# Lipschitz classes of A-harmonic functions in Carnot groups

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**Abstract**

The Hölder continuity of a harmonic function is characterized by the growth of its gradient. We generalize these results to solutions of certain subelliptic equations in domains in Carnot groups.

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## 1 Introduction

Theorem 1.1 follows from results in [7]

**Theorem 1.1** *Let $u$ be harmonic in the unit disk $\mathbb{D} \subset \mathbb{R}^2$ and $0 < \alpha \leq 1$. If there exists a constant $C_1$ such that*

$$|\nabla u(z)| \leq C_1(1 - |z|)^{\alpha - 1} \tag{1}$$

*for all $z \in \mathbb{D}$, then there exists a constant $C_2$, depending only on $\alpha$ and $C_1$, such that*

$$\sup\{\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} : x_1, x_2 \in \mathbb{D}, x_1 \neq x_2\} \leq C_2. \tag{2}$$

We give generalizations in Section 5. Theorem 5.1 characterizes local Lipschitz conditions for A-harmonic functions in domains in Carnot groups by the growth of a local average of the horizontal gradient. These functions are solutions to certain subelliptic equations. Theorem 5.2 gives global results in Lipschitz extension domains. In Section 2 we describe Carnot groups. Section 3 presents subelliptic equations and integral inequalities for their solutions. In Section 4 appears Lipschitz conditions and extension domains.

## 2 Carnot groups

A Carnot group is a connected, simply connected, nilpotent Lie group $G$ of topological $\dim G = N \geq 2$ equipped with a graded Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$ so that $[V_1, V_i] = V_{i+1}$ for i=1,2,...,r-1 and $[V_1, V_r] = 0$. As usual, elements of $\mathcal{G}$ will be identified with left-invariant vectors fields on $G$. We fix a left-invariant Riemannian metric $g$ on $G$ with $g(X_i, X_j) = \delta_{ij}$. We denote the inner product with respect to this metric, as well as all other inner products, by $\langle , \rangle$. We assume that $\dim V_1 = m \geq 2$ and fix an orthonormal basis of $V_1 : X_1, X_2, ..., X_m$. The horizontal tangent bundle of $G$, $HT$ , is the subbundle determined by $V_1$ with horizontal tangent space $HT_x$ the fiber $span[X_1(x), ..., X_m(x)]$. We use a fixed global coordinate system as $\exp : \mathcal{G} \to G$ is a diffeomorphism. We extend $X_1, ..., X_m$ to an orthonormal basis $X_1, ..., X_m, T_1, ..., T_{N-m}$ of $\mathcal{G}$. All integrals will be with respect to the bi-invariant Harr measure on $G$ which arises as the push-forward of the Lebesque measure in $\mathbb{R}^N$ under the exponential map. We denote by $|E|$ the measure of a measurable set $E$. We normalize the Harr measure so that the measure of the unit ball is one. We denote by $Q$ the homogeneous dimension of the Carnot group $G$ defined by $Q = \sum_{i=1}^{r} i\dim V_i$. We write $|v|^2 = \langle v, v\rangle$, $d$ for the distributional exterior derivative and $\delta$ for the codifferential adjoint. We use the following spaces where $U$ is an open set in $G$ :

$C_0^\infty(U)$: infinitely differentiable compactly supported functions in $U$,

$HW^{1,q}(U)$ : horizontal Sobolev space of functions $u \in L^q(U)$ such that the distributional derivatives $X_i u \in L^q(U)$ for $i = 1$,...,m.

When u is in the local horizontal Sobolev space $HW^{1,q}_{loc}(U)$ we write the horizontal differential as $d_0 u = X_1 u dx_1 + ... + X_m u dx_m$. ( The horizontal gradient $\nabla_0 u = X_1 u X_1 + ... + X_m u X_m$ appears in the literature. Notice that $|d_0 u| = |\nabla_0 u|$.)

The family of dilations on $G$, $\{\delta_t : t > 0\}$, is the lift to $G$ of the automorphism $\delta_t$ of $\mathcal{G}$ which acts on each $V_i$ by multiplication by $t^i$. A path in $G$ is called horizontal if its tangents lie in $V_1$. The (left-invariant) Carnot-Carathéodory distance , $d_c(x, y)$ , between $x$ and $y$ is the infimum of the lengths, measured in the Riemannian metric $g$, of all horizontal paths which join $x$ to $y$. A homogeneous norm is given by $|x| = d_c(0, x)$. All homogeneous norms on $G$ are equivalent as such $|\cdot|$ is equivalent to the homogeneous norms used below. We have $|\delta_t(x)| = t|x|$. We write $B(x, r) = \{y \in G : |x^{-1}y| < r\}$ for the ball centered at $x$ of radius $r$. Since the Jacobian determinant of the dilation $\delta_r$ is $r^Q$ and we have normalized the measure, $|B(x, r)| = r^Q$. For $\sigma \geq 1$ we write $\sigma B$ for the ball with the same center as $B$ and $\sigma$ times the radius.

We write $\Omega$ throughout for a connected open subset of $G$. We give some examples of Carnot groups.

**Example 2.1** *Euclidean space $\mathbb{R}^n$ with its usual Abelian group structure is a Carnot group. Here $Q = n$ and $X_i = \partial/\partial x_i$.*

**Example 2.2** *Each Heisenberg group $H_n$, $n \geq 1$, is homeomorphic to $\mathbb{R}^{2n+1}$. They form a family of noncomutative Carnot groups which arise as the nilpotent part of the Iwasawa decomposition of $U(n, 1)$, the isometry group of the complex n-dimensional hyperbolic space. Denoting points* *in $H_n$ by $(z,t)$ with $z=(z_1,...,z_n)\in\mathbb{C}^n$ and $t\in\mathbb{R}$ we have the group law given as*

$$(z,t)\circ(z',t')=(z+z',t+t'+2\sum_{j=1}^{n}Im(z_j\bar{z}_j')). \tag{3}$$

*With the notation $z_j=x_j+iy_j$, the horizontal space $V_1$ is spanned by the basis*

$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t} \tag{4}$$

$$Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t}. \tag{5}$$

*The one dimensional center $V_2$ is spanned by the vector field $T=\partial/\partial t$ with commutator relations $[X_j,Y_j]=-4T$. All other brackets are zero. The homogeneous dimension of $H_n$ is $Q=2n+2$. A homogeneous norm is given by*

$$N(z,t)=(|z|^4+t^2)^{1/4}. \tag{6}$$

**Example 2.3** *A Generalized Heisenberg group, or H-type group, is a Carnot group with a two-step Lie algebra $\mathcal{G}=V_1\oplus V_2$ and an inner product $\langle,\rangle$ in $\mathcal{G}$ such that the linear map $J:V_2\to EndV_1$ defined by the condition*

$$\langle J_z(u),v\rangle=\langle z,[u,v]\rangle, \tag{7}$$

*satifies*

$$J_z^2=-\langle z,z\rangle\mathbf{Id} \tag{8}$$

*for all $z\in V_2$ and all $u,v\in V_1$. For each $g\in G$, let $v(g)\in V_1$ and $z(g)\in V_2$ be such that $g=\exp(v(g)+z(g))$. Then*

$$N(g)=(|v(g)|^4+16|z(g)|^2)^{1/4} \tag{9}$$

*defines a homogeneous norm in $G$. For each $l\in\mathbb{N}$ there exist infinitely many generalized Heisenberg groups with $dimV_2=l$. These include the nilpotent groups in the Iwasawa decomposition of the simple rank-one groups $SO(n,1),SU(n,1),Sp(n,1)$ and $F_4^{-20}$.*

See [1] [14] and [5] for material about these groups.

# 3 Subelliptic equations

We consider solutions to equations of the form

$$\delta A(x,u,d_0u)=B(x,u,d_0u) \tag{10}$$

where $u\in HW^{1,p}(\Omega)$ and $A:\Omega\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^m$ , $B:\Omega\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$ are measurable and for some $p>1$ satisfy the structural equations :

$$|A(x,u,\xi)|\leq a_0|\xi|^{p-1}+(a_1(x)|u|)^{p-1},$$
$$\xi\cdot A(x,u,\xi)\geq|\xi|^p-(a_2(x)|u|)^p,$$
$$|B(x,u,\xi)|\leq b_1(x)|\xi|^{p-1}+(b_2(x))^p|u|^{p-1}$$

with $(x,u,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^N$. Here $a_0 > 0$ and $a_i(x), b_i(x), i = 1,2$, are measurable and nonnegative and are assumed to belong to certain subspaces of $L^t(\Omega)$, where $t = \max(p,Q)$. See [11]. We refer to these quantities as the structure constants.

A weak solution to (10) means that

$$\int_\Omega \{\langle A(x,u,d_0u), d_0\phi\rangle - \phi B(x,u,d_0u)\}dx = 0$$

for all $\phi \in C_0^\infty(\Omega)$.

We use the exponent $p > 1$ for this purpose throughout. We assume that u is a solution to (10) in $\Omega$ throughout. We may assume that u is a continuous representative [8]. We write $u_B$ for the average of $u$ over $B$.

We use the following results.

**Theorem 3.1** *Here $C$ is a constant independent of $u$.*
*a) (Poincaré-Sobolev inequality) If $0 < s < \infty$.*

$$\int_B |u - u_B|^s \le C|B|^{s/Q} \int_B |d_0u|^s. \tag{11}$$

*for all balls $B \subset \Omega$.*
*b) If $s > p - 1$, then*

$$|u(x) - c| \le C(\frac{1}{|B|}\int_{\sigma B} |u - c|^s)^{1/s} \tag{12}$$

*for all $x \in B$, $\sigma B \subset \Omega$ and any constant $c$.*
*c) If $0 < s, t < \infty$, then*

$$(\frac{1}{|B|}\int_B |u - u_B|^t)^{1/t} \le C(\frac{1}{|B|}\int_{\sigma B} |u - u_B|^s)^{1/s}. \tag{13}$$

*for any $\sigma B \subset \Omega$.*
*d) (A Caccioppoli inequality)*

$$\int_B |d_0u|^p \le C|B|^{-p/Q} \int_{\sigma B} |u - c|^p \tag{14}$$

*for any constant $c$ and $\sigma B \subset \Omega$.*

See [8],[2],[9], [6] and [11].

**Theorem 3.2** *There exists an exponent $p' > p$, depending only on $Q, p, s$ and the structure constants, and there exists a constant $C$, depending only on $Q, p, s, \sigma$ and the structure constants, such that*

$$(\frac{1}{|B|}\int_B |d_0u|^{p'})^{1/p'} \le C(\frac{1}{|B|}\int_{\sigma B} |d_0u|^s)^{1/s} \tag{15}$$

*for $s > 0$ and all balls $B$ with $\sigma B \subset \Omega$.*

Proof : We combine the Caccioppoli estimate (14), inequality (13) and the Poincaré-Sobolev inequalty (11),

$$(\frac{1}{|B|}\int_B |d_0u|^p)^{1/p} \leq C|B|^{-1/Q}(\frac{1}{|B|}\int_{\sqrt{\sigma}B}|u-u_{\sqrt{\sigma}B}|^p)^{1/p}$$

$$\leq C|B|^{-1/Q}\frac{1}{|B|}\int_{\sigma B}|u-u_{\sigma B}|$$

$$\leq C\frac{1}{|B|}\int_{\sigma B}|d_0u|.$$

This is a reverse Hölder inequlity. As such it improves to all positive exponents on the right hand side and to some exponent $p' > p$ on the left. See [9],[2] and [8].

For $E \subset G$ we write $osc(u, E) = sup_E u - inf_E u$.

**Theorem 3.3** *Let $0 < s < \infty$. There is a constant $C$, depending only on s,p,Q,$\sigma$ and the structure constants such that*

$$osc(u, B) \leq C|B|^{(s-Q)/sQ}(\int_{\sigma B}|d_0u|^s)^{1/s} \tag{16}$$

*for all balls $B$ with $\sigma B \subset \Omega$.*

Proof : Fix $B$ with $\sigma B \subset \Omega$ and $x, y \in B$. Using (12) with $s = p$, the Poincaré inequality (11) and (15),

$$|u(x) - u(y)| \leq |u(x) - u_{\sqrt{\sigma}B}| + |u(y) - u_{\sqrt{\sigma}B}|$$

$$\leq C(\frac{1}{|B|}\int_{\sqrt{\sigma}B}|u-u_{\sqrt{\sigma}B}|^p)^{1/p}$$

$$\leq C|B|^{(p-Q)/pQ}(\int_{\sqrt{\sigma}B}|d_0u|^p)^{1/p}$$

$$\leq C|B|^{(s-Q)/sQ}(\int_{\sigma B}|d_0u|^s)^{1/s}.$$

When $p > Q$ Theorem 3.3 holds for all $u \in HW^{1,p}(\sigma B)$, see [8].

The last result follows from Harnack's inequality and also appears in [8].

**Theorem 3.4** *There exists constants $\beta, 0 < \beta \leq 1$ and $C$, depending only on p,Q and the structure constants, such that*

$$osc(u, B) \leq C\sigma^{-\beta} osc(u, \sigma B) \tag{17}$$

*for all balls $B$ with $\sigma B \subset \Omega$ with $\sigma \geq 1$.*

# 4 Lipschitz classes and domains

We use the following notations for $f : \Omega \to \mathbb{R}^m$ and $0 < \alpha \leq 1$,

$$||f||^{\alpha} = \sup\{|f(x_1) - f(x_2)|/d_c(x_1, x_2)^{\alpha} : x_1, x_2 \in \Omega, x_1 \neq x_2\},$$

$$||f||^{\alpha}_{\partial} = \sup\{|f(x_1) - f(x_2)|/(d_c(x_1, x_2) + d_c(x_1, \partial\Omega))^{\alpha} : x_1, x_2 \in \Omega, x_1 \neq x_2\},$$

$$||f||^{\alpha}_{loc} = \sup\{|f(x_1) - f(x_2)|/d_c(x_1, x_2)^{\alpha} : x_1, x_2 \in \Omega, x_1 \neq x_2, \\ d_c(x_1, x_2) < d_c(x_1, \partial\Omega)\},$$

$$||f||^{\alpha}_{loc,\partial} = \sup\{|f(x_1) - f(x_2)|/(d_c(x_1, x_2) + d_c(x_1, \partial\Omega))^{\alpha} : x_1, x_2 \in \Omega, x_1 \neq x_2, \\ d_c(x_1, x_2) < d_c(x_1, \partial\Omega)\}.$$

Notice

$$||f||^{\alpha}_{loc,\partial} \leq \min(||f||^{\alpha}_{loc}, ||f||^{\alpha}_{\partial}) \leq \max(||f||^{\alpha}_{loc}, ||f||^{\alpha}_{\partial}) \leq ||f||^{\alpha}.$$

**Definition 4.1** *A domain $\Omega \subset G$ is uniform if there exists constants $a, b > 0$ such that each pair of points $x_1, x_2 \in \Omega$ can be joined by a horizontal curve $\gamma \subset \Omega$ satisfying :*

*a. $l(\gamma) \leq a d_c(x_1, x_2)$,*

*b. $min_{i,j} l(\gamma(x_j, x)) \leq b d_c(x, \partial\Omega)$ for all $x \in \gamma$.*

*Here $l(\gamma)$ is the length of $\gamma$ in the $d_c$-metric and $l(x_j, x)$ is this length between $x_j$ and $x$.*

We give some known examples.

1. Metric balls in the Heisenberg groups are uniform.

2. The Euclidean cube $\{(x_1, y_1, ..., t) \in \mathbb{H}^n | \max(|x_i|, |y_i|, |t|) < 1\}$ is a uniform domain in the Heisenberg groups $\mathbb{H}^n$ [4].

3. The hyperspace $\{(x_1, y_1, ..., t) \in \mathbb{H}^n | t > 0\}$ is a uniform domain in the Heisenberg groups $\mathbb{H}^n$ [4].

4. The hyperspace $\{x \in G | x_i > 0, i = 1, ..., m\}$ is a uniform domain in a Carnot group $G$ [4].

For domains in $\mathbb{R}^n$, the following definition appears in [10] and with $\alpha = \alpha'$ in [3].

**Definition 4.2** *A domain $\Omega$ is a $Lip_{\alpha,\alpha'}$-extension domain, $0 < \alpha' \leq \alpha \leq 1$, if there exists a constant $M$, independent of $f : \Omega \to \mathbb{R}^n$, such that*

$$||f||^{\alpha'} \leq M ||f||^{\alpha}_{loc} \tag{18}$$

When $\alpha = \alpha'$ we write $Lip_{\alpha}$-extension domain.

**Theorem 4.3** *For $0 < \alpha' \leq \alpha \leq 1$, $\Omega$ is a $Lip_{\alpha,\alpha'}$-extension domain if there exists a constant $N$ such that each pair of points $x_1, x_2 \in \Omega$ can be joined by a horizontal path $\gamma \subset \Omega$ for which*

$$\int_\gamma d_c(\gamma(s), \partial\Omega)^{\alpha-1} ds \leq N d_c(x_1, x_2)^{\alpha'}. \tag{19}$$

*If metric balls are uniform domains, then the converse holds.*

The proof is the same as the corresponding result in Euclidean space given in [3] with minor modification.

It follows that if $\Omega$ is a $Lip_{\alpha,\alpha'}$-extension domain, then

$$||f||_\partial^{\alpha'} \leq M||f||_{loc,\partial}^{\alpha}. \tag{20}$$

**Theorem 4.4** *If $\Omega$ is a uniform domain, then it is a $Lip_\alpha$-extension domain.*

The proof is similar to that in [3] in $\mathbb{R}^n$. We give the simple proof here to show the connection with uniform domains.

Proof : Let $\gamma$ join $x_1$ to $x_2$ in $\Omega$ satisfy Definition 5.1. We have,

$$\int_\gamma d_c(x, \partial\Omega)^{\alpha-1} ds$$

$$\leq b^{\alpha-1} \int_0^{l(\gamma)} \min(s, l(\gamma) - s)^{\alpha-1} ds$$

$$\leq 2b^{\alpha-1} \int_0^{l(\gamma)/2} s^{\alpha-1} ds$$

$$=\leq 2^{1-\alpha} \alpha^{-1} b^{\alpha-1} a^\alpha d_c(x_1, x_2)^\alpha.$$

We also require the following results which characterize the local Lipschitz classes. We assume from here on that metric balls are uniform domains.

**Theorem 4.5** *Assume that $f : \Omega \to \mathbb{R}$ and $0 < \eta < 1$.*
*The following are equivalent:*

1. *There exists a constant $C_1$, independent of $f$, such that*

$$|f(x_1) - f(x_2)| \leq C_1 |x_1 - x_2|^\alpha$$

*for all $x_1, x_2 \in \Omega$ with $|x_1 - x_2| \leq \eta d_c(x_1, \partial\Omega)$.*

2. *There exists a constant $C_2$, independent of $f$, such that*

$$||f||_{loc}^\alpha \leq C_2.$$

**Theorem 4.6** *Assume that $f : \Omega \to \mathbb{R}$ and $0 < \eta < 1$.*

*The following are equivalent.*

*1. There exists a constant $C_1$, independent of $f$, such that*

$$|f(x_1) - f(x_2)| \leq |x_1 - x_2|^\alpha$$

*for all $x_1, x_2 \in \Omega$ with $|x_1 - x_2| = \eta d_c(x_1, \partial\Omega)$.*

*2. There exists a constant $C_2$, independent of $f$, such that*

$$||f||^\alpha_{loc,\partial} \leq C_2$$

Again the proofs are similar to those given in [3] and [10].

# 5 Lipschitz classes of solutions

Recall we are assuming that $u$ is a solution to (10). In the Euclidean case Theorems 5.1 and 5.2 appear in [13].

**Theorem 5.1** *The following are equivalent :*

*1. There exists a constant $C_1$, independent of $u$, such that*

$$D_u(x) \leq C_1 d_c(x, \partial\Omega)^{\alpha-1}$$

*for all $x \in \Omega$.*

*2. There exists a constant $C_2$, independent of $u$, such that*

$$||u||^\alpha_{loc,\partial} \leq C_2.$$

Proof : Assume 1. Fix $x_1, x_2 \in \Omega$ with $|x_1 - x_2| = d_c(x_1, \partial\Omega)/4$ and let $B = B(x_1, 2|x_1 - x_2|)$. we have, using (16)

$$|u(x_1) - u(x_2)| \leq C|B|^{(p-Q)/pQ}(\int_B |d_0 u|^p)^{1/p}$$

$$= C|B|^{1/Q} D_u(x_1)$$
$$\leq C|x_1 - x_2|^\alpha.$$

Statement 2 then follows from Theorem 4.6.

Conversely, using the Caccioppoli inequality (14)

$$D_u(x_1) = |B|^{-1/p}(\int_B |d_0 u|^p)^{1/p}$$

$$\leq C|B|^{-(p+Q)/pQ}(\int_{2B} |u - u(x_1)|^p)^{1/p}$$
$$\leq d_c(x_1, \partial\Omega)^{\alpha-1}.$$

**Theorem 5.2** *Suppose that $\Omega$ is a $Lip_{\alpha,\alpha'}$-extension domain, $0 < \alpha' \leq \alpha \leq 1$. If there exists a constant $C_1$, independent of $u$, such that*

$$D_u(x) \leq C_1 d(x, \partial\Omega)^{\alpha-1}, \tag{21}$$

*then there is a constant $C_2$, independent of $u$, such that*

$$||u||_{\partial}^{\alpha'} \leq C_2. \tag{22}$$

*Moreover there are constants $\beta$ and $C_3$, independent of $u$, such that if in addition $\alpha' \leq \beta$, then*

$$||u||^{\alpha'} \leq C_3. \tag{23}$$

*Otherwise, (21) only implies that*

$$||u||^{\beta} \leq C(diam\Omega)^{\alpha'-\beta}. \tag{24}$$

The first implication follows from (20) and Theorem 5.1. The second part is a consequence of the next result.

**Theorem 5.3** *Assume along with $u$ being a solution in $\Omega$ that it is also continuous in $\bar{\Omega}$. There exists a constant $\beta$, depending only on $Q, p$ and the structure constants, such that if $\alpha \leq \beta$ and if there exists a constant $C_1$ such that*

$$|u(x_1) - u(x_2)| \leq C_1 |x_1 - x_2|^{\alpha} \tag{25}$$

*for all $x_1 \in \Omega$ and $x_2 \in \partial\Omega$, then*

$$||u||^{\alpha} \leq C_2 \tag{26}$$

*where $C_2$ depends only on $Q, p, C_1$ and the structure constants. If $\beta < \alpha$, (25) only implies that*

$$||u||^{\beta} \leq C_2(diam\Omega)^{\alpha-\beta}. \tag{27}$$

The proof is similar to the Euclidean case, see [12]. It requires here inequality (17) in the Carnot case with an appropriate choice of $\sigma$.

# References


[1] J. Berndt, F. Tricerri and L. Vanhecke, Generalized Heisenberg groups and Damek-Ricci harmonic spaces, Lecture Notes in Mathematics 1598, Springer-Verlag, 1995.

[2] S.M. Buckley, P. Koskela and G. Lu, Subelliptic Poincaré inequalities: the case $p < 1$, *Publ. Mat.*, **39**(1995), 313–334.

[3] F.W. Gehring and O. Martio, Lipschitz classes and quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. A.I. Math.*, **10** (1985), 203–277.

[4] A.V. Greshnov, Unifrom domains and NTA-domains in Carnot groups, *Siberian Math. Jour.*, (5) **42** (2001), 851–864

[5] M. Gromov , Carnot-Carathéodory spaces seen from within, *Institut des Hautes Etudes Scientifiques*, 6, 1994.

[6] P. Hajlasz and P. Koskela, Sobolev met Poincaré,*Max-Plank-Institut für math*,Leipzig, preprint 41, 1998.

[7] G.H. Hardy and J.E. Littlewood, Some properties of conjugate functions, *J. Reine Agnew. Math.*,**167**(1932),405–432.

[8] J. Heinonen and I. Holopainen, Quasiregular maps on Carnot groups, *J. Geom. Anal.*,**7**1(1997),109–148.

[9] T. Iwaniec and C.A. Nolder, Hardy-Littlewood inequality for quasiregular mappings in certain domains in $\mathbb{R}^n$, *Ann. Acad. Sci. Fenn. Series A.I. Math.*, **10**(1985), 267–282.

[10] V. Lappapainen, $\mathrm{Lip}_h$-extension domains, *Ann. Acad. Sci. Fenn. Ser. A.I. Math. Dissertationes*, **71** (1988).

[11] G. Lu, Embedding theorems into the Orlicz and Lipschitz classes and applications to quasilinear subelliptic equations, preprint (1994).

[12] C.A. Nolder, Hardy-Littlewood theorems for solutions of elliptic equations in divergence form, *Indiana Univ. Math. Jour.*, **40**(1991), no. 1, 149–160.

[13] C.A. Nolder, Lipschitz classes of solutions to certain elliptic equations, *Ann. Acad. Sci. Fenn. Series A.I. Math.*, **17** (1992), 211–219.

[14] P. Pansu, Metriqués de Carnot-Carathéodory et quasiisometries des espaces symetriqués de rang un, *Ann. of Math.*, (2)**129** (1989), 1–60.