_2, 1) = L(A_3, 1) = L(A_6, 1) = 0,
\]

and

\[
\frac{L(A_{20}, 1)}{\Omega_{A_{20}}} = \frac{5^2 \cdot 2^7}{97},
\]

where \( 2^7 \) is a power of 2. Using [AS02], we find that \( \# A_{20}(\mathbb{Q}) = 97 \) and the Tamagawa number of \( A_{20} \) at 389 is also 97. The BSD Conjecture then predicts that \( \# \text{III}(A_{20}) = 5^2 \cdot 2^7 \). The following proposition provides support for this conjecture.

**Proposition 4.1.** There is an inclusion

\[
(\mathbb{Z}/5\mathbb{Z})^2 \cong A_1(\mathbb{Q})/5A_1(\mathbb{Q}) \hookrightarrow \text{Vis}_J(\text{III}(A_{20}^\vee)).
\]

**Proof.** Let \( A = A_{20}^\vee, B = A_1^\vee = A_1 \) and \( J = A + B \subset J_0(389) \). Using algorithms in [AS02], we find that \( A \cap B \cong (\mathbb{Z}/4)^2 \times (\mathbb{Z}/5\mathbb{Z})^2 \), so \( B[5] \subset A \). Since 5 does not divide the numerator of \( (389 - 1)/12 \), it does not divide the Tamagawa numbers or the orders of the torsion subgroups of \( A, B, J, \) and \( J/B \) (we also verified this using a modular symbols computations), so Theorem 3.1 implies that there is an injective map

\[
A_1(\mathbb{Q})/5A_1(\mathbb{Q}) \hookrightarrow \text{Vis}_J(\text{III}(A_{20}^\vee)).
\]

To finish, note that Cremona [Cre97] has verified that \( A_1(\mathbb{Q}) \approx \mathbb{Z} \times \mathbb{Z}. \) \( \blacksquare \)
4.2. Invisible Elements that Becomes Visible at Higher Level

Consider the elliptic curve \( E \) of conductor 5389 = 17 \cdot 317 defined by the equation
\[
y^2 + xy + y = x^3 - 35590x - 2587197.
\]
In [CM00], Cremona and Mazur observe that the BSD conjecture implies that \#\( \text{III}(E) \) = 9, but they find that \( \text{Vis}_{J_0(5389)}(\text{III}(E)[3]) = \{0\} \). We will now verify, without assuming any conjectures, that 9 \mid \#\( \text{III}(E) \) and that these 9 elements of \( \text{III}(E) \) are visible in \( J_0(5389 \cdot 7) \).

First note that the mod 3 representation \( \rho_{E,3} \) attached to \( E \) is irreducible because \( E \) is semistable and admits no 3-isogeny (according to [Cre]). The newform attached to \( E \) is
\[
f_E = q + q^2 - 2q^3 - q^4 + 2q^5 - 2q^6 - 2q^7 + \cdots,
\]
and \( a_1^2 = (-2)^2 \equiv (7 + 1)^2 \pmod{3} \), so Ribet’s level-raising theorem [Rib90] implies that there is a newform \( g \) of level 7 \cdot 5389 that is congruent modulo 3 to \( f_E \). This observation led us to the following proposition.

**Proposition 4.2.** Map \( E \) to \( J_0(7 \cdot 5389) \) by the sum of the two maps on Jacobians induced by the two degeneracy maps \( X_0(7 \cdot 5389) \to X_0(5389) \). The image \( E' \) of \( E \) in \( J_0(7 \cdot 5389) \) is 2-isogenous to \( E \) and
\[
(Z/3Z)^2 \subset \text{Vis}_{J_0(7 \cdot 5389)}(\text{III}(E')).
\]

**Proof.** It is easy to see from the discussion in [Rib90] that the kernel of the sum of the two degeneracy maps \( J_0(5389) \to J_0(7 \cdot 5389) \) is a group of 2-power order, so \( E' \) is isogenous to \( E \) via an isogeny of degree a power of 2.

Consider the elliptic curve \( F \) defined by \( y^2 - y = x^3 + x^2 + 34x - 248 \). Using Cremona’s programs \texttt{tate} and \texttt{mwrank} we find that \( F \) has conductor 7 \cdot 5389, and that \( F(\mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Z} \). The Tamagawa numbers of \( F \) at 7, 17, and 317 are 1, 2, and 1, respectively. The newform attached to \( F \) is
\[
f_F = q - 2q^2 + q^3 + 2q^4 - q^5 - 2q^6 - q^7 + \cdots
\]
and, by [Stu87], we prove that \( f_E(q) + f_F(q^7) \equiv f_F \pmod{3} \) by checking this congruence for the first 7632 = \( |\text{SL}_2(\mathbb{Z}) : \Gamma_0(7 \cdot 5389)|/6 \) terms. Since \( 2 \leq k < 3 \) and \( 3 \nmid 7 \cdot 5389 \), the first part of the multiplicity one theorem of [Edi92, §9] implies that \( F[3] = E'[3] \).

Finally, we apply Theorem 3.1 with \( A = E' \), \( B = F \), \( J = A + B \subset J_0(7 \cdot 5389) \), \( N = 7 \cdot 5389 \), and \( n = 3 \). It is routine to check the hypothesis. For example, the hypothesis that \( J/B \) has no \( \mathbb{Q} \)-rational 3-torsion can be checked as follows. Cremona’s online tables imply that \( E \) admits no 3-isogeny, so \( E[3] \) is irreducible. Since \( J/B \) is isogenous to \( E \), the representation \( (J/B)[3] \) is also irreducible, so \( (J/B)(\mathbb{Q})[3] = \{0\} \). Thus, by
Theorem 3.1, we have $(\mathbb{Z}/3\mathbb{Z})^2 \subset \text{Vis}_J(\text{T}(E'))$. To finish the proof, note that $\text{Vis}_J(\text{T}(E')) \subset \text{Vis}_{J_{(7,53,89)}}(\text{T}(E'))$. 

Since $E'$ is 2-isogenous to $E$ and $9 \mid \#\text{T}(E')$, it follows that $9 \mid \#\text{T}(E)$, as predicted by the BSD conjecture.

REFERENCES


