The Emergence of Large Scale Coherent Structure under Small Scale Random Bombardments

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Abstract

We provide mathematical justification of the emergence of large scale coherent structure in a two dimensional fluid system under small scale random bombardments with small data assumptions. The analysis shows that the large scale structure emerging out of the small scale random forcing is not the one predicted by equilibrium statistical mechanics. But the error is very small which explains earlier successful prediction of the large scale structure based on equilibrium statistical mechanics.

Keywords: *One layer quasi-geostrophic equations, Navier-Stokes system, large coherent structure, random small scale forcing, invariance principle, statistical mechanics, invariant measure, random dynamical system, random attractor, Gaussian process*

1 Introduction

One of the ubiquitous features of geophysical flows is the existence and persistence of large scale coherent structures such as the meandering jet stream in the atmosphere, gulf streams in the oceans. The most dramatic example is perhaps the Great Red Spot on Jupiter which has persisted for hundreds of years. We are naturally interested in understanding the mechanism behind the emergence and persistence of such large scale coherent structures.

Due to the relative fast rotation for geophysical flows, two dimensional models are suitable for the study of such large scale coherent structures. In the inviscid, unforced environment, the emergence of large scale coherent structures may be explained via equilibrium statistical mechanics, and their persistence via nonlinear stability theory (see for instance Majda and Wang 2004b, Pedlosky 1978, Salmon 1998 among others). However, such equilibrium statistical theories are restricted to the idealized inviscid unforced case, and the statistical predictions (mean field) are derived under certain ad hoc maximum entropy principles. Realistic fluid problems are always under the influence of various forcing and damping mechanism. The damping and forcing becomes important in long time behavior (such as statistical behavior) in particular. This leads to the question of whether various equilibrium statistical theories are still applicable in a damped forced environment. Of course, one shouldn't expect to extend applicability of the equilibrium statistical theory without restriction. But it would make sense to test if the statistical theories can be applied to the weakly forced and damped case where the situation is close to the case of inviscid/unforced environment, and the flow is in a quasi-equilibrium state. This hypothesis has been tested through comparison of numerical experiments and equilibrium statistical predictions in a series of papers (Majda and Holen, 1997, Grote and Majda, 1997, 2000, DiBattista *et al*, 2001).

As in many realistic situations, the external forcing is of relative small scale and is usually un-resolved in the large scale geophysical model. For instance, Jupiter's weather layer (where we observe the Great Red Spot) is under constant bombardment of very small scale thermal plumes, the earth's atmosphere is subject to intense small scale forcing due to convective storms, and the oceans are subject to the influence of unresolved baroclinic instability processes. These small scale bombardments appear to be random in nature (related to turbulent behavior of the inner core of Jupiter and the atmosphere) which suggests that the weak small scale forcing may be taken as random. It then seems appropriate to include viscosity since small scale structures are involved.

Next we consider an extremely simplified (idealized) situation of a two dimensional fluid system in a square under the influence of random small scale vortices mimicking the above situations, in the presence of a (small) viscosity. More precisely, we consider the two dimensional Navier-Stokes system in a square with free-slip boundary condition and impulse forcing of small scale

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q = \nu \Delta q + \mathcal{F}, \tag{1}$$

$$q = \Delta \psi \tag{2}$$

equipped with initial condition

$$q|_{t=0} = q_0 \tag{3}$$

and no-penetration, free-slip boundary condition

$$\psi = q = 0, \text{on } \partial Q \tag{4}$$

where the fluid occupies the square

$$Q = [0, \pi] \times [0, \pi]. \tag{5}$$

The random small scale forcing is given by

$$\mathcal{F} = \sum_{j=1}^{\infty} \delta(t - j\Delta t)\, A\omega_r(\vec{x} - \vec{x}_j) \tag{6}$$

where A is the amplitude of the small scale bombardment, $\vec{x}_j$ is the (random) center, the small scale vortex ω_r takes the form,

$$\omega_r(\vec{x}) = \begin{cases} (1 - |\vec{x}|^2/r^2)^2, & |\vec{x}|^2 \le r^2 \quad, \\ 0, & |\vec{x}|^2 > r^2 \quad, \end{cases} \tag{7}$$

and the center of the small vortices, $\vec{x}_j$, satisfies a uniform distribution on $Q_{r_0} = [r_0, \pi - r_0] \times [r_0, \pi - r_0]$ where $r_0 (\ge r)$ is a fixed constant (see figure 1). Since ω_r is piecewise smooth with compact support and it is C^1, we see that

$$\omega_r(\vec{x} - \vec{x}_j) \in H_0^2(Q) \tag{8}$$

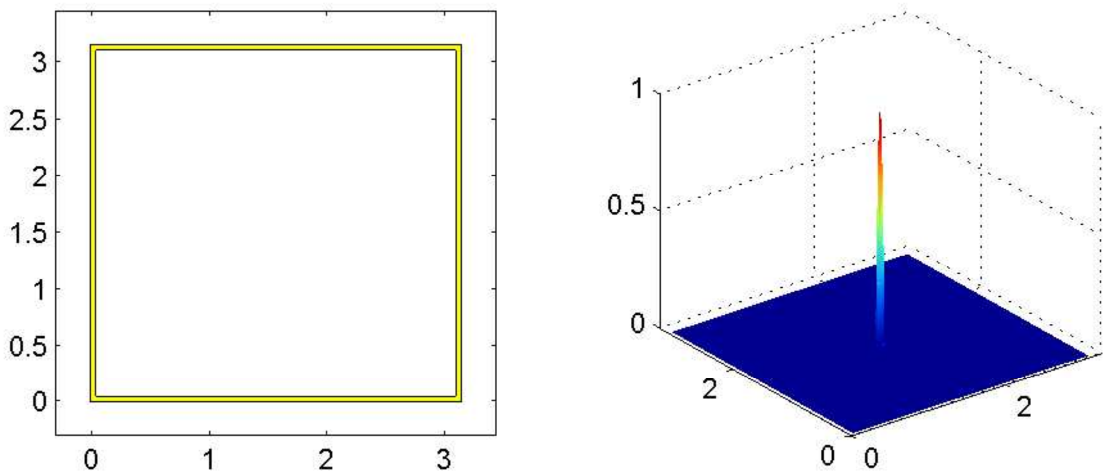

Figure 1: Left panel: the region of bombardment; Right panel: graph of a small vortex ω_r centered at the center of the domain.

with norm independent of j. In fact, $\omega_r(\vec{x}-\vec{x}_j) \in W^{2,\infty}(Q)$.

Here we used q as the notation for vorticity instead of the standard ω. This is because ω is a standard notation for point in probability space which is needed in our stochastic treatment in section 3, and q is the standard notation for potential vorticity in GFD which reduces to the usual vorticity in our classical fluid setting. Hopefully this will avoid some confusion instead of creating one.

It is then easy to see that there are two different stages in the dynamics, a stage of pure decay from $(j\Delta t)^+$ to $((j+1)\Delta t)^-$, governed by the decaying Navier-Stokes system

$$\frac{\partial q}{\partial t} + \nabla^\perp \psi \cdot \nabla q = \nu \Delta q, \tag{9}$$

$$q = \Delta \psi, \tag{10}$$

and a stage of instantaneous forcing

$$q((j\Delta t)^+) = q((j\Delta t)^-) + A\omega_r(\vec{x}-\vec{x}_j). \tag{11}$$

Numerical simulation in the regime of weak forcing and weak damping (Grote and Majda 1997, 2000, and figures 2, 3 and 4 in the present paper)

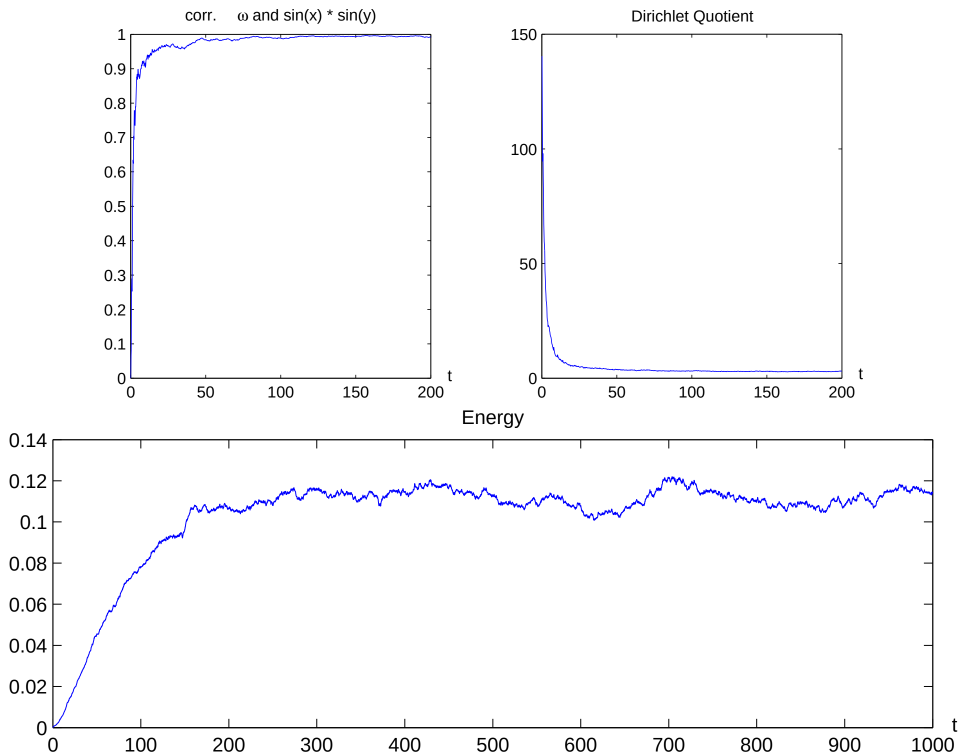

Figure 2: Upper left panel: correlation between the vorticity q (or ω) and the ground mode $\sin x \ \sin y$; Upper right panel: evolution of the Dirichlet quotient of the flow; Lower panel: evolution of the energy.

indicates the emergence and persistence of a large coherent structure. More precisely, numerical experiments demonstrate that the flow field reaches a quasi-equilibrium state in terms of energy (figure 2), enstrophy, circulation etc, and the contour plot of the vorticity field looks like a large vortex plus small (random) perturbation (see figures 3 and 4). For the special case of zero initial data, such a phenomenon is termed spin-up from rest (Grote and Majda 1997, Majda and Wang 2004). This large coherent structure resembles very much the ground state mode of the Laplace operator, i.e., $sin(x)sin(y)$, with a correlation between the vorticity field q and $sin(x)sin(y)$ above 0.97 (see figure 2).

The ground state mode is in fact the predicted most probable mean field of equilibrium statistical mechanics theory utilizing energy and enstrophy as conserved quantities (see for instance Majda and Wang 2004). Thus the

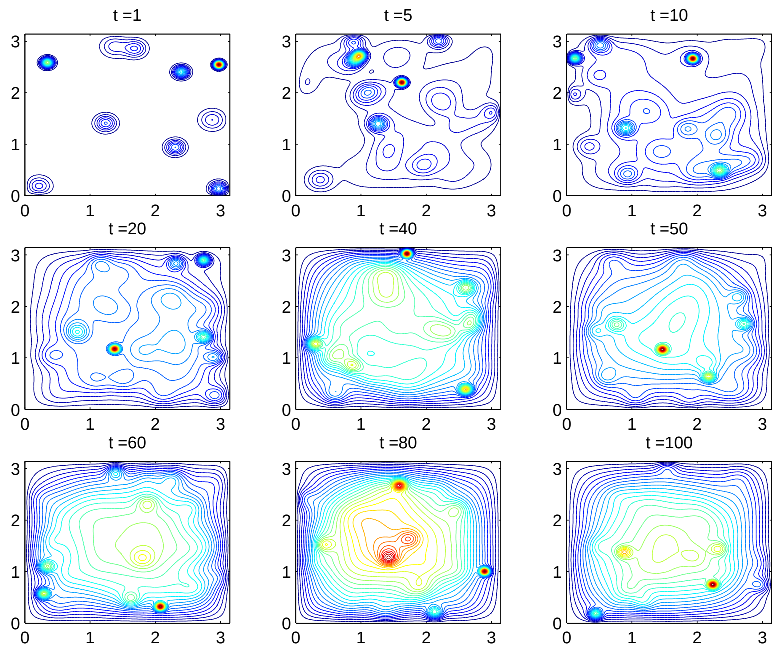

Figure 3: Contour plot of the vorticity field.

numerical evidence can be viewed as evidence toward applicability of equilibrium statistical mechanics in this damped driven case. If one applies a more sophisticated equilibrium statistical theory such as the point vortex energy-circulation theory which leads to a sinh-Poisson type mean field equation (see for instance Grote and Majda 1997, Majda and Wang 2004 among others), one gets better prediction which is not too much a surprise since we have more parameters with the sinh-Poisson equation and we recover the linear energy - enstrophy theory in the small amplitude limit. Moreover, Grote and Majda (1997) devised a so-called crude closure algorithm where they developed a simple algorithm for the time evolution of the energy and circulation without any detailed resolution of the Navier-Stokes equations. One then recovers the flow field via equilibrium statistical theory with energy and circulation as the conserved quantities (via solving the sinh-Poisson equation). Surprisingly, the Grote-Majda crude closure worked extremely well.

The purpose of this paper is to provide a rigorous theoretical underpinning of such success. More precisely, we will show, under appropriate assumptions on the small parameters (viscosity ν, time step Δt, amplitude A,

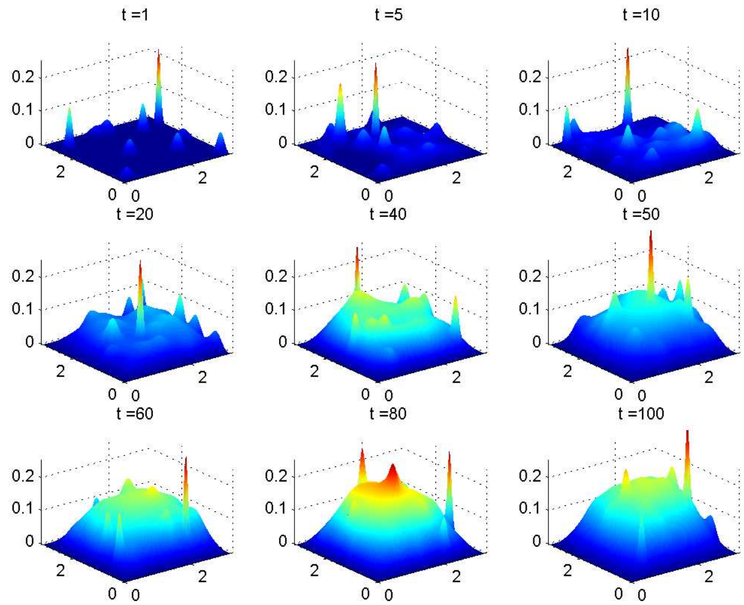

Figure 4: Surface plot of the vorticity field.

radius of small forcing vortex r), that the long time dynamics is that of a large coherent vortex q^0 which is close to (but not equal to) the ground state mode $sin(x)sin(y)$, plus small random fluctuations. Such a result indicates that neither the energy-enstrophy statistical theory (which predicts the ground state mode) nor the point vortex energy-circulation statistical theory (which predicts a sinh-Poisson type mean field equation not satisfied by q^0) predicts the exact statistical equilibrium. However the error is so small (less than 2%) which establishes the practical applicability of these equilibrium statistical mechanics theories to this damped driven situation.

The rest of the paper is organized as follows. In section 2 we consider a naive deterministic approach and derive time uniform bounds for the Dirichlet quotient and energy. The uniform bound on Dirichlet quotient indicates control of the small scales. However the bound we derived here is not close to the first eigenvalue of the Laplace operator which is the lowest value of the Dirichlet quotient corresponding to the ground state mode. Such a discrepancy is due to the deterministic approach where we must perform a worst scenario analysis. In section 3 we take a stochastic approach to the problem. We first observe that the random forcing can be decomposed into a mean field and a small fluctuation field under a natural assumption on the amplitude A in (6) which agrees with the existence of a quasi-equilibrium state. Utilizing an infinite dimensional version of Donsker's invariance principle, the external forcing can then be modeled formally as the sum of a deterministic forcing plus a small parameter times the time derivative of an infinite dimensional Gaussian process. We then prove under appropriate assumptions, that the mean field of the flow is captured by the Navier-Stokes equation forced by the deterministic part of the forcing. The asymptotic behavior of the mean field is derived under a further smallness assumption. The validation of the mean field equation is justified in several ways including almost sure pathwise convergence for finite time, expectation of the second moment of the difference, random attractor and invariant measures. All these support the applicability of appropriate equilibrium statistical theories. In the last section, i.e. section 4, we provide concluding remarks and present some issues that need to be resolved for physically more interesting cases.

2 Deterministic Estimates

Recall that numerical evidence suggest that the long time asymptotic of the flow is that of a large coherent vortex close to the ground state mode $\sin x \sin y$ plus small random fluctuations (see figure 2 and 3). One of most important and useful quantities in the analysis of fluid problems is the Dirichlet quotient

$$\Lambda(t) = \frac{\|\Delta\psi\|^2(t)}{\|\nabla\psi\|^2(t)} = \frac{\|q\|^2(t)}{\|\nabla\psi\|^2(t)} \tag{12}$$

which is the quotient of the enstrophy ($\mathcal{E} = \frac{1}{2}\|\Delta\psi\|^2$) over the energy ($E = \frac{1}{2}\|\nabla^\perp\psi\|^2$). Recall that the Dirichlet quotient controls the small scales in the flow. Indeed, the Dirichlet quotient attains its minimum, the first eigenvalue of the Laplace operator allowed by the geometry, if and only if the flow attains the maximum scale structure allowed by the geometry, i.e. the ground states. This is exactly what our numerics indicated: Dirichlet quotient close to the first eigenvalue (see figure 2). Moreover, flows with predominant small structures are characterized by large Dirichlet quotient, while flows with predominant large structures are characterized by small Dirichlet quotient. Therefore, an upper bound on the Dirichlet quotient for the flow is a partial justification for the emergence of the large scale structure. Such an upper bound on the Dirichlet quotient is the goal of this section.

In general, a flow governed by the quasi-geostrophic dynamics may not be able to maintain large scale structure under random small scale bombardments. Indeed, it is easy to construct a flow of the form $q_L + \varepsilon q_s$ where q_L is a large coherent structure (thus with small Dirichlet quotient), q_s is a small structure (thus with large Dirichlet quotient) and ε is a small parameter so that q_L dominates the flow. The bombardment of this flow by the a large vortex $-q_L$ results in the cancellation of the relatively large structure and leads to the extremely small structure q_s with a much higher Dirichlet quotient. This is supported by the numerical results. However, due to the special setting of the spin-up problem, we will see that the vorticity field will be non-negative for all the time due to a maximum principle. Such a positive vorticity field cannot be annihilated by the positive small scale bombardment and this is the main ingredient in the success in deriving an upper bound for the Dirichlet quotient.

There are two stages in the dynamics: a free decay stage and an instantaneous forcing stage. It is easy to see that the Dirichlet quotient is a monotonic decreasing function of time in the free decay stage from (9) (Chapter 3 of Majda and Wang, 2004). Thus it is only necessary to establish that the instantaneous forcing stage can not increase the Dirichlet quotient without bound.

Here we consider the special case of non-negative initial vorticity, i.e., $q_0 \geq 0$. One special feature of the external forcing given in (6) is positivity. When this positivity is combined with the positivity of the initial vorticity we obtain the positivity of future vorticity via a simple maximum principle argument as we may view (1) as a advection-diffusion equation for the vorticity q. Hence we have, see for instance Evans (1998),

$$q(\vec{x}, t) \geq 0, \text{ for all } t > 0. \tag{13}$$

As for the stream function, it can be solved from via the Poisson equation together with the zero boundary condition specified in (4). The maximum principle for Poisson equation (see for instance Evans (1998)) implies that the stream function is strictly negative inside the box Q unless $q \equiv 0$

$$\psi(\vec{x}, t) < 0, \text{ for all } t > 0. \tag{14}$$

Next we look into the evolution of the energy E, enstrophy $\mathcal{E}$ and circulation $\Gamma = \int_Q q$. It is easy to derive, after multiplying the quasi-geostrophic

(Navier-Stokes) equations (1) by $-\psi$ (q or 1 respectively) and integrating over the square Q,

$$\frac{dE}{dt} = -\nu \int_Q |q|^2 - \int_Q \psi \mathcal{F}, \quad (15)$$

$$\frac{d\mathcal{E}}{dt} = -\nu \int_Q |\nabla q|^2 + \int_Q q\mathcal{F}, \quad (16)$$

$$\frac{d\Gamma}{dt} = \nu \int_{\partial Q} \frac{\partial q}{\partial n}, + \int_Q \mathcal{F}. \quad (17)$$

where $\frac{\partial q}{\partial n}$ represents the normal derivative of the vorticity with respect to the unit outer normal at the boundary of the box Q. Since the vorticity is positive inside the box (13), the normal derivative of the vorticity is non-positive, i.e.

$$\frac{\partial q}{\partial n} \leq 0, \text{ at } \partial Q. \quad (18)$$

This implies that the Newtonian dissipation decreases the circulation. This is of course consistent with the intuitive idea of dissipation. A simple consequence of the equations (15,16, 17) together with (14) is the fact that the positive external forcing $\mathcal{F}$ increases energy, enstrophy and circulation. This partially justifies the notation of spin-up.

The dynamics of the Dirichlet quotient $\Lambda(t) = \frac{\mathcal{E}(t)}{E(t)}$ can be calculated easily using the dynamics of the energy and enstrophy. Indeed we have (see for instance Foias and Saut 1984, Majda, Shim and Wang 2000, Majda and Wang 2004, Montgomery *et al*, 1992, among others)

$$\begin{aligned} \frac{d\Lambda(t)}{dt} &= \frac{1}{E(t)}(\dot{\mathcal{E}}(t) - \Lambda(t)\dot{E}(t)) \\ &= -\frac{\nu\| - \Delta\vec{v} - \Lambda(t)\vec{v}\|_0^2}{E(t)} + \frac{\int_Q (q + \Lambda(t)\psi)\mathcal{F}}{E(t)}. \end{aligned} \quad (19)$$

Since the sign of $q + \Lambda(t)\psi$ is not definite, we are not sure if the positive external forcing increases or decreases the Dirichlet quotient. In fact numerical experiment suggests that it could be both ways.

Our goal here is to derive a time uniform bound on the Dirichlet quotient. The bound would imply that not much small scale structures are created even though the system is under constant bombardment of small scale vortices.

We introduce the notation

$$\omega_j = \omega_r(\vec{x} - \vec{x}_j), \qquad \phi_j = \Delta^{-1}\omega_j, \tag{20}$$

and

$$E_j = -\frac{1}{2}\int_Q \psi_j\omega_j, \qquad \mathcal{E}_j = \frac{1}{2}\int_Q \omega_j^2. \tag{21}$$

We see immediately, thanks to (11), the positivity of q and ω_j, and the negativity of ψ and ψ_j,

$$2E(t_j^-) + 2E_j \geq E(t_j^+) \quad \geq \quad E(t_j^-) + E_j, \tag{22}$$

$$2\mathcal{E}(t_j^-) + 2\mathcal{E}_j \geq \mathcal{E}(t_j^+) \quad \geq \quad \mathcal{E}(t_j^-) + \mathcal{E}_j, \tag{23}$$

where $t_j = j\ \Delta t$. We also have, by the definition of Dirichlet quotient and the instantaneous forcing effect (6),

$$\Lambda(t_j^+) = \frac{\mathcal{E}(t_j^-) + A\int_Q q(t_j^-)\omega_j + A^2\mathcal{E}_j}{E(t_j^-) - A\int_Q q(t_j^-)\psi_j + A^2E_j}. \tag{24}$$

It is clear from our choice of ω_j in (7) that there exists a constant λ_1 such that

$$\frac{\mathcal{E}_j}{E_j} \leq \lambda_1, \text{ for all } j. \tag{25}$$

In order to derive a uniform in time bound on the Dirichlet quotient, it is enough to prove the following claim

Claim: There exists a constant λ_2 such that

$$\omega_j \leq -\lambda_2\psi_j, \text{ for all } \ j. \tag{26}$$

We observe that with the validity of the claim we have

$$\begin{aligned} \mathcal{E}(t_j^-) + A\int_Q q(t_j^-)\omega_j + A^2\mathcal{E}_j &\leq \mathcal{E}(t_j^-) - \lambda_2 A\int_Q q(t_j^-)\psi_j + \lambda_1 A^2 E_j \\ &\leq \Lambda_{t_j^-}E(t_j^-) - \lambda_2 A\int_Q q(t_j^-)\psi_j + \lambda_1 A^2 E_j \\ &\leq \bar{\lambda}_j(E(t_j^-) - A\int_Q q(t_j^-)\psi_j + A^2E_j), \end{aligned}$$

where

$$\bar{\lambda}_j = \max\{\Lambda_{t_j^-}, \lambda_2, \lambda_1\}, \tag{27}$$

and hence

$$\Lambda_{t_j^+} \leq \bar{\lambda}_j. \tag{28}$$

Notice that during the pure decay stage $t_{j-1} < t < t_{j-1} + \Delta t = t_j$ the Dirichlet quotient is monotonically decreasing (see (19)) so we have

$$\Lambda_{t_j^-} \leq \Lambda_{t_{j-1}^+}.$$

When this is combined with (28), (27) and a simple iteration we deduce

$$\Lambda_{t_j^+} \leq \bar{\lambda}_1 \leq \bar{\lambda}_0 = \bar{\Lambda} \stackrel{def}{=} \max\{\Lambda_0, \lambda_2, \lambda_1\} \tag{29}$$

where

$$\Lambda_0 = \frac{\mathcal{E}(q_0)}{E(q_0)}. \tag{30}$$

This proves a uniform in time bound on the Dirichlet quotient.

It remains to prove the claim (26). By the special choice of our random forcing (6) and the small scale vortex (7) we see that the support of ω_j always overlaps with the interior region $Q_{r_0} = (r_0, \pi - r_0) \times (r_0, \pi - r_0)$ of Q since the center of ω_j lies in this subregion Q_{R-0}. This implies that there exist *finitely* many boxes $B_i, i = 1, \cdots, N$, such that $B_i \subset Q_{r_0}$ for all i, there exists a constant C_1 and for each ω_j there exists a B_i satisfying

$$\omega_j \geq C_1 \chi_i, \tag{31}$$

where

$$\chi_i(\vec{x}) = \begin{cases} 1 & \text{if } \vec{x} \in B_i, \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

is the indicator function of the set B_i.

Let ϕ_i be the solution of

$$\Delta \phi_i = \chi_i, \qquad \phi_i|_{\partial Q} = 0. \tag{33}$$

By a standard comparison principle we have

$$-\psi_j \geq -C_1 \phi_i. \tag{34}$$

By Hopf's strong maximum principle (Evans (1998)) we have

$$\frac{\partial \phi_i}{\partial n} > 0, \tag{35}$$

at the boundary ∂Q away from the four corners. More precisely, there exists a constant C_2 such that

$$\frac{\partial \phi_i}{\partial n} > C_2, \tag{36}$$

provided

$$|\vec{x}-(0,0)| \geq \frac{1}{4}r_0, |\vec{x}-(0,\pi)| \geq \frac{1}{4}r_0, |\vec{x}-(\pi,0)| \geq \frac{1}{4}r_0, |\vec{x}-(\pi,\pi)| \geq \frac{1}{4}r_0. \tag{37}$$

When this combined with the negativity of ϕ_i in the interior of the box Q and the choice of ω_j (7), we have

$$-\phi_i \geq C_3\omega_j, \text{ for all } i, \text{ and } j. \tag{38}$$

This combined with (34) yields the claim with

$$\lambda_2 = C_1 \, C_3. \tag{39}$$

To summarize, we have the following result:

Theorem 1 *For the Navier-Stokes system (1) with random kick forcing specified in (6) together with boundary conditions (4), there is a constant $\bar{\lambda}_0$ which depends on the non-negative initial data q_0, and the small scale random vortex ω_r such that*

$$\Lambda(t) \leq \bar{\Lambda} = \max\{\Lambda(q_0), \max \Lambda(\omega_j), \lambda_2\}, \tag{40}$$

for all time, provided that the initial vorticity is non-negative, i.e. $q_0 \geq 0$.

Remark As we mentioned earlier, such a uniform in time bound on the Dirichlet quotient of the flow is non-trivial and indicates some control of the small scales in the flow. Roughly speaking, no scales smaller than the initial small scale or the small scale determined by the forcing could emerge later on. The bound is optimal by considering the special case of zero forcing and the special case of zero initial condition. On the other hand, the bound is not very useful since it is not close to the minimum value of the Dirichlet quotient in our geometry (2). This is somehow expected since we haven't taken dissipation into consideration (we only used the part that the Dirichlet quotient is non-increasing during the decay stage), and we are doing a worst scenario analysis for a stochastic problem. This prompts us to take a stochastic approach in the next section.

3 Stochastic Estimates

Recall that the numerically observed emergence of large scale coherent structure is under the bombardment of random small scale forcing. The analysis in the last section was not sufficient since it is a worst scenario analysis not taking into consideration possible random effects. In this section we will study the problem from a probabilistic perspective. We will start with appropriate approximation of the random kick forcing.

It is easy to see that the random kick can be decomposed into a mean part and a random fluctuation part as

$$\omega_r(\vec{x} - \vec{x}_j) = \bar{\omega}_r + \omega_r'(j), \tag{41}$$

where the mean part is defined as

$$\bar{\omega}_r = \mathbf{E}\omega_r(\vec{x} - \vec{x}_j) \tag{42}$$

with $\mathbf{E}$ being the mathematical expectation operator over the random center, $\vec{x}_j$. It is easy to see that, thanks to (8),

$$\begin{aligned} \bar{\omega}_r &\in H_0^2(Q), \quad \omega_r'(j) \in H_0^2(Q). & (43)\\ \mathbf{E}(\omega_r'(j)) &= 0, & (44)\\ \mathbf{E}(\|\Delta\omega_r'(j)\|_{L^2}^2) &= \mathbf{E}(\|\Delta\omega_r(\vec{x} - \vec{x}_j)\|_{L^2}^2) + \mathbf{E}(\|\Delta\bar{\omega}_r\|_{L^2}^2) \\ &\quad -2\int_Q \Delta\bar{\omega}_r \mathbf{E}(\Delta\omega_r(\vec{x} - \vec{x}_j)) < \infty & (45) \end{aligned}$$

This means that the $\{\omega_r'(j)\}s$ are $H_0^2(Q)$ valued *i.i.d.* random variables.

Next we consider the cumulative effect of the forces. Notice at time t, the flow has been bombarded by

$$\lfloor \frac{t}{\Delta t} \rfloor$$

number of small scale random vortices. Thus the deterministic part of the cumulative forcing effect takes the form

$$\lfloor \frac{t}{\Delta t} \rfloor A\bar{\omega}_r.$$

Since we are in a non-trivial quasi-equilibrium state, it is natural to expect the deterministic part of the forcing to be of order one. This implies that

the amplitude A should scale like the time step Δt between forcing. We thus impose the following
Forcing Scaling Assumption:

$$A = c_A \Delta t \tag{46}$$

where c_A is now a derived parameter. In the rigorous analysis, we will make the additional smallness assumption on c_A as noted elsewhere in the remainder of this paper. The balance in (46) further implies that the deterministic part of the cumulative forcing converges to

$$\int_0^t \bar{\mathcal{F}} \to c_A t \bar{\omega}_r$$

and thus the deterministic instantaneous forcing may be approximated by

$$\bar{\mathcal{F}} = c_A \bar{\omega}_r \tag{47}$$

which is a steady state forcing.

We now consider the fluctuation part. The cumulative forcing up to time t for the fluctuating part takes the form, thanks to the amplitude time step scaling assumption (46)

$$\begin{aligned} \int_0^t \mathcal{F}' &= A \frac{\omega_r'(1) + \cdots + \omega_r'(\lfloor \frac{t}{\Delta t} \rfloor)}{\sqrt{\lfloor \frac{1}{\Delta t} \rfloor}} \sqrt{\frac{1}{\Delta t}} \\ &= c_A \varepsilon \frac{\omega_r'(1) + \cdots + \omega_r'(\lfloor \frac{t}{\Delta t} \rfloor)}{\sqrt{\lfloor \frac{1}{\Delta t} \rfloor}} \\ &\to c_A \varepsilon G(t) \end{aligned} \tag{48}$$

where we have applied an infinite dimensional version of Donsker's invariance principle[1], and $G(t)$ denotes an infinite dimensional Gaussian process and

$$\varepsilon = \sqrt{\Delta t} \tag{49}$$

[1]We are not able to locate a reference for this infinite dimensional version of Donsker's invariance principle. However such an infinite dimensional version is expected and can be verified via appropriate modification of the finite dimensional version provided we have enough decay (in Fourier spaces) and enough smoothness of the fluctuation ω_r'.

is a small parameter. This further implies that the fluctuating part of the random forcing may be modeled as

$$\mathcal{F}' = c_A \varepsilon \frac{dG}{dt}. \tag{50}$$

It is not hard to check, thanks to (45) and the invariance principle, that the Gaussian process $G(t)$ is $H^2(Q)\bigcap H_0^1(Q)$ valued. Indeed, assume that the infinite dimensional Gaussian process G takes the form

$$G(\vec{x}, t, \omega) = \sum_{\vec{k}} b_{\vec{k}} e_{\vec{k}}(\vec{x}) \beta_{\vec{k}}(t, \omega) \tag{51}$$

where $\{e_{\vec{k}}(\vec{x})\}$ is an orthonormal basis for $L^2(Q)$ given by $e_{\vec{k}} = \frac{\pi}{2}\sin(k_1 x)\sin(k_2 y)$ and the $\{\beta_{\vec{k}}(t,\omega)\}$s are standard one dimensional Brownian motions. These Brownian motions are not necessarily independent (in fact many of them are dependent). It is then easy to verify that, using a Galerkin approximation if necessary,

$$\mathbf{E}(\|\Delta G(t)\|_{L^2}^2) = \sum_{\vec{k}} t|\vec{k}|^4 |b_{\vec{k}}|^2 \leq t\mathbf{E}(\|\Delta \omega_r'(1)\|_{L^2}^2). \tag{52}$$

With the two approximations of the forcing introduced above, we may model the original problem as a stochastic partial differential equation in the following form.

Navier-Stokes Equations with Continuous in Time Small Random Forcing

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q = \nu \Delta q + c_A \bar{\omega}_r + c_A \varepsilon \frac{dG}{dt}, \tag{53}$$

$$q = \Delta \psi \tag{54}$$

together with the initial condition (3) and no-penetration, free-slip boundary condition (4). This is the Navier-Stokes equation with a random (white noise type) forcing. The well-posedness of this type of problems (existence and uniqueness of solution etc) is well-known (see for instance, Bensoussan and Temam 1973, Vishik and Fursikov 1988).

Here we are interested in the time asymptotic behavior of the solutions for parameters lying in appropriate regimes. Similar problems of random small perturbation of deterministic dynamical system have been studied mostly for the case of stochastic differential equations utilizing large deviation theory

(see for instance, Freidlin and Wentzell 1984). Our goal here is to show that the eventual state is a large coherent structure similar to the ground state mode $sin(x)sin(y)$ plus small random fluctuations under appropriate assumptions.

3.1 The Zero Noise Limit Problem and Coherent Structure

Since there is a small parameter ε in the continuous version (53) of our problem, we formally set ε to zero and deduce the following zero noise limit problem.

$$\frac{\partial q^0}{\partial t} + \nabla^\perp \psi^0 \cdot \nabla q^0 = \nu \Delta q^0 + c_A \bar{\omega}_r, \tag{55}$$

$$q^0 = \Delta \psi^0 \tag{56}$$

with the same initial/boundary condition. This is the usual deterministic Navier-Stokes equation and we know that for small enough c_A (which translates into small amplitude for individual random bombardments for fixed viscosity ν and size of random vortex r from (46)), the long time dynamics is determined by the unique steady state (see for instance, Temam 1997)

$$\nabla^\perp \psi^0_\infty \cdot \nabla q^0_\infty = \nu \Delta q^0_\infty + c_A \bar{\omega}_r, \tag{57}$$

$$q^0_\infty = \Delta \psi^0_\infty \tag{58}$$

together with the no-penetration / free slip boundary condition (4).

For even smaller relative amplitude c_A, the mean field equation is approximately linearized

$$-\nu \Delta q^0_\infty \approx c_A \bar{\omega}_r \tag{59}$$

$$q^0_\infty = \Delta \psi^0_\infty \tag{60}$$

whose solution is given by

$$q^0_\infty \approx \frac{c_A}{\nu} (-\Delta)^{-1} (\bar{\omega}_r). \tag{61}$$

It is clear that $\bar{\omega}_r$ is a constant within the sub-square $Q_{2r_0} = [2r_0, 2\pi - 2r_0] \times [2r_0, 2\pi - 2r_0]$ and it monotonically decreases to zero at the boundary as $\vec{x}$ moves to the boundary along outward normal direction. Thus $\bar{\omega}_r$ is approximately a constant on Q in L^2 space and the order of the constant is apparently related to r and can be estimated to be the order of r^2, i.e.

$$\bar{\omega}_r \approx r^2 \tag{62}$$

since

$$\omega_r(\vec{x}) \geq \frac{1}{4} \quad \text{for } |\vec{x}|^2 \leq \tfrac{r^2}{2},$$
$$\omega_r(\vec{x}) \leq 1 \quad \text{for } |\vec{x}|^2 \leq r^2.$$

This further implies, for very small relative amplitude c_A and small r_0 ($r_0 \geq r$),

$$q^0_\infty \approx \frac{c_A r^2}{\nu}(-\Delta)^{-1}(1). \tag{63}$$

It is then interesting to calculate $(-\Delta)^{-1}(1)$ and check if it is close to the large scale coherent structure $sin(x)sin(y)$ predicted by equilibrium statistical theory. A straight forward calculation shows

$$(-\Delta)^{-1}(1) = \sum_{l_1=0,l_2=0}^{\infty} \frac{16}{\pi^2(2l_1+1)(2l_2+1)((2l_1+1)^2+(2l_2+1)^2)} \sin((2l_1+1)x)\sin((2l_2+1)y) \tag{64}$$

It is then easy to check that there is an extremely strong correlation between $(-\Delta)^{-1}(1)$ and the ground state mode $\sin(x)\sin(y)$

$$\operatorname{corr}(\sin(x)\sin(y), (-\Delta)^{-1}(1)) \approx 0.99, \tag{65}$$

where the graphical correlation of two functions is given by

$$\operatorname{corr}(f,g) = \frac{\int_Q f\, g}{\|f\|_{L^2}\|g\|_{L^2}}.$$

Such a strong correlation would explain the success of the equilibrium statistical theory observed in numerical experiments provided the asymptotic behavior is governed by the limiting scaling (63). Our numerical experiments (see figure 5) strongly support our heuristic argument here. In figure 5, the top left panel is the graph of $(-\Delta)^{-1}(1)$ which clearly looks like the large scale coherent structure. The upper right panel is the correlation between the vorticity q and $(-\Delta)^{-1}(1)$ which is above 98% for large times. The

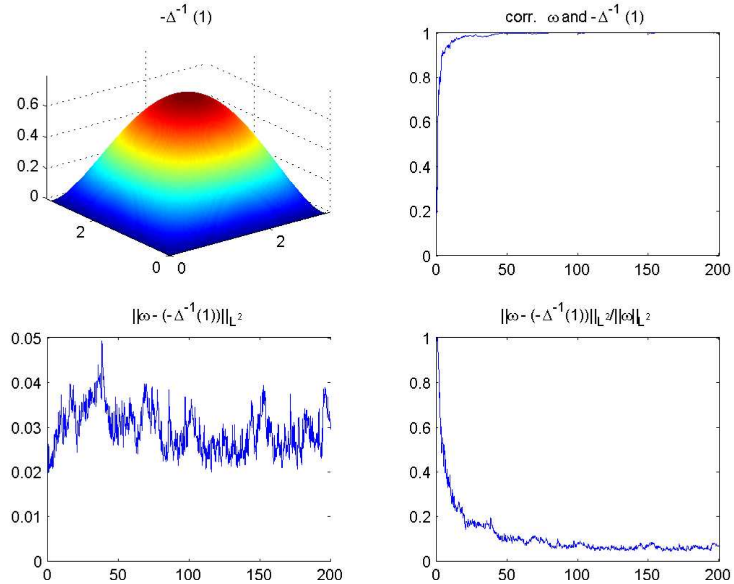

Figure 5: Upper left panel: plot of $(-\Delta)^{-1}(1)$; Upper right panel: correlation between the vorticity field q (or ω) and $(-\Delta)^{-1}(1)$; Lower left corner: L^2 error of the $(-\Delta)^{-1}(1)$theory; Lower right panel: relative L^2 error of the $(-\Delta)^{-1}(1)$ theory.

bottom left panel is the L^2 norm of the component of the vorticity q which is perpendicular to $(-\Delta)^{-1}(1)$. This is the error of the heuristic prediction and remains under 2.5% for large times. The bottom right panel is the relative error of the heuristic prediction which is under 10% for large times.

Strong correlation between the vorticity field q and the heuristic prediction $(-\Delta)^{-1}(1)$, together with strong correlation between $(-\Delta)^{-1}(1)$ and the ground state mode (prediction of statical theory) implies strong correlation between the vorticity field q and the ground state mode $sin(x)sin(y)$ which further validates the equilibrium statistical mechanics theory. Indeed, we have,

Lemma 1 *Assume that the functions g_1, g_2 and g_3 satisfy*

$$corr(g_1, g_3) \geq 1 - \delta_1, \qquad corr(g_2, g_3) \geq 1 - \delta_2. \tag{66}$$

Then

$$corr(g_1, g_2) \geq 1 - (\sqrt{\delta_1} + \sqrt{\delta_2})^2 \tag{67}$$

Proof The proof is elementary.

Without loss of generality, we assume g_1, g_2 and g_3 are unit vectors (their L^2 norms equal to 1). This implies

$$(g_1, g_3) = \mathrm{corr}(g_1, g_3) \geq 1 - \delta_1,$$
$$(g_2, g_3) = \mathrm{corr}(g_2, g_3) \geq 1 - \delta_2.$$

Now consider the orthogonal decomposition of g_1 and g_2 in the direction parallel to g_3 and the complementary direction perpendicular to g_3, i.e.

$$g_1 = r_1 g_3 + g_1', \quad (g_1', g_3) = 0,$$
$$g_2 = r_2 g_3 + g_2', \quad (g_2', g_3) = 0.$$

Thus we have

$$(g_1, g_3) = r_1 \geq 1 - \delta_1,$$
$$(g_2, g_3) = r_2 \geq 1 - \delta_2,$$

which further implies

$$\|g_1'\| = \sqrt{1 - r_1^2} \leq \sqrt{2\delta_1 - \delta_1^2}.$$

Similarly,

$$\|g_2'\| \leq \sqrt{2\delta_2 - \delta_2^2}.$$

Therefore,

$$\begin{aligned}
(g_1, g_2) &= (r_1 g_3 + g_1', r_2 g_3 + g_2') \\
&= r_1 r_2 + (g_1', g_2') \\
&\geq (1 - \delta_1)(1 - \delta_2) - \|g_1'\| \|g_2'\| \\
&\geq (1 - \delta_1)(1 - \delta_2) - \sqrt{(2\delta_1 - \delta_1^2)(2\delta_2 - \delta_2^2)} \\
&\geq 1 - \delta_1 - \delta_2 - 2\sqrt{\delta_1 \delta_2} \\
&\geq 1 - (\sqrt{\delta_1} + \sqrt{\delta_2})^2.
\end{aligned}$$

This ends the proof of the lemma.

In our particular application we have

$$\mathrm{corr}(q, (-\Delta)^{-1}(1)) \geq 1 - 0.02$$
$$\mathrm{corr}(\sin(x)\sin(y), (-\Delta)^{-1}(1)) \geq 1 - 0.02$$

which implies, thanks to the lemma,

$$\mathrm{corr}(q, \sin(x)\sin(y)) \geq 1 - (2\sqrt{0.02})^2 = 0.92$$

which is very good although not as good as what the numerics suggest.

It is interesting to note that there is some discrepancy between the proposed theory above and the numerical evidence. The amplitude of the forcing in our numerical experiments is not small as required in the theory (we have $\nu = 0.01, A = 1, \Delta t = 0.1$). It seems that there are two possible explanations to this numerical fact. First, the radius of the random forcing vortex, r, is small ($r = 1/64$) and thus the deterministic part of the forcing, $\bar{\omega}_r$ is small (scale like r^2, see (62)). Second, the long time dynamics is close to q^0_∞ which is close to the first eigenmode $\sin x \sin y$. It is well known that the first eigenmode is globally stable (see for instance Constantin, Foias and Temam 1988, Marchioro 1986, Majda and Wang 2004 among others). It is then expected that we have a less stringent condition for stability for a small perturbation of the ground eigenmode when compared to Serrin's stability requirement for general profiles. We can also view this as a consequence of the upper semi-continuity of global attractors .

Our goal now is to establish the validity of the zero-noise limit (55) in a rigorous fashion whose long time limiting behavior at small relative amplitude c_A and small radius of forcing vortex (r) is given by (63).

3.2 Justification of the Zero Noise Limit at Finite Times

Our first justification of the zero noise limit is in terms of finite time almost sure pathwise convergence to the zero noise limit (55).

The idea is simple. The continuous time stochastic version (53) does have a small random forcing term $c_A\varepsilon\frac{dG}{dt}$. But $\frac{dG}{dt}$ is not a function but a distribution (generalized function). In order to overcome this difficulty we consider the following change of variable

$$\tilde{q} = q - c_A\varepsilon G. \tag{68}$$

It is easy to check that $\tilde{q}$ satisfies the equation

$$\frac{\partial \tilde{q}}{\partial t} + \nabla^\perp(\tilde{\psi} + c_A\varepsilon\Delta^{-1}G)\cdot\nabla(\tilde{q} + c_A\varepsilon G) = \nu\Delta\tilde{q} + c_A\bar{\omega}_r + \nu c_A\varepsilon\Delta G \tag{69}$$

$$\tilde{q} = \Delta\tilde{\psi} \tag{70}$$

Next we take the difference between the new unknown $\tilde{q}$ and the zero noise solution q^0

$$q' = \tilde{q} - q^0. \tag{71}$$

We then deduce, q' satisfies the following equation

$$\begin{aligned}
&\frac{\partial q'}{\partial t} + \nabla^\perp \psi \cdot \nabla q' + c_A \varepsilon \nabla^\perp \tilde{\psi} \cdot \nabla G + c_A^2 \varepsilon^2 \nabla^\perp (\Delta^{-1} G) \cdot \nabla G + \nabla^\perp (\psi' + c_A \varepsilon \Delta^{-1} G) \cdot \nabla q^0 \\
&= \nu \Delta q' + \nu c_A \varepsilon \Delta G, && (72)\\
q' &= \Delta \psi', && (73)\\
q'|_{t=0} &= 0 = q'|_{\partial Q} = \psi'|_{\partial Q}. && (74)
\end{aligned}$$

Multiplying the equation by q' and integrating over Q we deduce

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|q'\|^2 + \nu \|\nabla q'\|^2 \leq\ & c_A \nu \varepsilon \|\nabla G\| \|\nabla q'\| \\
& + c_A \varepsilon \|\nabla^\perp \tilde{\psi}\| \|\nabla G\|_{L^\infty} \|q'\| \\
& + c_A^2 \varepsilon^2 \|\nabla^\perp \Delta^{-1} G\| \|\nabla G\|_{L^\infty} \|q'\| \\
& + \|\nabla^\perp \psi'\| \|\nabla q^0\|_{L^\infty} \|q'\| \\
& + c_A \varepsilon \|\nabla^\perp \Delta^{-1} G\| \|\nabla q^0\|_{L^\infty} \|q'\| \\
\leq\ & \frac{\nu}{2}\|\nabla q'\|^2 + \varepsilon^2 \frac{c_A^2 \nu}{2} \|\nabla G\|^2 + \frac{1}{2}\|q'\|^2 \\
& + \varepsilon^2 (2 c_A^2 \|\nabla^\perp \tilde{\psi}\|^2 \|\nabla G\|_{L^\infty}^2 + 2 c_A^4 \varepsilon^2 \|\nabla^\perp \Delta^{-1} G\|^2 \|\nabla G\|_{L^\infty}^2 \\
& + 2 c_A^2 \|\nabla^\perp \Delta^{-1} G\|^2 \|\nabla q^0\|_{L_\infty}^2) \\
& + \frac{\|\nabla q^0\|_{L^\infty}}{\sqrt{2}} \|q'\|^2
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{d}{dt}\|q'\|^2 + \nu \|\nabla q'\|^2 \leq\ & (1 + \sqrt{2}\|\nabla q^0\|_{L^\infty}) \|q'\|^2 \\
+\ & \varepsilon^2 (c_A^2 \nu \|\nabla G\|^2 + 4 c_A^2 \|\nabla^\perp \Delta^{-1} G\|^2 \|\nabla G\|_{L^\infty}^2 \\
& + 4 c_A^2 \|\nabla^\perp \Delta^{-1} G\|^2 \|\nabla q^0\|_{L^\infty}^2 + 4 c_A^2 \|\nabla^\perp \tilde{\psi}\|^2 \|\nabla G\|_{L^\infty}^2), \\
\|q'\|\Big|_{t=0} =\ & 0.
\end{aligned}$$

For any fixed time $T \geq 0$, for almost all ω in the probability space (probability one), $G(t, \omega)$ is a continuous function on $[0, T]$ and $G(0, \omega) = 0$, with values in $H^2(Q)\bigcap H_0^1(Q)$ (see (52)). Thus all terms involving G on the right hand side of the energy equation are bounded on $[0,T]$. The zero noise solutions are smooth (see for instance Constantin and Foias, 1988). Let us assume for the moment that $\|\nabla^\perp\tilde{\psi}\|$ is also a function bounded in time. We may then deduce from the energy inequality

$$\frac{d}{dt}\|q'\|^2 + \nu\|\nabla q'\|^2 \leq c_1\|q'\|^2 + c_2\varepsilon^2,$$

$$\|q'\|\Big|_{t=0} = 0,$$

where c_1, c_2 are constants independent of ε. This implies, thanks to the classical Gronwall inequality, that there exist constants κ_1 and κ_2, independent of ε, such that

$$\begin{aligned}\|q'\|_{L^\infty(0,T;L^2(Q))} &= \|q - c_A\varepsilon G - q^0\|_{L^\infty(0,T;L^2(Q))} \leq \kappa_1\varepsilon \\ \|\nabla q'\|_{L^2(0,T;L^2(Q))} &= \|\nabla q - c_A\varepsilon\nabla G - \nabla q^0\|_{L^2(0,T;L^2(Q))} \leq \kappa_2\varepsilon\end{aligned}$$

which further implies

Theorem 2 *For any fixed time T, for almost all ω in the probability space, i.e., with probability one, there exist constants κ_1, κ_2, independent of ε, such that*

$$\|q - q^0\|_{L^\infty(0,T;L^2(Q))} \leq \kappa_1\varepsilon,$$
$$\|\nabla q - \nabla q^0\|_{L^2(0,T;L^2(Q))} \leq \kappa_2\varepsilon.$$

Proof We have almost everything except the time uniform estimate on $\|\nabla^\perp\tilde{\psi}\|$.

For this purpose we multiply the equation for $\tilde{q}$ by $\tilde{q}$ and integrate over Q and we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{q}\|^2 + \nu\|\nabla\tilde{q}\|_{L^2}^2 &\leq c_A\|\bar{\omega}_r\|\|\tilde{q}\| + c_A\nu\varepsilon\|\nabla G\|\|\nabla\tilde{q}\| \\ &\quad + c_A\varepsilon\|\nabla^\perp\tilde{\psi}\|\|\nabla G\|_{L^\infty}\|\tilde{q}\| \\ &\quad + c_A^2\varepsilon^2\|\nabla^\perp\Delta^{-1}G\|\|\nabla G\|_{L^\infty}\|\tilde{q}\| \\ &\leq \frac{\nu}{2}\|\nabla\tilde{q}\|_2 + \frac{1}{2}(1 + 2c_A\varepsilon\|\nabla G\|_{L^\infty})\|\tilde{q}\|^2 \\ &\quad + c_A^2\|\bar{\omega}_r\|^2 + \frac{c_A^2\nu\varepsilon^2}{2}\|\nabla G\|^2 \\ &\quad c_A^4\varepsilon^4\|\nabla^\perp\Delta^{-1}G\|^2\|\nabla G\|_{L^\infty}^2.\end{aligned}$$

Therefore, for almost all ω (probability one)

$$\|\tilde{q}\|_{L^\infty(0,T;L^2(Q))} \leq \kappa$$
$$\|\tilde{q}\|_{L^2(0,T;H^1(Q))} \leq \kappa$$

where κ is a constant uniform for small $\varepsilon(\varepsilon \leq 1)$. This completes the proof of the theorem.

Remark This theorem gives us a clear indication that the deterministic zero noise model is the zero noise limit of the noisy model (53). The pathwise convergence rate of ε is the usual strong convergence rate in stochastic analysis (see for instance Bhattacharya and Waymire 1990, Karatzas and Shreve 1991). On the other hand, the convergence result is weak in the sense that all these constants depends on ω (path dependent) and it is for finite time.

3.3 Zero Noise Limit at Large Times

Our next justification of the heuristic limit will be for long time at the expense of imposing a relative smallness condition on data. We also have an estimate on the variation.

Indeed, the mathematically correct way of writing the continuous in time stochastic version of our problem (53) is

$$dq + (-\nu\Delta q + \nabla^\perp\psi \cdot \nabla q - c_A\bar{\omega}_r)dt = c_A\varepsilon dG \tag{75}$$

where we choose to use the Itô's formulation over the Stratonovich's formulation.

The difference between the noisy solution q and the zero noise solution q^0, i.e. $q' = q - q^0$, satisfies the following Itô differential equation.

$$dq' + (-\nu\Delta q' + \nabla^\perp\psi \cdot \nabla q' + \nabla^\perp\psi' \cdot \nabla q^0)dt = c_A\varepsilon dG.$$

We then formally apply Itô's celebrated formula to

$$\|q'\|^2 = \int_Q (q')^2 d\vec{x}$$

and we have

$$\begin{aligned} d(q')^2 &= 2q'dq' + \frac{1}{2}2c_A^2\varepsilon^2 \sum b_{\vec{k}}b_{\vec{l}}e_{\vec{k}}e_{\vec{l}}c_{\vec{k}\vec{l}}dt \\ &= (2\nu q'\Delta q' - 2\nabla^\perp\psi \cdot \nabla q'q' - 2\nabla^\perp\psi' \cdot \nabla q^0 q' + c_A^2\varepsilon^2 \sum b_{\vec{k}}b_{\vec{l}}e_{\vec{k}}e_{\vec{l}}c_{\vec{k}\vec{l}})dt + 2c_A\varepsilon q' dG \end{aligned}$$

where

$$c_{\vec{k}\vec{l}} = \mathbf{E}(\beta_{\vec{k}}\beta_{\vec{l}}) \tag{76}$$

is the covariance between $\beta_{\vec{k}}$ and $\beta_{\vec{l}}$.

Integrating over Q and utilizing the orthonormal property of $\{e_{\vec{k}}\}$, we have

$$\begin{aligned}
d\|q'\|^2 &= (-2\nu\|\nabla q'\|^2 + c_A^2\varepsilon^2\sum_{\vec{k}} b_{\vec{k}}^2)dt - 2(\int_Q \nabla^{\perp}\psi'\cdot\nabla q^0 q' d\vec{x})dt + 2c_A\varepsilon\int_Q (q', dG)d\vec{x} \\
&\leq (-2\nu\|\nabla q'\|^2 + c_A^2\varepsilon^2\sum_{\vec{k}} b_{\vec{k}}^2 + 2\|\nabla^{\perp}\psi'\|_{L^\infty}\|\nabla q'\|\|q^0\|)dt + 2c_A\varepsilon\int_Q (q', dG)d\vec{x} \\
&\leq (-2\nu\|\nabla q'\|^2 + c_A^2\varepsilon^2\sum_{\vec{k}} b_{\vec{k}}^2 + c_1\|q'\|^{\frac{1}{2}}\|\nabla q'\|^{\frac{3}{2}}\|q^0\|)dt + 2c_A\varepsilon\int_Q (q', dG)d\vec{x} \\
&\leq -2\nu\|\nabla q'\|^2(1 - \frac{c_2\|q^0\|}{\nu})dt + c_A^2\varepsilon^2\sum b_{\vec{k}}^2 dt + 2c_A\varepsilon\int_Q (q', dG)d\vec{x}.
\end{aligned}$$

Now we postulate the smallness condition

$$\frac{c_2\|q^0\|}{\nu} \leq \frac{1}{2}. \tag{77}$$

We then deduce, from the previous inequality,

$$d(e^{\nu t}\|q'(t)\|^2) \leq e^{\nu t}c_A^2\varepsilon^2\sum b_{\vec{k}}^2\, dt + 2c_A\varepsilon e^{\nu t}\int_Q (q', dG)d\vec{x}$$

which further implies

$$e^{\nu t}\|q'(t)\|^2 \leq e^{\nu t_0}\|q'(t_0)\|^2 + (e^{\nu t} - e^{\nu t_0})\frac{c_A^2\varepsilon^2}{\nu}\sum b_{\vec{k}}^2 + 2c_A\varepsilon\int_0^t e^{\nu s}\int_Q (q'(s), dG(s))d\vec{x}.$$

Next, we apply the mathematical expectation operator $\mathbf{E}$ and utilize the martingale property of $\int_0^t e^{\nu s}\int_Q (q', dG)$, we have

$$\mathbf{E}(\|q'(t)\|^2) \leq e^{-\nu(t-t_0)}\mathbf{E}(\|q'(t_0)\|^2) + \frac{c_A^2\varepsilon^2}{\nu}\sum b_{\vec{k}}^2. \tag{78}$$

As shown next, the estimates in (78) and (77) leads to

Theorem 3 *Assume relative weak amplitude of the forcing so that the smallness condition(77) is satisfied for any initial data at large time. Then there exists a constant κ independent of ε (but depend on initial data and all other parameters) such that*

$$\mathbf{E}(\|(q - q^0)(t)\|^2) \leq \kappa\varepsilon^2 \tag{79}$$

for all time.

Proof To complete the proof, we need to show that the first term in (78) is order ε^2.

For small enough c_A, it is easy to see that the zero noise solution q^0 (55) will satisfy the smallness condition (77) (see for instance Doering and Gibbon 1995, Foias, Manley, Rosa and Temam 2001, or Temam 2000 among others).

Indeed, simple calculation reveals

$$\frac{d}{dt}\|q^0\|^2 + \nu\|\nabla q^0\|^2 \leq \frac{c_A^2\|\bar{\omega}_r\|^2}{\nu}$$

Hence we have, after applying Poincaré inequality,

$$\frac{d}{dt}\|q^0\|^2 + 2\nu\|q^0\|^2 \leq \frac{c_A^2\|\bar{\omega}_r\|^2}{\nu}$$

which leads to

$$\|q^0(t)\|^2 \leq e^{-2\nu t}\|q_0\|^2 + \frac{c_A^2\|\bar{\omega}_r\|^2}{4\nu^2}. \tag{80}$$

Thus we will have the smallness condition (77) for large time satisfied provided

$$\frac{c_2^2 c_A^2\|\bar{\omega}_r\|^2}{\nu^4} \leq \frac{1}{2}. \tag{81}$$

Then for

$$t \geq T \stackrel{def}{=} \frac{1}{\nu}\ln\frac{4c_2\|q_0\|^2}{\nu} \tag{82}$$

the inequality (78) holds, which further leads to to our estimate over the interval $[T, \infty)$ provided we can control $\mathbf{E}(\|q'(T)\|^2)$ in the order of ε^2.

For the time interval $[0, T]$, we make a slight change in the inequalities in applying Itô's formula, and we have

$$\frac{d}{dt}\mathbf{E}(\|q'\|^2) \leq -\nu\mathbf{E}(\|\nabla q'\|^2) + \frac{c_3}{\nu^3}\|q^0\|^4\mathbf{E}(\|q'\|^2) + c_A^2\varepsilon^2\sum_{\vec{k}} b_{\vec{k}}^2. \tag{83}$$

This leads to a desired bound on $\mathbf{E}(\|q'\|^2)$ of the order of ε^2 on the time interval $[0, T]$.

The proof is complete after combining the above two estimates.

Remark The theorem states that the expected value of the distance in L^2 from the noisy solution to the heuristic limit of zero noise solution is of the order of ε for all time. This explains the emergence of large coherent structure very well when combined with our earlier discussion on the behavior of the solution to the zero noise equation.

3.4 Convergence of the Random Attractor

It is well-known that the two dimensional Navier-Stokes equations can be viewed as a dissipative dynamical system and it possesses a global attractor (see for instance Temam 1997). This deterministic theory can be extended to the stochastic case (see for instance Arnold 1998, Crauel, Debussche and Flandoli 1997 among others) in terms of random dynamical systems and random attractors. For the sake of exposition we quickly recall some of the relevant notations.

Given a probability space $(\Omega, \mathcal{F}, P)$ and a family of measure preserving maps θ_t on Ω satisfying

$$\theta_0 = id, \theta_{t+s} = \theta_t \theta_s, \tag{84}$$

a random dynamical system is a map ϕ

$$\begin{aligned} \phi : \quad & \mathcal{R}^+ \times \Omega \times H \to H \\ & (t, \omega, u) \mapsto \phi(t, \omega) u \end{aligned}$$

satisfying

$$\begin{aligned} \phi(0, \omega) &= id \\ \phi(t+s, \omega) &= \phi(t, \theta_s \omega) \cdot \phi(s, \omega) \quad \text{(co-cycle property)}. \end{aligned}$$

Here H is the physical phase space of the problem and in our case would be the L^2 space for the vorticity q.

A random set $\mathcal{A}(\omega)$ is called a random attractor of the random dynamical system ϕ if

i) $\mathcal{A}(\omega)$ is compact with "probability" one and $\text{dist}(\{u\}, \mathcal{A}(\omega))$ is measurable for all $u \in H$

ii) $\phi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t \omega)$, for all $t \geq 0$ (invariance)

ii) for any bounded set B in H

$$\lim_{t\longrightarrow\infty} \text{dist}(\phi(t,\theta_{-t}\omega)B, \mathcal{A}(\omega)) = 0$$

with probability one (attracting).

The theory of random attractor applies to our continuous in time stochastic version of our problem (53) in the sense that we can prove the existence of random attractors $\mathcal{A}_\varepsilon(\omega)$ for each $\varepsilon > 0$. Moreover, the random attractors $\mathcal{A}_\varepsilon(\omega)$ are related to the zero noise attractor $\mathcal{A}_0$ of (55) in an upper semi-continuous fashion just as in the deterministic case (see for instance Temam, 1997 or Hale 1988 among others). More precisely we have

Theorem 4 *Let $\mathcal{A}_\varepsilon(\omega)$ be the random attractor of the random dynamical system generated by (53) with noise level ε and let $\mathcal{A}_0$ be the global attractor of the deterministic Navier-stokes system (55). Then $\mathcal{A}_\varepsilon(\omega)$ converges to $\mathcal{A}_0$ in an upper semi-continuous fashion with probability one, i.e.*

$$\lim_{\varepsilon\to 0} dist(\mathcal{A}_\varepsilon(\omega), \mathcal{A}_0) = 0, \quad a.s.. \tag{85}$$

Proof The proof is an application of an upper-semi continuity of attractor for small random perturbation of dynamical system result obtained by Caraballo, Langa and Robinson (1998). In fact they already applied their result to small random perturbation of deterministic two dimensional Navier-stokes equations. The differences are: 1. they used velocity-pressure formulation of the NSE; 2. they only allowed one mode random perturbation, i.e., $G = \phi(\vec{x})\beta(t)$ where ϕ is a fixed mode not necessarily related to eigenmode of the the Laplace operator and β is a standard one dimensional Brownian motion.

There are two main ingredients in the application of the Caraballo, Langa, Robinson result, namely
1. uniform (from bounded initial data) convergence of trajectories with probability one(this is basically the first justification that we presented) and 2. the existence of uniform compact absorbing set. This can be accomplished with minor modification of the argument of Carabolla, Lange, Robinson. We leave out the details and the interested reader my consult the original work of Carabolla, Langa, Robinson (1998) for more information. The completes the proof of this theorem.

The result presented here again explains the emergence of the large scale coherent structure observed in the numerical experiments when we combine this theorem which says the numerical result (long time behavior) should be close to the solution of the zero-noise limit (55) which resembles, as shown earlier, the large scale coherent structure at small amplitude.

Since the long time behavior of the limit deterministic system (55) is unique for small relative amplitude c_A, and since the noisy solution q converges to the zero noise solution (both in terms of trajectories and in terms of attractors), we naturally ponder if there is any uniqueness for the noisy system (53) in terms of long time behavior for small noise (small ε) and small relative amplitude (small c_A). This suggest we should look into the question of whether the random attractors $\mathcal{A}_\varepsilon(\omega)$ generated by the stochastic PDE (53) has only one point (one single stochastic process). If the random attractor $\mathcal{A}_\varepsilon(\omega)$ consists of one stochastic process only, we then deduce that all statistical information must be encoded in this single stochastic process. Indeed, this implies that there exists a unique invariant measure. The support of the measure, which is exactly $\mathcal{A}_\varepsilon$, must be close to $\mathcal{A}_0$ which is a one point set $\{q^0_\infty\}$ for small enough c_A. This would offer an explanation of the emergence of large coherent structure in a way that is closer to traditional statistical theory, namely, there exists a unique invariant measure and the statistics with respect to the unique invariant measure yield a large coherent structure close to q^0_∞ which is asymptotically (up to a scaling) $(-\Delta)^{-1}(\bar{\omega}_r) \approx (-\Delta)^{-1}(1)$ who has an extremely strong correlation with $sin(x)sin(y)$. Indeed, all these heuristics are correct and we have the following theorem.

Theorem 5 *For small enough relative amplitude c_A of the forcing, the continuous time stochastic model (53) possesses a unique invariant measure and the random attractor consists of a single stochastic process $q_\infty(\omega, \varepsilon)$.*

Moreover, we have the following commutative diagram

$$\begin{array}{ccc} q(t,\omega,\varepsilon) & \overrightarrow{t \to \infty} & q_\infty(\omega,\varepsilon) \\ \downarrow \varepsilon \to 0 & & \downarrow \varepsilon \to 0 \\ q^0(t,\omega) & \overrightarrow{t \to \infty} & q^0_\infty \end{array} \tag{86}$$

Furthermore, we also have

$$\lim_{c_A \to 0, r_0 \to 0, r \to 0} corr(q^0_\infty, (-\Delta)^{-1}(1)) = 1. \tag{87}$$

Proof The second part of the theorem is clear since we have,

$$\lim_{c_A \to 0} \mathrm{corr}(q^0_\infty, (-\Delta)^{-1}(\bar{\omega}_r)) = 1 \tag{88}$$

and

$$\lim_{r_0\to 0, r\to 0} \operatorname{corr}((-\Delta)^{-1}(\bar{\omega}_r), (-\Delta)^{-1}(1)) = 1. \tag{89}$$

Therefore, according to lemma 1,

$$\lim_{c_A\to 0, r_0\to 0, r\to 0} \operatorname{corr}(q_\infty^0, (-\Delta)^{-1}(1)) = 1. \tag{90}$$

We have already shown that with probability one trajectories of the noisy system (53) converge to that of the deterministic zero noise limit system (55) as mentioned earlier. It is well-known that the deterministic system has a trivial attractor ($\mathcal{A}_0 = \{q_\infty^0\}$) for small enough relative amplitude c_A. We have also shown in the previous theorem, that the long time behavior of the noisy system (53) in terms of the attractors $\mathcal{A}_\varepsilon$ converge to the trivial attractor $\mathcal{A}_0$ as ε approaches zero. (In this case, upper semi-continuity implies continuity since $\mathcal{A}_0$ consists of one point only.) The convergence of attractors also follows from Theorem 3. Thus we only need to show that the random attractor of the noisy system consists of one single stochastic process q_∞^ε in order to establish the validity of the commutative diagram (86). In fact, if each random attractor at a fixed noise level consists of one stochastic process, the long time convergence result stated in Theorem 2 and the fact that the deterministic attractor of the zero noise system is trivial implies the convergence of random attractors without invoking theorem 3.

In order to show that the random attractor of the stochastic system (53) consists of one point only, we only need to show that there is contraction of the phase space under the dynamics. More precisely, we need to show that for almost all ω (probability one), and any two initial data q_{01}, q_{02}, the solutions starting from q_{01} and q_{02}, denoted $q^{(1)}$ and $q^{(2)}$ converge together, i.e., $\|q^{(2)} - q^{(1)}\| \to 0$, as $t \to \infty$.

Indeed, $q' = q^{(2)} - q^{(1)}$ satisfies the following equations

$$\begin{aligned}
\frac{\partial q'}{\partial t} + \nabla^\perp \psi^{(2)} \cdot \nabla q' + \nabla^\perp \psi' \cdot \nabla q^{(1)} &= \nu \Delta q', \\
q' &= \Delta \psi', \\
q'|_{t=0} &= q_{02} - q_{01}.
\end{aligned}$$

Multiplying the equation by q' and integrating over Q we deduce

$$\begin{aligned}
\frac{d}{dt}\|q'\|^2 &\leq -2\nu\|\nabla q'\|^2 + 2\|\nabla^\perp \psi'\|_{L^\infty}\|q'\|\|\nabla q^{(1)}\| \\
&\leq -(4\nu - c_4\|\nabla q^{(1)}\|)\|q'\|^2
\end{aligned} \tag{91}$$

which further implies

$$\|q'(t)\|^2 \leq \|q_{02}-q_{01}\|^2 \exp(-4\nu t(1-\frac{c_4}{4\nu t}\int_0^t \|\nabla q^{(1)}(s)\|\,ds)). \tag{92}$$

Therefore, we will have exponential contraction provided that

$$\frac{1}{t}\int_0^t \|\nabla q^{(1)}(s)\|\,ds \leq \frac{2\nu}{c_4} \tag{93}$$

is satisfied for large t.

In order to estimate $\frac{1}{t}\int_0^t \|\nabla q^{(1)}(s)\|\,ds$, we apply Itô's formula again utilizing (75)

$$\begin{aligned} dq^2 &= 2qdq + \frac{1}{2}2c_A^2\varepsilon^2 \sum b_{\vec{k}} b_{\vec{l}} e_{\vec{k}} e_{\vec{l}} c_{\vec{k}\vec{l}} dt \\ &= (2\nu q\Delta q - 2\nabla^\perp\psi\cdot\nabla q\cdot q + 2qc_A\overline{\omega}_r + c_A^2\varepsilon^2 \sum b_{\vec{k}} b_{\vec{l}} e_{\vec{k}} e_{\vec{l}} c_{\vec{k}\vec{l}})dt + 2c_A\varepsilon q dG. \end{aligned}$$

Integrating over Q we deduce

$$\begin{aligned} d\|q\|^2 &\leq (-2\nu\|\nabla q\|^2 + 2c_A\|\overline{\omega}_r\|\|q\| + c_A^2\varepsilon^2\sum b_{\vec{k}}^2)dt + 2c_A\varepsilon\sum_{\vec{k}} b_{\vec{k}}\hat{q}_{\vec{k}} d\beta_{\vec{k}} \\ &\leq (-\nu\|\nabla q\|^2 + \frac{c_A^2}{\nu}\|\overline{\omega}_r\|^2 + c_A^2\varepsilon^2\sum b_{\vec{k}}^2)dt + 2c_A\varepsilon\sum_{\vec{k}} b_{\vec{k}}\hat{q}_{\vec{k}} d\beta_{\vec{k}}, \end{aligned}$$

where

$$q(t) = \sum_{\vec{k}} \hat{q}_{\vec{k}}(t) e_{\vec{k}}(\vec{x}).$$

This further implies

$$\begin{aligned} \frac{1}{t}\int_0^t \|\nabla q(s)\|^2 ds &\leq \frac{1}{t\nu}\|q_0\|^2 + \frac{c_A^2}{\nu^2}\|\overline{\omega}_r\|^2 + \frac{c_A^2\varepsilon^2}{\nu}\sum b_{\vec{k}}^2 \\ &\quad + \frac{2c_A\varepsilon}{\nu}\frac{1}{t}\int_0^t \sum_{\vec{k}} b_{\vec{k}}\hat{q}_{\vec{k}} d\beta_{\vec{k}}. \end{aligned}$$

It is clear that the first three terms on the right hand side can be made small, i.e.

$$\frac{1}{t\nu}\|q_0\|^2 + \frac{c_A^2}{\nu^2}\|\overline{\omega}_r\|^2 + \frac{c_A^2\varepsilon^2}{\nu}\sum b_{\vec{k}}^2 \leq \frac{\nu^2}{c_4^2}, \tag{94}$$

provided we take t large (large time) for the first term, c_A small (small relative amplitude) for the second term and either small c_A (small relative amplitude) or small ε from the third term. Thus we are only left to deal with the last term

$$\frac{2c_A\varepsilon}{\nu}\sum_{\vec{k}}\frac{1}{t}M(t)$$

where

$$M(t)=\int_0^t\sum_{\vec{k}}b_{\vec{k}}\widehat{q}_{\vec{k}}d\beta_{\vec{k}}. \tag{95}$$

is a martingale.

We want to show that $\sup_{0\leq\tau\leq t}M(\tau)$ does not grow too fast. In fact, we want to show that

$$\sup_{0\leq\tau\leq t}M(\tau)\leq\frac{\nu^3 t}{c_4^2 c_A\varepsilon} \tag{96}$$

almost surely for large t, since this would imply the contraction because

$$\frac{1}{t}\int_0^t\|\nabla q(s)\|\,ds\leq(\frac{1}{t}\int_0^t\|\nabla q(s)\|^2\,ds)^{\frac{1}{2}}\leq\frac{\nu}{c_4}.$$

The supremum of a (local) martingale can be estimated utilizing Burkhölder's inequality (see for instance Bhattacharya and Waymire 1990, Karatzas and Shreve 1991). For this purpose we need to consider the quadratic variation of M, namely

$$\begin{aligned}
[M,M](t) &\leq \sum_{\vec{k},\vec{l}}\int_0^t|b_{\vec{k}}||b_{\vec{l}}||\widehat{q}_{\vec{k}}||\hat{q}_{\vec{k}}||c_{\vec{k},\vec{l}}|ds\\
&\leq \bar{b}_2^2\int_0^t\sum_{\vec{k},\vec{l}}\frac{1}{|\vec{k}|^2|\vec{l}|^2}|\hat{q}_{\vec{k}}||\hat{q}_{\vec{l}}|\,ds\\
&\leq \bar{b}_2^2\int_0^t(\sum_{\vec{k}}\frac{|\hat{q}_{\vec{k}}|}{|\vec{k}|^2})^2\,ds\\
&\leq c_5\bar{b}_2^2\int_0^t\sum_{\vec{k}}|\hat{q}_{\vec{k}}|^2ds\\
&= c_5\bar{b}_2^2\int_0^t\|q\|^2
\end{aligned}$$

where

$$\bar{b}_2 = \sup_{\vec{k}} |b_{\vec{k}}||\vec{k}|^2 < \infty, \tag{97}$$

$$c_5 = \sum_{\vec{k}} \frac{1}{|\vec{k}|^4} < \infty, \tag{98}$$

by direct calculation and (52).

Next we apply Itô formula to $\|q\|^{2p}$ and we have

$$\begin{aligned} d\|q(t)\|^{2p} &= 2p\|q(t)\|^{2(p-1)}(-\nu\|\nabla q(t)\|^2 + c_A\|\bar{\omega}_r\|\|q(t)\|)\,dt + c_A\varepsilon\int_Q q(t)\,dG\,d\vec{x} \\ &\quad +2p(p-1)\|q(t)\|^{2(p-2)}c_A^2\varepsilon^2\sum_{\vec{k},\vec{l}} b_{\vec{k}}\widehat{q}_{\vec{k}}\cdot b_{\vec{l}}q_{\vec{l}}c_{\vec{k},\vec{l}}dt \\ &\quad +p\|q(t)\|^{2(p-1)}c_A^2\varepsilon^2\sum b_{\vec{k}}^2 dt. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d}{dt}\mathbf{E}(\|q(t)\|^{2p}) &\le \mathbf{E}(-2\nu p\|q(t)\|^{2p} + 2pc_A\|\bar{\omega}_r\|\|q(t)\|^{2p-1} \\ &\quad +2p(p-1)c_A^2\varepsilon^2 c_5\bar{b}_2^2\|q(t)\|^{2p-2} \\ &\quad +pc_A^2\varepsilon^2\sum b_{\vec{k}}^2\|q(t)\|^{2p-2}) \\ &\le -\nu p\mathbf{E}(\|q(t)\|^{2p}) + (\frac{2pc_A\|\bar{\omega}_r\|}{(\nu p/2)^{\frac{2p-1}{2p}}})^{2p} \\ &\quad +(\frac{2p(p-1)c_A^2\varepsilon^2 c_5\bar{b}_2^2 + pc_A^2\varepsilon^2\sum b_{\vec{k}}^2}{(\nu p/2)^{\frac{p-1}{p}}})^p. \end{aligned}$$

Thus

$$\mathbf{E}(\|q(t)\|^{2p}) \le e^{-\nu pt}\|q_0\|^{2p} + \frac{1}{p\nu}\{(\frac{2pc_A\|\bar{\omega}_r\|}{(\nu p/2)^{\frac{2p-1}{2p}}})^{2p} + (\frac{2p(p-1)c_A^2\varepsilon^2 c_5\bar{b}_2^2 + pc_A^2\varepsilon^2\sum b_{\vec{k}}^2}{(\nu p/2)^{\frac{p-1}{p}}})^p\}.$$

Now combining these estimates, together with Burkhölder inequality, Chebyshev inequality and Hölder's inequality, we have

$$\begin{aligned}
\text{Prob}\{\sup_{0\le t\le n} M(t) \ge \delta n^{\alpha}\} &\le \frac{\mathbf{E}((\sup_{0\le t\le n} M(t))^{2p})}{(\delta n^{\alpha})^{2p}} \\
&\le c_6 \frac{\mathbf{E}([M,M](n))^p}{\delta^{2p} n^{2\alpha p}} \\
&\le \frac{c_6 c_5^p \bar{b}_2^{2p} n^{p-1}}{\delta^{2p} n^{2\alpha p}} \int_0^n \mathbf{E}(\|q\|^{2p}) \\
&\le \frac{c_6 c_5^p \bar{b}_2^{2p}}{\delta^{2p} n^{2\alpha p - p + 1}} \{e^{-\nu p n} \|q_0\|^{2p} \\
&\quad + \frac{1}{p\nu}((\frac{2pc_A \|\bar{\omega}_r\|}{(\nu p/2)^{\frac{2p-1}{2p}}})^{2p} + (\frac{2p(p-1)c_A^2 \varepsilon^2 c_5 \bar{b}_2^2 + pc_A^2 \varepsilon^2 \sum b_{\vec{k}}^2}{(\nu p/2)^{\frac{p-1}{p}}})^p)\}
\end{aligned}$$

Next, we set

$$\delta = \frac{\nu^3}{2c_4^2 c_A \varepsilon}, \quad \alpha = 1, \quad p = 2,$$

and we see that

$$\sum_n Prob\{\sup_{0\le t\le n} M(t) \ge \frac{\nu^3}{2c_4^2 c_A \varepsilon} n\} \le c_7 \sum \frac{1}{n^3} < \infty.$$

Thus the Borel-Cantelli lemma tells us that with probability one, for each ω, there exists an $N(\omega)$, such that

$$\frac{1}{n} \sup_{0\le t\le n} M(t) \le \frac{\nu^2}{2c_4^2 c_A \varepsilon} \text{ for all } n \ge N(\omega).$$

Hence if $N(\omega) \le n < t < n+1$

$$\frac{1}{t} \sup_{0\le \tau\le t} M(\tau) \le \frac{1}{t} \sup_{0\le t\le n+1} M(t) \le \frac{n+1}{t}(\frac{\nu^3}{2c_4^2 c_A \varepsilon}) \le \frac{n+1}{n}(\frac{\nu^3}{2c_4^2 c_A \varepsilon}) \le \frac{\nu^3}{c_4^2 c_A \varepsilon} \quad \forall t \ge N(\omega).$$

This completes the proof of the theorem.

Remark There have been an intensive effort on the study of the uniqueness of invariant measure for randomly forced PDEs (see for instance E 2001, E, Mattingly and Sinai 2001, Eckman and Hairer 2001, Kuksin and Shirikyan

2001, Masmoudi and Young 2002, Mattingly 1999, Schmalfuss 1998 among others. See also Da Prato and Zabczyk, 1992, 1996). The uniqueness of invariant measure part of our result resembles those of Mattingly 1999 and Schmalfuss 1998. The differences are: 1. We consider forcing with a deterministic part and a random fluctuation part while the other authors considered random fluctuation only. We believe our setting is closer to physical reality where the overall mean (expectation) of physically realistic systems does not necessarily vanish; 2. We consider dependent Brownian fluctuations. This is generic if the randomness is introduced in the physical space (not frequency space) as is discussed here.

4 Conclusion and remarks

We have demonstrated both numerically and theoretically that small scale random forcing may induce large scale coherent structure in two dimension flow problems. Moreover, the large scale coherent structure is well predicted by equilibrium statistical theory utilizing energy-enstrophy as conserved quantities or energy-circulation as conserved quantities (see Majda and Wang 2004, Grote and Majda 1997) although the mean field predicted by the rigorous theory is different from the mean field predicted by the equilibrium statistical theory.

The main result can be generalized in some straightforward fashion. For instance, we can allow different probability distributions for the center $\vec{x_j}$ of the random small scale forcing. Also we may allow the random vortices to change signs as long as the mean does not vanish (so that the deterministic part does not vanish). Of course, the emerging large scale vortex will be changed as well. Other generalization such as to systems on different domain/geometry and more general one layer/multi-layer system (see for instance Gill, 1982, Majda, 2003, Majda and Wang, 2004, Pedlosky, 1987 among others) can be considered as well without much difficulty. We are especially interested in the geophysical effects (β-plane, F-plane, topography, Ekman damping etc.) The crude closure numerical algorithms give a variety of interesting new behavior when such geophysical effects are included (Grote and Majda, 2000; DiBattista, Majda and Grote, 2001). There is a recent statistical theory which successfully predicts the Great Red Spot of Jupiter in a fashion consistent with the observations from the Galileo and Voyager missions (Turkington *et al* 2001; Majda and Wang 2004); the idea of small scale bombardment of random vortices creating a large scale coherent structure is crucial in that work; that assumption is also supported by the observational record (Ingersoll *et al*, 2000). One of the challenges is to deduce if realistic large scale coherent structures such the Great Reel Spot can be predicted rigorously using the approach of this paper. We are far from our goal at this time.

There are many other issues to be considered so far as the theoretical problem is considered. For instance, what if the smallness (of the relative amplitude) assumption is violated ? We are then close to the situation of

$$\frac{\partial q}{\partial t} + J(\phi, q) = \nu \Delta q + \overline{\mathcal{F}} + \varepsilon \frac{dG}{dt} \tag{99}$$

where $\overline{\mathcal{F}}$ is not small so that the deterministic system (zero noise system) has non-trivial long time dynamics (non-trivial global attractor). This scenario is similar to the classical SRB measure problem for a finite dimensional dynamical system. In that case, a unique (distinguished) invariant measure, the SRB measure, is the one selected by the vanishing noise limit (see Young 2001) with appropriate assumptions on the system and noise. In our infinite dimensional setting, we anticipate that invariant measure for such noisy systems remains unique when all determining modes are forced independently. This can be done via appropriate modification of the works of E, Mattingly and Sinai (2001) (see also E and Liu 2002). A more interesting issue is the limit of such invariant measures at vanishing noise. It seems that we are able to show that this set of invariant measures (with noise level ε as parameter) is tight, and any limit should be an invariant measures of the zero-noise deterministic system. It would be very interesting to determine if the limit is unique since if it is unique, then the limiting invariant measure has the distinguished role of an SRB measure in the infinite dimensional setting, and thus all statistics should be performed utilizing this distinguished invariant measure.

Going back to our theoretical problem, we would also like to know if the invariant measure still remains unique if the smallness condition is violated. Our situation differs from the ones available in existing literature in two ways: we have a non-trivial mean part, and more importantly, the Brownian motions on different modes may be dependent (in our case it is intuitively two dimensional only since the distribution is determined with two parameters only). Uniqueness of invariant measure when not all determining modes are independently randomly forced is a major open problem (see E 2001, E and

Mattingly 2001, Eckmann and Hairer 2001 among others). Another related issue is what happens when the mean (deterministic) part of the forcing vanishes. Then the problem resembles those studied by E, Mattingly and Sinai (2001), Kuksin (2002). Again we encounter the difficulty of mode-wise dependent random forcing.

Lastly, we treated the continuous problem here only. We may then naturally ponder what happens to the original kick forcing problem? Does the discrete problem also have a large scale coherent structure? Is the limit of the discrete problem the continuous problem as we studied here? We will provide the answer to some of these problems in the near future in a separate manuscript.

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