

String topology for stacks

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Abstract

We establish the general machinery of string topology for differentiable stacks. This machinery allows us to treat on an equal footing free loops in stacks and hidden loops. In particular, we give a good notion of a free loop stack, and of a mapping stack $\text{Map}(Y, \mathfrak{X})$, where Y is a compact space and \mathfrak{X} a topological stack, which is functorial both in \mathfrak{X} and Y and behaves well enough with respect to pushouts. We also construct a bivariant (in the sense of Fulton and MacPherson) theory for topological stacks: it gives us a flexible theory of Gysin maps which are automatically compatible with pullback, pushforward and products. We introduce oriented stacks, generalizing oriented manifolds, which are stacks on which we can do string topology. We prove that the homology of the free loop stack of an oriented stack is a BV-algebra and a Frobenius algebra, and the homology of hidden loops is a Frobenius algebra. Using our general machinery, we construct an intersection pairing for (non necessarily compact) almost complex orbifolds which is in the same relation to the intersection pairing for manifolds as Chen-Ruan orbifold cup-product is to ordinary cup-product of manifolds. We show that the string product of almost complex is isomorphic to the orbifold intersection pairing twisted by a canonical class.

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Introduction

String topology is a term coined by Chas-Sullivan [14] to describe the rich algebraic structure on the homology of the free loop manifold LM of an oriented manifold M . The algebraic structure in question is induced by geometric operations on loops such as gluing or pinching of loops. In particular, $H_*(LM)$ inherits a canonical product and coproduct yielding a structure of Frobenius algebra [14, 18]. Furthermore, the canonical action of S^1 on LM together with the multiplicative structure make $H_*(LM)$ a **BV**-algebra [14]. These algebraic structures, especially the loop product, are known to be related to many subjects in mathematics and in particular mathematical physics [46, 13, 17, 2, 21].

Many interesting geometric objects in (algebraic or differential) geometry or mathematical physics are *not* manifolds. There are, for instance, orbifolds, classifying spaces of compact Lie groups, or, more generally, global quotients of a manifold by a Lie group. All these examples belong to the realm of (geometric) stacks. A natural generalization of smooth manifolds, including the previous examples, is given by differentiable stacks [7] (on which one can still do differentiable geometry). Roughly speaking, differential stacks are Lie groupoids *up to Morita equivalence*.

One important feature of differentiable stacks is that they are *non-singular*, when viewed as stacks (even though their associated coarse spaces are typically singular). For this reason, differentiable stacks have an intersection product on their homology, and a loop product on the homology of their free loop stacks.

The aim of this paper is to establish the general machinery of string topology for differentiable stacks. This machinery allows us to treat on an equal footing free loops in stacks and *hidden* loops. The latter are loops inside the stack, which vanish on the associated coarse space. The stack of hidden loops in the stack \mathfrak{X} is the *inertia stack* of \mathfrak{X} , notation $\Lambda\mathfrak{X}$. The inertia stack $\Lambda\mathfrak{X} \rightarrow \mathfrak{X}$ is an example of a family of commutative (*sic!*) groups over the stack \mathfrak{X} , and the theory of hidden loops generalizes to arbitrary commutative families of groups over stacks.

In the realm of stacks several new difficulties arise whose solutions should be of independent interest.

First, we need a good notion of *free loop stack* $L\mathfrak{X}$ of a stack \mathfrak{X} , and more generally of mapping stack $\text{Map}(Y, \mathfrak{X})$ (the stack of stack morphisms $Y \rightarrow \mathfrak{X}$). For the general theory of mapping stacks, we do not need a differentiable structure on \mathfrak{X} ; we work with topological stacks. This is the content of Section 5.1. The issue here is to obtain a mapping stack with a topological structure which is functorial both in \mathfrak{X} and Y and behaves well enough with respect to pushouts

in order to get geometric operations on loops. For instance, a key point in string topology is the identification $\text{Map}(S^1 \vee S^1, \mathfrak{X}) \cong \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}$. Since pushouts are a delicate matter in the realm of stacks, extra care has to be taken in finding the correct class of topological stacks to work with (Section ?? and [40]). For this reason, we restrict our attention to the class of *Hurewicz* topological stacks. These are topological stacks which admit an atlas with a certain fibrancy property. Without restricting to this special class of topological stacks, $S^1 \vee S^1$ would not be the pushout of two copies of S^1 , in the category of stacks.

A crucial step in usual string topology is the existence of a canonical Gysin homomorphism $H_{\bullet}(\text{LM} \times \text{LM}) \rightarrow H_{\bullet-d}(\text{LM} \times_M \text{LM})$ when M is a d -dimensional manifold. In fact, the loop product is the composition

$$\begin{aligned} H_p(\text{LM}) \otimes H_q(\text{LM}) &\rightarrow \\ &\rightarrow H_{p+q}(\text{LM} \times \text{LM}) \rightarrow H_{p+q-d}(\text{LM} \times_M \text{LM}) \rightarrow H_{p+q-d}(\text{LM}), \end{aligned} \quad (0.1)$$

where the last map is obtained by gluing two loops at their base point.

Roughly speaking the Gysin map can be obtained as follows. The free loop manifold is equipped with a structure of Banach manifold such that the evaluation map $\text{ev} : \text{LM} \rightarrow M$ which maps a loop f to $f(0)$ is a surjective submersion. The pullback along $\text{ev} \times \text{ev}$ of a tubular neighborhood of the diagonal $M \rightarrow M \times M$ in $M \times M$ yields a normal bundle of codimension d for the embedding $\text{LM} \times_M \text{LM} \rightarrow \text{LM}$. The Gysin map can then be constructed using a standard argument on Thom isomorphism and Thom collapse [19].

This approach does not have a straightforward generalization to stacks. For instance, the free loop stack of a differentiable stack is not a Banach stack in general, and neither is the inertia stack. In order to obtain a flexible theory of Gysin maps, we construct a *bivariant theory* in the sense of Fulton-MacPherson [25] for topological stacks, whose underlying homology theory is singular homology. A bivariant theory is an efficient tool encompassing into a unified framework both homology and cohomology as well as many (co)homological operations, in particular Gysin homomorphisms. The Gysin maps of a bivariant theory are automatically compatible with pullback, pushforward, cup and cap-products (see [25]). (Our bivariant theory is somewhat weaker than that of Fulton-MacPherson, in that products are not always defined.)

In Section 8.1 we introduce *oriented stacks*. These are the stacks over which we are able to do string topology. Examples of oriented stacks include: oriented manifolds, oriented orbifolds, and quotients of oriented manifolds by compact Lie groups (if the action is orientation preserving and of finite orbit type). A topological stack \mathfrak{X} is *orientable* if the diagonal map $\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ factors as

$$\mathfrak{X} \xrightarrow{0} \mathfrak{N} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{X} \times \mathfrak{X}, \quad (0.2)$$

where \mathfrak{N} and \mathfrak{E} are orientable vector bundles over \mathfrak{X} and $\mathfrak{X} \times \mathfrak{X}$ respectively, and $\mathfrak{N} \rightarrow \mathfrak{E}$ is an isomorphism onto an open substack (there is also the technical assumption that \mathfrak{E} is metrizable, and $\mathfrak{X} \rightarrow \mathfrak{E}$ factors through the unit disk bundle). The embedding $\mathfrak{N} \rightarrow \mathfrak{E}$ plays the role of a tubular neighborhood. The dimension of \mathfrak{X} is $\text{rk } \mathfrak{N} - \text{rk } \mathfrak{E}$.

The factorization (0.2) gives rise to a bivariant class $\theta \in H(\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X})$, the *orientation* of \mathfrak{X} .

Sections 5-8 are devoted to the string topology operations, focusing on the Frobenius and **BV**-algebra structures. The bivariant formalism has the following consequence: if \mathfrak{X} is a an oriented stack of dimension d , then any cartesian square

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \mathfrak{Z} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

defines a canonical Gysin map $\Delta^! : H_\bullet(\mathfrak{Z}) \rightarrow H_{\bullet-d}(\mathfrak{Y})$. For example, the cartesian square

$$\begin{array}{ccc} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} & \longrightarrow & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

Gives rise to a Gysin map $\Delta^! : H_\bullet(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \rightarrow H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X})$, and we can construct a loop product

$$\star : H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \rightarrow H_{\bullet-d}(\mathrm{L}\mathfrak{X}),$$

as in 0.1, or [14, 19, 18].

We also obtain a coproduct

$$\delta : H_\bullet(\mathrm{L}\mathfrak{X}) \longrightarrow \bigoplus_{i+j=\bullet-d} H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X}).$$

Furthermore, $\mathrm{L}\mathfrak{X}$ admits a natural S^1 -action yielding the operator $D : H_\bullet(\mathrm{L}\mathfrak{X}) \rightarrow H_{\bullet+1}(\mathrm{L}\mathfrak{X})$ which is the composition:

$$H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{\times\omega} H_{\bullet+1}(\mathrm{L}\mathfrak{X} \times S^1) \longrightarrow H_{\bullet+1}(\mathrm{L}\mathfrak{X}),$$

where $\omega \in H_1(S^1)$ is the fundamental class. Thus we prove that $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ is a Frobenius algebra and that the shifted homology $(H_{\bullet+d}(\mathrm{L}\mathfrak{X}), \star, D)$ is a **BV**-algebra.

Since the inertia stack can be considered as the stack of hidden loops, the general machinery of Gysin maps yields, for any oriented stack \mathfrak{X} , a product and a coproduct on the homology $H_\bullet(\Lambda\mathfrak{X})$ of the inertia stack $\Lambda\mathfrak{X}$, making it a Frobenius algebra, too. Moreover in Section 11.4, we construct a natural map $\Phi : \Lambda\mathfrak{X} \rightarrow \mathrm{L}\mathfrak{X}$ inducing a morphism of Frobenius algebras in homology.

In Section 13, we consider almost complex orbifolds (not necessarily compact). Using Gysin maps and the obstruction bundle of Chen-Ruan [16], we construct the *orbifold intersection pairing* on the homology of the inertia stack.

It is in the same relation to the intersection pairing on the homology of a manifold as the Chen-Ruan orbifold cup-product [16] is to the ordinary cup product on the cohomology of a manifold.

The orbifold intersection pairing defines a structure of associative, graded commutative algebra on $H_{\bullet}^{\text{orb}}(\mathfrak{X})$ for any almost complex orbifold \mathfrak{X} . As a vector space the orbifold homology $H_{\bullet}^{\text{orb}}(\mathfrak{X})$ coincides with the homology of the inertia stack $\Lambda\mathfrak{X}$, but the grading is shifted according to the age as in [16, 23].

In the compact case, the orbifold intersection pairing is identified with the Chen-Ruan product, via orbifold Poincaré duality.

We also prove that the loop product, string product and intersection pairing (for almost complex orbifolds) can be twisted by a cohomology class in $H_{\bullet}(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X})$ or $H_{\bullet}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$, satisfying the 2-cocycle condition (see Propositions 9.11, 10.3, and 13.6). The notion of twisting provides a connection between the orbifold intersection pairing and the string product. In fact, we associate to an almost complex orbifold \mathfrak{X} a canonical vector bundle $\mathcal{O}_{\mathfrak{X}} \oplus \mathfrak{N}_{\mathfrak{X}}$ over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ and prove that the orbifold intersection pairing, twisted by the Euler class of $\mathcal{O}_{\mathfrak{X}} \oplus \mathfrak{N}_{\mathfrak{X}}$, is the string product of \mathfrak{X} .

Parallel to our work, the string product for global quotient orbifolds was studied in [35, 27]. Furthermore, a nice interpretation of the string product in terms of the Chen-Ruan product of the cotangent bundle was given by González et al. [27]. A loop product for global quotients of a manifold by a finite group was studied in [36, 35].

Conventions

Topological spaces

All topological spaces are compactly generated. The category of topological spaces endowed with the Grothendieck topology of open coverings is denoted Top . This is the *site* of topological spaces.

Manifolds

All manifolds are second countable and Hausdorff. In particular they are regular Lindelöf and paracompact.

Groupoids

We will commit the usual abuse of notation and abbreviate a groupoid to $\Gamma_1 \rightrightarrows \Gamma_0$. A *topological groupoid*, is a groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, where Γ_1 and Γ_0 are topological spaces, but no further assumptions is made on the source and target maps, except continuity. A topological groupoid is a *Lie groupoid* if Γ_1, Γ_0 are manifolds, all the structures maps are smooth and, in addition, the source and target maps are subjective submersions.

Stacks

For stacks, we use the words *equivalent* and *isomorphic* interchangeably. We will often omit 2-isomorphisms from the notation. For example, we may call morphisms equal if they are 2-isomorphic. The stack associated to a groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ we denote by $[\Gamma_0/\Gamma_1]$, because we think of it as the quotient. Also if G is a Lie group acting on a space Y , we simply denote $[Y/G]$ the stack associated to the transformation groupoid $Y \times G \rightrightarrows Y$.

Our terminology is different from that of [40]. The quotient stack \mathfrak{X} of a topological groupoid $[\Gamma_0/\Gamma_1]$ is called a *topological stack* in this paper, where as in [ibid.] these are called *pretopological stacks*. If the source and target map of $[\Gamma_0/\Gamma_1]$ are local Hurewicz fibrations, then we say that \mathfrak{X} is a *Hurewicz topological stack*; see Section S:topological.

(Co)homology

The coefficients of our (co)homology theories will be taken in a commutative unital ring k . All tensor products are over k unless otherwise specified.

We will write both $H(\mathfrak{X})$, $H_\bullet(\mathfrak{X})$ for the total homology groups $\bigoplus H_n(\mathfrak{X})$. We use the first notation when we deal with ungraded elements and ungraded maps, while we use the second when dealing with homogeneous homology classes and graded maps. Similarly, in Section 7, we use respectively the notations $H(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$ and $H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$ for the total bivariant cohomology groups when we want to deal with ungraded maps or with graded ones.

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1 Topological stacks

We review some basic facts about topological stacks. More details can be found in [40].

1.1 Stacks over Top

Throughout these notes, by a stack we mean a stack over the site \mathbf{Top} of compactly generated topological spaces with the standard Grothendieck topology. This means, a stack is a category \mathfrak{X} fibered in groupoids over \mathbf{Top} satisfying the decent condition. Alternatively, we can think of \mathfrak{X} as a presheaf of groupoids over \mathbf{Top} which satisfies decent.

We list some basic facts about stacks.

1. Stacks over \mathbf{Top} form a 2-category in which 2-morphisms are invertible. Therefore, given two stacks \mathfrak{X} and \mathfrak{Y} , we have the *groupoid* $\mathrm{Hom}(\mathfrak{Y}, \mathfrak{X})$ of morphisms between them. In the case where the source stack $\mathfrak{Y} = T$ is a topological space, we usually use the alternative notation $\mathfrak{X}(T)$ for the above hom-groupoid. This is sometimes referred to as the groupoid of *T-valued points* of \mathfrak{X} .

Although in practice one may really be interested only in the category of stacks which obtained by identifying 2-isomorphic 1-morphisms, the 2-category structure can not be ignored. For example, when we talk about *fiber products of stacks*, we exclusively mean the 2-fiber product in the 2-category of stacks.

2. The 2-category of stacks has fiber products and inner homs, so it is cartesian closed. The 2-fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is characterized by the property that, for every topological space T , its groupoid of T -valued points is given equivalent to

$$\mathfrak{X}(T) \times_{\mathfrak{Z}(T)} \mathfrak{Y}(T).$$

Given stacks \mathfrak{X} and \mathfrak{Y} be stacks over \mathbf{Top} , the inner hom between them, called the *mapping stack* $\mathbf{Hom}(\mathfrak{Y}, \mathfrak{X})$, is defined by the rule

$$T \in \mathbf{Top} \quad \mapsto \quad \mathrm{Hom}(T \times \mathfrak{Y}, \mathfrak{X}).$$

Note that we have a natural equivalence of groupoids

$$\mathbf{Hom}(\mathfrak{Y}, \mathfrak{X})(*) \cong \mathrm{Hom}(\mathfrak{Y}, \mathfrak{X}),$$

where $*$ is a point. The mapping stack has the exponential property. That is, given stacks \mathfrak{X} , \mathfrak{Y} , and \mathfrak{Z} , we have a natural equivalence of stacks

$$\mathbf{Hom}(\mathfrak{Z} \times \mathfrak{Y}, \mathfrak{X}) \cong \mathbf{Hom}(\mathfrak{Z}, \mathbf{Hom}(\mathfrak{Y}, \mathfrak{X})).$$

3. The category of topological spaces embeds fully faithfully in the 2-category of stacks. This means, given two topological spaces X and Y , viewed as stacks via the functor they represent, the hom-groupoid $\mathrm{Hom}(X, Y)$ is equivalent to a set, and this set is in a natural bijection with the set of continuous functions from X to Y .

This way, we can think of a topological space as a stack.

This embedding preserves the closed cartesian structure on \mathbf{Top} . This means that fiber products of spaces get sent to 2-fiber products of the corresponding stacks, and the mapping spaces (with the compact-open topology) get sent to mapping stacks.

4. The embedding of the category of topological spaces in the 2-category of stacks admits a left adjoint. That is, to every stack \mathfrak{X} one can associate a topological space, together with a natural map $\pi: \mathfrak{X} \rightarrow \mathfrak{X}_{mod}$ which is universal among maps from \mathfrak{X} to topological spaces. (That is, every map from \mathfrak{X} to a topological space T factors uniquely through π .) ; See [40], §4.3.

The space \mathfrak{X}_{mod} is called the *coarse moduli space* of \mathfrak{X} and it should be thought of as the “underlying space” of \mathfrak{X} .

In particular, the underlying set of \mathfrak{X}_{mod} is the set of isomorphism classes of the groupoid $\mathfrak{X}(*)$, where $*$ stands for a point. In other words, the points in \mathfrak{X}_{mod} are the 2-isomorphism classes of points of \mathfrak{X} , where by a *point* of \mathfrak{X} we mean a morphism $x: * \rightarrow \mathfrak{X}$.

The underlying set of the coarse moduli space of the mapping stack $\mathbf{Hom}(\mathfrak{Y}, \mathfrak{X})$ is the set of 2-isomorphism classes of morphisms from \mathfrak{Y} to \mathfrak{X} .

5. To a point $x: * \rightarrow \mathfrak{X}$ of a stack \mathfrak{X} there is associated a group I_x , called the *inertia group* of \mathfrak{X} at x . By definition, I_x is the group of 2-isomorphisms from the point x to itself. An element in I_x is sometimes referred to as a *ghost* or *hidden loop*; see ([40], §10).

The groups I_x assemble into a stack $\Lambda\mathfrak{X} \rightarrow \mathfrak{X}$ over \mathfrak{X} called the *inertia stack*. The inertia stack is defined by the following 2-fiber square

$$\begin{array}{ccc} \Lambda\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \Delta \\ \mathfrak{X} & \longrightarrow_{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

1.2 Morphisms of stacks

A morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of stacks is called **representable** if for every map $T \rightarrow \mathfrak{Y}$ from a topological space T , the fiber product $T \times_{\mathfrak{Y}} \mathfrak{X}$ is a topological space. This is, roughly speaking, saying that the fibers of f are topological spaces.

Any property \mathbf{P} of morphisms of topological spaces which is invariant under base change can be defined for an arbitrary representable morphism of stacks. More precisely, we say that a representable morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is \mathbf{P} , if for every map $T \rightarrow \mathfrak{Y}$ from a topological space T , the base extension $f_T: T \times_{\mathfrak{Y}} \mathfrak{X} \rightarrow T$ is \mathbf{P} as a map of topological spaces; see ([40], §4.1).

This way we can talk about *embeddings (closed, open, locally closed, or arbitrary) of stacks, proper morphisms, finite morphisms*, and so on.

We say that $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is an *epimorphism*, if it is an epimorphism in the sheaf theoretic sense. In the case where f is representable, this is equivalent to saying that every base extension f_T of f over a topological space T admits local sections.

1.3 Topological stacks

A **topological stack** ([40], Definition 7.1) is a stack \mathfrak{X} over \mathbf{Top} which admits a representable epimorphism $p: X \rightarrow \mathfrak{X}$ from a topological space X . Equivalently, \mathfrak{X} is equivalent to the quotient stack $[X_0/X_1]$ of a topological groupoid $X_1 \rightrightarrows X_0$. This quotient stack, by definition, is the stack associated to the presheaf of groupoids

$$T \mapsto X_1(T) \rightrightarrows X_0(T).$$

This stack is the equivalent to the stack of torsors for the groupoid $X_1 \rightrightarrows X_0$; see ([40], §12).

An important example to keep in mind is the case of a topological group G acting on a topological space X . The quotient stack of the topological groupoid $X \times G \rightrightarrows X$ associated to this action is denoted by $[X/G]$. For a topological space T , the groupoid $[X/G](T)$ of T -points of $[X/G]$ is the groupoid of pairs (P, φ) , where P is a principal G -bundle over T , and $\varphi: P \rightarrow X$ is a G -equivariant map. In the case where X is a point, the corresponding quotient stack $[*/G]$ is called the *classifying stack* of G . Its group of T -points is precisely the groupoid of principal G -bundles over T .

We list some basic facts about topological stacks.

1. Topological stacks form a full sub 2-category of the 2-category of stacks over \mathbf{Top} .
2. The 2-category of topological stacks is closed under fiber products. It, however, does not seem to have inner hom objects. That is, it does not seem to be the case in general that the mapping stack $\mathbf{Hom}(\mathfrak{Y}, \mathfrak{X})$ of two topological stacks \mathfrak{X} and \mathfrak{Y} is a topological stack. This *is* the case, however, if \mathfrak{Y} is the quotient stack of a groupoid $Y_1 \rightrightarrows Y_0$ such that Y_0 and Y_1 are compact topological spaces; see Proposition 5.1.
3. The stack associated to a topological space X is topological. It is, in fact, equivalent to the stack associated to the trivial groupoid $X \rightrightarrows X$. Thus, the category of topological spaces is a full subcategory of the 2-category of topological stacks.

4. Let $\mathfrak{X} = [X_0/X_1]$ be the quotient stack of a topological groupoid $X_1 \rightrightarrows X_0$. Then, the coarse moduli space \mathfrak{X}_{mod} of \mathfrak{X} is naturally homeomorphic to the coarse quotient space of the groupoid $X_1 \rightrightarrows X_0$. In particular, the coarse moduli space of the quotient stack $[X/G]$ is the orbit space X/G of the action of G on X . The coarse moduli space of the classifying stack $[*/G]$ of G is just a single point.
5. For a point $x: * \rightarrow \mathfrak{X}$ of a topological stack \mathfrak{X} , the inertia group I_x is naturally a topological group. The inertia stack $\Lambda\mathfrak{X}$ is a topological stack, and the natural map $\Lambda\mathfrak{X} \rightarrow \mathfrak{X}$ is representable.
6. Every morphism $T \rightarrow \mathfrak{X}$ from a topological space T to a topological stack \mathfrak{X} is representable.

1.4 Hurewicz topological stacks

As we will see in §1.5, in order to have nice gluing properties for maps into a stack \mathfrak{X} , we need to assume \mathfrak{X} is a Hurewicz stack. This will be needed later on when we work with loop stacks. We recall the definition of a Hurewicz stack.

A *Hurewicz fibration* is a map having the homotopy lifting property for all topological spaces. A map $f: X \rightarrow Y$ of topological spaces is a *local Hurewicz fibration* if for every $x \in X$ there are opens $x \in U$ and $f(x) \in V$ such that $f(U) \subseteq V$ and $f|_U \rightarrow V$ is a Hurewicz fibration. The most important example for us is the case of a topological submersion: a map $f: X \rightarrow Y$, such that locally U is homeomorphic to $V \times \mathbb{R}^n$, for some n .

Dually, we have the notion of *local cofibration*. It is known ([44]), that if $A \rightarrow Z$ is a closed embedding of topological spaces, it is a local cofibration if and only if there exists an open neighborhood $A \subset U \subset Z$ such that A is a strong deformation retract of U . If $A \rightarrow Z$ is a local cofibration, so is $A \times T \rightarrow Z \times T$ for every topological space T . Moreover, the following result is essential for our purposes ([45]):

Given a commutative diagram, with $A \rightarrow Z$ a local cofibration and $X \rightarrow Y$ a local fibration

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

then for every point $a \in A$ there exists an open neighborhood Z' of a in Z , such that there exists a lifting (the dotted arrow) giving two commutative triangles

$$\begin{array}{ccc} A' & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ Z' & \longrightarrow & Y \end{array}$$

where $A' = A \cap Z'$.

Definition 1.1 A topological stack \mathfrak{X} is called **Hurewicz** if it is equivalent to the quotient stack $[X_0/X_1]$ of a topological groupoid $X_1 \rightrightarrows X_0$ whose source and target maps are local Hurewicz fibrations.

Example 1.2 A topological space is a Hurewicz topological stack. The topological stack underlying any differentiable stack is a Hurewicz topological stack. In particular, any global quotient $[M/G]$ of a manifold by a Lie group is a Hurewicz topological stack.

1.5 Pushouts in the category of stacks

The following generalizes ([40], Theorem 16.2).

Proposition 1.3 *Let $A \rightarrow Y$ be a closed embedding of Hausdorff spaces, which is a local cofibration. Let $A \rightarrow Z$ be a finite proper map of Hausdorff spaces. Suppose we are given a pushout diagram in the category of topological spaces*

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z \vee_A Y \end{array}$$

Then this diagram remains a pushout diagram in the 2-category of Hurewicz topological stacks. In other words, for every Hurewicz topological stack \mathfrak{X} , the morphism

$$\mathfrak{X}(Z \vee_A Y) \longrightarrow \mathfrak{X}(Z) \times_{\mathfrak{X}(A)} \mathfrak{X}(Y)$$

is an equivalence of groupoids.

PROOF. Let us abbreviate the pushout by $U = Z \vee_A Y$.

The fully faithful property only uses that \mathfrak{X} is a topological stack and that U is a pushout. Let us concentrate on essential surjectivity. Because \mathfrak{X} is a stack and we already proved full faithfulness, the question is local in U . Assume given $Z \rightarrow \mathfrak{X}$ and $Y \rightarrow \mathfrak{X}$, and an isomorphism over A . Let $X_1 \rightrightarrows X_0$ be a groupoid presenting \mathfrak{X} , whose source and target maps are local fibrations.

Let us remark that both $Z \rightarrow U$ and $Y \rightarrow U$ are finite proper maps of Hausdorff spaces. Thus we can cover U by open subsets U_i , such that for every i , both $Z_i = U_i \cap Z$ and $Y_i = U_i \cap Y$ admit liftings to X_0 of their morphisms to \mathfrak{X} . We thus reduce to the case that we have $Z \rightarrow X_0$, $Y \rightarrow X_0$, and $A \rightarrow X_1$. Next, we need to construct the dotted arrow in

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & \swarrow \text{dotted} & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

We can cover Y by opens over which this arrow exists, because $A \rightarrow Y$ is a local cofibration and $X_1 \rightarrow X_0$ a local fibration. Then for a point $u \in U$ we

choose an open neighborhood in U small enough such that the preimage in Y is a disjoint union of sets over which the dotted arrow exists. Passing to such a neighborhood of u reduces to the case that the dotted arrow exists. Then there is nothing left to prove. \square

1.6 Geometric stacks

We will encounter other types of stacks. A **differentiable stack** is a stack on the category of C^∞ -manifolds, which is isomorphic to the quotient stack of a Lie groupoid. Every differentiable stack has an underlying topological stack. If the Lie groupoid $X_1 \rightrightarrows X_0$ represents the differentiable stack \mathfrak{X} , the underlying topological groupoid represents the underlying topological stack. Often we will tacitly pass from a differentiable stack to its underlying topological stack. For more on differentiable stacks, see [7].

An **almost complex stack** is a stack on the category of almost complex manifolds, which is isomorphic to the quotient stack of an almost complex Lie groupoid, i.e., a Lie groupoid $X_1 \rightrightarrows X_0$, where X_0 and X_1 are almost complex manifolds, and all structure maps respect the almost complex structure. Every almost complex stack has an underlying differentiable stack and hence also an underlying topological stack.

2 Homotopy type of a topological stack

Classifying space of a topological stack

Many facts about topological stacks can be reduced to the case of topological spaces by virtue of the following.

Theorem 2.1 *For every topological stack \mathfrak{X} , there exists a topological space X together with a morphism $\varphi: X \rightarrow \mathfrak{X}$ which has the property that, for every morphism $T \rightarrow \mathfrak{X}$ from a topological space T , the pull back $T \times_{\mathfrak{X}} X \rightarrow T$ is a weak homotopy equivalence.*

A topological space X with the above property is called a **classifying space** for \mathfrak{X} . A classifying space for \mathfrak{X} can be constructed by taking the fat realization of the nerve of a groupoid $X_1 \rightrightarrows X_0$ whose quotient stack is \mathfrak{X} . For more details see [41]. The above theorem implies that the classifying space is unique up to a unique isomorphism in the weak homotopy category of topological spaces (i.e., the category of topological spaces with weak homotopy equivalences inverted).

In the case where $\mathfrak{X} = [X/G]$ is the quotient stack of a group action, the Borel construction $X \times_G EG$ is a classifying space for \mathfrak{X} . Here EG is the total space of the universal principal G -bundle in the sense of Milnor.

2.1 (Co)homology theories for topological stacks

Theorem 2.1 allows one to extend every (generalized) (co)homology theory h to the 2-category of topological stacks. For instance, let us explain how to define

$h(\mathfrak{X}, \mathfrak{A})$ for a pair $(\mathfrak{X}, \mathfrak{A})$ of topological stacks.

Choose a classifying space $\varphi: X \rightarrow \mathfrak{X}$, and let $A := \varphi^{-1}\mathfrak{A}$. It follows that the pair (X, A) is well-defined in the weak homotopy category of pairs (i.e., is independent of the choice of a particular classifying space X). So, we can define $h(\mathfrak{X}, \mathfrak{A})$ to be $h(X, A)$. It can be easily verified that this construction is functorial in morphisms of pairs.

The cohomology theory thus defined on topological stacks will maintain all natural properties that it had on spaces. For example, it will be homotopy invariant (in particular, it will not distinguish 2-isomorphic morphisms), it will satisfy excision, it will maintain all the products (cap, cup, etc.) that it had on spaces, and so on.

In the case where $\mathfrak{X} = [X/G]$, we will recover the usual G -equivariant (co)homology of X defined using the Borel construction. That is, $h([X/G]) \cong h(X \times_G EG)$.

2.2 Paracompactness of the classifying space

In many applications, it is important to find a classifying space for \mathfrak{X} which is *paracompact*. There are various conditions on a groupoid $X_1 \rightrightarrows X_0$ which guarantee that the fat realization of the nerve of $X_1 \rightrightarrows X_0$ is paracompact. The following is one.

Definition 2.2 A topological stack \mathfrak{X} is called **regular Lindelöf** if it is equivalent to the quotient stack $[X_0/X_1]$ of a topological groupoid $X_1 \rightrightarrows X_0$ such that X_1, X_0 are regular Lindelöf spaces.

The proof of the following proposition will appear elsewhere.

Proposition 2.3 *If \mathfrak{X} is a regular Lindelöf stack, there exists a classifying space for \mathfrak{X} which is a regular Lindelöf space, in particular paracompact.*

Remark 2.4 Every differentiable stack is regular Lindelöf and hence has a paracompact classifying space.

2.3 Singular homology and cohomology

We will fix once and for all a coefficient ring and drop it from the notation consistently.

We assume all topological stacks are regular Lindelöf.

Singular homology and cohomology for spaces lifts to topological stacks. The singular (co)homology of the topological stack \mathfrak{X} can be defined to be the singular (co)homology of its classifying space. Alternatively, let $\Gamma: \Gamma_1 \rightrightarrows \Gamma_0$ be a topological groupoid presentation of \mathfrak{X} . We denote $\Gamma_p = \Gamma_1 \times_{\Gamma_0} \dots \times_{\Gamma_0} \Gamma_1$ (p -fold) the space of composable sequences of p arrows in the groupoid Γ . It yields a simplicial space Γ_\bullet .

$$\dots \Gamma_2 \rightrightarrows \Gamma_1 \rightrightarrows \Gamma_0. \quad (2.1)$$

The *singular chain complex* of Γ_\bullet is the total complex associated to the double complex $C_\bullet(\Gamma_\bullet)$ [5]. Here $C_q(\Gamma_p)$ is the linear space generated by the continuous maps $\Delta_q \rightarrow \Gamma_p$. Its homology groups $H_q(\Gamma_\bullet) = H_q(C_\bullet(\Gamma_\bullet))$ are called the *homology groups* of Γ . The *singular cochain complex* of Γ_\bullet is the dual of $C_\bullet(\Gamma_\bullet)$. In other words the total complex associated to the bicomplex $C^p(\Gamma_q)$. These groups are Morita invariant (i.e. only depend on the quotient stack $[\Gamma_0/\Gamma_1]$). By definition they are the (co)homology groups of the stack $[\Gamma_0/\Gamma_1]$. They coincide with the cohomology of the classifying space of \mathfrak{X} .

This Theory generalizes to singular cohomology theory $H(\mathfrak{X}, \mathfrak{A})$ for pairs $(\mathfrak{X}, \mathfrak{A})$ of topological stacks satisfying the Eilenberg-Steenrod axioms. This theory comes with cup products and coincided with the usual singular cohomology when $(\mathfrak{X}, \mathfrak{A})$ is a pair of topological spaces. In the case when $\mathfrak{X} = [X/G]$ is the quotient stack of a topological group action, H is G -equivariant cohomology. The following less standard facts are also true about H .

Proposition 2.5 *Let $\mathfrak{A} \hookrightarrow \mathfrak{B} \hookrightarrow \mathfrak{X}$ be closed embeddings of topological stacks. Then, there is a natural product*

$$H^n(\mathfrak{X}, \mathfrak{X} - \mathfrak{B}) \otimes H^m(\mathfrak{B}, \mathfrak{B} - \mathfrak{A}) \rightarrow H^{m+n}(\mathfrak{X}, \mathfrak{X} - \mathfrak{A})$$

which coincides with the cup product if $\mathfrak{B} = \mathfrak{X}$.

PROOF. One uses the fact the classifying space is paracompact (Proposition 2.3). It is a general fact (for instance see [32]) that if F is a sheaf over a paracompact space X and $Z \subset X$ is closed, then $\varinjlim_{U \supset Z} \Gamma(U, F) \xrightarrow{\sim} \Gamma(Z, F)$,

where U is open. Then the result follows from the same argument as for topological spaces in [25], § 3. \square

Proposition 2.6 *Let \mathfrak{X} be a topological stack and $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{X}$ substacks. Then, we have a cohomology long exact sequence*

$$\dots \rightarrow H^{n-1}(\mathfrak{A}, \mathfrak{A} \cap \mathfrak{B}) \rightarrow H^n(\mathfrak{X}, \mathfrak{A} \cup \mathfrak{B}) \rightarrow H^n(\mathfrak{X}, \mathfrak{B}) \rightarrow H^n(\mathfrak{A}, \mathfrak{A} \cap \mathfrak{B}) \rightarrow H^{n+1}(\mathfrak{X}, \mathfrak{A} \cup \mathfrak{B}) \dots$$

PROOF. By Excision $H^n(\mathfrak{A}, \mathfrak{A} \cap \mathfrak{B}) \cong H^n(\mathfrak{A} \cup \mathfrak{B}, \mathfrak{B})$. The result follows from the long exact cohomology sequence for the triple $(\mathfrak{X}, \mathfrak{A} \cup \mathfrak{B}, \mathfrak{B})$. \square

3 Vector bundles on stacks

We begin with the definition of a (representable) vector bundle on a stack.

Definition 3.1 Let \mathfrak{X} be a (topological) stack. A real **vector bundle** on \mathfrak{X} is a representable morphism of stacks $\mathfrak{E} \rightarrow \mathfrak{X}$ which makes \mathfrak{E} a vector space object relative to \mathfrak{X} . That is, we have an addition morphism $\mathfrak{E} \times_{\mathfrak{X}} \mathfrak{E} \rightarrow \mathfrak{E}$ and an \mathbb{R} -action $\mathbb{R} \times \mathfrak{E} \rightarrow \mathfrak{E}$, both relative to \mathfrak{X} , which satisfy the usual axioms. A complex vector bundle is defined analogously.

Remark 3.2 There are two alternative definitions for a vector bundle on a stack \mathfrak{X} that are equivalent to the one given above:

1. A vector bundle on \mathfrak{X} is a representable morphism of stacks $\mathfrak{E} \rightarrow \mathfrak{X}$ such that, for every $f: T \rightarrow \mathfrak{X}$ with T a topological space, the pull back $E_T \rightarrow T$ is endowed with the structure of a vector bundle. Here, $E_T := f^*\mathfrak{E} = T \times_{\mathfrak{X}} \mathfrak{E}$.

$$\begin{array}{ccc} E_T & \longrightarrow & \mathfrak{E} \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & \mathfrak{X} \end{array}$$

We also require, for every $g: T' \rightarrow T$, that the natural isomorphism

$$(f \circ g)^*\mathfrak{E} \rightarrow g^*(f^*\mathfrak{E})$$

be a bundle map.

2. Let $\mathbb{X} = [s, t: X_1 \rightrightarrows X_0]$ be a groupoid presentation for \mathfrak{X} . Then, a vector bundle on \mathfrak{X} is the same things as an \mathbb{X} -equivariant vector bundle. Recall that an \mathbb{X} -equivariant vector bundle consists of a vector bundle E over X_0 , and an isomorphism $\psi: s^*E \rightarrow t^*E$ of vector bundles over X_1 such that the three restrictions of ψ to $X_1 \times_{X_0} X_1$ satisfy the cocycle condition.

Let us briefly explain how these three definitions are related. Going from Definition 3.1 to (1) is obvious. Given a vector bundle in the sense of (1), we obtain a vector bundle $E := p^*\mathfrak{E}$ on X_0 , where $p: X_0 \rightarrow \mathfrak{X}$ is the quotient map. This is easily seen to be equivariant. Finally, given an \mathbb{X} -equivariant vector bundle E , we define \mathfrak{E} to be the quotient stack of the groupoid $[E_1 \rightrightarrows E_0]$, where $E_0 := E$ and $E_1 := s^*E = X_1 \times_{X_0} E$. The source map $E_1 \rightarrow E_0$ is the projection map $\text{pr}_2: X_1 \times_{X_0} E \rightarrow E_0$. The target map is $\text{pr}_2 \circ \psi$. It is easy to verify that \mathfrak{E} is a vector bundle over \mathfrak{X} in the sense of Definition 3.1.

3.1 Operations on vector bundles

The standard operations on vector bundles on spaces (e.g., direct sum, tensor product, exterior powers, and so on) can be carried out on vector bundles on stacks *mutatis mutandis*. This is more easily seen if we think of a vector bundle as in Remark 3.2 (1). In this case, we simply perform the desired operation simultaneously on the pull backs E_T , for varying T , and the resulting family of vector bundles, say F_T , will give rise to a vector bundle \mathfrak{F} on \mathfrak{X} .

In view of Remark 3.2 (2), operations on vector bundles on \mathfrak{X} correspond to operations on \mathbb{X} -equivariant vector bundles.

Similarly, we can define a **metric** on a vector bundle. More precisely, a metric on \mathfrak{E} is the same thing as a compatible family of metrics on E_T , for varying T . Given a presentation $\mathbb{X} = [X_1 \rightrightarrows X_0]$ for \mathfrak{X} , a metric on \mathfrak{E} is the

same thing as an invariant metric on the \mathbb{X} -equivariant vector bundle E . (The latter simply means a metric on the vector bundle E over X_0 such that the isomorphism $\psi: s^*E \rightarrow t^*E$ is an isometry.)

Example 3.3 Let X be a paracompact topological space (say, a manifold) and G a compact Lie group acting on it. Set $\mathfrak{X} := [X/G]$. Then every vector bundle \mathfrak{E} on \mathfrak{X} admits a metric. In fact, metrics on \mathfrak{E} are in bijection with G -invariant metrics on the vector bundle $E := p^*\mathfrak{E}$ over X . (Here $p: X \rightarrow \mathfrak{X}$ is the quotient map.)

4 Thom isomorphism

Definition 4.1 We say a vector bundle $p: \mathfrak{E} \rightarrow \mathfrak{X}$ of rank n on a topological stack \mathfrak{X} is **orientable**, if there is a class $\mu \in H^n(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ such that the map

$$\begin{array}{ccc} H^i(\mathfrak{X}) & \xrightarrow{\tau} & H^{i+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) \\ c & \mapsto & p^*(c) \cup \mu \end{array}$$

is an isomorphism for all $i \in \mathbb{Z}$. The class μ is called a **Thom class**, or an **orientation**, for $p: \mathfrak{E} \rightarrow \mathfrak{X}$.

Lemma 4.2 Let $\mathfrak{E} \rightarrow \mathfrak{X}$ be an oriented vector bundle and $\mu \in H^n(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ a Thom class for it. Let $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism of stacks. Then $f^*\mathfrak{E} \rightarrow \mathfrak{Y}$ is an oriented vector bundle and $f^*(\mu)$ is a Thom class for it.

Lemma 4.3 Let $\mathfrak{E} \rightarrow \mathfrak{X}$ be a vector bundle. Let $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ be a trivial fibration of topological stacks, and let ν be a Thom class for the vector bundle $f^*\mathfrak{E} \rightarrow \mathfrak{Y}$. Then, there is a unique Thom class μ for \mathfrak{E} such that $f^*(\mu) = \nu$.

Proposition 4.4 Let $p: \mathfrak{E} \rightarrow \mathfrak{X}$ be an orientable vector bundle of rank n , and let $\mu \in H^n(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ be a Thom class for it. Let $\mathfrak{K} \subset \mathfrak{X}$ be a closed substack. Then, the homomorphism

$$\begin{array}{ccc} H^*(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) & \xrightarrow{\tau} & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \\ c & \mapsto & p^*(c) \cup \mu \end{array}$$

is an isomorphism. Here, we have identified \mathfrak{K} with a closed substack of \mathfrak{E} via the zero section of $\mathfrak{E} \rightarrow \mathfrak{X}$.

PROOF. Let $\mathfrak{U} = \mathfrak{X} - \mathfrak{K}$. The map $c \mapsto p^*(c) \cup \mu$ induces a map between long exact sequences

$$\begin{array}{ccccccc} \cdots & \rightarrow & H^{\bullet+n}(\mathfrak{E}|_{\mathfrak{U}}, \mathfrak{E}|_{\mathfrak{U}} - \mathfrak{U}) & \rightarrow & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) & \rightarrow & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \rightarrow \cdots \\ & & \cong \uparrow & & \cong \uparrow & & \uparrow \\ \cdots & \longrightarrow & H^{\bullet}(\mathfrak{U}) & \longrightarrow & H^{\bullet}(\mathfrak{X}) & \longrightarrow & H^{\bullet+n}(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) \rightarrow \cdots \end{array}$$

(The top sequence is long exact by Proposition 2.6.) The claim follows from 5-lemma. \square

Proposition 4.5 *In Proposition 4.4, identify \mathfrak{X} with a closed substack of \mathfrak{E} via the zero section. Then, for every $c \in H^*(\mathfrak{X}, \mathfrak{X} - \mathfrak{K})$, we have $\tau(c) = c \cdot \mu$, where \cdot is the product of Proposition 2.5.*

Proposition 4.6 *In Proposition 4.4, assume that \mathfrak{E} is metrized, and let \mathfrak{D} denote its disc bundle of radius r . Set $\mathfrak{L} = p^{-1}(\mathfrak{K}) \cap \mathfrak{D}$. and let $\rho: H^\bullet(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \rightarrow H^\bullet(\mathfrak{E}, \mathfrak{E} - \mathfrak{L})$ be the restriction homomorphism. Then the homomorphism*

$$\begin{array}{ccc} H^\bullet(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) & \xrightarrow{\tau} & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{L}) \\ c & \mapsto & \rho(p^*(c) \cup \mu) \end{array}$$

is an isomorphism. In particular, the map ρ is an isomorphism.

PROOF. Let $\mathfrak{U} = \mathfrak{X} - \mathfrak{K}$. In the case where $\mathfrak{K} = \mathfrak{X}$, a standard deformation retraction argument shows that ρ is an isomorphism, so the result follows from Proposition 4.4. The general case reduces to this case by considering the map of long exact sequences induced by $c \mapsto \rho(p^*(c) \cup \mu)$,

$$\begin{array}{ccccccc} \cdots & \rightarrow & H^{\bullet+n}(\mathfrak{E}|_{\mathfrak{U}}, \mathfrak{E}|_{\mathfrak{U}} - \mathfrak{D}|_{\mathfrak{U}}) & \rightarrow & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{D}) & \rightarrow & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{L}) & \rightarrow & \cdots \\ & & \cong \uparrow & & \cong \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & H^\bullet(\mathfrak{U}) & \longrightarrow & H^\bullet(\mathfrak{X}) & \longrightarrow & H^{\bullet+n}(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) & \rightarrow & \cdots, \end{array}$$

and applying 5-lemma. \square

The following lemma strengthens Proposition 4.4.

Lemma 4.7 *Let $p: \mathfrak{E} \rightarrow \mathfrak{X}$ be an orientable vector bundle of rank n , and let $\mu \in H^n(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ be a Thom class for it. Let $\mathfrak{K} \subset \mathfrak{X}$ be a closed substack, and $\mathfrak{K}' \subset \mathfrak{E}$ a closed substack of \mathfrak{E} mapping isomorphically to \mathfrak{K} under p . Then, we have a natural isomorphism $H^\bullet(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) \cong H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}')$.*

Lemma 4.8 *Let $p: \mathfrak{E} \rightarrow \mathfrak{X}$ and $q: \mathfrak{F} \rightarrow \mathfrak{X}$ be vector bundles over \mathfrak{X} , and assume that \mathfrak{E} is oriented. Then, an orientation for \mathfrak{F} determines an orientation for $\mathfrak{E} \oplus \mathfrak{F}$, and vice versa. Indeed, if μ is an orientation for \mathfrak{E} , and ν an orientation for \mathfrak{F} , then $\mu \cdot p^*(\nu)$ is an orientation for $\mathfrak{E} \oplus \mathfrak{F}$. Here, \cdot is the product of Proposition 2.5.*

PROOF. We only prove one of the statements, namely, the case where \mathfrak{E} and $\mathfrak{E} \oplus \mathfrak{F}$ are oriented. We show that \mathfrak{F} is also oriented. Assume \mathfrak{E} and \mathfrak{F} have rank m and n , respectively, and let $\mu \in H^m(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ and $\nu \in H^{m+n}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{X})$ be orientations for \mathfrak{E} and $\mathfrak{E} \oplus \mathfrak{F}$. The class $q^*(\mu) \in H^m(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{F})$ is an orientation for the pull-back bundle $q^*(\mathfrak{E}) \cong \mathfrak{E} \oplus \mathfrak{F}$ over \mathfrak{F} ; note that the bundle map $q^*(\mathfrak{E}) \rightarrow \mathfrak{F}$ can be naturally identified with the second projection map

$\pi: \mathfrak{E} \oplus \mathfrak{F} \rightarrow \mathfrak{F}$. By Proposition 4.4, applied to the vector bundle $\pi: \mathfrak{E} \oplus \mathfrak{F} \rightarrow \mathfrak{F}$, we have an isomorphism

$$\begin{aligned} H^n(\mathfrak{F}, \mathfrak{F} - \mathfrak{X}) &\rightarrow H^{n+m}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{X}) \\ c &\mapsto \pi^*(c) \cup q^*(\mu). \end{aligned}$$

The inverse image of ν under this isomorphism is the desired orientation class in $H^n(\mathfrak{F}, \mathfrak{F} - \mathfrak{X})$. \square

In Lemma 4.8, we call the orientation on $\mathfrak{E} \oplus \mathfrak{F}$ the **sum** of the orientations of \mathfrak{E} and \mathfrak{F} , and the orientation on \mathfrak{F} the **difference** of the orientations on $\mathfrak{E} \oplus \mathfrak{F}$ and \mathfrak{E} .

Lemma 4.9 *Let $0 \rightarrow \mathfrak{F} \rightarrow \mathfrak{M} \rightarrow \mathfrak{E} \rightarrow 0$ be a short exact sequence of vector bundles over a topological stack \mathfrak{X} . Then, the choice of orientations on two of the three vector bundles uniquely determines an orientation on the third one.*

PROOF. Apply Lemma 4.3 to the trivial fibration $f: \mathfrak{M} \rightarrow \mathfrak{X}$ to reduce the problem the split case and then apply Lemma 4.8. \square

Lemma 4.10 *In Lemma 4.7, assume we are given another oriented vector bundle $\mathfrak{F} \rightarrow \mathfrak{X}$ of rank m , and endow $\mathfrak{E} \oplus \mathfrak{F}$ with the sum orientation. Let $\mathfrak{K}'' \subset \mathfrak{E} \oplus \mathfrak{F}$ be a closed substack mapping isomorphically to \mathfrak{K}' under the projection $\mathfrak{E} \oplus \mathfrak{F} \rightarrow \mathfrak{E}$. Then, the diagram*

$$\begin{array}{ccc} H^\bullet(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) & \xrightarrow{\cong} & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}') \\ \searrow \cong & & \swarrow \cong \\ & & H^{\bullet+n+m}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{K}'') \end{array}$$

commutes. (All the isomorphisms in this diagram are the ones of Lemma 4.7. So, in the case where $\mathfrak{K} = \mathfrak{K}' = \mathfrak{K}''$, the isomorphisms are simply the Thom isomorphisms of Proposition 4.4.)

Finally, we prove a lemma about compatibility of Thom isomorphism with excision.

Lemma 4.11 *Let X be a manifold, and let $E \rightarrow X$ and $N \rightarrow X$ be vector bundles of rank n . Assume that E is oriented. Let $i: N \rightarrow E$ be an open embedding which sends the zero section of N to the zero section of E . (Note that N is naturally isomorphic to E , hence oriented, via the isomorphisms $TX \oplus N \cong TE \cong TX \oplus E$.) Then, the following diagram commutes:*

$$\begin{array}{ccc} H^{\bullet+n}(N, N - X) & \xrightarrow[\cong]{\text{excision}} & H^{\bullet+n}(E, E - X) \\ \tau_N \swarrow \cong & & \cong \searrow \tau_E \\ & & H^\bullet(X) \end{array}$$

5 Loop stacks

5.1 Mapping stacks and the free loop stack

Let \mathfrak{X} and \mathfrak{Y} be stacks over \mathbf{Top} . We define the stack $\mathbf{Hom}(\mathfrak{Y}, \mathfrak{X})$, called the **mapping stack** from \mathfrak{Y} to \mathfrak{X} , by the rule

$$T \in \mathbf{Top} \quad \mapsto \quad \mathbf{Hom}(T \times \mathfrak{Y}, \mathfrak{X}),$$

where \mathbf{Hom} denotes the groupoid of stack morphisms. This is easily seen to be a stack. It follows from the exponential law for mapping spaces ([48]) that when X and Y are spaces, with Y Hausdorff, then $\mathbf{Hom}(Y, X)$ is representable by the usual mapping space from Y to X (endowed with the compact-open topology).

The mapping stacks $\mathbf{Hom}(\mathfrak{Y}, \mathfrak{X})$ are functorial in \mathfrak{X} and \mathfrak{Y} .

Proposition 5.1 *Let \mathfrak{X} be a topological stack and A a compact topological space. Then $\mathbf{Hom}(A, \mathfrak{X})$ is a topological stack.*

PROOF. To show that $\mathbf{Hom}(A, \mathfrak{X})$ is a topological stack, first we show that every morphism $p: S \rightarrow \mathbf{Hom}(A, \mathfrak{X})$ from a topological space S is representable. Our argument will indeed be valid for any topological space A and any stack \mathfrak{X} whose diagonal is representable. Let $q: T \rightarrow \mathbf{Hom}(A, \mathfrak{X})$ be an arbitrary morphism from a topological space T . Let $\tilde{p}: S \times A \rightarrow \mathfrak{X}$ be the defining map of p , and $\tilde{q}: T \times A \rightarrow \mathfrak{X}$ the one for q . Set $Y := (T \times A) \times_{\mathfrak{X}} (S \times A)$. This is a topological space which sits in the following cartesian diagram:

$$\begin{array}{ccc} \mathbf{Hom}(A, Y) & \longrightarrow & \mathbf{Hom}(A, S \times A) \\ \downarrow & & \downarrow \tilde{p}_* \\ \mathbf{Hom}(A, T \times A) & \xrightarrow{\tilde{q}_*} & \mathbf{Hom}(A, \mathfrak{X}) \end{array}$$

The claim now follows from the fact that p and q factor through \tilde{p}_* and \tilde{q}_* , respectively (both via the corresponding id).

We now construct an epimorphism $R \rightarrow \mathbf{Hom}(A, \mathfrak{X})$. First we need a bit of notation. Let $\mathbb{Y} = [Y_1 \rightrightarrows Y_0]$ and $\mathbb{X} = [X_1 \rightrightarrows X_0]$ be topological groupoids, with Y_0, Y_1 Hausdorff. We define $\mathbf{Hom}(\mathbb{Y}, \mathbb{X})$ to be the space of continuous groupoid morphisms from \mathbb{Y} to \mathbb{X} . This is topologized as a subspace of $\mathbf{Hom}(Y_1, X_1) \times \mathbf{Hom}(Y_0, X_0)$, and it represents the following functor

$$T \in \mathbf{Top} \quad \mapsto \quad \text{groupoid morphisms } T \times \mathbb{Y} \rightarrow \mathbb{X},$$

where $T \times \mathbb{Y}$ stands for the groupoid $[T \times Y_1 \rightrightarrows T \times Y_0]$. In particular, we have a universal family of groupoid morphisms $\mathbf{Hom}(\mathbb{Y}, \mathbb{X}) \times \mathbb{Y} \rightarrow \mathbb{X}$.

Let J be the set of all finite open covers of A . For $\alpha \in J$, let U_α denote the disjoint union of the open sets appearing in the open cover α . There is a natural map $U_\alpha \rightarrow A$. Let $\mathbb{A}_\alpha := [U_\alpha \times_A U_\alpha \rightrightarrows U_\alpha]$ be the corresponding topological groupoid. The spaces $U_\alpha \times_A U_\alpha$ and U_α are Hausdorff and the quotient stack of

\mathbb{A}_α is A . Fix a groupoid presentation $\mathbb{X} = [X_1 \rightrightarrows X_0]$ for \mathfrak{X} , and let $\pi: X_0 \rightarrow \mathfrak{X}$ be the corresponding chart for \mathfrak{X} . Set $R_\alpha = \mathbf{Hom}(\mathbb{A}_\alpha, \mathbb{X})$, and $R = \coprod_\alpha R_\alpha$.

The universal groupoid morphisms $R_\alpha \times \mathbb{A}_\alpha \rightarrow \mathbb{X}$ give rise to morphisms $R_\alpha \rightarrow \mathbf{Hom}(A, \mathfrak{X})$. Putting these all together we obtain a morphism $R \rightarrow \mathbf{Hom}(A, \mathfrak{X})$. We claim that this is an epimorphism. Here is where we use compactness of A . Let $p: T \rightarrow \mathbf{Hom}(A, \mathfrak{X})$ be an arbitrary morphism. We have to show that for every $t \in T$, there exists an open neighborhood W of t such that $p|_W$ lifts to R . Let $\tilde{p}: T \times A \rightarrow \mathfrak{X}$ be the defining morphism for p . Since $\pi: X_0 \rightarrow \mathfrak{X}$ is an epimorphism, we can find finitely many open sets V_i of $T \times A$ which cover $\{t\} \times A$ and such that $\tilde{p}|_{V_i}$ lifts to X_0 for every i . We may assume $V_i = A_i \times W$, where A_i are open subsets of A , and W is an open neighborhood of t independent of i . Let $\alpha := \{A_i\}$ be the corresponding open cover of A . Then $p|_W$ lifts to $R_\alpha \subset R$. \square

Let \mathfrak{X} be a topological stack. Then $\mathbf{L}\mathfrak{X} = \mathbf{Hom}(S^1, \mathfrak{X})$ is also a topological stack. It is called the **loop stack** of \mathfrak{X} . By functoriality of mapping stacks, for every $t \in S^1$ we have the corresponding evaluation map $\text{ev}_t: \mathbf{L}\mathfrak{X} \rightarrow \mathfrak{X}$. In particular, denoting by $0 \in S^1$ the standard choice of a base point, there is an evaluation map

$$\text{ev}_0: \mathbf{L}\mathfrak{X} \rightarrow \mathfrak{X}. \quad (5.1)$$

Similarly, the **path stack** of \mathfrak{X} , which is defined to be $\mathbf{Hom}(I, \mathfrak{X})$, is a topological stack.

For the next result, we need to assume that \mathfrak{X} is a Hurewicz topological stack.

Lemma 5.2 *Let A, Y , and Z be as in Proposition 1.3. Let \mathfrak{X} be a Hurewicz topological stack. Then the diagram*

$$\begin{array}{ccc} \mathbf{Hom}(Z \vee_A Y, \mathfrak{X}) & \longrightarrow & \mathbf{Hom}(Y, \mathfrak{X}) \\ \downarrow & & \downarrow \\ \mathbf{Hom}(Z, \mathfrak{X}) & \longrightarrow & \mathbf{Hom}(A, \mathfrak{X}) \end{array}$$

is a 2-cartesian diagram of topological stacks.

PROOF. We have to verify that for every topological space T the T -points of the above mapping stacks form a 2-cartesian diagram of groupoids. This follows from Proposition 1.3 applied to $A \times T, Y \times T$, and $Z \times T$. \square

We denote by ‘8’ the wedge $S^1 \vee S^1$ of two circles.

Corollary 5.3 *Let \mathfrak{X} be a Hurewicz topological stack, and let $L\mathfrak{X}$ be its loop stack. Then, the diagram*

$$\begin{array}{ccc} \text{Map}(8, \mathfrak{X}) & \rightarrow & L\mathfrak{X} \\ \downarrow & & \downarrow \\ L\mathfrak{X} & \longrightarrow & \mathfrak{X} \end{array}$$

is 2-cartesian.

5.2 Groupoid presentation

Let us now describe a particular groupoid presentation of the loop stack. For this we will assume that \mathfrak{X} is a *Hausdorff Hurewicz topological stack*. Thus \mathfrak{X} admits a groupoid presentation $\Gamma : \Gamma_1 \rightrightarrows \Gamma_0$, where Γ_0 and Γ_1 are Hausdorff topological spaces, $\Gamma_1 \rightarrow \Gamma_0 \times \Gamma_0$ is proper, and source and target maps are local fibrations. We will fix the groupoid Γ .

We will construct a groupoid $L\Gamma : L_1\Gamma \rightrightarrows L_0\Gamma$ out of Γ which represents $L\mathfrak{X}$. This groupoid presentation is useful in computations (see Section 11). Our construction resembles the construction of the fundamental groupoid of a groupoid [39].

Let $M\Gamma = [M_1\Gamma \rightrightarrows M_0\Gamma]$ be the morphism groupoid of Γ . Its object set is $M_0\Gamma = \Gamma_1$ and its morphism set $M_1\Gamma$ is the set of commutative squares in the underlying category of Γ :

$$\begin{array}{ccc} t(h) & \xleftarrow{g} & t(k) \\ \uparrow h & & \uparrow k \\ s(h) & \xleftarrow{h^{-1}gk} & s(k) \end{array} \quad (5.2)$$

The source and target maps are the horizontal arrows in square (5.2). The groupoid multiplication is by (vertical) superposition of such squares. Thus we have $M_1\Gamma \cong \Gamma_3 = \Gamma_1 \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} \Gamma_1$. The groupoid $M\Gamma$ is another presentation of the stack \mathfrak{X} and is Morita equivalent to Γ .

Let $P \subset S^1$ be a finite subset of S^1 which contains the base point of S^1 . The points of P are labeled according to increasing angle as P_0, P_1, \dots, P_n in such a way that $P_0 = P_n$ is the base point of S^1 . Write I_i for the closed interval $[P_{i-1}, P_i]$. Let S_0^P be the disjoint union $S_0^P = \coprod_{i=1}^n I_i$. There is a canonical map $S_0^P \rightarrow S^1$. Let S_1^P be the fiber product $S_1^P = S_0^P \times_{S^1} S_0^P$. There is an obvious topological groupoid structure $S_1^P \rightrightarrows S_0^P$. The compact-open topology induces a topological groupoid structure on $L^P\Gamma : L_1^P\Gamma \rightrightarrows L_0^P\Gamma$, where $L_0^P\Gamma$ is the set of continuous strict groupoid morphisms $[S_1^P \rightrightarrows S_0^P] \rightarrow [\Gamma_1 \rightrightarrows \Gamma_0]$ and $L_1^P\Gamma$ is the set of strict continuous groupoid morphisms $[S_1^P \rightrightarrows S_0^P] \rightarrow [M_1\Gamma \rightrightarrows M_0\Gamma]$.

The finite subsets of S^1 including the base point are ordered by inclusion. The ordering is *directed*. For $P \leq Q$ there is a canonical morphism of groupoids

$L^P\Gamma \longrightarrow L^Q\Gamma$. Using the fact that Γ_0 and Γ_1 are Hausdorff, it is not difficult to prove that $L^P\Gamma \rightarrow L^Q\Gamma$ is an isomorphism onto an open subgroupoid. Define the topological groupoid

$$L\Gamma = \varinjlim_{P \subset S^1} L^P\Gamma = \bigcup_{P \subset S^1} L^P\Gamma.$$

Proposition 5.4 *The groupoid $L\Gamma$ presents the loop stack $L\mathfrak{X}$.*

PROOF. First, we need to construct a morphism $L_0^P\Gamma \rightarrow L\mathfrak{X}$, for every P . The presentation $L_0\Gamma \rightarrow L\mathfrak{X}$ will then be obtained by gluing these morphisms using the stack property of $L\mathfrak{X}$ and the fact the $L_0^P\Gamma$ form an open covering of the topological space $L_0\Gamma$.

The structure map $L_0^P \times S_0^P \rightarrow \Gamma_0$ gives rise to a morphism $L_0^P \times S_0^P \rightarrow \mathfrak{X}$. This morphism descends to $L_0^P \times S^1 \rightarrow \mathfrak{X}$, by Proposition 1.3, because S^1 is obtained from S_0^P as a pushout covered by that proposition. By adjunction, we obtain the required morphism $L_0^P \rightarrow L\mathfrak{X}$.

The fact that $\bigcup_P L_0^P\Gamma \rightarrow L\mathfrak{X}$ is an epimorphism of stacks, follows as in Proposition 5.1.

The fact that $L_1\Gamma$ is the fibered product of $L_0\Gamma$ with itself over $L\mathfrak{X}$ reduces immediately to the case of $L_1^P\Gamma$, for which it is immediate. \square

It is easy to represent evaluation map and functorial properties of the free loop stack at the groupoid level with this model.

Remark 5.5 In particular, there is an equivalence of the underlying categories between $L\Gamma$ and the groupoid whose objects are the set of generalized morphisms from the space S^1 to Γ and has equivalences of such as arrows.

Corollary 5.6 *If \mathfrak{X} is a differentiable stack then $L\mathfrak{X}$ is regular Lindelöf.*

Target connected groupoid

Assume the groupoid Γ is target connected. This means that if T is a topological space, and $\phi : T \rightarrow \Gamma_1$ a continuous map, then for every point of T there exist an open neighborhood $T' \subset T$ and a homotopy $\Phi : T' \times I \rightarrow \Gamma_1$, such that $\Phi_0 = \phi$ and $\Phi_1 = t \circ \phi$, where $t : \Gamma_1 \rightarrow \Gamma_0$ is the target map. For example, any transformation groupoid with connected Lie group is target connected.

For every finite subset $P \subset S^1$ and $x \in L_0^P\Gamma$, there are arrows $g_i \in \Gamma_1$ with $t(g_i) = P_i \in I_i$ and $s(g_i) = P_i \in I_{i+1}$ (or $P_0 \in I_1$ if $i = n$). These arrows can be continuously deformed to the identity point $P_n \in I_n$. Thus there is an element $\tilde{x} \in \mathbf{Hom}(S^1, \Gamma_0) \subset L^{\{0\}}\Gamma_0$ and an arrow $\gamma \in L_1^P\Gamma$ with $s(\gamma) = \tilde{x}$ and $t(\gamma) = x$. From this observation, we deduce:

Proposition 5.7 *If Γ is target connected, then the groupoid $L\Gamma_1 \rightrightarrows L\Gamma_0$ with pointwise source map, target map and multiplication presents the loop stack*

$L\mathfrak{X}$. Here $L\Gamma_i$ is the usual free loop space of Γ_i endowed with the compact-open topology.

In particular, $L\Gamma$ is Morita equivalent to the groupoid $L\Gamma_1 \rightrightarrows L\Gamma_0$.

Example 5.8 If G is a connected Lie group acting on a manifold M , then Proposition 5.7 implies that $L[M/G] \cong [LM/LG]$.

Discrete group action

To the contrary, if G is a discrete group acting on a space M one can form the global quotient $[M/G]$ which is represented by the transformation groupoid $\Gamma : M \times G \rightrightarrows M$. For any $x \in L^P\Gamma_0$ one can easily find an arrow $\gamma \in L^P\Gamma_1$ such that $s(\gamma) = x$ and $t(\gamma) \in L^{\{0\}}\Gamma_0$. Furthermore, since G is discrete, an element of $L^P\Gamma_1$ is described by its source and one element $g_i \in G$ for $i = 0, \dots, |P|$. From these two observations one proves easily:

Proposition 5.9 *Let G be a discrete group acting on a space M . Then $L[M/G]$ is presented by the transformation groupoid*

$$\left(\coprod_{g \in G} \mathcal{P}_g M \right) \times G \rightrightarrows \coprod_{g \in G} \mathcal{P}_g M$$

where $\mathcal{P}_g M = \{f : [0, 1] \rightarrow M \text{ such that } f(0) = f(1).g\}$ and G acts by pointwise conjugation.

Note that if G is finite, one recovers the loop orbifold of [34].

6 Bounded proper morphisms of topological stacks

Definition 6.1 Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of topological stacks and \mathfrak{E} a metrizable vector bundle over \mathfrak{Y} . A lifting $i : \mathfrak{X} \rightarrow \mathfrak{E}$ of f ,

$$\begin{array}{ccc} & & \mathfrak{E} \\ & \nearrow i & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

is called **bounded** if there is a choice of metric on \mathfrak{E} such that i factors through the unit disk bundle of \mathfrak{E} . A morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ of topological stacks is called **bounded proper** if there exists a metrizable orientable vector bundle \mathfrak{E} on \mathfrak{Y} and a bounded lifting i as above such that i is a closed embedding.

Definition 6.2 A bounded proper morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is called **strongly proper** if every orientable metrizable vector bundle \mathfrak{E} on \mathfrak{X} is a direct summand of $f^*(\mathfrak{E}')$ for some orientable metrizable vector bundle \mathfrak{E}' on \mathfrak{Y} . (Note that, possibly after multiplying by a positive \mathbb{R} -valued function on \mathfrak{Y} , we can arrange the inclusion $b^*\mathfrak{E} \hookrightarrow (gb)^*(\mathfrak{E}')$ to be contractive, i.e., have norm at most one.)

Example 6.3 1. Every bounded proper map $f: X \rightarrow Y$ of a topological spaces with Y compact is strongly proper. In that case, one can use the fact that every vector bundle on a compact space is a subbundle of a trivial bundle.

2. Let \mathfrak{X} be a topological stack such that $\Delta: \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is bounded proper. Then Δ is strongly proper. This follows from the fact that every vector bundle on \mathfrak{X} can be naturally extended to $\mathfrak{X} \times \mathfrak{X}$. Similarly, the iterated diagonal $\Delta^{(n)}: \mathfrak{X} \rightarrow \mathfrak{X}^n$ is strongly proper.

3. Let X, Y be compact G -manifolds (with G compact) and $f: X \rightarrow Y$ be a G -equivariant map. Then the induced map of stacks $[f/G]: [X/G] \rightarrow [Y/G]$ is strongly proper.

It does not seem to be true in general that two bounded proper maps compose to a bounded proper map, but we have the following.

Lemma 6.4 *Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ and $g: \mathfrak{Y} \rightarrow \mathfrak{Z}$ be strongly proper morphisms. Then $gf: \mathfrak{X} \rightarrow \mathfrak{Z}$ is strongly proper.*

PROOF. It is trivial that every orientable metrizable bundle on \mathfrak{X} is a direct summand of one coming from \mathfrak{Z} . Let us now prove that gf is proper. Suppose given factorizations

$$\begin{array}{ccc} & \mathfrak{E} & \\ & \nearrow i & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array} \quad \begin{array}{ccc} & \mathfrak{F} & \\ & \nearrow j & \downarrow \\ \mathfrak{Y} & \xrightarrow{g} & \mathfrak{Z} \end{array}$$

for f and g . By enlarging \mathfrak{E} , and using that g is strongly proper, we may assume that $\mathfrak{E} = g^*(\mathfrak{E}')$, for some oriented metrized vector bundle \mathfrak{E}' on \mathfrak{Z} . Let $i': \mathfrak{X} \rightarrow \mathfrak{E}'$ be the composition $\text{pr} \circ i$ where $\text{pr}: \mathfrak{E} \rightarrow \mathfrak{E}'$ is the projection map. The following diagram shows that gf is proper:

$$\begin{array}{ccc} & \mathfrak{E}' \oplus \mathfrak{F} & \\ & \nearrow (i', jf) & \downarrow \\ \mathfrak{X} & \xrightarrow{gf} & \mathfrak{Z} \end{array}$$

□

6.1 Some technical lemmas

In this section we prove a few technical lemmas that will be needed in Section 7 to define bivariant groups.

Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of topological stack that admits a factorization

$$\begin{array}{ccc} & & \mathfrak{E} \\ & \nearrow i & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

For example, every bounded proper f has this property (Definition 6.1). The following series lemmas investigate certain properties of the relative cohomology groups $H^\bullet(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$.

Lemma 6.5 *Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of topological stacks, and assume we are given two different factorizations (i, \mathfrak{E}) and (i', \mathfrak{E}') for it. Then, there is a canonical isomorphism $H^{\bullet+\text{rk } \mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) \cong H^{\bullet+\text{rk } \mathfrak{E}'}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{X})$.*

PROOF. Embed \mathfrak{X} in $\mathfrak{E} \oplus \mathfrak{E}'$ via $(i, i'): \mathfrak{X} \rightarrow \mathfrak{E} \oplus \mathfrak{E}'$. Consider the diagram

$$(\mathfrak{E}', \mathfrak{E}' - \mathfrak{X}) \leftarrow (\mathfrak{E}' \oplus \mathfrak{E}, \mathfrak{E}' \oplus \mathfrak{E} - \mathfrak{X}) \rightarrow (\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$$

of pairs of topological stacks. It follows from Proposition 4.4 that we have natural isomorphisms

$$H^{\bullet+\text{rk } \mathfrak{E}'}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{X}) \xleftarrow{\sim} H^{\bullet+\text{rk } \mathfrak{E} + \text{rk } \mathfrak{E}'}(\mathfrak{E}' \oplus \mathfrak{E}, \mathfrak{E}' \oplus \mathfrak{E} - \mathfrak{X}) \xrightarrow{\sim} H^{\bullet+\text{rk } \mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}).$$

We can now apply Lemma 4.7. \square

Using a triple direct sum argument, it can be shown that given three factorizations (i, \mathfrak{E}) , (i', \mathfrak{E}') , and (i'', \mathfrak{E}'') for f , the corresponding isomorphisms defined in the above lemma are compatible. Also, if we switch the order of (i, \mathfrak{E}) and (i', \mathfrak{E}') we get the inverse isomorphism. Finally, when (i, \mathfrak{E}) and (i', \mathfrak{E}') are equal we get the identity isomorphism. Therefore, the group $H^\bullet(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ only depends on the morphism f .

Lemma 6.6 *Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of topological stacks, and let $\varphi := f \circ \text{pr}: \mathfrak{X} \times I \rightarrow \mathfrak{Y}$, where I is the unit interval and pr stands for projection. Suppose we are give a factorization*

$$\begin{array}{ccc} & & \mathfrak{E} \\ & \nearrow i & \downarrow \\ \mathfrak{X} \times I & \xrightarrow{\varphi} & \mathfrak{Y} \end{array}$$

for φ . Let $0 \leq a \leq 1$, and define $\iota_a: \mathfrak{X} \rightarrow \mathfrak{E}$ to be the restriction of ι to $\mathfrak{X} = \mathfrak{X} \times \{a\}$. Then, the natural map $\phi_a: H^\bullet(\mathfrak{E}, \mathfrak{E} - \iota_a(\mathfrak{X})) \rightarrow H^\bullet(\mathfrak{E}, \mathfrak{E} - \iota(\mathfrak{X} \times I))$ induced by the map of pairs $(\mathfrak{E}, \mathfrak{E} - \iota(\mathfrak{X} \times I)) \rightarrow (\mathfrak{E}, \mathfrak{E} - \iota_a(\mathfrak{X}))$ is an isomorphism and it is independent of a .

PROOF. We may assume that the image of ι does not intersect the zero section of \mathfrak{E} . (For example, we lift everything to $\mathfrak{E} \oplus \mathbb{R}$ via $(\iota, 1): \mathfrak{X} \rightarrow \mathfrak{E} \oplus \mathbb{R}$ and apply Proposition 4.7 to the vector bundle $\mathfrak{E} \oplus \mathbb{R} \rightarrow \mathfrak{E}$.)

Let $\mathfrak{E}' = \mathfrak{E} \oplus \mathbb{R}$ and define $\beta: \mathfrak{X} \times I \hookrightarrow \mathfrak{E}'$ by $\beta(x, t) = (\iota(x, 0), t)$. This is a closed embedding, so by Lemma 6.5, we have a commutative diagram

$$\begin{array}{ccc} H^\bullet(\mathfrak{E}, \mathfrak{E} - \iota(\mathfrak{X} \times I)) & \xrightarrow{\cong} & H^\bullet(\mathfrak{E}', \mathfrak{E}' - \beta(\mathfrak{X} \times I)) \\ \phi_a \uparrow & & \uparrow \phi'_a \\ H^\bullet(\mathfrak{E}, \mathfrak{E} - \iota_a(\mathfrak{X})) & \xrightarrow{\cong} & H^\bullet(\mathfrak{E}', \mathfrak{E}' - \beta_a(\mathfrak{X})) \end{array}$$

This reduces the problem to the case where our map is β instead of ι , in which case the result is obvious.

□

7 Bivariant theory for topological stacks

We define a bivariant cohomology theory [25] on the category of topological stacks whose associated covariant and contravariant theories are singular homology and cohomology, respectively. Our bivariant theory satisfies weaker axioms than those of [25] in that products are not always defined. We show, however, that there are enough products to enable us to define Gysin morphisms as in [25].

The underlying category of our bivariant theory is the category TopSt of topological stacks. The confined morphisms are all maps and independent squares are 2-cartesian diagrams.

Bivariant groups

To a morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of topological stacks, we associate a category $\mathcal{C}(f)$ as follows. The objects of $\mathcal{C}(f)$ are morphism $a: \mathfrak{K} \rightarrow \mathfrak{X}$ such that $fa: \mathfrak{K} \rightarrow \mathfrak{Y}$ is bounded proper (Definition 6.1). A morphism in $\mathcal{C}(f)$ between $a: \mathfrak{K} \rightarrow \mathfrak{X}$ and $b: \mathfrak{L} \rightarrow \mathfrak{X}$ is a homotopy class (relative to \mathfrak{X}) of morphisms $g: \mathfrak{K} \rightarrow \mathfrak{L}$ over \mathfrak{X} .

Lemma 7.1 *The category $\mathcal{C}(f)$ is cofiltered.*

Once and for all, we choose, for each object $a: \mathfrak{K} \rightarrow \mathfrak{X}$, a vector bundle $\mathfrak{E} \rightarrow \mathfrak{Y}$ through which fa factors, as in Definition 6.3.

We define the **bivariant singular homology** of an arbitrary morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ to be the \mathbb{Z} -graded abelian group

$$H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) = \varinjlim_{\mathcal{C}(f)} H^{\bullet+\mathrm{rk} \mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}).$$

The homomorphisms in this direct limit are defined as follows. Consider a morphism $\varphi: \mathfrak{K} \rightarrow \mathfrak{K}'$ in $\mathcal{C}(f)$. From this we will construct a natural graded pushforward homomorphism $\varphi_*: H^{\bullet+m}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \rightarrow H^{\bullet+n}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}')$, where $m = \mathrm{rk} \mathfrak{E}$ and $n = \mathrm{rk} \mathfrak{E}'$.

Let $\mathfrak{F} = \mathfrak{E} \oplus \mathfrak{E}'$ with the sum orientation. Let $p: \mathfrak{E}' \rightarrow \mathfrak{Y}$ be the projection map. Then, $p^*(\mathfrak{E})$ is an oriented vector bundle over \mathfrak{E}' . Note that the projection map $\pi: p^*(\mathfrak{E}) \rightarrow \mathfrak{E}'$ is naturally isomorphic to the second projection map $\mathfrak{F} = \mathfrak{E} \oplus \mathfrak{E}' \rightarrow \mathfrak{E}'$; this allows us to view \mathfrak{F} as an oriented vector bundle of rank m over \mathfrak{E}' . Let $\mathfrak{D} \subseteq \mathfrak{F}$ be the unit disc bundle. It follows from the assumptions that $\mathfrak{K} \subseteq \mathfrak{D}$, hence also $\mathfrak{K} \subseteq \mathfrak{L} := \pi^{-1}(\mathfrak{K}') \cap \mathfrak{D}$. The restriction homomorphism

$$\varphi_*: H^{\bullet+m+n}(\mathfrak{F}, \mathfrak{F} - \mathfrak{K}) \rightarrow H^{\bullet+m+n}(\mathfrak{F}, \mathfrak{F} - \mathfrak{L}) \cong H^{\bullet+n}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}'),$$

induced by the inclusion of pairs $(\mathfrak{F}, \mathfrak{F} - \mathfrak{L}) \rightarrow (\mathfrak{F}, \mathfrak{F} - \mathfrak{K})$ is the desired pushforward homomorphism; here, we have used the isomorphism of Proposition 4.6.

Next we have to show that the φ_* is independent of the homotopy class of φ . Consider $a \circ \mathrm{pr}: \mathfrak{K} \times I \rightarrow \mathfrak{X}$, and let $\rho_0, \rho_1: \mathfrak{K} \rightarrow \mathfrak{K} \times I$ be the times 0 and time 1 maps. Note that $a \circ \mathrm{pr}: \mathfrak{K} \times I \rightarrow \mathfrak{X}$ is an object of $\mathcal{C}(f)$. Since every homotopy (relative to \mathfrak{X}) between maps with domain \mathfrak{K} factors through $\mathfrak{K} \times I$, it is enough to show that $\rho_{0,*} = \rho_{1,*}$. This follows from Lemma 6.6.

Remark 7.2 Let $\mathfrak{K} \rightarrow \mathfrak{Y}$ be a bounded proper morphism. It follows from Lemma 6.5, that the cohomology $H^\bullet(\mathfrak{E}, \mathfrak{E} - \mathfrak{K})$ is independent of choice of the vector bundle \mathfrak{E} and the embedding $i: \mathfrak{K} \hookrightarrow \mathfrak{E}$, up to a canonical isomorphism. Furthermore, the pushforward maps constructed above are compatible with these canonical isomorphisms. So, $H^\bullet(f)$ is independent of all choices involved in its definition.

Lemma 7.3 *Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a bounded proper morphism and $\mathfrak{X} \xrightarrow{i} \mathfrak{E} \rightarrow \mathfrak{Y}$ a factorization for f , where i is a closed embedding (but \mathfrak{E} is not necessarily metrizable). Then we have a natural isomorphism*

$$H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \cong H^{\bullet+\mathrm{rk} \mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}).$$

PROOF. Follows from Lemma 6.5. \square

Independent pullbacks

Consider a cartesian diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{Y}' \\ \downarrow & & \downarrow h \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

We define the pullback $h^*: H(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \rightarrow H(\mathfrak{X}' \xrightarrow{f'} \mathfrak{Y}')$ as follows.

Pullback along h induces a functor $h^*: \mathcal{C}(f) \rightarrow \mathcal{C}(f')$, $\mathfrak{K} \mapsto h^*\mathfrak{K} := \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{K}$. Furthermore, we have a natural homomorphism

$$H^{\bullet+\text{rk}} \mathfrak{E}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \rightarrow H^{\bullet+\text{rk}} \mathfrak{E}(h^*\mathfrak{E}, h^*\mathfrak{E} - h^*\mathfrak{K})$$

induced by the map of pairs $(h^*\mathfrak{E}, h^*\mathfrak{E} - h^*\mathfrak{K}) \rightarrow (\mathfrak{E}, \mathfrak{E} - \mathfrak{K})$. Using Lemma 6.5, this induces the desired homomorphism of colimits

$$h^*: \varinjlim_{\mathcal{C}(f)} H^{\bullet+\text{rk}} \mathfrak{E}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \rightarrow \varinjlim_{\mathcal{C}(f')} H^{\bullet+\text{rk}} \mathfrak{E}'(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}').$$

Confined pushforwards

Let $h: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of topological stacks (Definition 6.1) fitting in a commutative triangle

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{h} & \mathfrak{Y} \\ f \searrow & & \swarrow g \\ & \mathfrak{Z} & \end{array}$$

We define the pushforward homomorphism $h_*: H(\mathfrak{X} \xrightarrow{f} \mathfrak{Z}) \rightarrow H(\mathfrak{Y} \xrightarrow{g} \mathfrak{Z})$ as follows.

There is a natural functor $\mathcal{C}(f) \rightarrow \mathcal{C}(g)$, which sends $a: \mathfrak{K} \rightarrow \mathfrak{X}$ to $ha: \mathfrak{K} \rightarrow \mathfrak{Y}$. A factorization for fa gives a factorization for gha in a trivial manner:

$$\begin{array}{ccc} \mathfrak{K} \hookrightarrow \mathfrak{E} & & \mathfrak{K} \hookrightarrow \mathfrak{E} \\ a \downarrow & \downarrow & \mapsto \quad ha \downarrow \quad \downarrow \\ \mathfrak{X} \xrightarrow{f} \mathfrak{Z} & & \mathfrak{Y} \xrightarrow{g} \mathfrak{Z} \end{array}$$

Using Lemma 6.5, this induces the desired homomorphism

$$h_*: \varinjlim_{\mathcal{C}(f)} H^{\bullet+\text{rk}} \mathfrak{E}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \rightarrow \varinjlim_{\mathcal{C}(g)} H^{\bullet+\text{rk}} \mathfrak{E}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}).$$

Products

Unfortunately, we are not able to define product for arbitrary pairs of composable morphisms f and g . However, under an extra assumption on g this will be possible.

Definition 7.4 A morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of topological stacks is called **adequate** if in the cofiltered category $\mathcal{C}(f)$ the subcategory consisting of $a: \mathfrak{K} \rightarrow \mathfrak{X}$ such that $fa: \mathfrak{K} \rightarrow \mathfrak{Y}$ is strongly proper is cofinal.

Example 7.5 1. Every strongly proper morphism is adequate. (Because in this case $\mathcal{C}(f)$ has a final object that is strongly proper over \mathfrak{Y} .)

2. A morphism $f: \mathfrak{X} \rightarrow Y$ in which Y is a paracompact topological space is adequate. (In this case every object in $\mathcal{C}(f)$ is strongly proper over Y ; see Example 6.3)

Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ and $g: \mathfrak{Y} \rightarrow \mathfrak{Z}$ be morphisms of topological stacks, and assume g is adequate. Then we can define products of any two classes $\alpha \in H(f)$ and $\beta \in H(g)$. The construction of the product is as follows. Consider objects $(\mathfrak{K}, a) \in \mathcal{C}(f)$ and $(\mathcal{L}, b) \in \mathcal{C}(g)$, and choose factorizations

$$\begin{array}{ccc} \mathfrak{K} \xrightarrow{i} \mathfrak{E} & & \mathcal{L} \xrightarrow{j} \mathfrak{F} \\ a \downarrow & & b \downarrow \\ \mathfrak{X} \xrightarrow{f} \mathfrak{Y} & & \mathfrak{Y} \xrightarrow{g} \mathfrak{Z} \end{array}$$

We may assume $gb: \mathcal{L} \rightarrow \mathfrak{Z}$ is strongly proper. There exists a metrizable oriented vector bundle \mathfrak{E}' over \mathfrak{Z} such that $b^*\mathfrak{E}$ is isomorphic to a subbundle of $(gb)^*(\mathfrak{E}')$ as vector bundles over \mathcal{L} . Note that, possibly after multiplying by a positive \mathbb{R} -valued function on \mathfrak{Z} , we can arrange the inclusion $b^*\mathfrak{E} \hookrightarrow (gb)^*(\mathfrak{E}')$ to be contractive (i.e., have norm at most one). Let us denote $b^*\mathfrak{E}$ by \mathfrak{E}_0 , $(gb)^*(\mathfrak{E}')$ by \mathfrak{E}_1 , and the codimension of \mathfrak{E}_0 in \mathfrak{E}_1 by c .

We define the product

$$H^r(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \otimes H^s(\mathfrak{F}, \mathfrak{F} - \mathcal{L}) \rightarrow H^{r+s+c}(\mathfrak{E}' \oplus \mathfrak{F}, \mathfrak{E}' \oplus \mathfrak{F} - \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L}).$$

as follows. (Note that $(\mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L}, a \circ \text{pr})$ belongs to $\mathcal{C}(gf)$ and we have a factorization

$$\begin{array}{ccc} \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L} \xrightarrow{(i,j)} \mathfrak{E}' \oplus \mathfrak{F} \\ \downarrow & & \downarrow \\ \mathfrak{X} \xrightarrow{gf} \mathfrak{Z} \end{array}$$

for it. We explain this in more detail shortly.) By pulling back the map i along $\varpi: \mathfrak{E}_0 \rightarrow \mathfrak{E}$, we obtain a closed embedding $\mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L} \hookrightarrow \mathfrak{E}_0$. On the other hand,

we have a closed embedding $\mathfrak{E}_1 \hookrightarrow \mathfrak{E}' \oplus \mathfrak{F}$; this is simply the pullback of j along the projection map $\pi: \mathfrak{E}' \oplus \mathfrak{F} \rightarrow \mathfrak{F}$. Using the inclusion $\mathfrak{E}_0 \hookrightarrow \mathfrak{E}_1$, we find a factorization

$$\begin{array}{ccccccc} & & & & \xrightarrow{(i,j)} & & \\ & & & & \searrow & & \\ \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L} & \hookrightarrow & \mathfrak{E}_0 & \hookrightarrow & \mathfrak{E}_1 & \hookrightarrow & \mathfrak{E}' \oplus \mathfrak{F}. \end{array}$$

Now, let $\alpha \in H^r(\mathfrak{E}, \mathfrak{E} - \mathfrak{K})$ and $\beta \in H^s(\mathfrak{F}, \mathfrak{F} - \mathcal{L})$ be two cohomology classes. We define $\alpha \cdot \beta \in H^{r+s+c}(\mathfrak{E}' \oplus \mathfrak{F}, \mathfrak{E}' \oplus \mathfrak{F} - \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L})$ to be $\tau(\varpi^*(\alpha)) \cdot \pi^*(\beta)$, where the latter \cdot is the product of Proposition 2.5. In more detail, we have $\pi^*(\beta) \in H^s(\mathfrak{E}' \oplus \mathfrak{F}, \mathfrak{E}' \oplus \mathfrak{F} - \mathfrak{E}_1)$, $\varpi^*(\alpha) \in H^r(\mathfrak{E}_0, \mathfrak{E}_0 - \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L})$, and $\tau: H^r(\mathfrak{E}_0, \mathfrak{E}_0 - \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L}) \rightarrow H^{r+c}(\mathfrak{E}_1, \mathfrak{E}_1 - \mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L})$ is the Thom isomorphism of Proposition 4.4 for the vector bundle \mathfrak{E}_1 over \mathfrak{E}_0 ; to obtain this Thom isomorphism, we have used that, since the bundles are metrizable, \mathfrak{E}_0 is a direct summand of \mathfrak{E}_1 and its complement is oriented (Lemma 4.8). Finally, our \cdot is the one of Proposition 2.5 with the inclusions $\mathfrak{K} \times_{\mathfrak{Y}} \mathcal{L} \hookrightarrow \mathfrak{E}_1 \hookrightarrow \mathfrak{E}' \oplus \mathfrak{F}$, $n = r + c$ and $m = s$.

Associated covariant and contravariant theories

By definition, the n^{th} graded piece of the contravariant theory associated to the bivariant theory H is give by

$$H^n(\mathfrak{X}) = H^n(\mathfrak{X} \xrightarrow{\text{id}} \mathfrak{X}) = \varinjlim_{\mathbf{C}(\text{id}_{\mathfrak{X}})} H^{n+\text{rk } \mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}).$$

The category $\mathbf{C}(\text{id}_{\mathfrak{X}})$ has a final object $(\mathfrak{X}, \mathfrak{X})$, so the above colimit is isomorphic to $H^n(\mathfrak{X}, \mathfrak{X} - \mathfrak{X}) = H^n(\mathfrak{X})$, the usual singular cohomology.

The n^{th} graded piece of the covariant theory associated to H is defined to be

$$H_n(\mathfrak{X}) = H^{-n}(\mathfrak{X} \rightarrow \text{pt}) = \varinjlim_{\mathbf{C}(\mathfrak{X})} H^{e-n}(E, E - K) \cong \varinjlim_{K \rightarrow \mathfrak{X}} H_n(K).$$

Here, $\mathbf{C}(\mathfrak{X})$ is the category whose object are pairs (E, K) where E is a Euclidean space of dimension e and K is a compact subspace of E together with a map $K \rightarrow \mathfrak{X}$. In the latter colimit, we have used the Spanier-Whitehead duality $H_n(K) \cong H^{e-n}(E, E - K)$, and the limit is taken over the category of all maps $K \rightarrow \mathfrak{X}$ with K a compact topological space that is embeddable in some Euclidean space. By the following proposition, the latter colimit is, indeed, isomorphic to the singular cohomology $H_n(\mathfrak{X})$.

Proposition 7.6 *Let \mathfrak{X} be a topological stack. Then, we have a natural isomorphism*

$$\varinjlim_{K \rightarrow \mathfrak{X}} H_n(K) \cong H_n(\mathfrak{X}),$$

where the limit is taken over the category of all maps $K \rightarrow \mathfrak{X}$ with K a compact topological space that is embeddable in some Euclidean space.

It is possible to generalize the axiomatic framework for (skew-symmetric) bivariant theories [25] to include the present case, where products are only defined for a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ if $Y \xrightarrow{g} Z$ belongs to a subclass of morphisms called adequate. See Appendix A for the axioms. Details will appear elsewhere.

8 Gysin maps

8.1 Normally nonsingular morphisms of stacks and oriented stacks

Definition 8.1 We say that a representable morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of stacks is **normally nonsingular**, if there exist vector bundles \mathfrak{N} and \mathfrak{E} over the stacks \mathfrak{X} and \mathfrak{Y} , respectively, and a commutative diagram

$$\begin{array}{ccc} \mathfrak{N} & \xrightarrow{i} & \mathfrak{E} \\ s \uparrow & & \downarrow p \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

where s is the zero section of \mathfrak{N} , i is an open immersion, and \mathfrak{E} is oriented. When \mathfrak{N} is also oriented, we say that the diagram is **oriented**. (For the definition of orientation on a morphism f see Definition 8.5 below.) The integer $c = \text{rk } \mathfrak{N} - \text{rk } \mathfrak{E}$ depends only on f and is called the **codimension** of f .

A diagram as above is called a *normally nonsingular diagram* for f . The vector bundle \mathfrak{N} is sometimes referred to as the *normal bundle* of \mathfrak{X} in \mathfrak{E} , and $i(\mathfrak{N})$ as a *tubular neighborhood* of \mathfrak{X} in \mathfrak{E} .

Proposition 8.2 *Let G be a compact Lie group, and X and Y smooth G -manifolds, with $\mathfrak{X} = [X/G]$ and $\mathfrak{Y} = [Y/G]$ the corresponding quotient stacks. Assume further that X is of finite orbit type. Then, for every G -equivariant smooth map $X \rightarrow Y$, the induced morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of quotient stacks is normally nonsingular.*

PROOF. First we claim that, there is a vector bundle $\mathfrak{V} \rightarrow BG$ and a smooth embedding $j: \mathfrak{X} \rightarrow \mathfrak{V}$, as in the following commutative diagram:

$$\begin{array}{ccccc} & & & & \mathfrak{V} \\ & & & & \downarrow p \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} & \xrightarrow{\pi_{\mathfrak{Y}}} & BG \end{array}$$

j (dotted arrow from \mathfrak{X} to \mathfrak{V})

This statement is equivalent to the fact that every G -manifold X of finite orbit type embeds G -equivariantly into a linear G -representation V ([10], § II, Theorem 10.1). We can arrange for the G -action on V to be orientation preserving by simply replacing V with $V \oplus V$.

Let $\mathfrak{E} := \mathfrak{Y} \times_{BG} \mathfrak{Y}$ be the pullback of \mathfrak{Y} over \mathfrak{Y} . We obtain the following commutative diagram

$$\begin{array}{ccc} & & \mathfrak{E} \\ & \nearrow^{(f,j)} & \downarrow p \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

Observe that (f, j) is a smooth closed embedding (this can be checked by pulling back the whole picture along a chart, say $* \rightarrow BG$, for BG). Let \mathfrak{N} be the normal bundle of $(f, j)(\mathfrak{X})$ in \mathfrak{E} . By the existence of G -equivariant tubular neighborhoods ([10], § VI, Theorem 2.2), we find a vector bundle \mathfrak{N} over \mathfrak{X} and an open embedding $i: \mathfrak{N} \rightarrow \mathfrak{E}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathfrak{N} & \xhookrightarrow{i} & \mathfrak{E} \\ \uparrow s & \nearrow^{(f,j)} & \downarrow p \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

This is exactly what we were looking for. \square

Example 8.3 The action of a finite group on a manifold has finite orbit type. More interestingly, the action of a compact Lie group on a manifold whose \mathbb{Z} -coefficient homology groups are finitely generated has finite orbit type. This is Mann's Theorem, see [10], § IV.10.

Definition 8.4 Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a strongly proper morphism. A bivariant class $\theta \in H(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$, not necessarily homogenous, is called a **strong orientation** if for every $g: \mathfrak{Z} \rightarrow \mathfrak{X}$, multiplication by θ is an isomorphism $H(\mathfrak{Z} \xrightarrow{gf} \mathfrak{Y}) \xrightarrow{\sim} H(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$.

Definition 8.5 A strongly proper morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of topological stacks is called **strongly oriented**, if it is normally nonsingular and it is endowed with a strong orientation $\theta_f \in H^c(f)$, where $c = \text{codim } f$; see Definition 8.1. A topological stack \mathfrak{X} is called strongly oriented if the diagonal $\Delta: \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is strongly oriented. In this case, we define $\dim \mathfrak{X} := \text{codim } \Delta$.

Lemma 8.6 Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ and $g: \mathfrak{Y} \rightarrow \mathfrak{Z}$ be strongly proper morphisms, and let $\theta \in H(f)$ and $\psi \in H(g)$ be strong orientation classes. Then, $\theta \cdot \psi$ is a strong orientation class for $gf: \mathfrak{X} \rightarrow \mathfrak{Z}$. (Note that the latter map is strongly proper by Lemma 6.4.)

Lemma 8.7 *Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a strongly proper map and $\theta \in H(f)$ a strong orientation class for it. Then multiplication by θ induces an isomorphism $H(\mathfrak{X}) \xrightarrow{\sim} H(f)$. If $\theta' \in H(f)$ is another orientation class for f , then there is a unique unit $u \in H(\mathfrak{X})$ such that $\theta' = u \cdot \theta$.*

The following result states that an oriented normally nonsingular diagram gives rise a canonical strong orientation.

Proposition 8.8 *Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a strongly proper morphism of topological stacks equipped with an oriented normally nonsingular diagram. Then, f has a canonical strong orientation class $\theta_f \in H^c(f)$ where $c = \text{codim } f$.*

The following proposition shows that any morphism between strongly oriented topological stacks has a natural strong orientation. Proposition 8.11 shows that this class is multiplicative.

Proposition 8.9 *Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a strongly proper normally nonsingular morphism of topological stacks, and assume that \mathfrak{X} and \mathfrak{Y} are both strongly oriented (Definition 8.5). Let $d = \dim \mathfrak{X}$ and $c = \dim \mathfrak{Y} - \dim \mathfrak{X}$. Then, there is a unique strong orientation class $\theta_f \in H^c(f)$ which satisfies the equality $\theta_f \cdot \theta_{\mathfrak{Y}} = (-1)^{cd} \theta_{\mathfrak{X}} \cdot (\theta_f \times \theta_f)$, as in the diagram*

$$\begin{array}{ccc}
 \mathfrak{X} & \xrightarrow[\quad f \quad]{\quad \theta_f \quad} & \mathfrak{Y} \\
 \theta_{\mathfrak{X}} \downarrow \Delta & & \Delta \downarrow \theta_{\mathfrak{Y}} \\
 \mathfrak{X} \times \mathfrak{X} & \xrightarrow[\quad \theta_f \times \theta_f \quad]{\quad f \times f \quad} & \mathfrak{Y} \times \mathfrak{Y}
 \end{array}$$

PROOF. By Proposition 8.8, there exists a strong orientation θ for f . It is easy to see that $\theta \times \theta$ is a strong orientation for $f \times f: \mathfrak{X} \times \mathfrak{X} \rightarrow \mathfrak{Y} \times \mathfrak{Y}$. By Lemma 8.6, $\theta \cdot \theta_{\mathfrak{Y}}$ and $\theta_{\mathfrak{X}} \cdot (\theta \times \theta)$ are both strong orientation classes for $\mathfrak{X} \rightarrow \mathfrak{Y} \times \mathfrak{Y}$. Therefore, by Lemma 8.7, there is a unit $u \in H^0(\mathfrak{X})$ such that $\theta \cdot \theta_{\mathfrak{Y}} = u \cdot \theta_{\mathfrak{X}} \cdot (\theta \times \theta)$. It follows that $\theta_f := (-1)^{cd} u \cdot \theta$ has the desired property; see Lemma 8.10 below. \square

Lemma 8.10 *Let \mathfrak{X} be a topological stack and $\theta \in H(\mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times \mathfrak{X})$. Let $u, v \in H^0(\mathfrak{X})$, and let $u \times v \in H^0(\mathfrak{X}) \times H^0(\mathfrak{X})$ be their exterior product. Then, $\theta \cdot (u \times v) = u \cdot v \cdot \theta$, as classes in $H(\Delta)$.*

Proposition 8.11 *Assume $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ and $g: \mathfrak{Y} \rightarrow \mathfrak{Z}$ are strongly proper normally nonsingular morphisms of strongly oriented topological stacks. Let $\theta_f \in H^c(f)$, $c = \text{codim } f$, and $\theta_g \in H^d(g)$, $d = \text{codim } g$, be the strong orientations constructed in Proposition 8.9. Then, gf is a strongly proper normally nonsingular. Furthermore, $\theta_f \cdot \theta_g = \theta_{gf}$.*

If \mathfrak{X} is strongly oriented (Definition 8.5), its iterated diagonals $\Delta^{(n)} : \mathfrak{X} \rightarrow \mathfrak{X}^n$ are strongly proper (Example 6.3).

Corollary 8.12 *Let \mathfrak{X} be a oriented stack. Then the diagonals $\Delta^{(n)} : \mathfrak{X} \rightarrow \mathfrak{X}^n$ are canonically strongly oriented.*

Proposition 8.13 *Notation being as in Proposition 8.2, assume further that X and Y are oriented and that the G -actions are orientation preserving. Then, every normally nonsingular diagram for $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is naturally oriented. In particular, when f is strongly proper, we have a strong orientation class $\theta_f \in H^c(f)$, $c = \dim Y - \dim X$. Furthermore, this class is independent of the choice of the normally nonsingular diagram.*

PROOF. Let us first fix a notation: given a manifold X with an action of G , we denote $[TX/G]$ by $T\mathfrak{X}$. (So, $T\mathfrak{X}$ does depend on X , and not just on \mathfrak{X} . Since in what follows all stacks are quotients of a G -action on a given manifold, this should not cause confusion.)

Consider a normally nonsingular diagram

$$\begin{array}{ccc} \mathfrak{N} & \xhookrightarrow{i} & \mathfrak{E} \\ s \uparrow & & \downarrow p \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

as in the proof of Proposition 8.2. We show that \mathfrak{N} is naturally oriented. By Lemma 4.9, there is a natural orientation on $T\mathfrak{E}$, because it fits in the following short exact sequence

$$0 \rightarrow p^*T\mathfrak{E} \rightarrow T\mathfrak{E} \rightarrow p^*T\mathfrak{X} \rightarrow 0.$$

In particular, we have an orientation on $f^*(T\mathfrak{E})$. We have an isomorphism of vector bundles over \mathfrak{X}

$$T\mathfrak{X} \oplus \mathfrak{N} \cong f^*(T\mathfrak{E}).$$

It now follows from Lemma 4.8 that \mathfrak{N} also carries a natural orientation. This proves the first part of the proposition. In particular, when f is proper, we obtain a class $\theta_f \in H^c(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$ as in Proposition 8.8.

Now, we show that the class θ_f is independent of the normally nonsingular diagram above. Consider another oriented normally nonsingular diagram for f

$$\begin{array}{ccc} \mathfrak{M} & \xhookrightarrow{j} & \mathfrak{F} \\ t \uparrow & & \downarrow q \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

We have to show that the following diagram commutes

$$\begin{array}{ccc}
H^\bullet(\mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+\mathrm{rk}\mathfrak{N}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) \\
= \downarrow & & \downarrow \cong \\
H^\bullet(\mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+\mathrm{rk}\mathfrak{M}}(\mathfrak{F}, \mathfrak{F} - \mathfrak{X})
\end{array}$$

where the horizontal isomorphisms are the one of Proposition 8.8, and the vertical isomorphism is the one of Lemma 6.5. First we prove a special case.

Special case. Assume $\mathfrak{E} = \mathfrak{F}$, and $is = tj$. In this case, we can choose a third vector bundle $\mathcal{L} \rightarrow \mathfrak{X}$ and an open embedding $k: \mathcal{L} \hookrightarrow \mathfrak{E}$ that factors through both \mathfrak{N} and \mathfrak{M} . The two orientations induced on \mathcal{L} from \mathfrak{M} and \mathfrak{N} , as in Lemma 4.11, are the same (because they are equal to the orientation induced from \mathfrak{E} , as described above). The claim now follows from the commutative diagram of Lemma 4.11 (applied once to the open embedding $\mathcal{L} \hookrightarrow \mathfrak{N}$ and once to the open embedding $\mathcal{L} \hookrightarrow \mathfrak{M}$).

General case. To prove the general case, we make use of the following auxiliary oriented nonsingular diagrams:

$$\begin{array}{ccc}
\mathfrak{N} \oplus f^*\mathfrak{F} & \xrightarrow{\hookrightarrow^{(i,\mathrm{pr})}} & \mathfrak{E} \oplus \mathfrak{F} & \quad & \mathfrak{M} \oplus f^*\mathfrak{E} & \xrightarrow{\hookrightarrow^{(j,\mathrm{pr})}} & \mathfrak{F} \oplus \mathfrak{E} \\
(s,jt) \uparrow & & \downarrow & & (t,is) \uparrow & & \downarrow \\
\mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} & & \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}
\end{array}$$

Here, the two maps pr stand for the projection maps $f^*\mathfrak{F} = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{F} \rightarrow \mathfrak{F}$ and $f^*\mathfrak{E} = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{E} \rightarrow \mathfrak{E}$. Let us denote the ranks of \mathfrak{E} , \mathfrak{F} , \mathfrak{N} , and \mathfrak{M} by e , f , n , and m . (Hopefully, presence of two different f in the notation will not cause confusion!) The first normally nonsingular diagram gives rise to the following commutative diagram of isomorphisms:

$$\begin{array}{ccccc}
& & \xrightarrow{\varphi} & & \\
H^\bullet(\mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+n+f}(\mathfrak{N} \oplus f^*\mathfrak{F}, \mathfrak{N} \oplus f^*\mathfrak{F} - \mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+n+f}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{X}) \\
= \downarrow & & \cong \downarrow & & \downarrow \cong \\
H^\bullet(\mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+n}(\mathfrak{N}, \mathfrak{N} - \mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+n}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})
\end{array}$$

The commutativity of the left square is because of Lemma 4.10, and the commutativity of the right square is because Thom isomorphism (vertical) commutes with excision (horizontal).

Similarly, the second normally nonsingular diagram gives rise to the following

commutative diagram of isomorphisms

$$\begin{array}{ccccc}
& & \xrightarrow{\psi} & & \\
H^\bullet(\mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+m+e}(\mathfrak{M} \oplus f^*\mathfrak{E}, \mathfrak{M} \oplus f^*\mathfrak{E} - \mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+m+e}(\mathfrak{F} \oplus \mathfrak{E}, \mathfrak{F} \oplus \mathfrak{E} - \mathfrak{X}) \\
\downarrow = & & \downarrow \cong & & \downarrow \cong \\
H^\bullet(\mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+m}(\mathfrak{M}, \mathfrak{M} - \mathfrak{X}) & \xrightarrow{\cong} & H^{\bullet+m}(\mathfrak{F}, \mathfrak{F} - \mathfrak{X})
\end{array}$$

On the other hand, using the special case that we just proved, the two normally nonsingular diagrams give rise to the following commutative diagram:

$$\begin{array}{ccc}
H^\bullet(\mathfrak{X}) & \xrightarrow[\cong]{\varphi} & H^{\bullet+n+f}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{X}) \\
\downarrow = & & \downarrow = \\
H^\bullet(\mathfrak{X}) & \xrightarrow[\cong]{\psi} & H^{\bullet+m+e}(\mathfrak{F} \oplus \mathfrak{E}, \mathfrak{F} \oplus \mathfrak{E} - \mathfrak{X})
\end{array}$$

The general case now follows from combining this diagram with (the other rectangles) of the previous two diagrams. \square

Corollary 8.14 *Let \mathfrak{X} be a stack that is equivalent to the quotient stack $[X/G]$ of smooth orientation preserving action of a compact Lie group G on a smooth oriented manifold X having finitely generated homology groups. Then, the diagonal $\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is naturally oriented. In particular, the diagonal of the classifying stack BG of a compact Lie group G is naturally oriented.*

Remark 8.15 Let \mathfrak{X} , \mathfrak{Y} and f be as in Proposition 8.13. There are two ways of giving a strong orientation to f . Either we can use Proposition 8.13 directly, or we first apply Corollary 8.14 to endow \mathfrak{X} and \mathfrak{Y} with a strong orientation, and then apply Proposition 8.9. The orientations we get are the same for f . We denote θ_f this strong orientation.

Proposition 8.16 *Let \mathfrak{X} be a paracompact oriented orbifold. Then the diagonal $\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is strongly oriented and in particular, \mathfrak{X} is naturally oriented.*

PROOF. Locally, we can find a tubular neighborhood for the diagonal. The result follows using partition of unity. \square

8.2 Construction of the Gysin maps

We recall the construction of Gysin homomorphisms associated to a bivariant class [25].

Fix an element $\theta \in H^i(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$. Let $u : \mathfrak{Y}' \rightarrow \mathfrak{Y}$ be an arbitrary morphism of topological stacks and $\mathfrak{X}' = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ the base change given by the cartesian square :

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{f'} & \mathfrak{Y}' \\ \downarrow & & \downarrow p \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{X}' \end{array} \quad (8.1)$$

Then θ determines **Gysin homomorphisms**

$$\theta^! : H_j(\mathfrak{Y}') \rightarrow H_{j-i}(\mathfrak{X}')$$

and

$$\theta_! : H^j(\mathfrak{X}') \rightarrow H^{j+i}(\mathfrak{Y}').$$

For the cohomology Gysin map, we need to assume that f is adequate. These homomorphisms are defined by

$$\theta^!(a) = (u^*(\theta)) \cdot a, \quad \text{for } a \in H_j(\mathfrak{Y}') = H^{-j}(\mathfrak{Y}' \rightarrow pt),$$

and

$$\theta_!(b) = f'_*(b \cdot u^*(\theta)), \quad \text{for } b \in H^j(\mathfrak{X}') = H^j(\mathfrak{X}' \xrightarrow{\text{id}} \mathfrak{X}').$$

The homology Gysin map is defined because the map $\mathfrak{X}' \rightarrow *$ is adequate (see Example 7.5).

Gysin homomorphisms associated to a bivariant theory satisfies many properties such as functoriality, naturality or commutation with product, see [25] for details.

Notation: When $\mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ is strongly oriented and $\theta = \theta_f$ is the strong orientation, we denote $f^! = (\theta_f)^!$ the Gysin map.

Let M, N be oriented compact manifolds and G a Lie group acting on M, N by orientation preserving diffeomorphisms. By Proposition 8.13, if $f : M \rightarrow N$ is a G -equivariant map, then f is canonically strongly oriented. Gysin maps for equivariant (co)homology were already considered, for example, by Atiyah and Bott [4].

Proposition 8.17 *The Gysin map $f_!, f^!$ associated to f in (co)homology coincides with the equivariant Gysin map in the sense of Atiyah and Bott [4].*

PROOF. Gysin map in [4] are obtained by the use of fiber integration and Thom classes over the spaces $M_G = M \times_G EG$ and $N_G = N \times_G EG$. These spaces are respectively classifying spaces of the stacks $[M/G]$ and $[N/G]$ and thus are respectively the pullbacks $[M/G] \times_{[* / G]} BG$, $[N/G] \times_{[* / G]} BG$. The pullback of the normally non singular diagram of Proposition 8.2 along the natural maps $[M/G] \times_{[* / G]} BG \rightarrow [M/G]$ and $[N/G] \times_{[* / G]} BG \rightarrow [N/G]$ yields a bundle $\mathfrak{N}_G = \mathfrak{N} \times_{[* / G]} BG$ over M_G and a bundle $\mathfrak{E}_G = \mathfrak{E} \times_{[* / G]} BG$ over N_G . This

defines a nonsingular diagram for the induced map $f_G : M_G \rightarrow N_G$. Unfolding the definition of bivariant classes, it is straightforward to check that the Gysin map associated to the strong orientation class of Proposition 8.13 is induced by the Thom isomorphism associated to the bundle \mathfrak{N}_G over M_G . \square

Proposition 8.18 (Excess formula) *Assume all topological stacks in the cartesian diagram*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{j} & \mathfrak{X}' \\ q \downarrow & & \downarrow p \\ \mathfrak{Y} & \xrightarrow{f} & \mathfrak{Y}' \end{array}$$

are as in Corollary 8.14 or as in Proposition 8.16. Moreover, assume f and j are immersions with normal bundle $\mathfrak{N}_f, \mathfrak{N}_j$ respectively. Then

$$p^*(\theta_f) = e(q^*(\mathfrak{N}_f)/\mathfrak{N}_j) \cdot \theta_j. \quad (8.2)$$

Here $e(\mathfrak{N}_f/\mathfrak{N}_j) \in H^\bullet(\mathfrak{X})$ is the Euler class of the bundle $\mathfrak{N}_f/\mathfrak{N}_j$.

PROOF. Since $\mathfrak{X}' \rightarrow \mathfrak{Y}'$ factors as the composition $\mathfrak{X}' \xrightarrow{(1,p)} \mathfrak{X}' \times \mathfrak{Y}' \rightarrow \mathfrak{Y}'$ of an embedding and a submersion, by functoriality of pullbacks, it is enough to consider the case where p is a submersion and j is an embedding separately. In the latter case, it is enough to prove it with $\mathfrak{X} \rightarrow \mathfrak{Y}$, $\mathfrak{X} \rightarrow \mathfrak{X}'$ assumed to be tubular neighborhood $\mathfrak{E}, \mathfrak{E}'$ of \mathfrak{X} and $\mathfrak{Y}' = \mathfrak{E} \oplus \mathfrak{E}' \oplus \mathfrak{N}$ where $\mathfrak{N} = \mathfrak{N}_f/\mathfrak{N}_j$ and f, p are the natural inclusion maps. Since we have assumed our stacks to be oriented, by Proposition 8.9 we have canonical orientation classes $\theta_1 \in H^\bullet(\mathfrak{E} \rightarrow \mathfrak{E} \oplus \mathfrak{E}' \oplus \mathfrak{N})$, $\theta_2 \in H^\bullet(\mathfrak{E} \rightarrow \mathfrak{E} \oplus \mathfrak{N})$ and $\theta_3 \in H^\bullet(\mathfrak{E} \oplus \mathfrak{N} \rightarrow \mathfrak{E} \oplus \mathfrak{E}' \oplus \mathfrak{N})$. By Lemma 6.5, the natural maps $i_1 : \mathfrak{N} \rightarrow \mathfrak{E} \oplus \mathfrak{N}$, $i_2 : \mathfrak{E}' \oplus \mathfrak{N} \rightarrow \mathfrak{E} \oplus \mathfrak{E}' \oplus \mathfrak{N}$ induces natural isomorphisms $i_1^* : H^\bullet(\mathfrak{E} \rightarrow \mathfrak{E} \oplus \mathfrak{N}) \xrightarrow{\sim} H^\bullet(\mathfrak{X} \rightarrow \mathfrak{N})$ and $i_2^* : H^\bullet(\mathfrak{E} \oplus \mathfrak{N} \rightarrow \mathfrak{E} \oplus \mathfrak{E}' \oplus \mathfrak{N}) \xrightarrow{\sim} H^\bullet(\mathfrak{X} \rightarrow \mathfrak{E}')$. It follows that Equation (8.2) is equivalent to

$$\theta_1 = \tilde{p}^*(\theta_2) \cdot p^*(\theta_3) \quad (8.3)$$

where $\tilde{p} : \mathfrak{X} \rightarrow \mathfrak{E} \oplus \mathfrak{N}$ is the natural inclusion map. Equation (8.3) holds since product and pullback commute. \square

Let G be a subgroup of a finite (discrete) group H . Let Y be a manifold endowed with a (right) H -action (and thus a G -action). Consider the quotient stacks $[Y/G]$ and $[Y/H]$. There are well known “transfer maps” $\text{tr}_H^G : H_*^H(Y) \rightarrow H_*^G(Y)$ (see [8])

Lemma 8.19 *When G is a finite group, the Gysin map associated to the cartesian square*

$$\begin{array}{ccc} [Y/G] & \longrightarrow & [Y/H] \\ \downarrow & & \downarrow \\ [*/G] & \longrightarrow & [*/H] \end{array}$$

where the lower map is induced by the inclusion $G \hookrightarrow H$, is the usual “transfer map” $H_*^H(Y) \rightarrow H_*^G(Y)$ in equivariant homology.

PROOF. The space $Y \times H$ is endowed with a natural right H -action given by $(y, h).k = (y.k, k^{-1}h)$ as well as a right G -action $(y, h).g = (y, hg)$. These two actions commutes hence we can form the quotient stack $[Y \times H/H \times G] \cong [Y \times (H/G)/H]$. Clearly the map $(y, h) \mapsto yh$ is equivariant with respect to the G action on the target and $H \times G$ -action on the source. One easily checks that this map induces an equivalence $[Y \times (H/G)/H] \cong [Y/G]$. We are thus left to study the Gysin map of an equivariant covering with fibers the set H/G . The argument of Proposition 8.17 easily shows that it coincides with the usual transfer maps for coverings by a finite group and thus with the transfer. \square

Assuming we take coefficient in a field of characteristic coprime with $|H|$ for the singular homology, we have

$$H_\bullet([Y/H]) \cong H_*^H(Y) \cong (H_*(Y))_H.$$

In that case the map $\text{tr}_H^G : (H_*(Y))_H \rightarrow (H_*(Y))_G$ is explicitly given by

$$\text{tr}_H^G(x) = \sum_{h \in H/G} h.x. \quad (8.4)$$

9 The loop product

In this section we consider *strongly oriented* stacks (Definition 8.5). We obtain a loop product on the homology of the free loop stack of an oriented stack which generalizes Chas Sullivan product for the homology of a loop manifolds [14]. Recall that a stack \mathfrak{X} is called strongly oriented if the diagonal $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ has a strong orientation class (Definition 8.4). For instance, oriented manifolds and oriented orbifolds are oriented stacks. More generally, the quotient stack of a compact Lie group acting by orientation preserving automorphisms on an oriented manifold is an oriented stack.

9.1 Construction of the loop product

Let \mathfrak{X} be an oriented stack of finite dimension d . The construction of the loop product

$$H_\bullet(\mathbb{L}\mathfrak{X}) \otimes H_\bullet(\mathbb{L}\mathfrak{X}) \rightarrow H_\bullet(\mathbb{L}\mathfrak{X})$$

is divided into 3 steps.

Step 1 There is a well-known external product (called the “cross product”)

$$H_p(\mathbb{L}\mathfrak{X}) \otimes H_q(\mathbb{L}\mathfrak{X}) \xrightarrow{S} H_{p+q}(\mathbb{L}\mathfrak{X}).$$

Step 2 The diagonal $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ and the evaluation map $\text{ev}_0 : \text{L}\mathfrak{X} \rightarrow \mathfrak{X}$ (5.1) yield the cartesian square

$$\begin{array}{ccc} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} & \longrightarrow & \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \\ \downarrow & & \downarrow (\text{ev}_0, \text{ev}_0) \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}. \end{array} \quad (9.1)$$

We will usually denote by $e : \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ the map $(\text{ev}_0, \text{ev}_0)$. Since \mathfrak{X} is Hurewicz, Corollary 5.3 implies that there is a natural equivalence of stacks

$$\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \cong \text{Map}(8, \mathfrak{X}),$$

where the figure “8” stands for the topological stack associated to the topological space $S^1 \vee S^1$. The wedge $S^1 \vee S^1$ is taken with respect to the basepoint 0 of S^1 . Since \mathfrak{X} is oriented, its diagonal $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is oriented normally nonsingular and according to Section 8.2, there is a Gysin map

$$\Delta^! : H_{\bullet}(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) \rightarrow H_{\bullet-d}(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) \cong H_{\bullet-d}(\text{Map}(8, \mathfrak{X})).$$

Step 3 The map $S^1 \rightarrow S^1 \vee S^1$ that pinches $\frac{1}{2}$ to 0, induces a natural map of stacks $m : \text{Map}(8, \mathfrak{X}) \rightarrow \text{L}\mathfrak{X}$, called the *Pontrjagin multiplication*. Hence we have an induced map on homology

$$m_* : H_{\bullet}(\text{Map}(8, \mathfrak{X})) \rightarrow H_{\bullet}(\text{L}\mathfrak{X}).$$

We define the *loop product* to be the following composition

$$H_p(\text{L}\mathfrak{X}) \otimes H_q(\text{L}\mathfrak{X}) \xrightarrow{S} H_{p+q}(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) \xrightarrow{\Delta^!} H_{p+q-d}(\text{Map}(8, \mathfrak{X})) \xrightarrow{m_*} H_{p+q-d}(\text{L}\mathfrak{X}). \quad (9.2)$$

Theorem 9.1 *Let \mathfrak{X} be an oriented stack of dimension d . The loop product induces a structure of associative and graded commutative algebra for the shifted homology $\mathbb{H}_{\bullet}(\text{L}\mathfrak{X}) := H_{\bullet+d}(\text{L}\mathfrak{X})$.*

The loop product is of degree $d = \dim(\mathfrak{X})$ because the Gysin map involved in Step 2 is of degree d . If we denote $\mathbb{H}_{\bullet}(\text{L}\mathfrak{X}) := H_{\bullet+\dim(\mathfrak{X})}(\text{L}\mathfrak{X})$ the shifted homology groups, then the loop product induces a degree 0 multiplication $\mathbb{H}_{\bullet}(\text{L}\mathfrak{X}) \otimes \mathbb{H}_{\bullet}(\text{L}\mathfrak{X}) \rightarrow \mathbb{H}_{\bullet}(\text{L}\mathfrak{X})$.

Indeed one can introduce a “twisted” version of loop product. Let α be a class in $\bigoplus_{r \geq 0} H^r(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X})$. The *twisted loop product* $\star_{\alpha} : H_{\bullet}(\text{L}\mathfrak{X}) \otimes H_{\bullet}(\text{L}\mathfrak{X}) \rightarrow H_{\bullet}(\text{L}\mathfrak{X})$ is defined, for all $x, y \in H_{\bullet}(\text{L}\mathfrak{X})$,

$$x \star_{\alpha} y = m_*(\Delta^!(x \times y) \cap \alpha).$$

Remark 9.2 The twisted product \star_α is not graded since we do not assume α to be homogeneous. However, if $\alpha \in H^r(\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X})$ is homogeneous of degree r , then $\star_e : H_\bullet(\mathbb{L}\mathfrak{X}) \otimes H_\bullet(\mathbb{L}\mathfrak{X}) \rightarrow H_{\bullet-d-r}(\mathbb{L}\mathfrak{X})$ is of degree $r + \dim(\mathfrak{X})$.

Let us introduce some notations. We denote, respectively, $p_{12}, p_{23} : \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}$ the projections on the first two and the last two factors. Also let $(m \times 1) : \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}$ and $(1 \times m) : \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}$ be the Pontrjagin multiplication of the two first factors and two last factors respectively. Furthermore, there are flip maps $\sigma : \mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X}$, $\tilde{\sigma} : \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}$ permuting the two factors of $\mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X}$.

Theorem 9.3 *Let α be a class in $\bigoplus_{r \geq 0} H^r(\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X})$.*

- *If α satisfies the 2-cocycle condition*

$$p_{12}^*(x) \cup (m \times 1)^*(\alpha) = p_{23}^*(\alpha) \cup (1 \times m)^*(\alpha) \quad (9.3)$$

in $H^\bullet(\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X})$, then $\star_e : H_\bullet(\mathbb{L}\mathfrak{X}) \otimes H_\bullet(\mathbb{L}\mathfrak{X}) \rightarrow H_\bullet(\mathbb{L}\mathfrak{X})$ is associative.

- *If α satisfies the flip condition $\tilde{\sigma}^*(\alpha) = \alpha$, then the twisted Loop product $\star_\alpha : \mathbb{H}(\mathbb{L}\mathfrak{X}) \otimes \mathbb{H}(\mathbb{L}\mathfrak{X}) \rightarrow \mathbb{H}(\mathbb{L}\mathfrak{X})$ is graded commutative.*

Example 9.4 If E is an oriented vector bundle over a stack \mathfrak{X} it has a Euler class $e(E)$. Note that the rank may vary on different connected components of \mathfrak{X} . In particular, any vector bundle E over $\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}$ defines a twisted loop product $\star_E := \star_{e(E)} : H(\mathbb{L}\mathfrak{X}) \otimes H(\mathbb{L}\mathfrak{X}) \rightarrow H(\mathbb{L}\mathfrak{X})$. Moreover, $\tilde{\sigma}^*(e(E)) \cong e(E)$ whenever $\tilde{\sigma}^*E \cong E$. Since identities between Euler classes are equivalent to identities in K -theory we have:

Corollary 9.5 *Let \mathfrak{X} be an oriented stack and E a vector bundle over $\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}$.*

- *If E satisfies the cocycle condition*

$$p_{12}^*(E) + (m \times 1)^*(E) = p_{23}^*(E) + (1 \times m)^*(E)$$

in K -theory, then \star_E is associative.

- *If $\tilde{\sigma}^*E \cong E$, then the twisted Loop product $\star_E : H(\mathbb{L}\mathfrak{X}) \otimes H(\mathbb{L}\mathfrak{X}) \rightarrow H(\mathbb{L}\mathfrak{X})$ is graded commutative.*

Remark 9.6 Let M be an oriented manifold and G a finite group acting on M by orientation preserving diffeomorphisms and $\mathfrak{X} = [M/G]$ be the associated global quotient orbifold. Using Proposition 5.9, Proposition 8.17 and the argument of the proof of Proposition 14.9 below to identify evaluation maps and Pontrjagin map, it is straightforward to prove the Loop product $\star : \mathbb{H}_\bullet(\mathbb{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathbb{L}\mathfrak{X}) \rightarrow \mathbb{H}_\bullet(\mathbb{L}\mathfrak{X})$ coincides with the one introduced in [35].

9.2 Proof of Theorems

The Pontrjagin multiplication $m : \text{Map}(S, \mathfrak{X}) \rightarrow \text{L}\mathfrak{X}$ is induced by the pinch map $S^1 \rightarrow S^1 \vee S^1$. The latter is homotopy coassociative, thus there is a chain homotopy equivalence between

$$m(m \times \text{id}) : C_\bullet(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) \rightarrow C_\bullet(\text{L}\mathfrak{X})$$

and $m(\text{id} \times m)$. This proves the next lemma:

Lemma 9.7 *The Pontrjagin multiplication satisfies*

$$m_*((\text{id} \times m)_*) = m_*((m \times \text{id})_*).$$

Proposition 9.8 *The loop product $H_\bullet(\text{L}\mathfrak{X}) \otimes H_\bullet(\text{L}\mathfrak{X}) \xrightarrow{\bullet} H_{\bullet-d}(\text{L}\mathfrak{X})$ is associative.*

PROOF. It is well known that the cross product is associative so that

$$S^{(2)} : H_\bullet(\text{L}\mathfrak{X}) \otimes H_\bullet(\text{L}\mathfrak{X}) \otimes H_\bullet(\text{L}\mathfrak{X}) \rightarrow H_\bullet(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \times \text{L}\mathfrak{X})$$

is equal to both $S(S \times 1)$ and $S(1 \times S)$. We write $m^{(2)}$ for the iterated map

$$m_*(m \times 1)_* = m_*(1 \times m)_*$$

as in Lemma 9.7 and $\Delta^{(2)}$ the iterated diagonal

$$\Delta(\Delta \times 1) = \Delta(1 \times \Delta) : \mathfrak{X} \rightarrow \mathfrak{X}^{\times 3}.$$

Also let $e^{(2)} : \text{L}\mathfrak{X}^{\times 3} \rightarrow \mathfrak{X}^{\times 3}$ denote the product $\text{ev}_0 \times \text{ev}_0 \times \text{ev}_0$ of the evaluation map on each component. It is enough to prove that, for all $x, y, z \in H_\bullet(\text{L}\mathfrak{X})$,

$$(x \bullet y) \bullet z = m^{(2)}(\Delta^{(2)\dagger}(x \times y \times z)) = x \bullet (y \bullet z).$$

The first equality is given by the commutativity of the following diagram:

$$\begin{array}{ccccccc}
 H(\text{L}\mathfrak{X}) \otimes H(\text{L}\mathfrak{X}) \otimes H(\text{L}\mathfrak{X}) & & & & & & \\
 \begin{array}{c} \downarrow S \otimes 1 \\ H(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) \otimes H(\text{L}\mathfrak{X}) \end{array} & \xrightarrow{S^{(2)}} & H(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) & \xrightarrow{\Delta^{(2)\dagger}} & H(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) & & \\
 \begin{array}{c} \downarrow \Delta^! \otimes 1 \\ H(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) \otimes H(\text{L}\mathfrak{X}) \end{array} & \xrightarrow{S} & H(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) & \xrightarrow{\Delta^!} & H(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) & \parallel & \\
 \begin{array}{c} \downarrow m_* \otimes 1 \\ H(\text{L}\mathfrak{X}) \otimes H(\text{L}\mathfrak{X}) \end{array} & \xrightarrow{S} & H(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) & \xrightarrow{\Delta^!} & H(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) & \xrightarrow{m_*} & H(\text{L}\mathfrak{X}). \\
 & & & & \downarrow \widetilde{m \times 1} & \nearrow p^{(2)} & \\
 & & & & H(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) & &
 \end{array}$$

(9.4)

The commutativity of bottom left square follows from the naturality of the cross product, and the bottom right triangle from the associativity of m_* according to Lemma 9.7. The three remaining squares commutes thanks to the following reasons:

(Square 1) There is a diagram of cartesian squares

$$\begin{array}{ccccc}
L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \longrightarrow & L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \times L\mathfrak{X} & \longrightarrow & L\mathfrak{X} \times L\mathfrak{X} \times L\mathfrak{X} . \\
\downarrow & & \downarrow \widetilde{ev_0} \times ev_0 & & \downarrow e^{(2)} \\
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} & \xrightarrow{\Delta \times 1} & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}
\end{array}$$

Thus the commutativity follows from the functoriality of Gysin maps.

(Square 2) Note that the map $\widetilde{ev_0}$ in square (1) is equal to $ev_0 \circ m$. The commutativity follows, by naturality of Gysin maps, from the tower of cartesian diagrams:

$$\begin{array}{ccc}
L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \longrightarrow & L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \times L\mathfrak{X} \\
\downarrow \widetilde{m \times 1} & & \downarrow m \times 1 \\
L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \longrightarrow & L\mathfrak{X} \times L\mathfrak{X} \\
\downarrow \widetilde{ev_0} & & \downarrow e \\
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} .
\end{array}$$

(Square 3) It is commutative by compatibility of Gysin maps with the cross product.

Hence it follows that, for all $x, y, z \in H(L\mathfrak{X})$, one has $(x \bullet y) \bullet z = m^{(2)}(\Delta^{(2)1}(x \times y \times z))$.

One proves in a similar way the identity $m^{(2)}(\Delta^{(2)1}(x \times y \times z)) = x \bullet (y \bullet z)$ from which the equation $(x \bullet y) \bullet z = \bullet(y \bullet z)$ follows. \square

Proposition 9.9 *The loop product $\mathbb{H}_\bullet(L\mathfrak{X}) \otimes \mathbb{H}_\bullet(L\mathfrak{X}) \xrightarrow{*} \mathbb{H}_*(L\mathfrak{X})$ is graded commutative.*

PROOF. Essentially, this result follows from the homotopy commutativity of the Pontrjagin map $m : L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} = \text{Map}(S, \mathfrak{X}) \rightarrow \mathfrak{X}$. More precisely we need to prove that, for $x \in \mathbb{H}_p(L\mathfrak{X}), y \in \mathbb{H}_q(L\mathfrak{X})$, we have

$$m_*(\Delta^1(x \times y)) = (-1)^{pq}(m_*(\Delta^1(y \times x))).$$

The tower of pullback squares

$$\begin{array}{ccc}
L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \longrightarrow & L\mathfrak{X} \times L\mathfrak{X} \\
\downarrow \tilde{\sigma} & & \downarrow \sigma \\
L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \xrightarrow{\text{id}} & L\mathfrak{X} \times L\mathfrak{X} \\
\downarrow & & \downarrow e \\
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}
\end{array}$$

implies that

$$\tilde{\sigma}_* \circ \Delta^!(x \times y) = (-1)^{pq} \Delta^!(y \times x).$$

Here $\tilde{\sigma} : \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \rightarrow \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X}$ and $\sigma : \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X} \rightarrow \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X}$ are flip maps. Hence the result follows from $m_* \circ \tilde{\sigma}_* = m_*$ in homology. The latter is an immediate consequence of the existence of a homotopy between the pinch map $p : S^1 \rightarrow S^1 \vee S^1$ and $\sigma \circ p : S^1 \rightarrow S^1 \vee S^1$ obtained by making the base point $0 \in S^1$ goes to $\frac{1}{2} \in S^1$. Passing to the mapping stack functor $\text{Map}(-, \mathfrak{X})$ yields a homotopy equivalence between $m \circ \tilde{\sigma}$ and m . \square

Remark 9.10 Note that the homotopy between the two pinch maps does not preserve the canonical basepoints. Hence it is crucial to work with the free loop stack (in other words with non pointed mapping stack functors) in this proof.

Proposition 9.11 *If $\alpha \in H^\bullet(\mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X})$ satisfies the cocycle equation (9.3), then the twisted loop product $\star_\alpha : H(\mathcal{L}\mathfrak{X}) \otimes H(\mathcal{L}\mathfrak{X}) \rightarrow H(\mathcal{L}\mathfrak{X})$ is associative.*

PROOF. We write $f_1 : \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X} \rightarrow \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X}$ and $f_3 : \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \rightarrow \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X}$ for the canonical projections. Also we denote $j_3 : \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \hookrightarrow \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X}$ and $j_1 : \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \hookrightarrow \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X}$ the canonical maps. Using the naturality of cup product and cross product, we can write an associativity diagram similar to (9.4) for \star_α , for which the only non obviously commuting square is the one labeled by (1) which becomes :

$$\begin{array}{ccc} H(\mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X}) & \xrightarrow{1 \times \Delta^!} & H(\mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X}) & \xrightarrow{\cap f_3^*(\alpha)} & H(\mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X}) \\ \downarrow \Delta \times 1^! & & & & \downarrow \Delta^! \\ H(\mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X}) & & & & H(\mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X}) \\ \downarrow \cap f_1^*(\alpha) & & & & \downarrow \cap p_{23}^*(\alpha) \\ H(\mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X}) & \xrightarrow{\Delta^!} & H(\mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X}) & \xrightarrow{\cap p_{12}^*(\alpha)} & H(\mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X}). \end{array}$$

Since Gysin maps commute with pullback, for any $y \in H_\bullet(\mathcal{L}\mathfrak{X} \times_{\mathfrak{X}} \mathcal{L}\mathfrak{X} \times \mathcal{L}\mathfrak{X})$,

$$\begin{aligned} \Delta^!(y \cap f_1^*(\alpha)) &= \Delta^!(y) \cap (f_1 \circ j_3)^*(\alpha) \\ &= \Delta^!(y) \cap (m \times 1)^*(\alpha). \end{aligned}$$

Similarly, $\Delta^!(y \cap f_3^*(\alpha)) = \Delta^!(y) \cap (1 \times m)^*(\alpha)$. From square (1) of diagram 9.4 we deduce that the commutativity of the square is equivalent to the identity

$$\begin{aligned} (\Delta^! \circ \Delta \times 1^!) \cap (m \times 1)^*(\alpha) \cap p_{12}^*(\alpha) &= (\Delta^! \circ 1 \times \Delta^!) \cap (1 \times m)^*(\alpha) \cap p_{23}^*(\alpha) \\ \iff \Delta^{(2)1} \cap ((m \times 1)^*(\alpha) \cup p_{12}^*(\alpha)) &= \Delta^{(2)1} \cap ((1 \times m)^*(\alpha) \cup p_{23}^*(\alpha)). \end{aligned}$$

The last equality follows immediately from the 2-cocycle condition (9.3). \square

Proposition 9.12 *If $\tilde{\sigma}^*(\alpha) = \alpha$, then the twisted loop product $\star_\alpha : \mathbb{H}(\mathcal{L}\mathfrak{X}) \otimes \mathbb{H}(\mathcal{L}\mathfrak{X}) \rightarrow \mathbb{H}(\mathcal{L}\mathfrak{X})$ is commutative.*

PROOF. The proof of Propositions 9.9 applies verbatim. \square

10 String product for family of groups over a stack

The Chas-Sullivan product generalizes the intersection product for a manifold M . Indeed, the embedding of M as the space of constant loop in LM makes $H_\bullet(M)$ a subalgebra of the loop homology and the restriction of the loop product to this subalgebra is the intersection product [14].

In the context of stacks, there are more interesting "constant" loops, that is loops which are constant on the coarse space. From a mathematical physics point of view, these spaces of loops are sometimes called *ghost loops*. The canonical ghost loop stack is the inertia stack.

10.1 String product

In this section we construct a string product for the inertia stack. From the categorical point of view the **inertia stack** $\Lambda\mathfrak{X}$ of a stack \mathfrak{X} is the stack of pairs (X, φ) where X is an object of \mathfrak{X} and φ an automorphism of X . If \mathfrak{X} is a Hurewicz topological stack then so is $\Lambda\mathfrak{X}$. However, if \mathfrak{X} is differentiable, $\Lambda\mathfrak{X}$ is not necessarily differentiable. Let Γ be a topological groupoid representing \mathfrak{X} . Let $S\Gamma = \{g \in \Gamma_1 \mid s(g) = t(g)\}$ be the space of closed loops. There is a natural action of Γ on $S\Gamma$ by conjugation. Thus one forms the transformation groupoid $\Lambda\Gamma : S\Gamma \times \Gamma \rightrightarrows S\Gamma$, which is always a topological groupoid, called the *inertia groupoid*. It is a presentation of the inertia stack. and denoted $\Lambda\mathfrak{X}$. Indeed one obtains the following morphism of groupoids

$$\begin{array}{ccc} \Lambda\Gamma & \rightrightarrows & S\Gamma \\ \downarrow & & \downarrow \\ \Gamma_1 & \rightrightarrows & \Gamma_0. \end{array} \tag{10.1}$$

There is a morphism of topological groupoids $\text{ev}_0 : \Lambda\Gamma \rightarrow \Gamma$ given, for $(x, g) \in S\Gamma \times \Gamma$, by $\text{ev}_0((x, g)) = g \in \Gamma_1$. This groupoid morphism $\text{ev}_0 : \Lambda\Gamma \rightarrow \Gamma$ induces the evaluation map

$$\text{ev}_0 : \Lambda\mathfrak{X} \rightarrow \mathfrak{X} \tag{10.2}$$

on the corresponding stacks.

The construction of the string product can be made in 3 steps.

Step 1 The external product induces a map:

$$H_p(\Lambda\mathfrak{X}) \otimes H_q(\Lambda\mathfrak{X}) \xrightarrow{S} H_{p+q}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}).$$

Step 2 We can form the pullback of the evaluation map $\text{ev}_0 : \Lambda\mathfrak{X} \rightarrow \mathfrak{X}$ along

the diagonal $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$, thus obtaining the cartesian square

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \\ \downarrow & & \downarrow^{(ev_0, ev_0)} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}. \end{array} \quad (10.3)$$

Again we denote by $e : \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ the map (ev_0, ev_0) . Since \mathfrak{X} is strongly oriented, so is its diagonal $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$. Hence we have a Gysin map:

$$\Delta^! : H_{\bullet}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \rightarrow H_{\bullet-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}).$$

Step 3 The stack $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ is known as the *double inertia stack*. Its objects are triples (X, φ, ψ) where X is an object of \mathfrak{X} and φ, ψ are automorphisms of X . On the groupoid level the stack $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ is presented by the transformation groupoid $(S\Gamma \times_{\Gamma_0} S\Gamma) \rtimes \Gamma_1 \rightrightarrows S\Gamma \times_{\Gamma_0} S\Gamma$ where Γ_1 acts on $S\Gamma \times_{\Gamma_0} S\Gamma$ by conjugation diagonally. The double inertia stack is endowed with a ‘‘Pontrjagin’’ multiplication map $m : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X}$ given by $m(X, \varphi, \psi) = (X, \varphi\psi)$. It induces a morphism on homology

$$m_* : H_{\bullet}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \rightarrow H_{\bullet}(\Lambda\mathfrak{X}).$$

Composing the three maps in the above steps one obtains a product

$$\star : H_p(\Lambda\mathfrak{X}) \otimes H_q(\Lambda\mathfrak{X}) \rightarrow H_{p+q-d}(\Lambda\mathfrak{X}),$$

called the string product:

$$H_p(\Lambda\mathfrak{X}) \otimes H_q(\Lambda\mathfrak{X}) \xrightarrow{S} H_{p+q}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \xrightarrow{\Delta^!} H_{p+q-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{m_*} H_{p+q-d}(\Lambda\mathfrak{X}). \quad (10.4)$$

As for the loop product, the string product is a degree 0 multiplication on the shifted homology groups: $\mathbb{H}_{\bullet}(\Lambda\mathfrak{X}) = H_{\bullet+d}(\Lambda\mathfrak{X})$.

Theorem 10.1 *Let \mathfrak{X} be an oriented stack of dimension d . The shifted homology $\mathbb{H}_{\bullet}(\Lambda\mathfrak{X})$ of the inertia stack is an associative graded commutative algebra.*

Before proving Theorem 10.1, let us remark that the ‘‘Pontrjagin’’ map $m : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X}$ corresponds to the multiplication is associative. Thus, passing to homology one has the lemma:

Lemma 10.2 *$m_* : H_{\bullet}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \rightarrow H_{\bullet}(\Lambda\mathfrak{X})$ satisfies the associativity condition:*

$$m_*((\text{id} \times m)_*) = m_*((m \times \text{id})_*).$$

Less obvious is that it is also commutative : indeed there is a 2-arrow α

$$\begin{array}{ccc}
 \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \xrightarrow{m} & \Lambda\mathfrak{X} \\
 \text{flip} \downarrow & \searrow^{\alpha} & \nearrow_m \\
 \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & &
 \end{array} \tag{10.5}$$

which associates to (X, φ, ψ) in the double inertia the isomorphism φ^{-1} :

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi\psi} & X \\
 \varphi^{-1} \downarrow & \cong & \downarrow \varphi^{-1} \\
 X & \xrightarrow{\psi\varphi} & X
 \end{array} \tag{10.6}$$

between $(X, \varphi\psi)$ and $(X, \psi\varphi)$ in $\Lambda\mathfrak{X}$.

PROOF OF THEOREM 10.1. Associativity follows *mutatis mutandis* from the proof of Theorem 9.8, substituting $L\mathfrak{X}$ with $\Lambda\mathfrak{X}$ in the argument. Similarly, the proof of Theorem 9.9 leaves us to proving that the induced map $m_* \circ \tilde{\sigma} : H_{\bullet}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \rightarrow H_{\bullet}(\Lambda\mathfrak{X})$ in homology is equal to m_* . Here again $\tilde{\sigma}$ is the flip map. Passing to any groupoid Γ representing \mathfrak{X} and denoting $\Lambda\Gamma \times_{\Gamma} \Lambda\Gamma = (S\Gamma \times_{\Gamma_0} S\Gamma) \rtimes \Gamma_1 \rightrightarrows (S\Gamma \times_{\Gamma_0} S\Gamma)$, it is enough to check that the induced map

$$m_* \circ \tilde{\sigma}_* : H_{\bullet}(\Lambda\Gamma \times_{\Gamma} \Lambda\Gamma) \rightarrow H_{\bullet}(\Lambda\Gamma)$$

in groupoid homology is equal to m_* . At level of groupoids, the 2-arrow α of diagram (10.5) yields the identity

$$\begin{aligned}
 m_*(\sigma(n_1, n_2)) &= \mu(n_2, n_1) \\
 &= (\mu(n_1, n_2))^{n_2^{-1}}
 \end{aligned}$$

for all $x = (n_1, n_2, \gamma) \in (S\Gamma \times_{\Gamma_0} S\Gamma) \rtimes \Gamma_1$. Here $\mu : S\Gamma \times_{\Gamma_0} S\Gamma \rightarrow S\Gamma$ is the restriction of the groupoid multiplication of Γ . Thus $m_*(\sigma(n_1, n_2))$ is canonically conjugate to $m_*(n_1, n_2)$ and in a equivariant way. It follows that after passing to groupoid homology, one has $m_* = m_* \circ \tilde{\sigma}$. An explicit homotopy $h : C_n(\Lambda\Gamma \times_{\Gamma} \Lambda\Gamma) \rightarrow C_{n+1}(\Lambda\Gamma)$ between m_* and $m_* \circ \tilde{\sigma}$ at the chain level is given by $h : \sum_{i=0}^n (-1)^i h_i$ where

$$\begin{aligned}
 h_i((n_1, n_2), g_1, \dots, g_n) &= \left(\mu(n_1, n_2) \right)^{n_2^{-1}}, g_1, \dots \\
 &\quad \dots, g_i, (g_1 \dots g_i)^{-1} n_2 (g_1 \dots g_i), g_{i+1}, \dots, g_n
 \end{aligned}$$

for $i > 0$ and $h_0((n_1, n_2), g_1, \dots, g_n) = \left(\mu(n_1, n_2) \right)^{n_2^{-1}}, n_2, g_1, \dots, g_n$. \square

If α is a cohomology class in $\bigoplus_{r \geq 0} H^r(\Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X})$, one defines the twisted string product

$$\star_{\alpha} : H_{\bullet}(\Lambda \mathfrak{X}) \otimes H_{\bullet}(\Lambda \mathfrak{X}) \rightarrow H_{\bullet}(\Lambda \mathfrak{X})$$

as follows. For any $x, y \in H_{\bullet}(\Lambda \mathfrak{X})$,

$$x \star_{\alpha} y = m_{*}(\Delta^!(x \times y) \cap \alpha).$$

We use similar notations as for Theorem 9.3: denote $p_{12}, p_{23} : \Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X} \rightarrow \Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X}$ the projections on the first two and the last two factors

Proposition 10.3 *Let α be a class in $\bigoplus_{r \geq 0} H^r(\Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X})$.*

- *If α satisfies the cocycle condition:*

$$p_{12}^{*}(\alpha) \cup (m \times 1)^{*}(\alpha) = p_{23}^{*}(\alpha) \cup (1 \times m)^{*}(\alpha) \quad (10.7)$$

in $H^{\bullet}(\Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X})$, then $\star_{\alpha} : H(\Lambda \mathfrak{X}) \otimes H(\Lambda \mathfrak{X}) \rightarrow H(\Lambda \mathfrak{X})$ is associative.

- *If α satisfies the flip condition $\tilde{\sigma}^{*}(\alpha) = \alpha$, then the twisted string product $\star_{\alpha} : \mathbb{H}(\Lambda \mathfrak{X}) \otimes \mathbb{H}(\Lambda \mathfrak{X}) \rightarrow \mathbb{H}(\Lambda \mathfrak{X})$ is graded commutative.*

PROOF. The argument of Proposition 9.11 and Proposition 9.12 applies. \square

Corollary 9.5 has an obvious counterpart for inertia stack.

Corollary 10.4 *Let \mathfrak{X} be an oriented stack and E a vector bundle over $\Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X}$.*

- *If E satisfies the cocycle condition*

$$p_{12}^{*}(E) + (m \times 1)^{*}(E) = p_{23}^{*}(E) + (1 \times m)^{*}(E)$$

in K -theory, then \star_E is associative.

- *If $\tilde{\sigma}^{*}E \cong E$, then the twisted Loop product $\star_E : H(\Lambda \mathfrak{X}) \otimes H(\Lambda \mathfrak{X}) \rightarrow H(\Lambda \mathfrak{X})$ is graded commutative.*

10.2 Family of commutative groups and crossed modules

The string product can be defined for more general "ghost loops" stacks than the mere inertia stack. In fact, we can replace the commutative family $\Lambda \mathfrak{X} \rightarrow \mathfrak{X}$ by an arbitrary commutative family of groups.

A **family of groups** over a (topological) stack \mathfrak{X} is a (topological) stack \mathfrak{G} together with a morphism of (topological) stacks $\text{ev} : \mathfrak{G} \rightarrow \mathfrak{X}$ and an associative multiplication $m : \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \rightarrow \mathfrak{G}$. A family of groups $\mathfrak{G} \rightarrow \mathfrak{X}$ (over \mathfrak{X}) is said

to be a **commutative family of groups** (over \mathfrak{X}) if there exists an invertible 2-arrow α making the following diagram

$$\begin{array}{ccc}
 \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} & \xrightarrow{m} & \mathfrak{G} \\
 \text{flip} \downarrow & \searrow \alpha & \nearrow m \\
 \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} & &
 \end{array} \tag{10.8}$$

commutative. Clearly, the inertia stack is a commutative family of groups (see Equation 10.6).

In the groupoid language, a commutative family of groups can be represented as follows. A *crossed module* of (topological) groupoids is a morphism of groupoids

$$\begin{array}{ccc}
 N_1 & \xrightarrow{i} & \Gamma_1 \\
 \Downarrow & & \Downarrow \\
 N_0 & \xrightarrow{=} & \Gamma_0
 \end{array}$$

which is the identity on the base spaces (in particular $N_0 = \Gamma_0$) and where $N_1 \rightrightarrows N_0$ is a family of groups (i.e. source and target are equal), together with a right action $(\gamma, n) \rightarrow n^\gamma$ of Γ on N by automorphisms satisfying:

1. for all $(n, \gamma) \in N \times_{\Gamma_1}$, $i(n^\gamma) = \gamma^{-1}i(n)\gamma$;
2. for all $(x, y) \in N \times_{\Gamma_0} N$, $x^{i(y)} = y^{-1}xy$.

Note that the equalities in (1) and (2) make sense because N is a family of groups. We use the short notation $[N \xrightarrow{i} \Gamma]$ for a crossed module.

Remark 10.5 In the literature, groupoids for which source equals target are sometimes called bundle of groups. Since we do not assume the source to be locally trivial, we prefer the terminology family of groups.

Since a crossed module $[N \xrightarrow{i} \Gamma]$ comes with an action of Γ on N , one can form the transformation groupoid $\Lambda[N \xrightarrow{i} \Gamma] : N_1 \times_{\Gamma_1} N_1 \rightrightarrows N_1$, which is a topological groupoid. Furthermore, the projection $N_1 \times_{\Gamma_0} \Gamma_1 \rightarrow \Gamma_1$ on the second factor induces a (topological) groupoid morphism $\text{ev} : \Lambda[N \xrightarrow{i} \Gamma] \rightarrow \Gamma$. Let \mathfrak{G} and \mathfrak{X} be the quotient stack $[N_1/N_1 \times_{\Gamma_1}]$ and $[\Gamma_0/\Gamma_1]$ respectively. Then $\text{ev} : \mathfrak{G} \rightarrow \mathfrak{X}$ is a commutative family of groups over \mathfrak{X} . Clearly, the inertia stack $\Lambda\mathfrak{X}$ corresponds to the crossed module $[S\Gamma \hookrightarrow \Gamma]$ for any groupoid presentation Γ of \mathfrak{X} . Obviously $\Lambda[S\Gamma \hookrightarrow \Gamma]$ is the inertia groupoid $\Lambda\Gamma$. The inertia stack is universal among commutative family of groups over \mathfrak{X} :

Lemma 10.6 *Let $\text{ev} : \mathfrak{G} \rightarrow \mathfrak{X}$ be a commutative family of groups over \mathfrak{X} . There exists a unique factorization*

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{e} & \Lambda\mathfrak{X} \\ & \searrow \text{ev} & \downarrow \text{ev}_0 \\ & & \mathfrak{X}. \end{array}$$

In fact, for any crossed module $[N \xrightarrow{i} \Gamma]$, there is a unique map $e : \Lambda[N \xrightarrow{i} \Gamma] \rightarrow \Lambda\Gamma$ making the following diagram commutative:

$$\begin{array}{ccc} \Lambda[N \xrightarrow{i} \Gamma] & \xrightarrow{e} & \Lambda\Gamma \\ & \searrow \text{ev} & \downarrow \text{ev}_0 \\ & & \Gamma. \end{array}$$

Example 10.7 Let \mathfrak{X} be an abelian orbifold, that is an orbifold which can be locally represented by quotients $[X/G]$ where G is (finite) abelian. Then the k -twisted sectors of [16] carries a natural crossed module structure $[S_\Gamma^k \xrightarrow{\mu} \Gamma]$ where μ is the $k-1$ -fold multiplication $S_\Gamma^k \rightarrow S_\Gamma$ followed by the inclusion ι . Of course, for $k=1$, it is well-known that the induced stack is the inertia stack and that the abelian hypothesis can be dropped. The associated commutative family of groups is $\Lambda_k\mathfrak{X} \rightarrow \mathfrak{X}$ where $\Lambda_k\mathfrak{X} = \Lambda\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ is the k^{th} -inertia stack.

Let $\mathfrak{G} \rightarrow \mathfrak{X}$ be a commutative family of groups over a stack \mathfrak{X} . If \mathfrak{X} is strongly oriented, Section 8.2 yields a canonical Gysin map

$$\Delta^! : H_\bullet(\mathfrak{G} \times \mathfrak{G}) \rightarrow H_{\bullet-d}(\mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G}).$$

Thus one can form the composition

$$\star : H_p(\mathfrak{G}) \otimes H_q(\mathfrak{G}) \xrightarrow{S} H_{p+q}(\mathfrak{G} \times \mathfrak{G}) \xrightarrow{\Delta^!} H_{p+q-d}(\mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G}) \xrightarrow{m_\star} H_{p+q-d}(\mathfrak{G}) \quad (10.9)$$

Since $m : \mathfrak{G} \times_{\mathfrak{X}} \mathfrak{G} \rightarrow \mathfrak{G}$ is associative and commutative as for the inertia stack in Section 10.1, Step 3, the argument of Theorem 10.1 yields easily

Proposition 10.8 *Let \mathfrak{G} be a commutative family of groups over an oriented stack \mathfrak{X} (with $\dim(\mathfrak{X}) = d$). The multiplication \star (see Equation (10.9)) endows the shifted homology groups $\mathbb{H}_\bullet(\mathfrak{G}) \cong H_{\bullet+d}(\mathfrak{G})$ with a structure of associative, graded commutative algebra.*

Remark 10.9 It is easy to define twisted ring structures on $\mathbb{H}_\bullet(\mathfrak{G})$ along the lines of Theorem 10.3. Details are left to the reader.

Remark 10.10 If \mathfrak{X} is a oriented stack and if $\mathfrak{G} \rightarrow \mathfrak{X}$ is a family of groups which is not supposed to be commutative, the product \star (Equation (10.9)) is still defined. Moreover the proof of Theorem 10.1 shows that $(\mathbb{H}_\bullet(\mathfrak{G}), \star)$ is an associative algebra. The k^{th} -inertia stack $\Lambda_k\mathfrak{X} = \Lambda\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ is an example of non (necessarily) commutative family of groups.

11 Frobenius algebra structures

The loop homology (with coefficients in a field) of a manifold carries a rich algebraic structure besides the loop product. It is known [18] that there exists also a coproduct, which makes it a Frobenius algebra (without counit).

It is natural to expect that such a structure also exists on $H_\bullet(\mathbb{L}\mathfrak{X})$ for an oriented stack \mathfrak{X} . In Section 11.2 we show that this is indeed the case. We also prove a similar statement for the homology of inertia stacks.

In this section we assume that our coefficient ring k is a field, since we will use the Künneth formula $H_\bullet(\mathfrak{X} \otimes \mathfrak{Y}) \xrightarrow{\sim} H_\bullet(\mathfrak{X}) \otimes H_\bullet(\mathfrak{Y})$.

11.1 Quick review on Frobenius algebras

Let k be a field and A a k -vector space. Recall that A is said to be a *Frobenius algebra* if there is an associative commutative multiplication $\mu : A^{\otimes 2} \rightarrow A$ and a coassociative cocommutative comultiplication $\delta : A \rightarrow A^{\otimes 2}$ satisfying the following compatibility condition

$$\delta \circ \mu = (\mu \otimes 1) \circ (1 \otimes \delta) = (1 \otimes \mu) \circ (\delta \circ 1) \quad (11.1)$$

in $\text{Hom}(A^{\otimes 2}, A^{\otimes 2})$. Here we do not require the existence of a unit nor a counit. Also we allow A to be graded and the maps μ and δ to be graded as well. When both maps are of the same degree d , we say that A is a Frobenius algebra of *degree d* . The tensor product of two Frobenius algebras A and B is naturally a Frobenius algebra with the multiplication $(\mu \otimes \mu) \circ (\tau_{23})$ and comultiplication $\tau_{23}^{-1} \circ (\delta \otimes \delta)$ where $\tau_{23} : A \otimes B \otimes A \otimes B \rightarrow A^{\otimes 2} \otimes B^2$ is the map permuting the two middle components.

Warning We need a few words of caution concerning our definition of Frobenius algebras. In the literature, one often encounters (commutative) Frobenius algebras which are both unital and counital such that, if $c : A \rightarrow k$ is the counit, then $c \circ \mu : A \otimes A \rightarrow k$ is a nondegenerate pairing.

11.2 Frobenius algebra structure for loop stacks

In this subsection we prove the existence of a Frobenius algebra structure on the homology of the free loop stack of an oriented stack. Let $\text{ev}_0, \text{ev}_{1/2} : \mathbb{L}\mathfrak{X} \rightarrow \mathfrak{X}$ be the evaluation maps defined as in Equation (5.1), where \mathfrak{X} is a Hurewicz topological stack. To simplify the notations, let \check{e} be the evaluation map $(\text{ev}_0, \text{ev}_{1/2}) : \mathbb{L}\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$.

Lemma 11.1 *The stack $\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}$ fits into a cartesian square*

$$\begin{array}{ccc} \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} & \xrightarrow{m} & \mathbb{L}\mathfrak{X} \\ \downarrow & & \downarrow \check{e} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array} \quad (11.2)$$

where $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is the diagonal.

PROOF. Since S^1 is compact and \mathfrak{X} is a Hurewicz topological stack, Lemma 5.2 ensures that the pushout diagram of topological spaces

$$\begin{array}{ccc} \text{pt} \amalg \text{pt} & \xrightarrow{0 \amalg \frac{1}{2}} & S^1 \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & S^1 \vee S^1 \end{array}$$

becomes a pullback diagram after applying the mapping stack functor $\text{Map}(-, \mathfrak{X})$. This is precisely diagram (11.2). \square

Remark 11.2 The argument of Lemma 11.1 can be applied to iterated diagonals as well. In particular, $\text{L}\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \text{L}\mathfrak{X}$ (with n -terms) is the mapping stack $\text{Map}(S^1 \vee \cdots \vee S^1, \mathfrak{X})$ (with n copies of S^1) and moreover there is a cartesian square

$$\begin{array}{ccc} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \text{L}\mathfrak{X} & \longrightarrow & \text{L}\mathfrak{X} \\ \downarrow & & \downarrow (\text{ev}_0, \text{ev}_{1/n}, \dots, \text{ev}_{n-1/n}) \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \cdots \times \mathfrak{X}. \end{array} \quad (11.3)$$

Now assume that \mathfrak{X} is oriented of dimension d . According to Section 8.2, the cartesian square (11.2) yields a Gysin map

$$\Delta^! : H_{\bullet}(\text{L}\mathfrak{X}) \longrightarrow H_{\bullet-d}(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}).$$

By diagram (9.1), there is a canonical map $\text{Map}(8, \mathfrak{X}) \cong \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \xrightarrow{j} \text{L}\mathfrak{X} \times \text{L}\mathfrak{X}$. Thus we obtain a degree d map

$$\delta : H_{\bullet}(\text{L}\mathfrak{X}) \xrightarrow{\Delta^!} H_{\bullet-d}(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) \xrightarrow{j^*} H_{\bullet-d}(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) \cong \bigoplus_{i+j=\bullet-d} H_i(\text{L}\mathfrak{X}) \otimes H_j(\text{L}\mathfrak{X}) \quad (11.4)$$

Theorem 11.3 *Let \mathfrak{X} be an oriented stack of dimension d . Then $(H_{\bullet}(\text{L}\mathfrak{X}), \star, \delta)$ is a Frobenius algebra, where both operations \star and δ are of degree d .*

PROOF. It remains to prove the coassociativity, cocommutativity of the co-product and the Frobenius compatibility relation. Denote by $\delta_S : H_{\bullet}(\mathfrak{X} \times \mathfrak{Y}) \rightarrow H_{\bullet}(\mathfrak{X}) \otimes H_{\bullet}(\mathfrak{Y})$ the inverse of the cross product induced by the Künneth isomorphism and $\delta_S^{(n)}$ for its iteration.

i) Coassociativity Let $\check{e}^{(2)} : \text{L}\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$ be the iterated evaluation map $(\text{ev}_0, \text{ev}_{1/3}, \text{ev}_{2/3})$. According to Corollary 8.12, the iterated diagonal $\Delta^{(2)} : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$ is naturally normally nonsingular oriented. Thus, Remark 11.2 implies that there is a Gysin map $\Delta^{(2)!}$. Similarly there is a canonical map $j^{(2)} : \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \cong \text{Map}(S^1 \vee S^1 \vee S^1, \mathfrak{X}) \rightarrow \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \times \text{L}\mathfrak{X}$. The argument

of the proof of Theorem 9.8 shows that it is sufficient to prove that the following diagram is commutative (which is, in a certain sense, the dual of diagram (9.4)).

$$\begin{array}{ccccccc}
H(\mathbb{L}\mathfrak{X}) \otimes H(\mathbb{L}\mathfrak{X}) \otimes H(\mathbb{L}\mathfrak{X}) & & & & & & \\
\delta_S \otimes 1 \uparrow & \swarrow \delta_S^{(2)} & & & & & \\
H(\mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X}) \otimes H(\mathbb{L}\mathfrak{X}) & \xleftarrow{\delta_S} & H(\mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X}) & \xleftarrow{j_*^{(2)}} & H(\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}) & & \\
j_* \otimes 1 \uparrow & & \uparrow (j \times 1)_* & & \parallel & & \\
H(\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}) \otimes H(\mathbb{L}\mathfrak{X}) & \xleftarrow{1 \times \delta_S} & H((\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}) \times \mathbb{L}\mathfrak{X}) & \xleftarrow{(1 \times j)_*} & H(\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}) & & \\
\Delta^! \otimes 1 \uparrow & & \Delta^! \uparrow & & \Delta^! \uparrow & \swarrow \Delta^{(2)'} & \\
H(\mathbb{L}\mathfrak{X}) \otimes H(\mathbb{L}\mathfrak{X}) & \xleftarrow{\delta_S} & H(\mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X}) & \xleftarrow{j_*} & H(\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}) & \xleftarrow[\Delta^!]{(4)} & H(\mathbb{L}\mathfrak{X})
\end{array} \tag{11.5}$$

where p and \tilde{p} denote, respectively, the projections $\mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X}$ and $\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X}$ on the first factor. Square (5) is commutative by naturality of the cross coproduct δ_S and the upper left triangle by its coassociativity. We are left to study the three remaining squares (1), (2), (3) and triangle (4).

Square (1) commutes in view of the identity $j^{(2)} = (j \times 1) \circ (1 \times j)$ which follows from the natural isomorphism $(\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X}) \times \mathbb{L}\mathfrak{X} \cong \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} (\mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X})$. Here the map $\mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X} \rightarrow \mathfrak{X}$ is the composition $\text{ev}_0 \circ p$. In the sequel, we use this isomorphism without further notice.

Square (2) Since $\check{e} \circ \tilde{p} = (\check{e} \circ p) \circ j : \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$, the commutativity of square (2) follows immediately, by naturality of Gysin maps, from the tower of cartesian diagrams

$$\begin{array}{ccc}
\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} & \twoheadrightarrow & \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \\
1 \times j \downarrow & & \downarrow j \\
\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X} & \xrightarrow{m \times 1} & \mathbb{L}\mathfrak{X} \times \mathbb{L}\mathfrak{X} \\
\downarrow & & \downarrow \check{e} \circ p \\
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}.
\end{array}$$

Square (3) commutes by the same argument as for square (3) in diagram (9.4).

Triangle (4) The sequence of cartesian diagrams

$$\begin{array}{ccccc}
\mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} & \xrightarrow{m \times 1} & \mathbb{L}\mathfrak{X} \times_{\mathfrak{X}} \mathbb{L}\mathfrak{X} & \xrightarrow{m} & \mathbb{L}\mathfrak{X} & (11.6) \\
\downarrow & & \downarrow \check{e} \circ \tilde{p} & & \downarrow (\text{ev}_0, \text{ev}_{\frac{1}{2}}, \text{ev}_{\frac{1}{4}}) \\
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} & \xrightarrow{\Delta \times 1} & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}.
\end{array}$$

implies, by naturality of Gysin maps, that

$$\Delta^! \circ (\Delta \times 1)^! = \Delta^{(2)'} \tag{11.7}$$

There is an homeomorphism $h : S^1 \xrightarrow{\sim} S^1$ which, together with the flip map σ , induces a commutative diagram

$$\begin{array}{ccccc}
L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \xrightarrow{\quad} & L\mathfrak{X} & \xrightarrow{\quad} & L\mathfrak{X} & (11.8) \\
\downarrow & & \downarrow \varepsilon^{(2)} & & \downarrow (\text{ev}_0, \text{ev}_{\frac{1}{2}}, \text{ev}_{\frac{1}{4}}) & \\
\mathfrak{X} & \xrightarrow{\Delta^{(2)}} & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} & \xrightarrow{1 \times \sigma} & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} &
\end{array}$$

As $h^* = \text{Map}(-, \mathfrak{X})(h)$ is a homeomorphism and $(1 \times \sigma) \circ \Delta^{(2)} = \Delta^{(2)}$, diagram (11.8) identifies $\Delta^{(2)}$ with the Gysin map (denoted $\Delta^{(2)}$ by abuse of notation) associated to Diagram (11.6). Since $(\Delta \times 1)^! = \Delta^!$ the commutativity of Triangle (4) follows from Equation (11.7).

ii) Let's turn to the point of cocommutativity. It is sufficient to prove that

$$\Delta^! = \tilde{\sigma}_* \circ \Delta^!, \quad (11.9)$$

where $\tilde{\sigma} : L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \rightarrow L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X}$ is the flip map. There is a natural homotopy $F : I \times L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \rightarrow L\mathfrak{X}$ between $m \circ \tilde{\sigma}$ and m (see the proof of Theorem 9.9). Equation (11.9) follows easily by naturality of Gysin maps applied to the cartesian squares below (where $t \in I$)

$$\begin{array}{ccc}
L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \xrightarrow{F(t, -)} & L\mathfrak{X} \\
(t, 1) \downarrow & & \downarrow (t, 1) \\
I \times L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \xrightarrow{(1 \circ p, F)} & I \times L\mathfrak{X} \\
\downarrow & & \downarrow \tilde{\varepsilon} \\
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}.
\end{array}$$

The map $(t, 1) : L\mathfrak{X} \rightarrow I \times L\mathfrak{X}$ is the map $L\mathfrak{X} \xrightarrow{\sim} \{t\} \times L\mathfrak{X} \rightarrow I \times L\mathfrak{X}$. The left upper vertical map is similar. The maps $(t, 1)$ are homotopy equivalences inverting the canonical projections $I \times L\mathfrak{X} \rightarrow L\mathfrak{X}$, $I \times L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \rightarrow L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X}$.

iii) It remains to prove the Frobenius relation (11.1). To avoid confusion between different Gysin maps, we now denote $m^! := \Delta^! : H_{\bullet}(L\mathfrak{X}) \rightarrow H_{\bullet-d}(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X})$ and $j^! := \Delta^! : H_{\bullet}(L\mathfrak{X} \times L\mathfrak{X}) \rightarrow H_{\bullet-d}(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X})$ the Gysin maps inducing the product and coproduct. The cartesian squares

$$\begin{array}{ccc}
L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \xrightarrow{1 \times m} & L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & & L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \xrightarrow{\tilde{j}} & L\mathfrak{X} \times L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \\
\downarrow & & \downarrow \tilde{\varepsilon} \circ \tilde{p} & & \downarrow & & \downarrow (\text{ev}_0 \times \text{ev}_0) \circ (1 \times \tilde{p}) \\
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}, & & \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}
\end{array}$$

give rise to Gysin maps (see Section 8.2)

$$\begin{aligned}
(1 \times m)^! : H_{\bullet}(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X}) &\rightarrow H_{\bullet-d}(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X}) & \text{and} \\
\tilde{j}^! : H_{\bullet}(L\mathfrak{X} \times L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X}) &\rightarrow H_{\bullet-d}(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X}).
\end{aligned}$$

There is a canonical map $\tilde{j} : \text{Map}(S^1 \vee S^1 \vee S^1, \mathfrak{X}) \cong \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \rightarrow \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times \text{L}\mathfrak{X}$ sitting in the pullback diagram

$$\begin{array}{ccc} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} & \xrightarrow{\tilde{j}} & (\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) \times \text{L}\mathfrak{X} \\ \downarrow & & \downarrow \widetilde{\text{ev}_0 \times \text{ev}_0} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

Consider the following diagram

$$\begin{array}{ccccc} H_{\bullet}(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) & \xrightarrow{(1 \times m)^!} & H_{\bullet-d}(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) & \xrightarrow{(1 \times j)_*} & H_{\bullet-d}(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) \\ \downarrow j^! & \searrow (a) \quad m_{23}^! & \downarrow \tilde{j}^! & \searrow (b') & \downarrow (j \times 1)^! \\ H_{\bullet-d}(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) & \xrightarrow{1 \times_{\mathfrak{X}} m^!} & H_{\bullet-2d}(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) & \xrightarrow{\tilde{j}_*} & H_{\bullet-2d}(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) \\ \downarrow m_* & \searrow (b) & \downarrow m_* \times_{\mathfrak{X}} 1 & \searrow (c) & \downarrow m_* \times 1 \\ H_{\bullet-d}(\text{L}\mathfrak{X}) & \xrightarrow{m^!} & H_{\bullet-2d}(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}) & \xrightarrow{j_*} & H_{\bullet-2d}(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) \end{array} \quad (11.10)$$

where $m_{23}^! : H_{\bullet}(\text{L}\mathfrak{X} \times \text{L}\mathfrak{X}) \rightarrow H_{\bullet-2d}(\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X})$ is the Gysin map determined by the cartesian square (applying Corollary 8.12)

$$\begin{array}{ccc} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} & \xrightarrow{j \circ (1 \times_{\mathfrak{X}} m)} & \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \\ \downarrow & & \downarrow \text{ev}_0 \times \tilde{\epsilon} \\ \mathfrak{X} & \xrightarrow{\Delta^{(2)}} & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}. \end{array}$$

The triangle (a) in diagram (11.10) is commutative because we have a sequence of cartesian squares

$$\begin{array}{ccccc} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} & \xrightarrow{1 \times_{\mathfrak{X}} m} & \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} & \xrightarrow{j} & \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \\ \downarrow & & \downarrow \tilde{\epsilon} \circ \bar{p} & & \downarrow \text{ev}_0 \times \tilde{\epsilon} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} & \xrightarrow{\Delta \times 1} & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}. \end{array} \quad (11.11)$$

Similarly, triangle (a') is commutative, i.e., $\tilde{j}^! \circ (1 \times m)^! = m_{23}^!$. By naturality of Gysin maps, the towers of cartesian squares

$$\begin{array}{ccc} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \xrightarrow{1 \times_{\mathfrak{X}} m} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} & & \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \longrightarrow \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \\ \downarrow 1 \times_{\mathfrak{X}} m & & \downarrow \tilde{j} \\ \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \xrightarrow{m} \text{L}\mathfrak{X} & & \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \xrightarrow{j \times 1} \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} \\ \downarrow & & \downarrow \widetilde{\text{ev}_0 \circ (1 \times p)} \\ \mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times \mathfrak{X} & & \mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times \mathfrak{X} \end{array}$$

give the commutativity of squares (b) and (b') in diagram (11.10). The commutativity of Square (c) is trivial. Thus diagram (11.10) is commutative. Up to the identification $H_\bullet(\mathbf{L}\mathfrak{X} \times \mathbf{L}\mathfrak{X}) \cong H_\bullet(\mathbf{L}\mathfrak{X}) \otimes H_\bullet(\mathbf{L}\mathfrak{X})$, the composition of the bottom horizontal map and the left vertical one in diagram (11.10) is the composition $\delta(-\star-)$. The composition of the right vertical map with the upper arrow is $(a\star b^{(1)}) \otimes b^{(2)}$. Finally commutation of the Gysin maps with the cross product yields the identity

$$\delta(a\star b) = a\star b^{(1)} \otimes b^{(2)}.$$

The proof of identity $\delta(a\star b) = a^{(1)} \otimes a^{(2)} \star b$ is similar. \square

11.3 Frobenius algebra structure for inertia stacks

In this section we show that the homology of the inertia stack is also a Frobenius algebra, similarly to Theorem 11.3.

Let \mathfrak{X} be a Hurewicz topological stack of dimension d and Γ a topological groupoid representing \mathfrak{X} . Thus its inertia stack $\Lambda\mathfrak{X}$ is the stack associated to the inertia groupoid $\Lambda\Gamma := S\Gamma \rtimes \Gamma \rightrightarrows S\Gamma$, where $S\Gamma$ is the space of closed loops. Any loop $S^1 \rightarrow X$ on a topological space X can be evaluated in 0 but also in $1/2$. It is folklore to think of $\Lambda\mathfrak{X}$ as a ghost loop stack. Hence evaluation map at 0 and $1/2$ should make sense as well. We first construct these evaluation maps for the inertia stack which leads to the construction of the Frobenius structure on $H_\bullet(\Lambda\mathfrak{X})$ when \mathfrak{X} is oriented.

First of all, let us introduce another groupoid $\widetilde{\Lambda}\Gamma$ which is Morita equivalent to $\Lambda\Gamma$. Objects of $\widetilde{\Lambda}\Gamma$ consist of all diagrams

$$x \xleftarrow{g_1} y \xleftarrow{g_2} x \tag{11.12}$$

in Γ . Note that the composition $g_1 g_2$ is a loop over x . Arrows of $\widetilde{\Lambda}\Gamma$ consist of commutative diagrams

$$\begin{array}{ccccc} & & x & \xleftarrow{g_1} & y & \xleftarrow{g_2} & x & & \\ & & \uparrow h_0 & & \uparrow h_{1/2} & & \uparrow h_0 & & \\ & & x' & \xleftarrow{\quad} & y' & \xleftarrow{\quad} & x' & & \end{array}$$

Note that the left and right vertical arrows are the same. The target map is the top row

$$x \xleftarrow{g_1} y \xleftarrow{g_2} x$$

while the source map is the bottom row

$$x' \xleftarrow{h_0^{-1} g_1 h_{1/2}} y' \xleftarrow{h_{1/2}^{-1} g_2 h_0} x' .$$

The unit map is obtained by taking identities as vertical arrows. The composition is obtained by superposing two diagrams and deleting the middle row of the diagram, i.e.

$$\begin{array}{ccc}
 x & \xleftarrow{g_1} & y & \xleftarrow{g_2} & x & * & x' & \xleftarrow{\quad} & y' & \xleftarrow{\quad} & x' \\
 h_0 \uparrow & & h_{1/2} \uparrow & & h_0 \uparrow & & h'_0 \uparrow & & h'_{1/2} \uparrow & & h'_0 \uparrow \\
 x' & \xleftarrow{\quad} & y' & \xleftarrow{\quad} & x' & & x'' & \xleftarrow{\quad} & y'' & \xleftarrow{\quad} & x''
 \end{array}$$

is mapped to

$$\begin{array}{ccc}
 x & \xleftarrow{g_1} & y & \xleftarrow{g_2} & x \\
 h_0 h'_0 \uparrow & & h_{1/2} h'_{1/2} \uparrow & & h_0 h'_0 \uparrow \\
 x'' & \xleftarrow{\quad} & y'' & \xleftarrow{\quad} & x''
 \end{array}$$

In other words, $\widetilde{\Lambda}\Gamma$ is the transformation groupoid $\widetilde{S}\Gamma \rtimes_{\Gamma_0 \times \Gamma_0} (\Gamma_1 \times \Gamma_1)$, where $\widetilde{S}\Gamma = \{(g_1, g_2) \in \Gamma_2 \mid t(g_1) = s(g_2)\}$, the momentum map $\widetilde{S}\Gamma \rightarrow \Gamma_0 \times \Gamma_0$ is (t, t) , and the action is given, for all compatible $(h_0, h_{1/2}) \in \Gamma_1 \times \Gamma_1$, $(g_1, g_2) \in \widetilde{S}\Gamma$, by

$$(g_1, g_2) \cdot (h_0, h_{1/2}) = (h_0^{-1} g_1 h_{1/2}, h_{1/2}^{-1} g_2 h_0).$$

One defines evaluation maps taking by the vertical arrows of $\widetilde{S}\Gamma$, i.e. $\forall (g_1, g_2, h_0, h_{1/2}) \in \widetilde{\Lambda}\Gamma_1$ define

$$\text{ev}_0 : (g_1, g_2, h_0, h_{1/2}) \mapsto h_0, \quad \text{ev}_{1/2} : (g_1, g_2, h_0, h_{1/2}) \mapsto h_{1/2}.$$

It is simple to prove

Lemma 11.4 *Both evaluation maps $\text{ev}_0 : \widetilde{\Lambda}\Gamma \rightarrow \Gamma$ and $\text{ev}_{1/2} : \widetilde{\Lambda}\Gamma \rightarrow \Gamma$ are groupoid morphisms.*

There is a map

$$p : \widetilde{\Lambda}\Gamma \rightarrow \Lambda\Gamma \tag{11.13}$$

obtained by sending a diagram in $\widetilde{\Lambda}\Gamma_1$ to the composition of the horizontal arrows, i.e.,

$$\begin{array}{ccc}
 x & \xleftarrow{g_1} & y & \xleftarrow{g_2} & x \\
 h_0 \uparrow & & h_{1/2} \uparrow & & h_0 \uparrow \\
 x' & \xleftarrow{\quad} & y' & \xleftarrow{\quad} & x'
 \end{array}
 \quad \text{is mapped to} \quad
 \begin{array}{ccc}
 x & \xleftarrow{g_1 g_2} & x \\
 h_0 \uparrow & & h_0 \uparrow \\
 x' & \xleftarrow{\quad} & x'
 \end{array}$$

In other words $p(g_1, g_2, h_0, h_{1/2}) = (g_1 g_2, h_0)$.

Lemma 11.5 *The map $p : \widetilde{\Lambda}\Gamma \rightarrow \Lambda\Gamma$ is a Morita morphism.*

PROOF. The map $p_0 : \widetilde{\Lambda}\Gamma_0 \rightarrow \Lambda\Gamma_0$ is a surjective submersion with a section given by $g \mapsto (g, 1_{s(g)})$ for $g \in S\Gamma$. Let $g, g' \in S\Gamma$. Assume given $(g_1, g_2) \in \Gamma_2$ with $g_1 g_2 = g$ and $(g'_1, g'_2) \in \Gamma_2$ with $g'_1 g'_2 = g'$. Then any arrow in $\widetilde{\Lambda}\Gamma$ from

$x \xleftarrow{g_1} y \xleftarrow{g_2} x$ to $x \xleftarrow{g'_1} y \xleftarrow{g'_2} x$ is uniquely determined by $h_0 \in \Gamma_1$ satisfying $h_0^{-1}g_1g_2h_0 = g'_1g'_2$. Indeed, $h_{1/2}$ is given by $h_{1/2} = g_2h_0g'_2{}^{-1}$. \square

As a consequence the groupoid $\widetilde{\Lambda\Gamma}$ also represents the inertia stack $\Lambda\mathfrak{X}$, and Lemma 11.4 implies that there are two stack maps $\text{ev}_0, \text{ev}_{1/2} : \Lambda\mathfrak{X} \rightarrow \mathfrak{X}$.

We now proceed to construct the string coproduct. As in Section 10.1 above, $\Lambda\Gamma \times_{\Gamma} \Lambda\Gamma$ is the transformation groupoid $(S\Gamma \times_{\Gamma_0} S\Gamma) \rtimes \Gamma \rightrightarrows S\Gamma \times_{\Gamma_0} S\Gamma$, where Γ acts on $S\Gamma \times_{\Gamma_0} S\Gamma$ by conjugations diagonally. Its corresponding stack is $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

Lemma 11.6 *The stack $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ fits into the cartesian square*

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \xrightarrow{m} & \Lambda\mathfrak{X} \\ \downarrow & & \downarrow (\text{ev}_0, \text{ev}_{1/2}) \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array} .$$

As in Section 11.3, we denote $\check{e} := (\text{ev}_0, \text{ev}_{1/2}) : \Lambda\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ the right vertical map in the diagram of lemma 11.6.

PROOF. We use $\widetilde{\Lambda\Gamma}$ as a groupoid representative of $\Lambda\mathfrak{X}$. By the definition of the evaluation maps, the fiber product

$$\begin{array}{ccc} \Gamma \times_{\Gamma \times \Gamma} \widetilde{\Lambda\Gamma} & \longrightarrow & \widetilde{\Lambda\Gamma} \\ \downarrow & & \downarrow (\text{ev}_0, \text{ev}_{1/2}) \\ \Gamma & \xrightarrow{\Delta} & \Gamma \times \Gamma \end{array}$$

can be identified with the subgroupoid of $\widetilde{\Lambda\Gamma}$, which consists of $(g_1, g_2, h_0, h_{1/2})$ such that $h_0 = h_{1/2}$. The latter is simply the transformation groupoid

$$S\Gamma \times_{\Gamma_0} S\Gamma \rtimes \Gamma_1 \rightrightarrows S\Gamma \times_{\Gamma_0} S\Gamma$$

which is precisely $\Lambda\Gamma \times_{\Gamma} \Lambda\Gamma$. Moreover the composition

$$\Lambda\Gamma \times_{\Gamma} \Lambda\Gamma \cong \Gamma \times_{\Gamma \times \Gamma} \widetilde{\Lambda\Gamma} \rightarrow \widetilde{\Lambda\Gamma} \xrightarrow{p} \Lambda\Gamma,$$

where p is defined by equation (11.13), is precisely the ‘‘Pontrjagin’’ map $m : \Lambda\Gamma \times_{\Gamma} \Lambda\Gamma \rightarrow \Lambda\Gamma$ in Section 10.1. \square

Remark 11.7 It is not hard to generalize the above construction to any finite number of evaluation maps and obtain the following cartesian square (see the proof of Theorem 11.3 below)

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X} \times \cdots \times \mathfrak{X}. \end{array}$$

If \mathfrak{X} is oriented of dimension d , the cartesian square of Lemma 11.6 yields a Gysin map (Section 8.2)

$$\Delta^! : H_\bullet(\Lambda\mathfrak{X}) \longrightarrow H_{\bullet-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}).$$

As shown in Section 10.1, there is also a canonical map $j : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$.

Theorem 11.8 *Assume \mathfrak{X} is an oriented stack of dimension d . The composition*

$$H_n(\Lambda\mathfrak{X}) \xrightarrow{\Delta^!} H_{n-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{j} H_{n-d}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \cong \bigoplus_{i+j=n-d} H_i(\Lambda\mathfrak{X}) \otimes H_j(\Lambda\mathfrak{X})$$

yields a coproduct $\delta : H_\bullet(\Lambda\mathfrak{X}) \rightarrow \bigoplus_{i+j=\bullet-d} H_i(\Lambda\mathfrak{X}) \otimes H_j(\Lambda\mathfrak{X})$ which is a coassociative and graded cocommutative coproduct on the shifted homology $\mathbb{H}_\bullet(\Lambda\mathfrak{X}) := H_{\bullet+d}(\Lambda\mathfrak{X})$, called the string coproduct of $\Lambda\mathfrak{X}$.

PROOF. The proof is very similar to that of Theorem 11.3. We only explain the difference.

i) First we introduce a third evaluation map $\text{ev}_{2/3} : \Lambda\mathfrak{X} \rightarrow \mathfrak{X}$ similar to $\text{ev}_{1/2}$. Taking a representative Γ of \mathfrak{X} , the idea is to replace $\Lambda\Gamma$ by another groupoid $\widetilde{\Lambda\Gamma}$, where $\widetilde{\Lambda\Gamma}_1$ consists of commutative diagrams:

$$\begin{array}{ccccc} & & g_1 & & \\ & & \curvearrowright & & \\ x & \longleftarrow & y & \longleftarrow & z & \longleftarrow & x \\ & & g_2 & & g_3 & & \\ & & \curvearrowright & & \curvearrowright & & \\ h_0 \uparrow & & h_{1/2} \uparrow & & h_{2/3} \uparrow & & \uparrow h_0 \\ & & & & & & \\ x' & \longleftarrow & y' & \longleftarrow & z' & \longleftarrow & x' \end{array}$$

The source and target maps are, respectively, given by the bottom and upper lines. The multiplication is by superposition of diagrams. There are evaluation maps $\text{ev}_0, \text{ev}_{1/2}, \text{ev}_{2/3} : \widetilde{\Lambda\Gamma} \rightarrow \Gamma$, respectively, given by $h_0, h_{1/2}, g_{2/3}$. A proof similar to those of Lemmas 11.5 and Lemmas 11.6 gives the following facts :

1. the groupoid $\widetilde{\Lambda\Gamma}$ is Morita equivalent to $\Lambda\Gamma$. Hence it also represents the stack $\Lambda\mathfrak{X}$.
2. The evaluation maps induce a cartesian square

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X} \\ \downarrow & & \downarrow \varepsilon^{(2)} \\ \mathfrak{X} & \xrightarrow{\Delta^{(2)}} & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \end{array}$$

which yields a Gysin map $\Delta^{(2)!} : H_\bullet(\Lambda\mathfrak{X}) \rightarrow H_{\bullet-2d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$.

It follows that one can form a diagram similar to (11.5) for $\Lambda\mathfrak{X}$ and prove that all its squares (1),(2), (3), (5) are commutative *mutatis mutandis*. The proof

of the commutativity of triangle (4) is even easier: it follows immediately from the sequence of cartesian square

$$\begin{array}{ccccc}
\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \xrightarrow{m} & \Lambda\mathfrak{X} \\
\downarrow & & \downarrow \tilde{e} \circ \tilde{p} & & \downarrow \tilde{e}^{(2)} \\
\mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} & \xrightarrow{\Delta \times 1} & \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}.
\end{array}$$

ii) Since $p \circ \tilde{\sigma}$ is conjugate to p , the proof of the cocommutativity of δ is similar to the proof of Proposition 11.3 and of Proposition 10.1. \square

Theorem 11.9 *The homology groups $(H_{\bullet}(\Lambda\mathfrak{X}), \bullet, \delta)$ form a (non unital, non counital) Frobenius algebra of degree d .*

PROOF. According to Theorems 10.1, 11.3 it suffices to prove the compatibility condition between the string product and string coproduct. The argument of the proof of Theorem 11.3.iii) applies. \square

Remark 11.10 If \mathfrak{X} has finitely generated homology groups in each degree, then by universal coefficient theorem, $H^{\bullet}(\Lambda\mathfrak{X})$ inherits a Frobenius coalgebra structure which is unital iff $(H_{\bullet}(\Lambda\mathfrak{X}), \delta)$ is counital.

11.4 The canonical morphism $\Lambda\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X}$

There is a morphism of stacks $\Phi : \Lambda\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X}$ generalizing the canonical inclusion of a space into its loop space (as a constant loop).

Remark 11.11 Objects of $\Lambda\mathfrak{X}$ are pairs (X, φ) where X is an object of \mathfrak{X} and φ an automorphism of X . The morphism Φ may be thought to maps $(X, \varphi) \in \Lambda\mathfrak{X}$ to the isotrivial family over S^1 , which is obtained from the constant family X_I over the interval by identifying the two endpoints via φ .

We show in this Section that Φ induces a morphism of Frobenius algebras in homology.

Let Γ be a groupoid representing the oriented stack \mathfrak{X} (of dimension d) and $\Lambda\Gamma$ its inertia groupoid representing $\Lambda\mathfrak{X}$. Proposition 5.4 gives a groupoid $\mathbb{L}\Gamma$ representing the free loop stack $\mathbb{L}\mathfrak{X}$. We use the notations of Section 5.2. Recall that the topological groupoid $\mathbb{L}\Gamma$ is a limit of topological groupoids $L^P\Gamma$ where P is a finite subset of S^1 . We take $P = \{1, 1\} \subset S^1$ the trivial subset of S^1 . We will construct a morphism of groupoids $\Lambda\Gamma \rightarrow L^P\Gamma$ inducing the map $\Lambda\mathfrak{X} \rightarrow \mathbb{L}\mathfrak{X}$.

Any $(g, h) \in S\Gamma \times \Gamma = \Lambda\Gamma_1$ (i.e. $g \in \Gamma_1$ with $s(g) = t(g)$) determines a commutative diagram $\Phi(g, h)$ in the underlying category of the groupoid Γ :

$$\begin{array}{ccc}
t(h) & \xleftarrow{g} & t(h) \\
h \uparrow & & \uparrow h \\
s(h) & \xleftarrow{h^{-1}gh} & s(h) .
\end{array} \tag{11.14}$$

The square $\Phi(g, h)$ (defined by diagram (11.14)) being commutative, it is an element of $M_1\Gamma$. Since P is a trivial subset of S^1 , a morphism $[S_1^P \rightrightarrows S_0^P] \rightarrow [M_1\Gamma \rightrightarrows M_0\Gamma]$ is given by a path $f : [0, 1] \rightarrow \Gamma_1$ and elements $k, k' \in \Gamma_1$ such that the diagram

$$\begin{array}{ccc} t(f(0)) & \xleftarrow{k} & t(f(1)) \\ f(0) \uparrow & & \uparrow f(1) \\ s(f(0)) & \xleftarrow{k'} & s(f(1)) \end{array}$$

commutes. In particular, the diagram $\Phi(g, h) \in M_1\Gamma$ yields a (constant) groupoid morphism $[S_1^P \rightrightarrows S_0^P] \rightarrow [M_1\Gamma \rightrightarrows M_0\Gamma]$ defined by $t \mapsto f(t) = h$. The map $(g, h) \mapsto \Phi(g, h)$ is easily seen to be a groupoid morphism. We denote by $\Phi : \Lambda\Gamma \rightarrow \text{L}\Gamma$ its composition with the inclusion $\text{L}^P\Gamma \rightarrow \text{L}\Gamma$. It is still a morphism of groupoids. Hence we have the following

Lemma 11.12 *The map $\Phi : \Lambda\Gamma \rightarrow \text{L}\Gamma$ induces a functorial map of stacks $\Lambda\mathfrak{X} \rightarrow \text{L}\mathfrak{X}$.*

In particular there is an induced map $\Phi_* : H_\bullet(\Lambda\mathfrak{X}) \rightarrow H_\bullet(\text{L}\mathfrak{X})$ in homology.

Theorem 11.13 *Let \mathfrak{X} be an oriented stack. The map $\Phi_* : (H_\bullet(\Lambda\mathfrak{X}), \bullet, \delta) \rightarrow (H_\bullet(\text{L}\mathfrak{X}), \star, \delta)$ is a morphism of Frobenius algebras.*

PROOF. Let Γ be a groupoid representing \mathfrak{X} . For any $(g, h) \in S\Gamma \times \Gamma_1 = \Lambda\Gamma_1$, one has

$$\text{ev}_0(\Phi(g, h)) = h = \text{ev}_0(g, h)$$

where ev_0 stands for both evaluation maps $\text{L}\Gamma \rightarrow \Gamma$, $\Lambda\Gamma \rightarrow \Gamma$. Thus the cartesian square of Step (2) in the construction of the string product factors through the one of the loop product and we have a tower of cartesian squares:

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} & (11.15) \\ \tilde{\Phi} \downarrow & & \downarrow \Phi \times \Phi & \\ \text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X} & \longrightarrow & \text{L}\mathfrak{X} \times \text{L}\mathfrak{X} & \\ \downarrow & & \downarrow \text{ev}_0 \times \text{ev}_0 & \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} & \end{array}$$

where $\tilde{\Phi}$ is induced by $\Phi \times \Phi$. The square (11.15) shows that

$$\Delta^! \circ (\Phi \times \Phi)_* = \tilde{\Phi}_* \circ \Delta^! \quad (11.16)$$

Since $\text{L}\Gamma$ is a presentation of $\text{L}\mathfrak{X}$, the cartesian square $\text{L}\Gamma \times_{\Gamma} \text{L}\Gamma$ represents the stack $\text{L}\mathfrak{X} \times_{\mathfrak{X}} \text{L}\mathfrak{X}$. Given any (g_1, g_2, h) in $(S\Gamma \times_{\Gamma_0} S\Gamma) \times \Gamma_1 = \Lambda\Gamma \times_{\Gamma} \Lambda\Gamma$,

one can form a commutative diagram $\tilde{\Phi}(g_1, g_2, h)$:

$$\begin{array}{ccccc}
t(h) & \xleftarrow{g_1} & t(h) & \xleftarrow{g_2} & t(h) \\
h \uparrow & & \uparrow_h & & \uparrow_h \\
s(h) & \xleftarrow{h^{-1}g_1h} & s(h) & \xleftarrow{h^{-1}g_2h} & s(h),
\end{array}$$

which induces canonically an arrow of $\mathrm{L}\Gamma \times_{\Gamma} \mathrm{L}\Gamma$ as in the construction of Φ . The map $(g_1, g_2, h) \mapsto \tilde{\Phi}(g_1, g_2, h)$ represents the stack morphism $\tilde{\Phi}$. Since

$$m(\tilde{\Phi}(g_1, g_2, h)) = \Phi(g_1 g_2, h)$$

the diagram

$$\begin{array}{ccc}
\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \xrightarrow{m} & \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \\
\tilde{\Phi} \downarrow & & \downarrow m \\
\Lambda\mathfrak{X} & \xrightarrow{\Phi} & \mathrm{L}\mathfrak{X}
\end{array} \tag{11.17}$$

is commutative. Hence, diagram (11.17) and Equation (11.16) implies that Φ_* is an algebra morphism. Similarly Φ_* is a coalgebra morphism since the diagram

$$\begin{array}{ccc}
\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \xrightarrow{m} & \Lambda\mathfrak{X} \\
\tilde{\Phi} \downarrow & & \downarrow \Phi \\
\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} & \xrightarrow{m} & \mathrm{L}\mathfrak{X} \\
\downarrow & & \downarrow (\mathrm{ev}_0, \mathrm{ev}_{1/2}) \\
X & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}.
\end{array}$$

is commutative. \square

Remark 11.14 If the stack \mathfrak{X} is actually a manifold X , then its inertia stack is X itself and $\mathrm{L}\mathfrak{X} = LX$ the free loop space of X . It is clear that the map Φ becomes the usual inclusion $X \hookrightarrow LX$ identifying X with constant loops. For manifolds, the map Φ_* is injective but not surjective (except in trivial cases). However, for general stacks, Φ_* is not necessary injective nor surjective. See Section 14.4.

12 The BV-algebra on the homology of free loop stack

The circle S^1 acts on itself by left multiplication. By functoriality of the mapping stack, the S^1 -action on itself confers a S^1 -action to $\mathrm{L}\mathfrak{X}$ for any topological stack

\mathfrak{X} . This action endows $H_\bullet(\mathbf{L}\mathfrak{X})$ with a degree one operator D as follows. Let $[S^1] \in H_1(S^1)$ be the fundamental class. Then a linear map $D : H_\bullet(\mathbf{L}\mathfrak{X}) \rightarrow H_{\bullet+1}(\mathbf{L}\mathfrak{X})$ is defined by the composition

$$H_\bullet(\mathbf{L}\mathfrak{X}) \xrightarrow{\times[S^1]} H_{\bullet+1}(\mathbf{L}\mathfrak{X} \times S^1) \xrightarrow{\rho_*} H_{\bullet+1}(\mathbf{L}\mathfrak{X}),$$

where the last arrow is induced by the action $\rho : S^1 \times \mathbf{L}\mathfrak{X} \rightarrow \mathbf{L}\mathfrak{X}$.

Lemma 12.1 *The operator D satisfies $D^2 = 0$, i.e. is a differential.*

PROOF. Write $m : S^1 \times S^1 \rightarrow S^1$ for the group multiplication on S^1 . The naturality of the cross product implies, for any $x \in \mathbb{H}_\bullet(\mathbf{L}\mathfrak{X})$, that

$$D^2(x) = \rho_*(m_*([S^1] \times [S^1]) \times x) = 0$$

since $m_*([S^1] \times [S^1]) \in H_2(S^1) = 0$. \square

In order to prove that D together with the loop product makes the shifted homology $\mathbb{H}_\bullet(\mathbf{L}\mathfrak{X})$ a **BV**-algebra (for an oriented stack \mathfrak{X}), we follow a very efficient procedure due to Cohen-Jones [19, 21]. It relies on the fact that $\mathbf{L}\mathfrak{X}$ is an algebra on the homology of the cacti operad and results of Getzler [26] and Voronov [47].

Theorem 12.2 *Let \mathfrak{X} be an oriented stack of dimension d and assume that k is a field of characteristic different from 2. Then the shifted homology $\mathbb{H}_\bullet(\mathbf{L}\mathfrak{X}) = H_{\bullet+d}(\mathbf{L}\mathfrak{X})$ admits a **BV**-algebra structure given by the loop product $\star : \mathbb{H}_\bullet(\mathbf{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathbf{L}\mathfrak{X}) \rightarrow \mathbb{H}_\bullet(\mathbf{L}\mathfrak{X})$ and the operator $D : \mathbb{H}_\bullet(\mathbf{L}\mathfrak{X}) \rightarrow \mathbb{H}_{\bullet+1}(\mathbf{L}\mathfrak{X})$.*

PROOF. According to [19], the result follows from the following fact about the cacti operad which is an operad in the category of topological spaces (see [47]).

Claim $\mathbb{H}_\bullet(\mathbf{L}\mathfrak{X})$ is an algebra over the homology of the cacti operad.

A key-point is that a cactus, i.e. an element of the cacti operad, has an underlying topological space c which is obtained by gluing together a finite numbers of oriented circles along with an orientation preserving map $p : S^1 \rightarrow c$ called the pinching map. In particular the space c is compact, thus the mapping stack $\text{Map}(c, \mathfrak{X})$ is a well-defined topological stack and there is a functorial stack morphism $p^* : \text{Map}(c, \mathfrak{X}) \rightarrow \mathbf{L}\mathfrak{X}$. Furthermore, any choice of k distinct points in c yield evaluation maps $\text{Map}(c, \mathfrak{X}) \rightarrow \mathfrak{X}^k$. It is then straightforward to check, using Lemma 5.2, the Gysin map given by the bivariant theory (see Section 8.2, and the technics of the proofs of Theorem 9.1 and Theorem 11.3, that the proof of the claim for manifolds [19] goes through the category of oriented stacks. Details are left to the reader. \square

Remark 12.3 As in [21], one can prove that Theorem 12.2 follows from a cactus algebra structure of the free loop stack $\mathbf{L}\mathfrak{X}$ in the category of *correspondences*

of topological stacks (and not of topological stacks). The proof of [21] applies *verbatim* to the framework of stacks. Note however, that there is a little subtlety here to apply directly the proof of [21]: one needs carefully avoid the use of mapping stacks with a non-compact topological space as the source.

Remark 12.4 One could try to apply this operadic framework to inertia stack as well. However it does not seem there is a non-trivial S^1 -action on $\Lambda\mathfrak{X}$.

13 Orbifold intersection pairing

Let \mathfrak{X} be an almost complex orbifold. Then $\Lambda\mathfrak{X}$ is again an almost complex orbifold. In particular, \mathfrak{X} and $\Lambda\mathfrak{X}$ are oriented orbifolds. Care has to be taken, because even if \mathfrak{X} is connected and has constant dimension, $\Lambda\mathfrak{X}$ usually has many components of varying dimension (the so-called twisted sectors).

Warning In this section, all (co)homology groups are taken with coefficients in \mathbb{C} , the field of complex numbers. In particular this is true for singular homology $H_\bullet(\mathfrak{X})$, de Rham cohomology (denoted $H_{\text{DR}}^\bullet(\mathfrak{X})$) and compactly supported de Rham cohomology (denoted $H_{\text{DR},c}^\bullet(\mathfrak{X})$).

13.1 Poincaré duality and Orbifolds

For oriented orbifolds, there is the **Poincaré duality homomorphism** $\mathcal{P} : H_i(\mathfrak{X}) \rightarrow H^{d-i}(\mathfrak{X})$ [5]. Here \mathfrak{X} is an oriented orbifold which has constant (real) dimension $d = \dim(\mathfrak{X})$. Let us recall briefly the definition of the Poincaré duality homomorphism, see [5] for details. There is the canonical inclusion $H_i(\mathfrak{X}) \hookrightarrow (H^i(\mathfrak{X}))^*$ which is an isomorphism if $H_i(\mathfrak{X})$ is finite dimensional. Since \mathfrak{X} is of dimension d , there is the Poincaré duality isomorphism $(H_{\text{DR}}^i)^* \xrightarrow{\sim} H_{\text{DR},c}^{d-i}$ [5]. Let $\text{inc} : H_{\text{DR},c}^\bullet(\mathfrak{X}) \rightarrow H_{\text{DR}}^\bullet(\mathfrak{X})$ be the canonical map. The Poincaré duality homomorphism \mathcal{P} is the composition

$$H_i(\mathfrak{X}) \longrightarrow (H^i(\mathfrak{X}))^* \xrightarrow{\sim} (H_{\text{DR}}^i(\mathfrak{X}))^* \xrightarrow{\sim} H_{\text{DR},c}^{n-i}(\mathfrak{X}) \xrightarrow{\text{inc}} H_{\text{DR}}^{n-i}(\mathfrak{X}) \xrightarrow{\sim} H^{n-i}(\mathfrak{X}).$$

If the orbifold \mathfrak{X} is proper, then $\mathcal{P} : H_\bullet(\mathfrak{X}) \rightarrow H^{d-\bullet}(\mathfrak{X})$ is an isomorphism.

Recall that the inertia stack $\Lambda\mathfrak{X}$ has usually many components of varying dimension. The inverse map $I : \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X}$ is the isomorphism defined for any object (X, φ) in $\Lambda\mathfrak{X}$, where X is an object of \mathfrak{X} and φ an automorphism of X , by $I(X, \varphi) = (X, \varphi^{-1})$. In the language of groupoids, if \mathfrak{X} is presented by a Lie groupoid Γ , the map I is presented by the map $(\gamma, \alpha) \mapsto (\gamma^{-1}, \beta)$ for any $(\gamma, \alpha) \in S\Gamma \times_{\Gamma_0} \Gamma_1$.

The age is a locally constant function $\text{age} : \Lambda\mathfrak{X} \rightarrow \mathbb{Q}$. If $\mathfrak{X} = [M/G]$ is a global quotient with G a finite group, then

$$\Lambda\mathfrak{X} = \left[\left(\coprod_{g \in G} M^g \right) / G \right]$$

and for $x \in M^g$, the age is equal to $\sum k_j$ if the eigenvalues of g on $T_x M$ are $\exp(2i\pi k_j)$ with $0 \leq k_j < 1$. The age does not depend on which way \mathfrak{X} is considered as a global quotient. So it is well-defined on $\Lambda\mathfrak{X}$ for any arbitrary almost complex orbifold, because any such \mathfrak{X} can be locally written as a global quotient $[M/G]$. Similarly, the dimension is a locally constant function $\dim : \Lambda\mathfrak{X} \rightarrow \mathbb{Z}$. The age and the dimension are related by the formula (for instance see [16, 23])

$$\dim = d - 2 \text{age} - 2I \circ \text{age} \quad (13.1)$$

where $I : \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X}$ is the inverse map. The **orbifold homology** of \mathfrak{X} is

$$H_{\bullet}^{\text{orb}}(\mathfrak{X}) = H_{\bullet - 2 \text{age}}(\Lambda\mathfrak{X}) = \bigoplus_{n \in \mathbb{Q}} H_{\bullet - 2n}([\Lambda\mathfrak{X}]_{\text{age}=n})$$

where $[\Lambda\mathfrak{X}]_{\text{age}=n}$ is the component of $\Lambda\mathfrak{X}$ for which the age is equal to n . Since $\Lambda\mathfrak{X}$ is an oriented orbifold, there is the Poincaré duality homomorphism $\mathcal{P} : H_{\bullet}(\Lambda\mathfrak{X}) \rightarrow H^{\bullet}(\Lambda\mathfrak{X})$. The orbifold cohomology is $H_{\text{orb}}^{\bullet}(\mathfrak{X}) = H^{\bullet - 2 \text{age}}(\Lambda\mathfrak{X})$ (see [16, 23]).

Lemma 13.1 *The composition*

$$H_{\bullet}(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{P}} H^{\bullet}(\Lambda\mathfrak{X}) \xrightarrow{I^*} H^{\bullet}(\Lambda\mathfrak{X})$$

maps $H_i^{\text{orb}}(\mathfrak{X})$ into $H_{\text{orb}}^{d-i}(\mathfrak{X})$. We call it the **orbifold Poincaré duality homomorphism** $\mathcal{P}^{\text{orb}} : H_i^{\text{orb}}(\mathfrak{X}) \rightarrow H_{\text{orb}}^{d-i}(\mathfrak{X})$.

PROOF. It follows from formula (13.1). \square

13.2 Orbifold intersection pairing and string product

Recall that, if \mathfrak{X} is a manifold, then the homology $H_{\bullet}(\mathfrak{X})$ has the intersection pairing and the cohomology $H^{\bullet}(\mathfrak{X})$ has the cup-product. The Poincaré duality homomorphism is an algebra map. However, if \mathfrak{X} is not compact, the intersection ring and cohomology ring may be very different from each other (for instance, if \mathfrak{X} is not compact, $H_{\bullet}(\mathfrak{X})$ has no unit).

Chen-Ruan [16] defined the orbifold cup-product on the cohomology $H_{\text{orb}}^{\bullet}(\mathfrak{X})$, generalizing the cup-product for manifolds. We will define the analogue of Chen-Ruan orbifold product in homology. Our construction generalizes the intersection pairing for manifolds. Note that we do not assume our orbifolds to be compact.

Our definition of the orbifold intersection pairing is as follows: there are the canonical maps $j : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$ and $m : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X}$ (see Section 10.1) and a Gysin homomorphism $j^! : H(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \rightarrow H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$ because j is strongly oriented. The main ingredient in the definition of Chen-Ruan orbifold cup-product is the so called *obstruction bundle* whose construction is explained in details in [16] and [23]. It is a bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ denoted

$\mathfrak{D}_{\mathfrak{X}}$. The inverse map $I : \Lambda\mathfrak{X} \xrightarrow{\sim} \Lambda\mathfrak{X}$ induces the "inverse" obstruction bundle $\mathfrak{D}_{\mathfrak{X}}^{-1} = (I \times_{\mathfrak{X}} I)^*(\mathfrak{D}_{\mathfrak{X}})$ which is also a bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. We denote $\epsilon_{\mathfrak{X}} = e(\mathfrak{D}_{\mathfrak{X}}^{-1})$ the Euler class of $\mathfrak{D}_{\mathfrak{X}}^{-1}$. The **orbifold intersection pairing** is the composition:

$$H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \xrightarrow{\times} H(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \xrightarrow{j^!} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{\cap \epsilon_{\mathfrak{X}}} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{m_*} H(\mathfrak{X}).$$

Theorem 13.2 *Suppose \mathfrak{X} is an almost complex orbifold of (real) dimension d .*

1. *The orbifold intersection pairing defines a bilinear pairing*

$$H_i^{\text{orb}}(\mathfrak{X}) \otimes H_j^{\text{orb}}(\mathfrak{X}) \xrightarrow{\mathfrak{m}} H_{i+j-d}^{\text{orb}}(\mathfrak{X}).$$

2. *The orbifold intersection pairing \mathfrak{m} is associative and graded commutative.*

3. *The orbifold Poincaré duality map $\mathcal{P}^{\text{orb}} : H_{\bullet}^{\text{orb}}(\mathfrak{X}) \longrightarrow H_{\text{orb}}^{d-\bullet}(\mathfrak{X})$ is a homomorphism of \mathbb{C} -algebras, where $H_{\text{orb}}^{d-\bullet}(\mathfrak{X})$ is equipped with the orbifold cup-product [16].*

Recall that graded commutative means that, for any $x \in H_i([\Lambda\mathfrak{X}]_{\text{age}=k}) \subset H_{i+2k}^{\text{orb}}(\mathfrak{X})$ and $y \in H_j([\Lambda\mathfrak{X}]_{\text{age}=\ell}) \subset H_{j+2\ell}^{\text{orb}}(\mathfrak{X})$, one has $x \mathfrak{m} y = (-1)^{(i+2k)(j+2\ell)} y \mathfrak{m} x$.

PROOF.

1. By Riemann-Roch, the obstruction bundle $\mathfrak{D}_{\mathfrak{X}}$ satisfies the following well-known formula (see [23] Lemma 1.12 and [16] Lemma 4.2.2):

$$\text{rank}(\mathfrak{D}_{\mathfrak{X}}) = 2(\text{age} \circ p_1 + \text{age} \circ p_2 - \text{age} \circ m) + \dim_2 - \dim \circ m \quad (13.2)$$

where $p_1, p_2 : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X}$ are the projections on the first and second factor respectively, $\dim_2 : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \mathbb{Z}$ is the dimension function of the orbifold $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ and $\text{rank} : \mathfrak{D}_{\mathfrak{X}} \rightarrow \mathbb{Z}$ is the rank function of the vector bundle $\mathfrak{D}_{\mathfrak{X}}$ (as a real vector bundle). Since $j : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$ has codimension equal to $\dim \circ p_1 + \dim \circ p_2 - \dim$, the result follows from formula (13.2) and formula (13.1).

2. Since $\text{flip}(\mathfrak{D}_{\mathfrak{X}}) \cong \mathfrak{D}_{\mathfrak{X}}$ (for instance see [23]) and $\epsilon_{\mathfrak{X}}$ is of even degrees (thus strictly commutes with any class), the commutativity follows as in the proof of 10.1. It remains to prove the associativity. Consider the cartesian

diagrams

$$\begin{array}{ccc}
 & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} & \\
 j_{(12)3} \nearrow & & \searrow m_{12} \\
 \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \\
 m_{12} \searrow & & \nearrow j \\
 & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} &
 \end{array} \tag{13.3}$$

$$\begin{array}{ccc}
 & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \\
 j_{1(23)} \nearrow & & \searrow m_{23} \\
 \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \\
 m_{23} \searrow & & \nearrow j \\
 & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} &
 \end{array} \tag{13.4}$$

The map $j_{(12)3}, j_{1(23)}$ are the canonical embeddings induced by $j : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$ (applied, respectively, to the last two and first two factors). The maps m_{ii+1} ($i = 1, 2$) are induced by multiplication of the components $i, i + 1$. We also denote $p_{ij} : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X}$ ($i \neq j$) the map (p_i, p_j) induced by the projections on the component i and j and $j_{12} = j \times \text{id} : \Lambda\mathfrak{X}^{\times 3} \rightarrow \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$, $j_{23} = \text{id} \times j : \Lambda\mathfrak{X}^{\times 3} \rightarrow \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. The so-called excess bundle \mathfrak{E}_{12} associated to diagram (13.3) is defined as follows. There is a canonical map from the normal bundle $N_{j_{(12)3}}$ of $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{j_{(12)3}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$ to the restriction $m_{12}^* N_j$ of the normal bundle of $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{j} \Lambda\mathfrak{X}$. By definition $\mathfrak{E}_{12} = \text{Coker}(N_{j_{(12)3}} \hookrightarrow m_{12}^* N_j)$. Similarly, there is the excess bundle $\mathfrak{E}_{23} = \text{Coker}(N_{j_{1(23)}} \hookrightarrow m_{23}^* N_j)$ associated to diagram (13.4). The proof of Theorem 9.8 together with the commutativity of $\mathfrak{e}_{\mathfrak{X}}$ with any class, shows that

$$\begin{aligned}
 (\alpha \frown \beta) \frown \gamma &= m_* (j^! (m_{12*} (j_{12}^! (\alpha \times \beta \times \gamma) \cap p_{12}^* \mathfrak{e}_{\mathfrak{X}})) \cap \mathfrak{e}_{\mathfrak{X}}) \\
 &= m_* (m_{12*} (j_{(12)3}^! ((j_{12}^! (\alpha \times \beta \times \gamma) \cap p_{12}^* \mathfrak{e}_{\mathfrak{X}}) \cap e(\mathfrak{E}_{12})) \cap \mathfrak{e}_{\mathfrak{X}}) \\
 &= m_*^{(2)} (j^{(2)1} (\alpha \times \beta \times \gamma) \cap p_{12}^* \mathfrak{e}_{\mathfrak{X}} \cap e(\mathfrak{E}_{12}) \cap m_{12}^* \mathfrak{e}_{\mathfrak{X}}).
 \end{aligned}$$

The second line follows from the excess bundle formula (see Proposition 8.18) applied to diagram (13.3). Similarly,

$$\alpha \frown (\beta \frown \gamma) = m_*^{(2)} (j^{(2)1} (\alpha \times \beta \times \gamma) \cap p_{23}^* \mathfrak{e}_{\mathfrak{X}} \cap e(\mathfrak{E}_{23}) \cap m_{23}^* \mathfrak{e}_{\mathfrak{X}}).$$

Hence we need to prove that the bundles $\mathfrak{D}_{\mathfrak{X}}$ and \mathfrak{E}_{ij} satisfy the following identity

$$p_{12}^* (\mathfrak{D}_{\mathfrak{X}}^{-1}) + m_{12}^* (\mathfrak{D}_{\mathfrak{X}}^{-1}) + \mathfrak{E}_{12} = p_{23}^* (\mathfrak{D}_{\mathfrak{X}}^{-1}) + m_{23}^* (\mathfrak{D}_{\mathfrak{X}}^{-1}) + \mathfrak{E}_{23} \tag{13.5}$$

in the K -theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

The main property of the obstruction bundle $\mathcal{D}_{\mathfrak{X}}$ is that it satisfies an "affine cocycle condition" see Equation (13.9) below. In fact, there are two cartesian squares (for $i = 1, 2$), analogous to (13.3), (13.4)

$$\begin{array}{ccc}
 & & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \\
 & \nearrow^{p_{ii+1}} & \\
 \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & & \Lambda\mathfrak{X} \\
 & \searrow_{m_{ii+1}} & \\
 & & \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}
 \end{array} \quad (13.6)$$

Since $p_{12} = p_{12} \circ j_{(12)3}$ and $p_1 = p_1 \circ j$, it is easy to check that the "excess" bundles associated to diagram (13.6) for $i = 1, 2$ coincide with \mathfrak{E}_{12} and \mathfrak{E}_{23} respectively. Indeed, there are the following identities

$$\mathfrak{E}_{12} = p_{12}^* m^* T_{\Lambda\mathfrak{X}} + T_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} - p_{12}^* T_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} - m_{12}^* T_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} \quad (13.7)$$

$$\mathfrak{E}_{23} = p_{23}^* m^* T_{\Lambda\mathfrak{X}} + T_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} - p_{23}^* T_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} - m_{23}^* T_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} \quad (13.8)$$

in the K -theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ associated to the diagram (13.6) defined by $\mathfrak{E}_{ii+1} = \text{Coker}(N_{p_{ii+1}} \rightarrow m_{ii+1}^* N_{p_{ii+1}})$. It follows from Lemma 4.3.2 and Proposition 4.3.4 in [16] (also see Lemma 1.20 and Proposition 1.25 of [23] for more details) that $\mathcal{D}_{\mathfrak{X}}$ satisfies the following "associativity" equation

$$p_{12}^*(\mathcal{D}_{\mathfrak{X}}) + m_{12}^*(\mathcal{D}_{\mathfrak{X}}) + \mathfrak{E}_{12} = p_{23}^*(\mathcal{D}_{\mathfrak{X}}) + m_{23}^*(\mathcal{D}_{\mathfrak{X}}) + \mathfrak{E}_{23} \quad (13.9)$$

in the K -theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. Let σ_{13} be the automorphism of $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ given by permutation of the factors 1 and 3, *i.e.*, $\sigma_{13}(X, \varphi, \psi, \theta) = (X, \theta, \psi, \varphi)$. Then $(I \times_{\mathfrak{X}} I) \circ m_{12} \circ \sigma_{13} = m_{23} \circ (I \times_{\mathfrak{X}} I \times_{\mathfrak{X}} I)$ and $(I \times_{\mathfrak{X}} I) \circ p_{12} \circ \sigma_{13} = \sigma_{12} \circ p_{32} \circ (I \times_{\mathfrak{X}} I \times_{\mathfrak{X}} I)$. Since $\sigma_{12}^* \mathcal{D}_{\mathfrak{X}} \cong \mathcal{D}_{\mathfrak{X}}$, the pullback of the left-hand side in Equation (13.9) along the map $I \times_{\mathfrak{X}} I \times_{\mathfrak{X}} I$ is easily seen to be

$$(I \times_{\mathfrak{X}} I \times_{\mathfrak{X}} I)^*(p_{12}^*(\mathcal{D}_{\mathfrak{X}}) + m_{12}^*(\mathcal{D}_{\mathfrak{X}}) + \mathfrak{E}_{12}) = p_{23}^*(\mathcal{D}_{\mathfrak{X}}^{-1}) + m_{23}^*(\mathcal{D}_{\mathfrak{X}}^{-1}) + \mathfrak{E}_{23},$$

i.e., the right-hand side in Equation (13.5). Similarly, the pullback of the right-hand side in (13.9) is the left-hand side (13.5). Hence Equation (13.9) is equivalent to Equation (13.5); the associativity of \mathfrak{m} follows.

3. Since $\dim : \Lambda\mathfrak{X} \rightarrow \mathbb{Z}$ is always even, \mathcal{P} commutes with the cross product. Using general argument on the Poincaré duality homomorphism in [5], Proposition 8.17 and tubular neighborhood, it is straightforward that $\mathcal{P} \circ f^! = f^* \mathcal{P}$ for any strongly oriented map of orbifolds $f : \mathfrak{X} \rightarrow \mathfrak{Y}$. Hence the following diagram is commutative

$$\begin{array}{ccccccccc}
 H^\bullet(\Lambda\mathfrak{X}) \otimes H^\bullet(\Lambda\mathfrak{X}) & \xrightarrow{\times} & H^\bullet(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) & \xrightarrow{j^*} & H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) & \xrightarrow{\cup e(\mathcal{D}_{\mathfrak{X}})} & H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) & \xrightarrow{m_!} & H^\bullet(\mathfrak{X}) \\
 I^* \uparrow & & I^* \uparrow & & I^* \uparrow & & I^* \uparrow & & I^* \uparrow \\
 H^\bullet(\Lambda\mathfrak{X}) \otimes H^\bullet(\Lambda\mathfrak{X}) & \xrightarrow{\times} & H^\bullet(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) & \xrightarrow{j^*} & H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) & \xrightarrow{\cup e_{\mathfrak{X}}} & H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) & \xrightarrow{m_!} & H^\bullet(\mathfrak{X}) \\
 \mathcal{P} \uparrow & & \mathcal{P} \uparrow & & \mathcal{P} \uparrow & & \mathcal{P} \uparrow & & \mathcal{P} \uparrow \\
 H_\bullet(\Lambda\mathfrak{X}) \otimes H_\bullet(\Lambda\mathfrak{X}) & \xrightarrow{\times} & H_\bullet(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) & \xrightarrow{j^!} & H_\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) & \xrightarrow{\cap e_{\mathfrak{X}}} & H_\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) & \xrightarrow{m_*} & H_\bullet(\mathfrak{X}).
 \end{array}$$

Now the result follows from Lemma 13.4 below. \square

Remark 13.3 If \mathfrak{X} is compact, the orbifold Poincaré duality map is a linear isomorphism, thus an isomorphism of algebras according to Theorem 13.2.3.

Lemma 13.4 *The Chen-Ruan orbifold cup-product [16] is the composition*

$$H^\bullet(\Lambda\mathfrak{X}) \otimes H^\bullet(\Lambda\mathfrak{X}) \xrightarrow{\times} H^\bullet(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \xrightarrow{i^*} H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{\cup e(\mathfrak{D}_{\mathfrak{X}})} H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{m^!} H^\bullet(\Lambda\mathfrak{X}).$$

PROOF. The Chen-Ruan pairing in [16] is defined, for compact orbifolds, by the formula

$$\langle \alpha \cup_{\text{orb}} \beta, \gamma \rangle_{\text{orb}} = \int_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} p_1^*(\alpha) \cup p_2^*(\beta) \cup m^*(I^*(\gamma)) \cup f. \quad (13.10)$$

Until the end of this proof, let us write μ for the pairing given by the formula of Proposition 13.4. We compute $\langle \mu(\alpha, \beta), \gamma \rangle_{\text{orb}}$. Denoting $\int_{\Lambda\mathfrak{X}}$ the orbifold integration map defined in [16], we find

$$\begin{aligned} \langle \mu(\alpha, \beta), \gamma \rangle_{\text{orb}} &= \int_{\Lambda\mathfrak{X}} \mu(\alpha, \beta) \cup I^*(\gamma) \\ &= \int_{\Lambda\mathfrak{X}} m^!(p_1^*(\alpha) \cup p_2^*(\beta) \cup e(\mathfrak{D}_{\mathfrak{X}})) \cup I^*(\gamma) \\ &= \int_{\Lambda\mathfrak{X}} m^!(p_1^*(\alpha) \cup p_2^*(\beta) \cup e(\mathfrak{D}_{\mathfrak{X}}) \cup m^*(I^*(\gamma))) \\ &= \int_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} p_1^*(\alpha) \cup p_2^*(\beta) \cup m^*(I^*(\gamma)) \cup e(\mathfrak{D}_{\mathfrak{X}}). \end{aligned}$$

By nondegeneracy of the orbifold pairing, we get $\alpha \cup_{\text{orb}} \beta = \mu(\alpha, \beta)$. \square

Similarly to the twisted string product 10.1, we now introduce orbifold intersection pairing twisted by a cohomology class.

Definition 13.5 1. Let $\alpha \in H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$ be a (not necessarily homogeneous) cohomology class. The *orbifold intersection pairing twisted by α* , denoted \mathfrak{m}^α , is the composition

$$H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \xrightarrow{\times} H(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \xrightarrow{j^!} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{\cap (\epsilon_{\mathfrak{X}} \cup \alpha)} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{m^*} H(\mathfrak{X}).$$

2. Let \mathfrak{E} be a vector bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. We call $\mathfrak{m}^{e(\mathfrak{E})}$ the orbifold intersection pairing twisted by \mathfrak{E} .

With similar notations as for Theorem 9.3, we prove

Proposition 13.6 1. *If α satisfies the cocycle condition:*

$$p_{12}^*(\alpha) \cup (m \times 1)^*(\alpha) = p_{23}^*(\alpha) \cup (1 \times m)^*(\alpha)$$

in $H^\bullet(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$, then $\mathfrak{m}^\alpha : H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \rightarrow H(\Lambda\mathfrak{X})$ is associative.

2. If \mathfrak{E} is a bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ which satisfies the cocycle condition

$$p_{12}^*(\mathfrak{E}) + (m \times 1)^*(\mathfrak{E}) = p_{23}^*(\mathfrak{E}) + (1 \times m)^*(\mathfrak{E}) \quad (13.11)$$

in the K -theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$, then $\mathfrak{m}^{e(\mathfrak{E})}$ is associative.

PROOF. It follows as Theorem 10.3 and Theorem 13.2.2. \square

Let $\mathfrak{N}_{\mathfrak{X}}$ be the normal bundle of the map $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{m} \Lambda\mathfrak{X}$.

Theorem 13.7 For any almost complex orbifold \mathfrak{X} , the string product coincides with the orbifold intersection pairing twisted by $\mathfrak{D}_{\mathfrak{X}} \oplus \mathfrak{N}_{\mathfrak{X}}$, i.e., for any $x \in H_{\bullet}(\Lambda\mathfrak{X})$,

$$x \star y = x \mathfrak{m}^{e(\mathfrak{D}_{\mathfrak{X}} \oplus \mathfrak{N}_{\mathfrak{X}})} y.$$

The proof reduces to the following lemmas.

The full excess bundle $\mathfrak{F}_{\mathfrak{X}}$ is the excess bundle associated to the cartesian diagram

$$\begin{array}{ccc} & \Lambda\mathfrak{X} & \\ p_1 \nearrow & & \searrow \text{ev} \\ \Lambda\mathfrak{X}_{\mathfrak{X}} \Lambda\mathfrak{X} & & \mathfrak{X} \\ p_2 \searrow & & \nearrow \text{ev} \\ & \Lambda\mathfrak{X} & \end{array}$$

$$\text{i.e., } \mathfrak{F}_{\mathfrak{X}} = \text{Coker} \left(N_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{p_1} \Lambda\mathfrak{X}} \hookrightarrow N_{\Lambda\mathfrak{X} \xrightarrow{\text{ev}} \mathfrak{X}} \right).$$

Lemma 13.8 Let \mathfrak{X} be an almost complex orbifold. The string product $\star : H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \rightarrow H(\Lambda\mathfrak{X})$ is equal to the composition

$$H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \xrightarrow{\times} H(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \xrightarrow{j^!} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{\cap e(\mathfrak{F}_{\mathfrak{X}})} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{m^*} H(\mathfrak{X}).$$

PROOF. Apply the excess formula (Proposition 8.18). \square

Lemma 13.9 The obstruction bundle satisfies the identity

$$\mathfrak{D}_{\mathfrak{X}} + \mathfrak{N}_{\mathfrak{X}} + \mathfrak{D}_{\mathfrak{X}}^{-1} = \mathfrak{F}_{\mathfrak{X}}$$

in the K -theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

PROOF. Recall that $\mathfrak{D}_{\mathfrak{X}}$ is solution of the equation (13.9):

$$p_{12}^*(\mathfrak{D}_{\mathfrak{X}}) + m_{12}^*(\mathfrak{D}_{\mathfrak{X}}) + \mathfrak{E}_{12} = p_{23}^*(\mathfrak{D}_{\mathfrak{X}}) + m_{23}^*(\mathfrak{D}_{\mathfrak{X}}) + \mathfrak{E}_{23}$$

in the K -theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$.

For any permutation $\tau \in \Sigma_3$ of the set $\{1, 2, 3\}$, there is a map $\mathcal{J}_\tau : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ defined as the composition

$$\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{(p_1, p_2, I \circ m)} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{\tilde{\tau}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \xrightarrow{p_{12}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X},$$

where $\tilde{\tau}$ is the permutation of factors induced by τ . It is well-known (see [16], [23] Lemma 1.10) that $\mathcal{J}_\tau^*(\mathfrak{D}_{\mathfrak{X}}) \cong \mathfrak{D}_{\mathfrak{X}}$.

Let r be the map $(p_1, p_2, I \circ p_2) : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. Note that

$$p_{12} \circ r = \text{id}, \quad (m_{12} \circ r) = I \circ \mathcal{J}_{(13)}, \quad p_{23} \circ r = (p_2, I \circ p_2).$$

and furthermore $r^* m_{23}^* T_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} \cong p_1^*(T_{\Lambda\mathfrak{X}})$. It follows (using Equation (13.7), Equation (13.8) and $\mathfrak{N}_{\mathfrak{X}} = m^* T_{\Lambda\mathfrak{X}} - T_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}}$) that the pullback of Equation (13.9) along $r : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \rightarrow \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$, yields the identity

$$\begin{aligned} \mathfrak{D}_{\mathfrak{X}} + \mathfrak{D}_{\mathfrak{X}}^{-1} + \mathfrak{N}_{\mathfrak{X}} &= \text{ev}^* T_{\mathfrak{X}} - T_{\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}} - p_1^* T_{\Lambda\mathfrak{X}} - p_2^* T_{\Lambda\mathfrak{X}} \\ &\quad + r^* p_{23}^*(\mathfrak{D}_{\mathfrak{X}}) + r^* m_{23}^*(\mathfrak{D}_{\mathfrak{X}}) \end{aligned}$$

in the K -theory group of vector bundles over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$. Since the right-hand side of the first line is isomorphic to $\mathfrak{F}_{\mathfrak{X}}$, it suffices to prove that $r^* p_{23}^*(\mathfrak{D}_{\mathfrak{X}})$ and $r^* m_{23}^*(\mathfrak{D}_{\mathfrak{X}})$ have rank 0. It is an easy application of the Riemann-Roch formula (13.2). \square

PROOF OF THEOREM 13.7. By Lemma 13.8, it suffices to prove that $e(\mathfrak{F}_{\mathfrak{X}}) = \epsilon_{\mathfrak{X}} \cup e(\mathfrak{D}_{\mathfrak{X}} \oplus \mathfrak{N}_{\mathfrak{X}})$ which is trivial by Lemma 13.9. \square

Remark 13.10 According to Theorem 13.7, Theorem 13.2.3 and Remark 13.3, if \mathfrak{X} is compact, the orbifold Poincaré duality homomorphism \mathcal{P}^{orb} induces an isomorphism of algebras between the string algebra $(\mathbb{H}_{\bullet}(\Lambda\mathfrak{X}), \star)$ and the orbifold cohomology equipped with Chen-Ruan orbifold cup-product twisted by the class $e(\mathfrak{D}_{\mathfrak{X}}^{-1} \oplus \mathfrak{N}_{\mathfrak{X}})$. A nice interpretation of this isomorphism has recently been found by González, Lupercio, Segovia and Uribe [27]. They proved that the string product of compact complex orbifolds is isomorphic to the Chen Ruan product of the cotangent bundle $T^*\mathfrak{X}$ of \mathfrak{X} .

14 Examples

14.1 The case of manifolds

Smooth manifolds form a special class of differentiable stacks with normally non-singular diagonal (Definition 8.1). Denote by the same letter M a manifold and its associated (topological) stack. The diagonal $\Delta : M \rightarrow M \times M$ is strongly oriented iff the manifold M is oriented.

Proposition 14.1 *Let M be an oriented manifold. The **BV**-algebra and Frobenius algebra structures of $H_{\bullet}(LM)$ given by Theorem 12.2 and Theorem 11.3 coincide with Chas-Sullivan [14], Cohen-Jones [19] and Cohen-Godin [18] ones.*

PROOF. By Proposition 5.7, the free loop stack of M is isomorphic to the free loop space LM . It follows from Proposition 8.17 and Proposition 8.13 (in the case $G = \{1\}$), that the Gysin maps of Sections 9, 11, 12 coincide with the Gysin maps in [19] Section 1 (also see [24] Section 3.1). \square

Remark 14.2 When M is an oriented manifold, the string product on $H_\bullet(\Lambda M) \cong H_\bullet(M)$ is simply the usual intersection pairing.

14.2 String (co)product for global quotient by a finite group

A special important class of oriented orbifolds is the global quotient $[M/G]$, where G is a finite group, M is an oriented manifold together with an action of G by orientation preserving diffeomorphisms. In this case, the homology of the inertia stack $H([M/G])$ is well known. Assume that our coefficient ring k is a field of characteristic coprime with $|G|$ (or 0). The inertia stack of $[M/G]$ is represented by the transformation groupoid

$$\coprod_{g \in G} M^g \times G \rightrightarrows \coprod_{g \in G} M^g \quad (14.1)$$

where the action of $h \in G$ moves $y \in M^g$ to $y \cdot h \in M^{h^{-1}gh}$. Furthermore, $\Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X} \cong [\coprod_{g,h \in G} M^{g,h}/G]$, where $M^{g,h} = M^g \cap M^h$, and the ‘‘Pontrjagin’’ map $m : \Lambda \mathfrak{X} \times_{\mathfrak{X}} \Lambda \mathfrak{X} \rightarrow \Lambda \mathfrak{X}$ is induced by the embeddings $i_{g,h} : M^{g,h} \hookrightarrow M^{gh}$. Since $|G|$ is coprime with $\text{char}(k)$, the homology groups of the inertia stack $\Lambda[M/G]$ are

$$H_\bullet(\Lambda[M/G]) \cong H_\bullet \left(\coprod_{g \in G} M^g \right)_G \cong \left(\bigoplus_{g \in G} H_\bullet(M^g) \right)_G.$$

The excess bundle $Ex(M, X, X')$ of the diagram of embeddings

$$\begin{array}{ccc} & X & \\ & \nearrow & \searrow \\ Z = X \cap X' & & M \\ & \searrow & \nearrow \\ & X' & \end{array}$$

is the cokernel of the bundle map $N_{Z \hookrightarrow X} \hookrightarrow (N_{X' \hookrightarrow M})_{/Z}$. Thus $Ex(M, X, X')$ is the virtual bundle $T_M - T_X - T_{X'} + T_Z$ (each component being restricted to Z). For $g, h \in G$, we denote $Ex(g, h) := Ex(M, M^g, M^h)$. The bundles $Ex(g, h)$ induce a bundle Ex on $\Lambda[M/G] \times_{[M/G]} \Lambda[M/G]$ whose Euler class is denoted $e(Ex)$. Since the diagonal $G \rightarrow G \times G$ is a group monomorphism, there is a transfer map $\text{tr}_{G \times G}^G : (\bigoplus_{g,h \in G} H_\bullet(M^g) \otimes H_\bullet(M^h))_{G \times G} \rightarrow (\bigoplus_{g,h \in G} H_\bullet(M^g) \otimes H_\bullet(M^h))_G$ explicitly given (see Equation (8.4)) by

$$\text{tr}_{G \times G}^G(x) = \sum_{g \in G} x \cdot (g, 1).$$

The maps $i_g : M^{g,h} \hookrightarrow M^g$, $i_h : M^{g,h} \hookrightarrow M^h$ yield Gysin morphisms $(i_g \times i_h)^! : H_\bullet(M^g \times M^h) \rightarrow H_\bullet(M^{g,h})$.

Proposition 14.3 *The string product $\star : H(\Lambda[M/G]) \otimes H(\Lambda[M/G]) \rightarrow H(\Lambda[M/G])$ is the composition*

$$\begin{aligned} \left(\bigoplus_{g \in G} H(M^g) \right)_G \otimes \left(\bigoplus_{h \in G} H(M^h) \right)_G &\rightarrow \left(\bigoplus_{g,h \in G} H(M^g \times M^h) \right)_{G \times G} \xrightarrow{\text{tr}_{G \times G}^G} \left(\bigoplus_{g,h \in G} H(M^g \times M^h) \right)_G \\ &\xrightarrow{\oplus (i_g \times i_h)^!} \left(\bigoplus_{g,h \in G} H(M^{g,h}) \right)_G \xrightarrow{\cap e(E_x)} \left(\bigoplus_{g,h \in G} H(M^{g,h}) \right)_G \xrightarrow{m_*} \left(\bigoplus_{k \in G} H(M^k) \right)_G \end{aligned}$$

The proof of Proposition 14.3 relies on Lemma 14.4 below, which is of independent interest. Note that there is a oriented stack morphism

$$\varphi : \Lambda[M/G] \times_{[M/G]} \Lambda[M/G] \cong \left[\coprod_{g,h \in G} M^{g,h}/G \right] \rightarrow \Lambda[M/G] \times \Lambda[M/G] \quad (14.2)$$

induced by the groupoid map $(x, g) \mapsto (i_g(x), g, i_h(x), g)$.

Lemma 14.4 *The Gysin map $\varphi^!$ is the composition*

$$\left(\bigoplus_{g,h \in G} H(M^g \times M^h) \right)_{G \times G} \xrightarrow{\text{tr}_{G \times G}^G} \left(\bigoplus_{g,h \in G} H(M^g \times M^h) \right)_G \xrightarrow{\oplus (i_g \times i_h)^!} \left(\bigoplus_{g,h \in G} H(M^{g,h}) \right)_G$$

PROOF. The G -equivariant map $M^{g,h} \rightarrow M^g \times M^h$, given by $x \mapsto (i_g(x), i_h(x))$, induces a oriented stack morphism $\psi : \Lambda[M/G] \times_{[M/G]} \Lambda[M/G] \rightarrow \Lambda[M/G] \times_{[* / G]} \Lambda[M/G]$. By Proposition 8.17, $\psi^! = \oplus (i_g \times i_h)^!$. Then, the result follows from the functoriality of Gysin maps and Lemma 8.19. \square

PROOF OF PROPOSITION 14.3. We use the notations of Section 10.1. The cartesian diagram (10.3) (where $\mathfrak{X} = [M/G]$) and the excess formula 8.18 shows that,

$$\Delta^! = \varphi^!(x) \cap e(E_x).$$

Thus the result follows from Lemma 14.4. \square

Similarly we compute the string coproduct. For any $g \in G$, the unit $1_g \in H_0(M^g)$ induces a map $1_g : H(M^g) \rightarrow H_0(M^g) \otimes H(M^g) \rightarrow H(M^g \times M^g)$.

Proposition 14.5 *The string coproduct is induced (after passing to G -invariant) by the composition*

$$\begin{aligned} \bigoplus_{g \in G} H(M^g) &\xrightarrow{\oplus 1_g} \bigoplus_{g \in G} H(M^g \times M^g) \xrightarrow{\text{tr}_{G \times G}^G} \bigoplus_{g,h \in G} H(M^h \times M^g) \xrightarrow{\oplus i_{g,h}^!} \bigoplus_{g,h \in G} H(M^{g,h}) \\ &\xrightarrow{\cap e(E_x)} \bigoplus_{g,h \in G} H(M^{g,h}) \xrightarrow{(i_g, i_h)^*} \bigoplus_{g,h \in G} H(M^g \times M^h) \cong \bigoplus_{g,h \in G} H(M^g) \otimes H(M^h). \end{aligned}$$

PROOF. Let Γ be the transformation groupoid $M \rtimes G \rightrightarrows M$. Unfolding the definition of the groupoid $\widetilde{\Lambda}\Gamma$ (see Section 11.3), one finds that $\widetilde{\Lambda}\Gamma$ is the transformation groupoid $(G \times \coprod_{h \in G} M^h) \rtimes G^2 \rightrightarrows G \times \coprod_{h \in G} M^h$, where the action of $(h_0, h_{1/2}) \in G^2$ on $(g, m) \in g \times M^h$ is $(h_0^{-1}gh_{1/2}, m \cdot h_0)$. The Morita map $p : \widetilde{\Lambda}\Gamma \rightarrow \Lambda[M/G]$ (Equation (11.13)) has a section κ defined, for $m \in M^h$ and $h_0 \in G$, by $\kappa(m, h_0) = (h, m, h_0, h_0)$. In particular κ induces an isomorphism in homology and commutes with Gysin maps. Thus the Gysin map $\Delta^!$ of Section 11.3 is the composition of κ_* with the Gysin map associated to the sequence of cartesian diagrams

$$\begin{array}{ccccc} \coprod M^{g,h}/G & \longrightarrow & [G \times \coprod M^h/G] & \longrightarrow & [G \times \coprod M^h/G \times G] \\ \downarrow & & \downarrow & & \downarrow \\ [M/G] & \longrightarrow & [M \times M/G] & \longrightarrow & [M/G] \times [M/G] \end{array} \quad (14.3)$$

By Proposition 8.18 and Lemma 8.19, the Gysin maps associated to the left square and the right square are, respectively, $i_{g,h}^!(-) \cap e(Ex(g, h))$ and $\mathrm{Tr}_{G \times G}^G$. Since $\kappa_* = \oplus 1_g$, the result follows. \square

Example 14.6 Consider $[*/G]$ where G is a finite group. By Proposition 5.9, the stack morphism $\Phi : \Lambda[*/G] \rightarrow \mathrm{L}[*/G]$ (see Lemma 11.12) is an isomorphism. Let k be a field of characteristic coprime with $|G|$. Then

$$H_\bullet(\Lambda[*/G]) = \left(\bigoplus_{g \in G} k \right)_G \cong \left(\bigoplus_{g \in G} k \right)^G \cong Z(k[G])$$

where $Z(k[G])$ is the center of the group algebra $k[G]$. By Propositions 14.3, the isomorphism $H_\bullet(\Lambda[*/G]) \cong Z(k[G])$ is an isomorphism of algebras. By Proposition 14.5, the string coproduct is given by $\delta([g]) = \sum_{hk=g} [h] \otimes [k]$. Thus the Frobenius algebra structure coincides with the one given by Dijkgraaf-Witten [22].

14.3 String topology of $[S^{2n+1}/(\mathbb{Z}/2\mathbb{Z})^{n+1}]$

Let S^{2n+1} be the euclidian sphere $\{|z_0|^2 + \dots + |z_n|^2 = 1, z_i \in \mathbb{C}\}$ acted upon by $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ identified with the group generated the reflections across the hyperplanes $z_i = 0$ ($0 \leq i \leq n$). Let $\mathfrak{X} = [S^{2n+1}/(\mathbb{Z}/2\mathbb{Z})^{n+1}]$ be the induced quotient stack which is obviously an oriented orbifold of dimension $2n + 1$. We now describe the Frobenius algebras associated to $\Lambda\mathfrak{X}$ and $\mathrm{L}\mathfrak{X}$. Until the end of this section we denote $R = (\mathbb{Z}/2\mathbb{Z})^{n+1}$.

The string product has a very simple combinatorial description. Let Δ^n be a n -dimensional standard simplex. Denote v_0, \dots, v_n its $n + 1$ -vertex and F_0, \dots, F_n its n -faces of dimension $n - 1$. In other words $F_i = \Delta(v_0, \dots, \widehat{v}_i, \dots, v_n)$ is the convex hull of all vertices but v_i . More generally we denote $F_{i_1 \dots i_k} := F_{i_1} \cap \dots \cap F_{i_k}$ the subface of dimension $n - k$ given by the convex hull of all vertices but v_{i_1}, \dots, v_{i_k} . We assign the degree $2n - 2k + 1$ to a face $F_{i_1 \dots i_k}$ of dimension $n - k$.

Proposition 14.7 *Let k be a ring with $1/2 \in k$. Then $H_\bullet(\Lambda\mathfrak{R})$ is the free k -module with basis indexed by elements $r \in R - \{1\}$ of degree 0 and all faces $F_{i_1 \dots i_k}$ in degree $2(n-k)+1$ (in particular $F_\emptyset = \Delta^n$ has degree $2n+1$), i.e.,*

$$H_\bullet(\Lambda\mathfrak{R}) \cong k^{|R|-1} \oplus \left(\bigoplus_{\substack{k=0 \dots n \\ 0 \leq i_1 < \dots < i_k \leq n}} k.F_{i_1 \dots i_k} \right).$$

The string product \star is defined on the basis by the identities

$$F_{i_1 \dots i_k} \star F_{j_1 \dots j_l} = F_{i_1 \dots i_k} \cap F_{j_1 \dots j_l}$$

if the two subfaces have transversal intersection in Δ^n , and is 0 otherwise. The element $\Delta^n = F_\emptyset$ is set to be the unit and all other products involving a generator of $k^{|R|-1}$ are trivial.

In other words, $H_0(\Lambda\mathfrak{R}) = k^{|R|-1}$, and $H_{2i+1}(\Lambda\mathfrak{R})$ is the free module generated by the subfaces of dimension i of the simplex Δ^n . The product is given by transverse intersection in Δ^n .

PROOF. Write s_i ($i=0 \dots n$) for the reflection across the hyperplane $z_i = 0$. Then, for $0 \leq k \leq n$,

$$(S^{2n+1})^{s_{i_1} \dots s_{i_k}} \cong \left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} / \sum_{j \neq i_1, \dots, i_k} |z_j|^2 = 1 \right\} \cong S^{2n-2k+1}.$$

Thus $H_\bullet((S^{2n+1})^{s_{i_1} \dots s_{i_k}}) \cong k V'_{s_{i_1} \dots s_{i_k}} \oplus k F'_{i_1 \dots i_k} [2(n-k)+1]$. Since these generators are R -invariant, $|R|$ is invertible in k and $(S^{2n+1})^{s_0 \dots s_n} = \emptyset$, one has

$$H_\bullet(\Lambda\mathfrak{R}) \cong \bigoplus_{g \in R} H_\bullet((S^{2n+1})^g)_R \cong \bigoplus_{g - \{1\} \in R} H_\bullet((S^{2n+1})^g)$$

By Proposition 14.3, the string product is the composition of $\text{tr}_{R \times R}^R$ with

$$H((S^{2n+1})^g \times (S^{2n+1})^h) \xrightarrow{(i_g \times i_h)^!} H((S^{2n+1})^{g,h}) \xrightarrow{\cap e(E_x(g,h))} H((S^{2n+1})^{g,h}) \xrightarrow{m_*} H((S^{2n+1})^{gh})$$

Clearly $\text{tr}_{R \times R}^R$ is multiplication by the order of R . Furthermore $F_{i_1 \dots i_k}$ and $F_{j_1 \dots j_l}$ are transversal iff the sets $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_l\}$ are disjoint iff the submanifolds $(S^{2n+1})^{s_{i_1} \dots s_{i_k}}$ and $(S^{2n+1})^{s_{j_1} \dots s_{j_l}}$ are transversal in (S^{2n+1}) . In particular, if $F_{i_1 \dots i_k}$ and $F_{j_1 \dots j_l}$ are transversal, $(S^{2n+1})^{s_{i_1} \dots s_{i_k}, s_{j_1} \dots s_{j_l}} = (S^{2n+1})^{s_{i_1} \dots s_{i_k} \cdot s_{j_1} \dots s_{j_l}}$, the excess bundle is of rank 0, $m_* = \text{id}$ and by Poincaré duality,

$$(i_{s_{i_1} \dots s_{i_k}} \times i_{s_{j_1} \dots s_{j_l}})^!(F'_{i_1 \dots i_k} \times F'_{j_1 \dots j_l}) = F'_{i_1 \dots i_k j_1 \dots j_l}.$$

If $F_{i_1 \dots i_k}$ and $F_{j_1 \dots j_l}$ are not transversal, one finds

$$(i_{s_{i_1} \dots s_{i_k}} \times i_{s_{j_1} \dots s_{j_l}})^!(F'_{i_1 \dots i_k} \times F'_{j_1 \dots j_l}) = F'_{i_1 \dots i_k} \cap F'_{j_1 \dots j_l} = F'_{\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_l\}}$$

and $(S^{2n+1})^{s_{i_1} \dots s_{i_k} s_{j_1} \dots s_{j_l}}$ contains $(S^{2n+1})^{s_{i_1} \dots s_{i_k}, s_{j_1} \dots s_{j_l}}$ as a submanifold of codimension > 0 . It follows that $m_* \left(F'_{\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_l\}} \cap e(Ex) \right) = 0$ for degree reason. Similarly, $F'_{i_1, \dots, i_k} \star g = 0$ for any $g \in R$. The result follows by identifying F_{i_1, \dots, i_k} with $2^{-n-1} F'_{i_1, \dots, i_k}$ as basis element. \square

Remark 14.8 It is easy to show that the string coproduct is trivial. Indeed, for degree reason, only the class of F_\emptyset might be non zero. Proposition 14.5 shows the string coproduct is induced by the composition

$$H(S^{2n+1}) \xrightarrow{\sum_i i_g^!} \bigoplus H((S^{2n+1})^g) \xrightarrow{\cap_{\oplus e((S^{2n+1})^h)}} \bigoplus H((S^{2n+1})^g).$$

Since $(S^{2n+1})^h$ is an odd dimensional sphere, its Euler class is 2-torsion, hence trivial by our assumption on k .

Since $R = (\mathbb{Z}/2\mathbb{Z})^{n+1}$ is abelian, its group algebra is a Frobenius algebra (see Example 14.6 above).

Proposition 14.9 *Let k be a field of characteristic different from 2. There is an isomorphism of **BV**-algebras as well as Frobenius algebras*

$$H_\bullet(\mathbb{L}\mathfrak{R}) \cong H_\bullet(\mathbb{L}S^{2n+1}) \otimes_k k[(\mathbb{Z}/2\mathbb{Z})^{n+1}]. \quad (14.4)$$

The **BV**-operator on the right hand side is $B \otimes \text{id}$ where $B : H_\bullet(\mathbb{L}S^{2n+1}) \rightarrow H_{\bullet+1}(\mathbb{L}S^{2n+1})$ is the **BV**-operator of the loop homology of S^{2n+1} .

PROOF. According to Proposition 5.9, the free loop stack $\mathbb{L}\mathfrak{R}$ is presented by the groupoid $\coprod_{g \in R} \mathcal{P}_g S^{2n+1} \times R \rightrightarrows \coprod_{g \in R} \mathcal{P}_g S^{2n+1}$. Hence

$$H_\bullet(\mathbb{L}\mathfrak{X}) = \left(\bigoplus_{g \in R} H_\bullet(\mathcal{P}_g S^{2n+1}) \right)_R.$$

Since R is a subgroup of the connected Lie group $SO(2n+2)$, which acts on S^{2n+1} , for all $g \in R$ there is a continuous path $\rho : [0, 1] \rightarrow SO(2n+2)$ connecting g to the identity (that is $\rho(0) = g$, $\rho(1) = 1$). In particular, any path $f \in \mathcal{P}_g S^{2n+1}$ can be composed with the path $f(0) \cdot \rho(t)$ yielding a loop $\Upsilon_g(f) \in \mathbb{L}S^{2n+1}$. It is a general fact that $\Upsilon_g : \mathcal{P}_g S^{2n+1} \rightarrow \mathbb{L}S^{2n+1}$ is a G -equivariant homotopy equivalence (see [35] for details). We write $\Upsilon : \coprod_{g \in R} \mathcal{P}_g S^{2n+1} \rightarrow \coprod_{g \in R} \mathbb{L}S^{2n+1}$ for the map induced by the maps Υ_g for $g \in R$. Since the G -action on $\mathbb{L}M = P_e M$ is trivial, the isomorphism (14.4) follows.

It remains to prove that the linear isomorphism (14.4) is an isomorphism of Frobenius algebras and **BV**-algebras. To do so, we need the evaluation map $\text{ev}_0 : \mathbb{L}\mathfrak{R} \rightarrow \mathfrak{R}$ at the groupoid level. One checks that ev_0 is represented by the maps $\text{ev}_g : \mathcal{P}_g S^{2n+1} \times R \rightarrow S^{2n+1} \times R$ defined by $\text{ev}_g((f, h)) = (f(1), h)$. Let $f, g \in \mathcal{P}_r S^{2n+1} \times \mathcal{P}_h S^{2n+1}$ such that $f(1) = g(1)$. The composition of the path $f(-)$ and $g(-) \cdot h$ gives an element $m(f, g) \in \mathcal{P}_{rh} S^{2n+1}$. This composition induces the stack morphism $m : \mathbb{L}\mathfrak{R} \times_{\mathfrak{R}} \mathbb{L}\mathfrak{R} \rightarrow \mathbb{L}\mathfrak{R}$. Denote \tilde{m} the

map $\coprod_{g,h \in R} \mathbb{L}S^{2n+1} \times_{S^{2n+1}} \mathbb{L}S^{2n+1} \xrightarrow{\tilde{m}} \coprod_{g \in R} \mathbb{L}S^{2n+1}$ which maps an element $(\gamma, \gamma') \in \mathbb{L}S^{2n+1} \times_{S^{2n+1}} \mathbb{L}S^{2n+1}$ in the component (g, h) to the element $m(\gamma, \gamma')$ in the component gh . Here m is the usual composition of paths. The map $\Upsilon_g : \mathcal{P}_g S^{2n+1} \rightarrow \mathbb{L}S^{2n+1}$ induces a commutative diagram of R -equivariant maps

$$\begin{array}{ccc} \coprod_{g,h} P_g S^{2n+1} \times P_h S^{2n+1} & \longleftarrow \coprod_{g,h} P_g S^{2n+1} \times_{S^{2n+1}} P_h S^{2n+1} \xrightarrow{m} & \coprod_g P_g S^{2n+1} \\ \downarrow \coprod \Upsilon_g \times \Upsilon_h & & \downarrow \coprod \Upsilon_g \\ \coprod_{g,h} \mathbb{L}S^{2n+1} \times \mathbb{L}S^{2n+1} & \longleftarrow \coprod_{g,h} \mathbb{L}S^{2n+1} \times_{S^{2n+1}} \mathbb{L}S^{2n+1} \xrightarrow{\tilde{m}} & \coprod_g \mathbb{L}S^{2n+1}. \end{array}$$

Since $\mathbb{L}S^{2n+1} \rightarrow S^{2n+1}$, $\mathcal{P}_g S^{2n+1} \rightarrow S^{2n+1}$ are fibration, the vertical arrows are R -homotopy equivalences. It follows easily that the map

$$\frac{1}{|R|} \Upsilon : H_\bullet([\coprod_{g \in R} \mathbb{L}S^{2n+1}/R]) \rightarrow H_\bullet(\mathbb{L}S^{2n+1}) \otimes k[R]$$

is a morphism of algebras. One proves similarly that $\frac{1}{|R|} \Upsilon$ is a coalgebra map.

Now we need to identify the **BV**-operator. Denote $\mathbb{L}\Gamma$ the transformation groupoid $\coprod_{g \in R} \mathcal{P}_g S^{2n+1} \rtimes R \rightrightarrows \coprod_{g \in R} \mathcal{P}_g S^{2n+1}$. Since the stack S^1 is canonically identified with the quotient stack $[\mathbb{R}/\mathbb{Z}]$, the homology $H_\bullet(S^1)$ coincides with the homology of the groupoid $\Gamma' : \mathbb{R} \rtimes \mathbb{Z} \rightrightarrows \mathbb{R}$. The 0-dimensional simplex $(0, 1) \in \mathbb{R} \rtimes \mathbb{Z} = \Gamma'_1$ defines an element in $C_0(\Gamma'_1) \subset C_1(\Gamma')$ which is the generator of $H_1(S^1)$. The map $\Gamma' \times \mathbb{L}\Gamma \xrightarrow{\theta} \mathbb{L}\Gamma$ defined, for $(x, n) \in \mathbb{R} \times \mathbb{Z}$, $f \in \mathcal{P}_g$ and $h \in R$, by $\theta(x, n, f, h)(t) = f(t+x).h^n$ is a groupoid morphism representing the S^1 -action on $\mathbb{L}\mathcal{X}$. Since $\Upsilon(\theta((0, 1), f)) = f$, Υ commutes with the **BV**-operator. \square

Remark 14.10 For the sake of completeness, we recall [20] that, $\mathbb{H}_\bullet(\mathbb{L}S^{2n+1}) \cong k[u, v]$ with $|v| = -2n - 1$ and $|u| = 2n$ for $n > 0$, and $\mathbb{H}_\bullet(\mathbb{L}S^{2n+1}) \cong k[[u, u^{-1}]] [v]$ if $n = 0$. Thus

$$\mathbb{H}_\bullet(\mathbb{L}[S^{2n+1}/(\mathbb{Z}/2\mathbb{Z})^{n+1}]) \cong k[(\mathbb{Z}/2\mathbb{Z})^{n+1}][u, v] \text{ if } n > 0, \quad \text{and}$$

$$\mathbb{H}_\bullet(\mathbb{L}[S^1/\mathbb{Z}/2\mathbb{Z}]) \cong k[[u, u^{-1}]] [\tau, v] / (\tau^2 = 1) \quad \text{with } |v| = 1, |u| = 0 \text{ if } n = 0.$$

Remark 14.11 The stack morphism $\Phi : \Lambda\mathcal{X} \rightarrow \mathbb{L}\mathcal{X}$ of Section 11.13 is represented at the groupoid level by $\coprod_{g \in R} (S^{2n+1})^g \hookrightarrow \coprod_{g \in R} \mathcal{P}_g S^{2n+1}$ where $x \in (S^{2n+1})^g$ is identified with a constant path. It follows easily that the Frobenius algebra morphism is given by $\Phi(F_\emptyset) = e$, $\Phi(F_{i_1 \dots i_k}) = 0$ and $\Phi(g) = gv$.

14.4 String topology of $\mathbb{L}[* / G]$ when G is a compact Lie group

Any topological group G naturally defines a topological stack corresponding to the groupoid $G \rightrightarrows \{*\}$, which is denoted by $[* / G]$. In this section we study

the Frobenius structures on the homology of its loop stack and inertia stack assuming that G is a compact and connected Lie group. It turns out that in this case the two Frobenius structures obtained are indeed isomorphic since $\Lambda[* / G]$ and $L[* / G]$ are homotopy equivalent. In this section, we assume that G is of dimension d and we will work with real coefficients for (co)homology groups.

First we will identify the homology groups $H_\bullet(\Lambda[* / G])$ and $H_\bullet(L[* / G])$.

Lemma 14.12 *The inertia stack $\Lambda[* / G]$ is represented by the transformation groupoid $G \rtimes G \rightrightarrows G$, where G acts on itself by conjugation, while the stack $\Lambda[* / G] \times_{[* / G]} \Lambda[* / G]$ is represented by the groupoid $(G \times G) \rtimes G \rightrightarrows G \times G$ with the diagonal conjugacy action.*

The following result is well known [12].

Lemma 14.13 *The map $\Lambda[* / G] \xrightarrow{\Phi} L[* / G]$ is an homotopy equivalence.*

PROOF. Since G is connected, by Proposition 5.7, $L[* / G]$ can be represented by the loop group $LG \rightrightarrows \{*\}$. On the other hand, $BLG \cong LBG$ is homotopy equivalent to $EG \times_G G$. Identifying BLG with $\{f \in \text{Map}(I, EG) / \exists g \in G \text{ such that } f(0) = f(1).g\} / G$, this equivalence is induced by the map $f \mapsto (f(0), g)$ [8], [12]. The map Φ of Lemma 11.12 and the isomorphism in between $L[* / G]$ and $[* / LG]$ (Proposition 5.7) gives a map $B\Phi : B[* / G] = EG \times_G G \rightarrow BLG$ which is indeed easily checked to establish the homotopy equivalence. \square

As an immediate consequence, we have

Corollary 14.14 *The map $\Phi_* : H_\bullet(\Lambda[* / G]) \rightarrow H_\bullet(L[* / G])$ is an isomorphism of Frobenius algebras.*

Thus it is sufficient to study the Frobenius structure on the homology of the inertia stack $\Lambda[* / G]$.

According to Remark 11.10, there is a dual Frobenius structure induced on $(H^\bullet(\Lambda[* / G]), \star, \delta)$. We refer to $\delta : H^\bullet(\Lambda[* / G]) \rightarrow H^\bullet(\Lambda[* / G]) \otimes H^\bullet(\Lambda[* / G])$ and $\star : H^\bullet(\Lambda[* / G]) \otimes H^\bullet(\Lambda[* / G]) \rightarrow H^\bullet(\Lambda[* / G])$ as the dual string coproduct and dual string product respectively. Since, it is technically easier, we will describe the Frobenius structure of $H^\bullet(\Lambda[* / G])$. The following result is standard [38]. We write EG for a free G -space which is contractible and $BG = EG \times_G *$ its classifying space so that $H^\bullet([* / G]) = H^\bullet(BG) = H_G^\bullet(*)$.

Proposition 14.15 *1. The cohomology of G , as a topological space, is*

$$H^\bullet(G) = (\Lambda \mathfrak{g}^*)^G \cong \Lambda(y_1, y_2, \dots, y_l)$$

2. The cohomology of $[/ G]$ is*

$$H^\bullet([* / G]) = (S^*(\mathfrak{g}^*))^G \cong S(x_1, x_2, \dots, x_l)$$

3. The cohomology of $[G/G]$ is

$$H^\bullet([G/G]) = (S^*(\mathfrak{g}^*))^G \otimes (\Lambda\mathfrak{g}^*)^G \cong S(x_1, x_2, \dots, x_l) \otimes \Lambda(y_1, y_2, \dots, y_l)$$

4. The cohomology of $[G \times G/G]$ is

$$\begin{aligned} H^\bullet([G \times G/G]) &= (S^*(\mathfrak{g}^*))^G \otimes (\Lambda(\mathfrak{g}^* \oplus \mathfrak{g}^*))^G \\ &\cong S(x_1, x_2, \dots, x_l) \otimes \Lambda(y_1, y_2, \dots, y_l, y'_1, y'_2, \dots, y'_l), \end{aligned}$$

5. The cohomology of $[G \times G/G \times G]$ is

$$\begin{aligned} H^\bullet([G \times G/G \times G]) &= (S^*(\mathfrak{g}^* \oplus \mathfrak{g}^*))^G \otimes (\Lambda(\mathfrak{g}^* \oplus \mathfrak{g}^*))^G \\ &\cong S(x_1, x_2, \dots, x_l, x'_1, x'_2, \dots, x'_l) \\ &\quad \otimes \Lambda(y_1, y_2, \dots, y_l, y'_1, y'_2, \dots, y'_l) \end{aligned}$$

Here $l = \text{rank}(G)$, $\text{deg}(y_i) = \text{deg}(y'_i) = 2d_i + 1$, $\text{deg}(x_i) = \text{deg}(x'_i) = 2d_i$ and d_i are the exponents of G .

To compute the Frobenius structure of $H^\bullet(\Lambda[*]/G)$, we need an explicit construction of some Gysin maps.

Let M be an oriented manifold with a smooth $(G \times G)$ -action. Consider G as a subgroup of $G \times G$ by embedding it diagonally. In this way, M becomes a G -space and we have a morphism of stacks $[M/G] \rightarrow [M/G \times G]$, which is indeed a G -principle bundle. According to Section 8.2, there is a cohomology Gysin map $\Delta_! : H^\bullet[M/G] \rightarrow H^{\bullet-d}[M/G \times G]$, which should be in a certain sense fibration integration.

Recall that when G is a compact connected Lie group, the cohomology of the quotient stack $H^\bullet([M/G])$ with real coefficients can be computed using the Cartan model $(\Omega_G(M), d_G)$, where $\Omega_G(M) := (S(\mathfrak{g}^*) \otimes \Omega(M))^G$ is the space of G -equivariant polynomials $P : \mathfrak{g} \rightarrow \Omega(M)$, and

$$d_G(P)(\xi) := d(P(\xi)) - \iota_\xi P(\xi), \quad \forall \xi \in \mathfrak{g}.$$

Here d is the de Rham differential and ι_ξ is the contraction by the generating vector field of ξ . Given a Lie group K and a Lie subgroup $G \subset K$, let G act on K from the right by multiplication and K act on itself from the left by multiplication. The submersion $K \rightarrow K/G$ is a principal K -equivariant right G -bundle. There is an isomorphism of stacks $[M/G] \xrightarrow{\sim} [K \times_G M/K]$ which induces an isomorphism in cohomology. It is known [38] that, on the Cartan model, this isomorphism can be described by an induction map $\text{Ind}_K^G : \Omega_G(M) \rightarrow \Omega_K(K \times_G M)$. Here G acts on $K \times M$ by

$$(k, m) \cdot g = (k \cdot g, g^{-1} \cdot m).$$

The induction map is the composition

$$\Omega_G(M) \xrightarrow{\text{Pul}} \Omega_{K \times G}(K \times M) \xrightarrow{\text{Car}} \Omega_K(K \times_G M),$$

where $\Omega_G(M) \xrightarrow{\text{Pul}} \Omega_{K \times G}(K \times M)$ is the natural pullback map, induced by the projections on the second factor $K \times G \rightarrow G$, and $\Omega_{K \times G}(K \times M) \xrightarrow{\text{Car}} \Omega_K(K \times_G M)$ is the Cartan map corresponding to a K -invariant connection for the G -bundle $K \rightarrow K/G$ [38]. We now recall the description of this map.

Let $\Theta \in \Omega^1(K) \otimes \mathfrak{g}$ be a K -invariant connection on the G -bundle $K \rightarrow K/G$. The associated principal G -bundle

$$G \rightarrow K \times M \rightarrow \frac{K \times M}{G} \cong K \times_G M$$

carries a pullback connection, denoted by the same symbol Θ . We denote $F^\Theta = d\Theta + \frac{1}{2}[\Theta, \Theta]$ its curvature, which is an element in $\Omega_K^2(K \times M) \otimes \mathfrak{g}$. The equivariant momentum map $\mu^\Theta \in (\mathfrak{k}^* \otimes \Omega^0(K))^K \otimes \mathfrak{g}$ is defined by

$$\xi \in \mathfrak{k} \mapsto \mu^\Theta(\xi) = -\iota_\xi \Theta$$

where ι_ξ is the contraction along $\hat{\xi} \in (\mathcal{K})$, the generating vector field of ξ . Then $F^\Theta + \mu^\Theta$ is the equivariant curvature of Θ [9]. Observe that $\Omega_{K \times G}(K \times M) \cong (S(\mathfrak{g}^*) \otimes \Omega_K(K \times M))^G$ that is the space of G -equivariant polynomial functions from \mathfrak{g} to $\Omega_K(K \times M)$. Hence if $x \in \mathfrak{g} \otimes \Omega_K^i(K \times M)$ and P is a homogeneous degree q polynomial on \mathfrak{g} , then by substitution of variables, we get an element $P(x)$ in $\Omega_K^{2q+qi}(K \times M)$. The Cartan map $\Omega_{K \times G}(K \times M) \rightarrow \Omega_K(K \times_G M)$ is the composition

$$\begin{aligned} P \otimes \omega \in (S(\mathfrak{g}^*) \otimes \Omega_K(K \times M))^G &\mapsto P(F^\Theta + \mu^\Theta)\omega \in \Omega_K(K \times M) \\ &\mapsto \text{Hor}(P(F^\Theta + \mu^\Theta)\omega) \in \Omega_K(K \times_G M), \end{aligned}$$

where $\text{Hor} : \Omega(K \times M) \rightarrow \Omega(K \times_G M)$ is the horizontal projection with respect to Θ .

If moreover $F^\Theta = 0$ and $K \times M \rightarrow K \times_G M$ admits a horizontal section $\sigma : K \times_G M \rightarrow K \times M$, we have the following lemma.

Lemma 14.16 *Let $P \otimes \omega$ be an element in $(S(\mathfrak{g}^*) \otimes \Omega(M))^G \cong \Omega_G(M)$. Then, $\text{Ind}_K^G(P \otimes \omega) \in \Omega(K \times_G M)$ is the K -equivariant polynomial on \mathfrak{k} with value in $\Omega(K \times_G M)$ defined, for any $\xi \in \mathfrak{k}$, by*

$$\text{Ind}_K^G(P \otimes \omega) : \xi \mapsto \sigma^*(P(\mu^\Theta(\xi))\text{pr}_2^*(\omega)).$$

PROOF. First of all, $\text{Pul}(P \otimes \omega) \in (S((\mathfrak{k} \oplus \mathfrak{g})^*) \otimes \Omega(K \times M))^{K \times G}$ is the map $\xi \oplus y \mapsto P(y)\text{pr}_2^*(\omega)$ for any $\xi \in \mathfrak{k}$ and $y \in \mathfrak{g}$. Then, by hypothesis,

$$\text{Hor}(P(F^\Theta + \mu^\Theta)\omega) = \sigma^*(P(\mu^\Theta(\xi))\text{pr}_2^*(\omega)) \in (S(\mathfrak{k}^*) \otimes \Omega(K \times_G M))^K$$

and the lemma follows. \square

Now let K be the cartesian product group $G \times G$. We view G as the diagonal subgroup of K . The K action on itself by left multiplication commutes with the right G -action. We have a principal right G -bundle

$$\begin{aligned} G &\longrightarrow K(= G \times G) \longrightarrow G \\ &\quad (g, h) \qquad \mapsto gh^{-1}. \end{aligned}$$

The left Maurer-Cartan form $\Theta_{MC}^L \in \Omega^1(G) \otimes \mathfrak{g}$ on G yields a K -invariant one-form $\Theta = \text{pr}_2^*(\Theta_{MC}^L) \in \Omega^1(K) \otimes \mathfrak{g}$ by pullback along the projection on the second factor. Then Θ is a $K(= G \times G)$ -invariant connection. Moreover it is flat, thus its equivariant curvature reduces to the equivariant momentum $\mu^\Theta : \mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g} \rightarrow \Omega^0(K) \otimes \mathfrak{g}$.

Lemma 14.17 *For any $(\alpha, \beta) \in \mathfrak{k}(= \mathfrak{g} \oplus \mathfrak{g})$, and $(g, h) \in K(= G \times G)$ one has*

$$\mu^\Theta(\alpha, \beta)|_{(g, h)} = -\text{Ad}_{h^{-1}} \beta.$$

PROOF. The generating vector field for the left G -action on G is given, for all $\beta \in \mathfrak{g}$ by

$$\hat{\beta}|_h = \left. \frac{\partial}{\partial t} \right|_{t=0} \exp(t\beta)h = L_h(\text{Ad}_{h^{-1}} \beta).$$

It follows, for any $(g, h) \in K = G \times G$, that

$$\begin{aligned} \mu^\Theta(\alpha, \beta)|_{(g, h)} &= -\iota_{(\hat{\alpha}, \hat{\beta})}(\Theta|_{(g, h)}) \\ &= -\iota_{\hat{\beta}} \Theta_{MC}^L|_h \\ &= -\Theta_{MC}^L|_1(\text{Ad}_{h^{-1}} \beta) = -\text{Ad}_{h^{-1}} \beta. \end{aligned}$$

□

Let M be a $K(= G \times G)$ space. It is then a G -space. Thus we have an induction map

$$\text{Ind}_{G \times G}^G : \Omega_G(M) \rightarrow \Omega_{G \times G}((G \times G) \times_G M) \cong \Omega_{G \times G}(G \times M).$$

The group $G \times G$ acts on $G \times M$ by

$$(k_1, k_2) \cdot (g, m) = (k_1 g k_2^{-1}, (k_1, k_2) \cdot m).$$

Lemma 14.18 *1. The map*

$$(G \times G) \times_G M \rightarrow G \times M, \quad (k_1, k_2, m) \mapsto (k_1 k_2^{-1}, (k_1, k_2) \cdot m)$$

is a $(G \times G)$ -equivariant diffeomorphism.

2. The map

$$\sigma : G \times M \rightarrow K \times M, \quad \sigma(g, m) = (g, 1, (g^{-1}, 1) \cdot m)$$

is a horizontal section for the principal G -bundle $G \rightarrow K \times M \rightarrow K \times_G M \cong G \times M$.

As a consequence, we have an isomorphism

$$\Omega_{G \times G}((G \times G) \times_G M) \xrightarrow{\sim} \Omega_{G \times G}(G \times M).$$

Thus there is an induction map

$$\text{Ind}_{G \times G}^G : \Omega_G(M) \rightarrow \Omega_{G \times G}(G \times M).$$

To obtain the Gysin map $H^\bullet([M/G]) \rightarrow H^{\bullet-d}([M/G \times G])$, one simply composes the induction map $\text{Ind}_{G \times G}^G : H^\bullet([M/G]) \rightarrow H^\bullet([G \times M/G \times G])$ with the equivariant fiber integration map [4] $H^\bullet([G \times M/G \times G]) \rightarrow H^{\bullet-d}([M/G \times G])$ over the first factor G .

Proposition 14.19 *Given a $(G \times G)$ -manifold M , the Gysin map*

$$H_G^\bullet(M) \rightarrow H_{G \times G}^{\bullet-d}(M)$$

is given, on the Cartan model, by the chain map $\Psi : \Omega_G(M) \rightarrow \Omega_{G \times G}(M)$, $\forall P \otimes \omega \in (S(\mathfrak{g}^) \otimes \Omega(M))^G$,*

$$\Psi(P \otimes \omega) = \left((\xi_1, \xi_2) \mapsto \int_G P(-\xi_2) \varphi^*(\omega) \right), \quad \forall \xi_1, \xi_2 \in \mathfrak{g}, \quad (14.5)$$

where $\varphi : G \times M \rightarrow M$ is the map $(g, m) \mapsto (g^{-1}, 1) \cdot m$, and \int_G stands for the fiber integration over the first factor G .

PROOF. The induction map $\text{Ind}_{G \times G}^G : \Omega_G(M) \rightarrow \Omega_{G \times G}(G \times M)$ is a chain level representative of the stacks isomorphisms

$$[M/G] \xleftarrow{\sim} [G \times G \times M/G \times G \times G] \xrightarrow{\sim} [G \times G \times_G M/G \times G].$$

induced by Morita equivalences of groupoids. Thus the Gysin map $\Delta_! : H^\bullet[M/G] \rightarrow H^{\bullet-d}[M/G \times G]$ is the composition of $\text{Ind}_{G \times G}^G$ with the Gysin map $H^\bullet([G \times M/G \times G]) \rightarrow H^{\bullet-d}([M/G \times G])$ which, by Proposition 8.17 is the equivariant fiber integration.

We now need to express the induction map more explicitly. Recall that, for any $\alpha \in \Omega_G(M)$, $\text{Ind}_{G \times G}^G(\alpha) \in \Omega_{G \times G}(G \times M)$. That is, $\text{Ind}_{G \times G}^G(\alpha)$ is a polynomial function on $\mathfrak{k}(= \mathfrak{g} \oplus \mathfrak{g})$ valued in $\Omega(G \times M)$. Write $\varphi : G \times M \rightarrow M$ for the composition $\varphi = \text{pr}_2 \circ \sigma$. Thus $\varphi(g, m) = (g^{-1}, 1) \cdot m$. According to Lemma 14.16, it suffices to compute $\sigma^*(P(\mu^\Theta(\xi_1, \xi_2)) \text{pr}_2^*(\omega))$. By Lemma 14.18.3 and Lemma 14.17 we find that

$$\sigma^*(P(\mu^\Theta(\xi_1, \xi_2))) = P(-\xi_2).$$

Now the very definition of φ yields that for any $\alpha = P \otimes \omega \in \Omega_G(M) \cong (S(\mathfrak{g}^*) \otimes \Omega(M))^G$, and $\forall \xi_1, \xi_2 \in \mathfrak{g}$,

$$\text{Ind}_{G \times G}^G(\alpha)(\xi_1, \xi_2) = P(-\xi_2) \varphi^*(\omega).$$

This concludes the proof. \square

Remark 14.20 If we identify an element of $\Omega_G(M)$ with a G -equivariant polynomial $Q : \mathfrak{g} \rightarrow \Omega(M)$, then Equation (14.5) can be written as follows. $\forall(\xi_1, \xi_2) \in \mathfrak{k} = \mathfrak{g} \oplus \mathfrak{g}$,

$$\Psi(Q)(\xi_1, \xi_2) = \int_G \varphi^*(Q(-\xi_2)).$$

We now go back to our special case. Denote by $m : G \times G \rightarrow G$ and $\Delta : G \rightarrow G \times G$ the group multiplication and the diagonal map respectively. The diagonal map induces a stack map $\Delta : [G \times G/G] \rightarrow [G \times G/G \times G]$ and thus a Gysin map

$$\Delta_! : H^\bullet([G \times G/G]) \rightarrow H^{\bullet-d}([G \times G/G \times G]),$$

which is given by Proposition 14.19. Similarly the group multiplication m induces a stack map $m : [G \times G/G] \rightarrow [G/G]$ and thus a Gysin map

$$m_! : H^\bullet([G \times G/G]) \rightarrow H^{\bullet-d}([G/G]).$$

Since m is G -equivariant, this is the usual G -equivariant Gysin map on manifolds according to Proposition 8.17.

Note that $H_G^\bullet(G)$ is a free module over $H^\bullet([*/G]) \cong S(x_1, \dots, x_l)$. In fact, $H_G^\bullet(G) = H^\bullet([*/G])[y_1, \dots, y_l]$ (the y_j s are of odd degrees). Thus elements of $H_G^\bullet(G)$ are linear combinations of monomials $y_1^{\epsilon_1} \dots y_l^{\epsilon_l}$, where each ϵ_j is either 0 or 1. Similarly $H_G^\bullet(G \times G)$ is the free $H_G^\bullet(G)$ -module generated by the monomials $y_1^{\epsilon_1} \dots y_l^{\epsilon_l} y_1^{\epsilon'_1} \dots y_l^{\epsilon'_l}$.

Lemma 14.21 *The map $m_!$ is a $H^\bullet([*/G])$ linear map defined by*

$$m_!(y_1^{\epsilon_1} \dots y_l^{\epsilon_l} y_1^{\epsilon'_1} \dots y_l^{\epsilon'_l}) = y_1^{\epsilon_1 + \epsilon'_1 - 1} \dots y_l^{\epsilon_l + \epsilon'_l - 1}$$

with the convention that $y_j^{-1} = 0$.

PROOF. Since $m : G \times G \rightarrow G$ is G -equivariant, the Gysin map $m_! : H^*([G \times G/G]) \rightarrow H^*([G/G])$ is a map of $H^\bullet([*/G])$ -module, and by Proposition 8.17, it is the equivariant fiber integration of the principal bundle $G \times G \rightarrow G$. It can be represented on the Cartan cochain complex by integration of forms, see [29] for details. In particular $m_!(y_1^{\epsilon_1} \dots y_l^{\epsilon_l} y_1^{\epsilon'_1} \dots y_l^{\epsilon'_l})$ is determined by the equation

$$\int_{G \times G} m^*(\alpha) \wedge (y_1^{\epsilon_1} \dots y_l^{\epsilon_l} y_1^{\epsilon'_1} \dots y_l^{\epsilon'_l}) = \int_G \alpha \wedge m_!(y_1^{\epsilon_1} \dots y_l^{\epsilon_l} y_1^{\epsilon'_1} \dots y_l^{\epsilon'_l}) \quad (14.6)$$

Since the volume form on G and $G \times G$ are respectively given by $y_1 \dots y_l$ and $y_1 \dots y_l y_1' \dots y_l'$, Equation (14.6) implies that $m_! : H^*(G \times G) \rightarrow H^{*-d}(G)$ sends $y_1^{\epsilon_1} \dots y_l^{\epsilon_l} y_1^{\epsilon'_1} \dots y_l^{\epsilon'_l}$ to $y_1^{\epsilon_1 + \epsilon'_1 - 1} \dots y_l^{\epsilon_l + \epsilon'_l - 1}$. This finishes the proof. \square

The string product and coproduct on $H_\bullet([G/G])$, by universal coefficient theorem, induces a degree $-d$ coproduct $\delta : H^\bullet([G/G]) \rightarrow H^\bullet([G/G]) \otimes H^\bullet([G/G])$ and degree $-d$ product $\star : H^\bullet([G/G]) \otimes H^\bullet([G/G]) \rightarrow H^\bullet([G/G])$ which makes $H^\bullet([G/G])$ into a Frobenius algebra, called the *dual Frobenius structure* on $H^\bullet([G/G])$.

More explicitly, these two operations are given by the following compositions:

$$\begin{aligned} \delta : H^\bullet([G/G]) &\xrightarrow{m^*} H^\bullet([G \times G/G]) \xrightarrow{\Delta!} H^{\bullet-d}([G \times G/G \times G]) \\ &\rightarrow \bigoplus_{i+j=\bullet-d} H^i([G/G]) \otimes H^j([G/G]), \end{aligned}$$

and

$$\star : H^\bullet([G/G]) \otimes H^\bullet([G/G]) \cong H^\bullet([G \times G/G \times G]) \xrightarrow{\Delta^*} H^\bullet([G \times G/G]) \xrightarrow{m!} H^{\bullet-d}([G/G]).$$

Theorem 14.22 *Let G be a compact connected Lie group. The dual string coproduct on $H^\bullet([G/G])$ is trivial. And the dual string product on $H^\bullet([G/G])$ is given as follows. For any $P(x_1, \dots, x_l)y_1^{\epsilon_1} \dots y_l^{\epsilon_l}$ and $Q(x_1, \dots, x_l)y_1^{\epsilon'_1} \dots y_l^{\epsilon'_l}$ in $H_\bullet([G/G])$, we have*

$$\begin{aligned} (P(x_1, \dots, x_l)y_1^{\epsilon_1} \dots y_l^{\epsilon_l}) \star (Q(x'_1, \dots, x'_l)y_1^{\epsilon'_1} \dots y_l^{\epsilon'_l}) \\ = (PQ)(x_1, \dots, x_l)y_1^{\epsilon_1 + \epsilon'_1 - 1} \dots y_l^{\epsilon_l + \epsilon'_l - 1} \end{aligned}$$

with the convention that $y_j^{-1} = 0$.

PROOF. On the Cartan model, by Proposition 14.19, the string coproduct is given by the following composition of chain maps:

$$\Omega_G(G) \xrightarrow{p^*} \Omega_G(G \times G) \xrightarrow{\Psi^*} \Omega_{G \times G}(G \times G) \xrightarrow{\cong} \Omega_G(G) \otimes \Omega_G(G).$$

Here the last map is Künneth formula, and the first map

$$p^* : \Omega_G(G) \cong (S(\mathfrak{g}^*) \otimes \Omega(G))^G \rightarrow \Omega_G(G \times G) \cong (S(\mathfrak{g}^*) \otimes \Omega(G \times G))^G$$

is $S(\mathfrak{g}^*)$ -linear and given by

$$p^*(P \otimes \omega) = P \otimes m^*(\omega), \quad \forall P \otimes \omega \in (S(\mathfrak{g}^*) \otimes \Omega(G))^G.$$

Note that the space $\Omega_G(G) \otimes \Omega_G(G) \cong (S(\mathfrak{g}^*) \otimes \Omega(G))^G \otimes (S(\mathfrak{g}^*) \otimes \Omega(G))^G$ has a $S(\mathfrak{g}^*)^G$ -module structure, which is given by multiplication on the second factor: i.e. $\forall Q \in S(\mathfrak{g}^*)$, $P_1 \otimes \omega_1 \otimes P_2 \otimes \omega_2 \in \Omega_G(G) \otimes \Omega_G(G)$, one defines

$$Q \cdot (P_1 \otimes \omega_1 \otimes P_2 \otimes \omega_2) = \{(\xi_1, \xi_2) \mapsto P_1(\xi_1) \otimes \omega_1 \otimes Q(-\xi_2)P_2(\xi_2) \otimes \omega_2 \in \Omega(G) \otimes \Omega(G)\}.$$

By Proposition 14.19, we know that the Gysin map $\Psi^* : \Omega_G(G \times G) \rightarrow \Omega_{G \times G}(G \times G)$ is indeed a $S(\mathfrak{g}^*)^G$ -module map.

There are two kinds of elements $P \otimes \omega$ in $H^\bullet([G/G]) = (S(\mathfrak{g}^*))^G \otimes \Lambda(\mathfrak{g})^G$. One consists of those where ω is a top degree form, i.e. a multiple of $y_1 \wedge \dots \wedge y_l$, and the others are those where ω corresponds to a form in $\Omega^{* < d}(G)$. In the latter case, $\Psi(P \otimes \omega)$ vanishes after fiber integration for degree reasons. In the first case, the G -action on G is by conjugation. Since the conjugacy action is trivial in cohomology, $\int_G \varphi^*(\omega) = 0$ and by Proposition 14.19, $\Psi(P \otimes \omega)$ vanishes. Hence the dual string coproduct is trivial.

We now compute the dual string product. First, by a simple computation, we know that, on the Cartan model, the map $\Delta^* : H_{G \times G}^*(G \times G) \rightarrow H_G^*(G \times G)$ is given by

$$\begin{aligned} \Delta^*(P(x_1, \dots, x_l, y_1, \dots, y_l, x'_1, \dots, x'_l, y'_1, \dots, y'_l)) \\ = P(x_1, \dots, x_l, y_1, \dots, y_l, x_1, \dots, x_l, y'_1, \dots, y'_l). \end{aligned}$$

In other words, the map Δ^* is an algebra map that leaves the odd degree generators y_i, y'_j unchanged and send both generators x_i, x'_i ($i = 1 \dots r$) to the generator x_i . By Lemma 14.21, one obtains that

$$(m_l \circ \Delta^*)(y_1^{\epsilon_1} \dots y_l^{\epsilon_l}, y'_1{}^{\epsilon'_1} \dots y'_l{}^{\epsilon'_l}) = y_1^{\epsilon_1 + \epsilon'_1 - 1} \dots y_l^{\epsilon_l + \epsilon'_l - 1}.$$

The dual string product now follows from the explicit $S(\mathfrak{g}^*)$ -module structure. \square

Remark 14.23 It follows that the string product on $H_\bullet([G/G])$ is trivial while the string coproduct has a counit given by the fundamental class of G , which is dual of the cohomology class $y_1 \dots y_l$.

15 Concluding Remarks

1. It is well-known [43, 3] that a structure of 1 + 1-dimensional Topological Quantum Field Theory on A is equivalent to a structure of unital and counital Frobenius algebra on A such that the pairing $c \circ \mu : A \otimes A \rightarrow k$, where c is the counit and μ the multiplication, is non-degenerate. Theorem 11.3 implies that $\mathbb{H}_\bullet(\mathcal{L}\mathfrak{X})$ is a 1 + 1-positive boundary TQFT in the sense of [18]. Positive boundary TQFT are obtained by considering only cobordism Σ with boundary $\partial\Sigma = -S_1 \amalg S_2$ such that $S_1, S_2 \neq \emptyset$ (see [18] for details). In particular, one can define operation $\mathbb{H}_\bullet(\mathcal{L}\mathfrak{X})^{\otimes p} \rightarrow \mathbb{H}_\bullet(\mathcal{L}\mathfrak{X})^{\otimes q}$ for any bordism $\Sigma : \amalg_{i=1}^p S^1 \rightarrow \amalg_{i=1}^q S^1$. Cohen-Godin [18] proved that these operations are parameterized by the so-called Sullivan chord diagrams of type (p, q) . Let \mathfrak{X} be a oriented stack. Using the general machinery developed in Section 5.1, Section 8.2 and the proof of Theorem 11.3, one can show along similar lines as in [18], that each chord diagram c of type (p, q) determines a linear map $\mu(c) : H(\mathcal{L}\mathfrak{X})^{\otimes p} \rightarrow H(\mathcal{L}\mathfrak{X})^{\otimes q}$. Furthermore the maps $\mu(c)$ are compatible with the gluing of chord diagrams.

2. A higher dimensional analog of String topology, called Brane topology, was studied by Sullivan-Voronov (see [21] Chapter 5) and Hu-Kriz-Voronov [31]. Brane topology is concerned with the algebraic structure of the homology of $M^{S^n} = \text{Map}(S^n, M)$, where M is an oriented manifold and S^n the standard n -dimensional sphere. Sullivan-Voronov ([21] Chapter 5) proved that $H^{\bullet+d}(M^{S^n}, \mathbb{Q})$ is a BV_{n+1} -algebra ([21] Definition 5.3.1) which is an analog of a **BV**-algebra where the BV -operator is of degree n . The proof is based on the action of the n -dimensional *cacti* operad, on M^{S^n} in a way similar to the $n = 1$ case. For a topological stack \mathfrak{X} , since the standard spheres S^n are compact, one can define $\mathfrak{X}^{S^n} = \mathbf{Hom}(S^n, \mathfrak{X})$ as in Section 5.1, as well as the stack $\mathbf{Hom}(S^n \vee S^n, \mathfrak{X})$. Corollary 5.3 can be easily generalized to arbitrary n . Applying the general machinery of this paper, more precisely Section 12, one can define Brane topology for oriented stacks.
3. Let G be either a compact Lie group or a discrete group. Then the stack $[* / G]$ is strongly oriented. Thus according to Theorem 11.3, $H_{\bullet}(L[* / G])$ is a Frobenius algebra. An alternative approach to string topology for $[* / G]$ has been carried out in Gruher-Salvatore [28]. It would be interesting to find a precise link between the result of Section 14.4 with those of [28]. Similarly, it would be interesting to find the connection between the construction relate of our string product (Theorem 10.1) with that of Abbaspour-Cohen-Gruher [1] for Poincaré duality groups.

A Generalized Fulton-MacPherson bivariant theories

In this section we recall the axioms of a Fulton-MacPherson bivariant theory. Indeed we need a slight generalization of it because products of bivariant classes are not always defined. This generalization is what we need to define Gysin homomorphisms, see Section 8.2.

The underlying category

The underlying category of a generalized bivariant theory is a category \mathbf{C} with fiber products and a final object. The category \mathbf{C} is equipped with the following structure:

- A class of commutative triangles called **confined triangles**

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \searrow u & & \swarrow v \\
 & S &
 \end{array}$$

We usually write this triangle as $X \xrightarrow{f} Y \xrightarrow{v} S$. We sometime refer to the above triangle as a *morphism* $f: X \rightarrow Y$ *confined relative to* S .

- a class of squares called **independent squares**

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

Note: we will distinguish the above square from its transpose, so the transpose of an independent square may not be independent.

- a class of morphisms called **adequate**.

We require the following axioms to be satisfied:

A1. A triangle $X \xrightarrow{f} Y \xrightarrow{v} Z$ in which f is an identity map is confined.

A2. If the inside triangles in

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & \searrow u & \downarrow v & \swarrow w & \\ & & S & & \end{array}$$

are confined, then so is the outside triangle.

B1. Any commutative square in which the top and the bottom morphisms are identity maps is independent.

B2. Any square obtained from juxtaposition (vertical, or horizontal) of independent squares is independent.

C. If in the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{v'} & S' \\ g' \downarrow & & \downarrow g & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{v} & S \end{array}$$

the left square (or its transpose) is independent and f is confined relative to S , then f' is confined relative to S' .

D. All isomorphisms are adequate.

Lemma A.1 *Given*

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow W$$

if f is confined relative to W then f is confined relative to Z .

PROOF. Use Axioms **B1** and **C**. \square

Axioms for a bivariant theory

A bivariant theory T on such a category \mathbf{C} assigns to every morphism $f: X \rightarrow Y$ in \mathbf{C} a graded abelian group $T(X \xrightarrow{f} Y)$, or $T(f)$ for short. We denote the i^{th} graded component, $i \in \mathbb{Z}$, of T by T^i . We sometimes denote an element $\alpha \in T(X \xrightarrow{f} Y)$ by

$$X \xrightarrow[\alpha]{} Y .$$

The functor T support three types of operations:

- *Product.* For every $f: X \rightarrow Y$ and adequate $g: Y \rightarrow Z$, there is a product

$$T^i(X \xrightarrow{f} Y) \otimes T^j(Y \xrightarrow{g} Z) \xrightarrow{\quad} T^{i+j}(X \xrightarrow{g \circ f} Z).$$

- *Pushforward.* Given a confined triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow u & \swarrow v \\ & & S \end{array}$$

there is a pushforward homomorphism

$$f_* : T^i(X \xrightarrow{u} S) \longrightarrow T^i(Y \xrightarrow{v} S).$$

- *Pullback.* For every independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

there is a pullback homomorphism

$$g^* : T^i(X \xrightarrow{f} Y) \longrightarrow T^i(X' \xrightarrow{f'} Y').$$

(Observe the abuse of notation.)

These operations should satisfy the following compatibility axioms:

A1. *Product is associative.* Given a diagram

$$X \xrightarrow[\textcircled{\alpha}]f Y \xrightarrow[\textcircled{\beta}]g Z \xrightarrow[\textcircled{\gamma}]h W$$

where g , h and $h \circ g$ are adequate, we have

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

in $T(h \circ g \circ f)$.

A2. *Pushforward is functorial.* If the triangles in

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & \searrow u & \downarrow v & \swarrow w & \\ & & S & & \end{array}$$

are confined, then

$$(g \circ f)_* = g_* \circ f_* : T^i(X \xrightarrow{u} S) \longrightarrow T^i(Z \xrightarrow{w} S).$$

A3. *Pullback is functorial.* If the squares in

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ f'' \downarrow & & f' \downarrow & & f \downarrow \\ Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y \end{array}$$

are independent, then

$$(g \circ h)^* = h^* \circ g^* : T^i(X \xrightarrow{f} Y) \longrightarrow T^i(X'' \xrightarrow{f''} Y'').$$

A12. *Product and pushforward commute.* Given

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

$\textcircled{\alpha}$

with f confined relative to W and h adequate, we have

$$f_*(\alpha \cdot \beta) = f_*(\alpha) \cdot \beta$$

in $T(h \circ g)$.

A13. *Product and pullback commute.* Given

$$\begin{array}{ccc}
 X' & \xrightarrow{h''} & X \\
 f' \downarrow & & f \downarrow \textcircled{\alpha} \\
 Y' & \xrightarrow{h'} & Y \\
 g' \downarrow & & g \downarrow \textcircled{\beta} \\
 Z' & \xrightarrow{h} & Z
 \end{array}$$

with independent squares, g and g' adequate, we have

$$h^*(\alpha \cdot \beta) = h'^*(\alpha) \cdot h^*(\beta)$$

in $T(g' \circ f')$.

A23. *Pushforward and pullback commute.* Given

$$\begin{array}{ccc}
 X' & \xrightarrow{h''} & X \\
 f' \downarrow & & f \downarrow \\
 Y' & \xrightarrow{h'} & Y \\
 g' \downarrow & & g \downarrow \\
 Z' & \xrightarrow{h} & Z
 \end{array} \textcircled{\alpha}$$

with independent squares and f confined relative to Z , we have

$$f'_*(h^*\alpha) = h_* f_*(\alpha)$$

in $T(g')$.

A123. *Projection formula.* Given

$$\begin{array}{ccccc}
 X' & \xrightarrow{g'} & X & & \\
 f' \downarrow & & f \downarrow \textcircled{\alpha} & & \\
 Y' & \xrightarrow{g} & Y & \xrightarrow{h} & Z \\
 & \searrow & \swarrow & \searrow & \\
 & & & \textcircled{\beta} &
 \end{array}$$

with independent square, g adequate and confined relative to Z and $h \circ g$ adequate, we have

$$\alpha \cdot g_*(\beta) = g'_*(g^*\alpha \cdot \beta)$$

in $T(h \circ f)$.

We say a bivariant theory T has **unital** if for every $X \in \mathbf{C}$ there is an element $1_X \in T^0(X \xrightarrow{id} X)$ with the following properties:

- For every $f: W \rightarrow X$ and every $\alpha \in T(W \xrightarrow{f} X)$, we have $\alpha \cdot 1_X = \alpha$.
- For every $g: X \rightarrow Y$ and every $\beta \in T(X \xrightarrow{g} Y)$, we have $1_X \cdot \beta = \beta$.
- For every $g: X' \rightarrow X$, we have $g^*(1_X) = 1_{X'}$.

A bivariant theory T is called **skew-commutative** (respectively, **commutative**), if for any square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \textcircled{a} \\ Y' & \xrightarrow[g]{\textcircled{b}} & Y \end{array}$$

that is independent or its transpose is independent, g and f are adequate, we have

$$g^*(\alpha) \cdot \beta = (-1)^{\deg(\alpha) \deg(\beta)} f^*(\beta) \cdot \alpha$$

(respectively, $g^*(\alpha) \cdot \beta = f^*(\beta) \cdot \alpha$).

Note that we don't assume the class of adequate morphisms to be closed; that is, if f, g are adequate, $f \circ g$ might not be adequate. However, in practice, it is convenient to specify a (large) closed subclass of adequate maps, called the **strongly adequate** morphisms. In particular, the product of bivariant classes are always defined and associative on the subclass of strongly adequate morphisms.

Using the definitions of Section 7 and results of Sections 4, 6, 6.1, it is straightforward to prove

Theorem A.2 *The bivariant theory of Section 7 is a generalized Fulton-MacPherson bivariant theory.*

Note that, in view of Lemma 6.4 and Example 7.5.1, we can choose the class of strongly adequate morphisms to be the class of strongly proper maps.

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