Some calculations regarding torsion and component groups

Note: What I am calling observations below are really hunches based on part of the data; they have to be tested on more data. Let f be a newform of weight 2 on $\Gamma_0(N)$, and let $A = A_f$ be the quotient of $J_0(N)$ associated to f. For the moment, suppose N is square-free, and let K be the full cuspidal subgroup of $J_0(N)$, which is necessarily rational.

Observation 0.1. $A^\vee(\mathbf{Q})_{\rm tor} \subseteq K$.

I suppose to start with, one can check if $|A^\vee(\mathbf{Q})_{\rm tor}|$ divides $|K|$, say for elliptic curves, using your table `http://modular.ucsd.edu/Tables/cuspgroup/index.html`. I would not be too surprised if things do not work out at the primes 2, 3 and the primes dividing N.

When N is not square-free, I guess the above may be true with K replaced by the rational part of the full cuspidal subgroup (i.e., $A^\vee(\mathbf{Q})_{\rm tor} \subseteq K \cap J_0(N)(\mathbf{Q})$). But I have not looked at the data. Suppose now on that N is any integer (but at times one may need to assume N is square-free).

The rational divisor $(0) - (\infty)$ generates a finite subgroup of $J_0(N)(\mathbf{Q})$, which we denote C. The image of C under the quotient map $J_0(N) \to A$ is a cyclic subgroup of $A(\mathbf{Q})_{\rm tor}$; we denote this subgroup by C_A and call it the *cuspidal subgroup of* A (note that this is *not* the subgroup generated by the images of all the cuspidal divisors, but by the image of just $(0) - (\infty)$). Let w_p denote the eigenvalue of the Atkin-Lehner involution W_p acting on f. The product of the W_p's for $p\,|\,N$ is the Fricke involution W_N, whose eigenvalue is denoted w_N.

The following observations are based on Cremona's data and your table
`http://modular.fas.harvard.edu/Tables/non_zeroinf_tor.txt`.

Observation 0.2. If an odd prime ℓ divides the order of the torsion subgroup $A^\vee(\mathbf{Q})_{\rm tor}$ but does not divide the order of the cuspidal subgroup C_A, then there is a prime $p\,|\,N$ such that $w_p = -1$ and f is congruent modulo ℓ to a newform of level dividing N/p,

I don't recall why I went to $A^\vee(\mathbf{Q})$ rather than $A(\mathbf{Q})$, considering that C_A is in $A(\mathbf{Q})$. If A is an elliptic curve, then it does not matter.

Similar things should happen for the orders of the arithmetic component groups:

Observation 0.3. If an odd prime ℓ divides $c_p(A)$ for some $p\,|\,N$ but ℓ does not divide the order of the torsion group $A(\mathbf{Q})_{\rm tor}$, then $w_p = -1$ and f is congruent modulo ℓ to a newform of level dividing N/p.

Assume in this paragraph that $L_A(1) \neq 0$. Then we showed in our paper "Visible evidence..." that the odd part of the denominator of $\frac{L_A(1)}{\Omega_A}$ divides $|C_A|$. Thus in the BSD conjectural formula

$$\frac{L_A(1)}{\Omega_A} \stackrel{?}{=} \frac{|\text{Ш}_A| \cdot \prod_p c_p(A)}{|A(\mathbf{Q})_{\rm tor}| \cdot |A^\vee(\mathbf{Q})_{\rm tor}|}\ , \tag{1}$$

one expects significant cancellation on the right side. By Emerton, when the level N is prime, $c_N(A) = |A(\mathbf{Q})_{\rm tor}| = |A^\vee(\mathbf{Q})_{\rm tor}| = |C_A|$, and hence the denominator on the right side is just $|C_A|$. When the level is not prime, if the BSD conjectural formula is true, then there should again be enough cancellation so that the odd part of the denominator divides $|C_A|$. Thus the contributions to the torsion group that are *not* explained by the cuspidal group (generated by $(0) - (\infty)$) should cancel with the arithmetic component group (when the analytic rank is zero). Recall that we expect that these extra contributions to torsion and arithmetic component groups come at the same set of odd primes: those ℓ such that for some $p\,|\,N$, $w_p = -1$ and f is congruence modulo ℓ

to a newform of level dividing N/p. So part of the observation is that more should be true: the exponents of these primes in $|A(\mathbf{Q})_{\mathrm{tor}}| \cdot |A^{\vee}(\mathbf{Q})_{\mathrm{tor}}|$ should be less than or equal to (and I guess the latter) the exponents in $\prod_p c_p(A)$ when A has analytic rank zero. Note that there there are some congruences that contribute to the arithmetic group but not the torsion (coming from "visibility").

I noted the following in your table:

Observation 0.4. Suppose a prime ℓ divides the order of the torsion group, but not the order of the cuspidal group. If $w_N = -1$, then there exist distinct primes p and q dividing N such that $w_p = w_q = -1$ and f is congruent modulo ℓ to a form of level dividing N/pq. If $w_N = 1$, then there usually exists only one prime p dividing N such that $w_p = -1$ and f is congruent modulo ℓ to a newform of level dividing N/p.

For example, the elliptic curve $E = 66C1$ has trivial cuspidal group, $|E(\mathbf{Q})_{\mathrm{tor}}| = 10$, and $w_N = -1$. One finds that $w_2 = w_3 = -1$ and $c_2(E) = 10$, $c_3(E) = 5$, $c_{11}(E) = 1$. Thus the odd parts of $c_2(E) \cdot c_3(E) \cdot c_{11}(E)$ and $|E(\mathbf{Q})_{\mathrm{tor}}|^2$ are the same, i.e., the contributions from congruences with lower level cancel on the right hand side of (1). This is consistent with the BSD formula by the discussion above. For the elliptic curve $E = 123A1$, which has trivial cuspidal subgroup, $|E(\mathbf{Q})_{\mathrm{tor}}| = 5$, but $w_N = 1$, one finds that $w_3 = w_{41} = -1$ and $c_3(E) = 5$, $c_{41}(E) = 1$; so $\frac{c_3(E) \cdot c_{41}(E)}{|E(\mathbf{Q})_{\mathrm{tor}}|^2} = \frac{1}{5}$. Thus the contributions from congruences with lower level on the right hand side of the BSD formula (1) need *not* cancel when $w_N = 1$. So the behavior is different based on the parity of the analytic racnk.

Finally, one would also like to know how the contributions from the cuspidal group distribute among $\prod_p c_p(A)$ and $|A(\mathbf{Q})_{\mathrm{tor}}| \cdot |A^{\vee}(\mathbf{Q})_{\mathrm{tor}}|$. When the analytic rank is zero, considering that the denominator of $L_A(1)/\Omega_A$ divides $|C_A|$ and in view of the BSD formula, one expects that $|C_A|$ divides each of $\prod_p c_p(A)$, $|A(\mathbf{Q})_{\mathrm{tor}}|$, and $|A^{\vee}(\mathbf{Q})_{\mathrm{tor}}|$ only once ("generically" speaking). This should probably hold more generally when $w_N = -1$. When $w_N = 1$, the cuspidal group of A is trivial (since W_N acts as -1 on $(0) - (\infty)$).

–Amod